March 2010 Problems

Please send your solutions or questions to Janet Vassilev (jvassil@math.unm.edu) or Dimiter Vassilev (vassilev@math.unm.edu). We are looking forward to hearing from you.

Let $A_1, A_2, \ldots, A_n$ be n points in the plane to each of which we attach a number $m_i > 0$, called the weight of the corresponding point, with total weight $m = m_1 + m_2 + \cdots + m_n$. Let A_0 be the center of mass of the given points, which is the point with coordinates (x_0, y_0) where

$$x_0 = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m}, \qquad y_0 = \frac{m_1y_1 + m_2y_2 + \cdots + m_ny_n}{m},$$

where (x_i, y_i) are the coordinates of the point A_i. For example, if all the weights are equal to one the total weight is n= the number of points, and the coordinates of the center of mass are the averages of corresponding coordinates of the given n points. *You should try $n = 2, 3, 4$ in any of the following problems if you have difficulties working with n points. Also, you can assume the claims of problems 1. and 4. and try the rest. On the other hand, if the questions are too easy, then replace "plane" with 3-D space (or any dimension) everywhere below.* In the following we will use $|AB|$ to denote the distance between two points A and B.

1. Let B be any point in the plane. Show that
$$\frac{m_1|BA_1|^2 + \cdots + m_n|BA_n|^2}{m} = |A_0B|^2 + \frac{m_1|A_0A_1|^2 + \cdots + m_n|A_0A_n|^2}{m}.$$
In particular, if all weights are equal to one, $m_1 = \cdots = m_n = 1$, then we obtain a formula relating the averages of the sum of the square of the distances of B to the given points, and the same quantity computed when B is the center of mass.
2. Given three points in the plane find the location of the point in the plane which minimizes the sum of squares of the distances of this point to the given three points. In other words, given the points A_1, A_2 and A_3 find the location of the point C such that
$$|CA_1|^2 + |CA_2|^2 + |CA_3|^2$$
is as small as possible. Of course, this is equivalent to making the average $\frac{|CA_1|^2+|CA_2|^2+|CA_3|^2}{3}$ as small as possible.
3. Can you solve the above problem using distances instead of their squares, i.e., minimize $\frac{|CA_1|+|CA_2|+|CA_3|}{3}$?
4. Let B be any point in the plane. Show that
$$\frac{m_1|BA_1|^2 + \cdots + m_n|BA_n|^2}{m} = |A_0B|^2 + \sum \frac{m_im_j}{m^2}|A_iA_j|^2,$$
where the sum in the right-hand side contains all possible (unordered) pairs (so there are $n(n-1)/2$ of them). In particular, if we have three points, all weights are equal to one, and we take $B = A_0$-the centroid of the triangle $A_1A_2A_3$, then we obtain
$$\frac{|A_0A_1|^2 + |A_0A_2|^2 + |A_0A_3|^2}{3} = \frac{|A_1A_2|^2 + |A_1A_3|^2 + |A_2A_3|^2}{9}.$$
5. Find the shortest distance between two parallel planes between which we can fit any triangular pyramid with a fixed sum of squares of its edges equal to S.
6. Given n points in the plane, what is the location of all points for which the sum of the squares of its distances from the given points is a given constant, say equal to one?