February 2010 Problems

Please send your solutions or questions to Janet Vassilev (jvassil@math.unm.edu) or Dimiter Vassilev (vassilev@math.unm.edu). We are looking forward to hearing from you.

Following are the 2008-2009 UNM-PNM 2nd round problems.
UNM - PNM STATEWIDE MATHEMATICS CONTEST XLII
February 6, 2010 Second Round

1. During a school year twenty nine boys each went to a bookstore on six different days. Every two boys were at the shop on the same day for at least one of their visits. Show that six of the boys must have been in the bookstore on the same day.

2. A math teacher has a bag containing $n > 3$ identically sized and shaped weights weighing 1 kilogram to n kilograms. You are to weigh them one at a time on a digital scale. If the successive weighings of any two weights differed by one kilogram, the teacher will pay you a dollar.
   (a) What is the minimum and maximum payment you may receive from the teacher?
   (b) Can you achieve all possible payments between this minimum and maximum in doing this experiment?

3. Sitting around a table are a group of teens who either always lie or always tell the truth. Because of previous conversations, each teen at the table knows which of the teens at the table are truth tellers or liars. A visiting math teacher walks over to the table not knowing anything about the teens at the table except that each always tells the truth or always lies. The teacher asks each teen if his or her right neighbor at the table is a liar and all but one teen says yes.
   (a) Can we say something about the number of teens seated around the table?
   (b) Can the teacher tell which teens tell the truth and which ones lie?
   (c) Explain why or why not and if not think of a question that the teacher can ask to determine the truthfulness of each student.

4. A regular octagon has side one. All points whose distance from at least one vertex is less than one are deleted. Find the remaining area.

5. A triangle $\triangle ABC$ is made of a uniform plastic (so any two pieces of equal area weigh the same). Let M be the intersection of all segments connecting a vertex with the midpoint of the opposite side (you can assume that these three segments intersect at one point). M is also called the center of mass of the triangle $\triangle ABC$ since you can balance it on the head of pin placed at M. Show that the triangles $\triangle ABM$, $\triangle BCM$ and $\triangle CAM$ have the same areas.

6. The sequence $1, 82, 163, 244, \ldots$ is an arithmetic progression, which means that for suitable numbers a and k the terms of the sequence are a, $a+k$, $a+2k$, $a+3k$, $a+4k, \ldots$.
(a) Find a and k for the given sequence.
(b) Find the first power of 10 appearing in the progression and which term it is.
(c) Show that this arithmetic progression contains infinitely many powers of 10.

7. (a) Show that for any two non-negative numbers a and b we have $\frac{a^2+b^2}{2} \geq \left(\frac{a+b}{2}\right)^2$.
(b) Suppose you are given a triangle made of hard plastic with sum of squares of its sides equal to S. You have to pass this triangle through a pipe. Let d be the diameter of the pipe. What is the smallest value of d that will work with absolute certainty?

8. The positive integers are grouped in sums as follows:

$$1+2,\ 3+4+5+6,\ 7+\cdots+12,\ 13+\cdots+20,\ 21+\cdots+30,\ 31+\cdots+42, \ldots.$$

(a) Find the value of the 9th sum.
(b) Find the value of the nth sum.

9. Let f be a real-valued function on the plane such that for every equilateral triangle ABC in the plane, the average value of f at the vertices is the same constant k (The average value of f at points A, B, C is $\dfrac{f(A)+f(B)+f(C)}{3}$).

(a) Show that f is constant. (For all P in the plane, $f(P)$ takes on the same value.)

(b) Find this constant.

10. Among all polygons inscribed in a given circle of radius R, which polygon has the largest sum of squares of its sides? Explain your answer.