Heterotic String Solutions with non-constant dilaton.

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Main Result

The seven dimensional quaternionic Heisenberg group $\boldsymbol{G}\left(\mathbb{H}\right)$ has applications in the construction of non-trivial solutions to the so called *Strominger system* in supersymmetric heterotic string theory.

Heterotic string backgrounds

The bosonic geometry is of the form $\mathbb{R}^{1,9-d} \times M^d$, where the bosonic fields are non-trivial only on M^d, $d \leq 8$.
The bosonic fields of the ten-dimensional supergravity which arises as low energy effective theory of the heterotic string are

- the spacetime metric g with ∇^g the Levi-Civita connection;
- the NS (three-form) field strength (flux) H;
- the dilaton (function) ϕ;
- the gauge connection A with curvature 2-form F^A on a vector bundle E over M (instanton bundle).

One considers the two *metric* connections $\nabla^{\pm} = \nabla^g \pm \frac{1}{2}H$.
Both connections have totally skew-symmetric torsion $\pm H$. We denote by R^g, $R^{\pm}$ the corresponding curvature.

The Strominger system

A heterotic geometry preserves supersymmetry iff in 10-Ds there exists at least one Majorana-Weyl spinor ϵ such that the Killing-spinor equations hold [Strominer '86, Bergshoeff-Roo '89]

- gravitino, $\nabla^+\epsilon = 0$;
- dilatino, $(d\phi - \frac{1}{2}H)\cdot\epsilon = 0$;
- gaugino (instanton), $F^A\cdot\epsilon = 0$,

where $\cdot$ means Clifford action of forms on spinors.

- The Green-Schwarz anomaly cancellation condition up to the first order of α'

$$dH = \frac{\alpha'}{4}8\pi^2(p_1(\nabla^-) - p_1(E)) = \frac{\alpha'}{4}\Big(Tr(R^-\wedge R^-) - Tr(F^A\wedge F^A)\Big),$$

where $p_1(\nabla^-)$ and $p_1(E)$ are the first Pontrjagin forms with respect to the connections ∇^- and A.

Types of G_2 structures and the gravitino & dilatino eq's

(Θ, g) a G_2-structure with Lee form $\theta = -\frac{1}{3} * (*d\Theta \wedge \Theta)$.

1. balanced, $\theta = 0$;
2. conformally balanced, $d\theta = 0$;
3. integrable, $d * \Theta = \theta \wedge *\Theta$;
4. co-calibrated if balanced and integrable;
5. pure type if integrable and $d\Theta \wedge \Theta = 0$.

- gravitino eqn, $\nabla^+ \epsilon = 0 \Leftrightarrow$ there is a non-trivial parallel spinor w.r.t. a G_2-connection w/ torsion 3-form $T \Leftrightarrow$ there is an integrable G_2 structure;
  NOTE: There is a unique connection ∇^+ whose torsion a 3-form s.t. $\nabla^+ \Theta = 0$ i.e. $Hol(\nabla^+) \subset G_2$.
- gravitino and dilationo equations $\Leftrightarrow$ there is a G_2-structure which is locally conformally balanced, $\theta = 2d\phi$, and of pure type.

Theorem (Gauntlett-Kim-Martelli-Waldram, Friedrich-Ivanov)

There exists a non-trivial solution to both dilatino and gravitino Killing spinor equations in dimension d=7 iff there exists a globally conformally co-calibrated G_2-structure (Θ, g) of pure type and the Lee form $\theta = -\frac{1}{3} * (*d\Theta \wedge \Theta) = \frac{1}{3} * (*d * \Theta \wedge *\Theta)$ is exact, i.e., a G_2-structure (Θ, g) satisfying the equations

$$d * \Theta = \theta \wedge *\Theta, \quad d\Theta \wedge \Theta = 0, \quad \theta = -2d\phi.$$

The torsion 3-form (the flux H) is given by

$$H = T = - * d\Theta - 2 * (d\phi \wedge \Theta).$$

NOTE: $\bar{\Theta} = e^{-3/2\phi}\Theta$, $\bar{g} = e^{-\phi} g$ is co-calibrated and of pure type.

7-D case - the gaugino (instanton) equation,

The gaugino (instanton) equation means $\Leftrightarrow$ F^A is contained in a Lie algebra of a Lie group which is a stabilizer of a non-trivial spinor. In 7D's the largest such a group is G_2. Denoting by Θ the non-degenerate three-form defining the G_2 structure,

- the G_2-instanton condition has the form

$$\sum_{k,l=1}^{7} (F^A)^i_j(e_k, e_l)\Theta(e_k, e_l, e_m) = 0.$$

The geometric model [Fernandez-Ivanov-Ugarte-Villacampa (2011)]

A geometric model which fits the above structures is given by a certain $\mathbb{T}^3$-bundle over a Calabi-Yau surface Z. Let ω_1 be the (closed) Kähler form and $\omega_2 + \sqrt{-1}\omega_3$ the holomorphic volume form on Z.
Suppose Γ_i, $i = 1, 2, 3$, are closed ASD 2-forms on Z, which represent integral cohomology classes.
Then, there is a compact 7-dimensional manifold M which is the total space of a $\mathbb{T}^3$-bundle over Z and has a G_2-structure $\Theta = \omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2 - \omega_3 \wedge \eta_3 + \eta_1 \wedge \eta_2 \wedge \eta_3$,solving the first two Killing spinor equations with *constant dilaton* in dimension 7, where η_i, $1 \leq i \leq 3$, is a 1-form on M such that $d\eta_i = \Gamma_i$, $i = 1, 2, 3$.

For any smooth function f on Z, the G_2-structure on M given by

$$\Theta_f = e^{2f}\Big[\omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2 - \omega_3 \wedge \eta_3\Big] + \eta_1 \wedge \eta_2 \wedge \eta_3$$

solves the first two Killing spinor equations with a non-constant dilaton $\phi = -2f$. The metric has the form

$$g_f = e^{2f} g_{CY} + \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3.$$

To achieve a smooth solution to the Strominger system *we still have to determine* an instanton bundle with a G_2-instanton in order to satisfy the anomaly cancellation condition.

The quaternionic Heisenberg group

The Lie algebra $\mathfrak{g}(\mathbb{H})$ of the seven dimensional group $\boldsymbol{G}\,(\mathbb{H})$ is

$$d\gamma^i = 0,\ i = 1, \ldots, 4,$$
$$d\gamma^5 = \gamma^{12} - \gamma^{34}, \quad d\gamma^6 = \gamma^{13} + \gamma^{24}, \quad d\gamma^7 = \gamma^{14} - \gamma^{23}.$$

where $\gamma_1, \ldots, \gamma_7$ is a basis of left invariant 1-forms on $\boldsymbol{G}\,(\mathbb{H})$. In particular, the quaternionic Heisenebrg group $\boldsymbol{G}\,(\mathbb{H})$ in dimension seven is an R^3-bundle over the flat Calabi-Yau space R^4 and therefore fits the geometric model described earlier.

Non-constant dilaton, [FIUV '14)]

In order to obtain results in dimensions less than seven through contractions of $\boldsymbol{G}\,(\mathbb{H})$ it will be convenient to consider the orbit of $G(\mathbb{H})$ under the natural action of $GL(3,\mathbb{R})$ on the span of $\{\gamma^5,\gamma^6,\gamma^7\}$.

Thus, for $A \in GL(3,\mathbb{R})$, let K_A be a 7-D real Lie group with Lie bracket $[x,x']_A = A[A^{-1}x, A^{-1}x']$ defined by a basis of left-invariant 1-forms $\{e^1,\ldots,e^7\}$ s.t. $e^i = \gamma^i$ for $1 \le i \le 4$ and $(e^5\ e^6\ e^7)^t = A\,(\gamma^5\ \gamma^6\ \gamma^7)^t$. The structure equations of the Lie algebra $\mathfrak{K}_A$ of the group K_A are $de^1 = de^2 = de^3 = de^4 = 0$, $de^{4+i} = \sum_{j=1}^3 a_{ij}\,\sigma_j$, $i = 1,2,3$, where

$$\sigma_1 = e^{12} - e^{34},\ \sigma_2 = e^{13} + e^{24},\ \sigma_3 = e^{14} - e^{23}$$

are the three ASD 2-forms on $\mathbb{R}^4$ and $A = \{a_{ij}\}$ is a 3 by 3 matrix.

Since $\mathfrak{K}_A$ is isomorphic to $\mathfrak{g}(\mathbb{H})$, if K_A is connected and simply connected it is isomorphic to $G(\mathbb{H})$. Any lattice Γ_A gives rise to a (compact) nilmanifold $M_A = K_A/\Gamma_A$, which is a $\mathbb{T}^3$-bundle over a $\mathbb{T}^4$ with connection 1-forms of ASD curvature on the four torus.

Beginning of the geometric setup

Consider the G_2 structure on the Lie group K_A defined by the 3-form

$$\Theta = \omega_1 \wedge e^7 + \omega_2 \wedge e^5 - \omega_3 \wedge e^6 + e^{567},$$

where

$$\omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} - e^{24}, \quad \omega_3 = e^{14} + e^{23}$$

are the three closed SD 2-forms on $\mathbb{R}^4$. The corresponding Hodge dual 4-form $*\Theta$ is given by

$$*\Theta = \omega_1 \wedge e^{56} + \omega_2 \wedge e^{67} + \omega_3 \wedge e^{57} + \frac{1}{2}\omega_1 \wedge \omega_1.$$

It is easy to check using $\sigma_i \wedge \omega_j = 0$ for $1 \leq i, j \leq 3$ that Θ is co-calibrated of pure type,

$$d * \Theta = 0, \quad d\Theta \wedge \Theta = 0,$$

hence this G_2 structure solves the gravitino and dilatino equations with constant dilaton.

For a smooth function f on $\mathbb{R}^4$, we consider the G_2 structure on K_A defined by the 3-form

$$\bar{\Theta} = e^{2f}\Big[\omega_1 \wedge e^7 + \omega_2 \wedge e^5 - \omega_3 \wedge e^6\Big] + e^{567},$$

The corresponding metric $\bar{g}$ on K_A has an orthonormal basis of 1-forms given by
$\bar{e}^i = e^f\, e^i, i = 1, 2, 3, 4, \qquad \bar{e}^5 = e^5, \quad \bar{e}^6 = e^6, \quad \bar{e}^7 = e^7$ with SD 2-forms $\bar{\omega}_k = e^{2f}\omega_k$ and ASD 2-forms $\bar{\sigma}_k = e^{2f}\sigma_k$, k=1,2,3.
From the general anszatz the G_2 structure $\bar{\Theta}$ solves the gravitino and dilatino equations with non-constant dilaton $\phi = -2f$.

Furthermore, with $f_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}$, $1 \le i, j \le 4$, we obtain the next formula for the torsion $\bar{T}$ of $\bar{\Theta}$, $T = - * d\Theta - 2 * (d\phi \wedge \Theta)$

$$\bar{T} = \bar{*}(2df \wedge e^{567} - de^{567})$$
$$= e^{-f}\Big[- 2f_1\, \bar{e}^{234} + 2f_2\, \bar{e}^{134} - 2f_3\, \bar{e}^{124} + 2f_4\, \bar{e}^{123}\Big]$$
$$+e^{-2f}\Big[(a_{11}\,\bar{\sigma_1}+a_{12}\,\bar{\sigma_2}+a_{13}\,\bar{\sigma_3})\wedge\bar{e}^5+(a_{21}\,\bar{\sigma_1}+a_{22}\,\bar{\sigma_2}+a_{23}\,\bar{\sigma_3})\wedge\bar{e}^6$$
$$+ (a_{31}\,\bar{\sigma_1} + a_{32}\,\bar{\sigma_2} + a_{33}\,\bar{\sigma_3}) \wedge \bar{e}^7\Big],$$

where $f_i = \frac{\partial f}{\partial x_i}$, $1 \le i \le 4$,. Hence,

$$d\bar{T} = -e^{-4f}\left[\triangle e^{2f} + 2|A|^2\right] \bar{e}^{1234} = -\left[\triangle e^{2f} + 2|A|^2\right] e^{1234},$$

where $\triangle e^{2f} = (e^{2f})_{11} + (e^{2f})_{22} + (e^{2f})_{33} + (e^{2f})_{44}$ is the Laplacian on $\mathbb{R}^4$.

The first Pontrjagin form of the $(-)$-connection

The connection 1-forms of a connection ∇ are determined by $\nabla_X e_j = \sum_{s=1}^{7} \omega_j^s(X) e_s$. From Koszul's formula, we have that the Levi-Civita connection 1-forms $(\omega^{\bar{g}})_{\bar{j}}^{\bar{\imath}}$ of the metric $\bar{g}$ are given by

$$\begin{aligned}(\omega^{\bar{g}})_{\bar{j}}^{\bar{\imath}}(\bar{e}_k) &= -\tfrac{1}{2}\Big(\bar{g}(\bar{e}_i, [\bar{e}_j, \bar{e}_k]) - \bar{g}(\bar{e}_k, [\bar{e}_i, \bar{e}_j]) + \bar{g}(\bar{e}_j, [\bar{e}_k, \bar{e}_i])\Big) \\ &= \tfrac{1}{2}\Big(d\bar{e}^i(\bar{e}_j, \bar{e}_k) - d\bar{e}^k(\bar{e}_i, \bar{e}_j) + d\bar{e}^j(\bar{e}_k, \bar{e}_i)\Big)\end{aligned}$$

taking into account $\bar{g}(\bar{e}_i, [\bar{e}_j, \bar{e}_k]) = -d\bar{e}^i(\bar{e}_j, \bar{e}_k)$.
Hence, the connection 1-forms $(\omega^-)_{\bar{j}}^{\bar{\imath}}$ of the connection ∇^- are

$$(\omega^-)_{\bar{j}}^{\bar{\imath}} = (\omega^{\bar{g}})_{\bar{j}}^{\bar{\imath}} - \frac{1}{2}(\bar{T})_{\bar{j}}^{\bar{\imath}}, \qquad \text{where} \qquad (\bar{T})_{\bar{j}}^{\bar{\imath}}(\bar{e}_k) = \bar{T}(\bar{e}_i, \bar{e}_j, \bar{e}_k).$$

The Pontrjagin form of ∇^-

A long calculation shows that the first Pontrjagin form of ∇^- is a scalar multiple of e^{1234},

$$\pi^2 p_1(\nabla^-) = \left[\mathcal{F}_2[f] + \triangle_4 f - \frac{3}{8}|A|^2 \triangle e^{-2f}\right] e^{1234},$$

where $\mathcal{F}_2[f]$ is the 2-Hessian of f, i.e., the sum of all principle 2×2-minors of the Hessian, and $\triangle_4 f = div(|\nabla f|^2 \nabla f)$ is the 4-Laplacian of f on $\mathbb{R}^4$.

This formula shows, in particular, that even though the curvature 2-forms of ∇^- are quadratic in the gradient of the dilaton, the Pontrjagin form of ∇^- is also quadratic in these terms. Furthermore, if f depends on two of the variables then $\mathcal{F}_2[f] = det(Hessf)$ while if f is a function of one variable $\mathcal{F}_2[f]$ vanishes.

The anomaly cancellation condition

What remains is to solve the anomaly cancellation condition. We use the G_2-instanton D_Λ which depends on a 3 by 3 matrix $\Lambda = (\lambda_{ij}) \in \mathfrak{gl}_3(\mathbb{R})$, and whose possibly non-zero 1-forms are given as follows

$$(\omega^{\mathrm{D}_\Lambda})^{\bar{1}}_{\bar{2}} = -(\omega^{\mathrm{D}_\Lambda})^{\bar{2}}_{\bar{1}} = -(\omega^{\mathrm{D}_\Lambda})^{\bar{3}}_{\bar{4}} = (\omega^{\mathrm{D}_\Lambda})^{\bar{4}}_{\bar{3}} = \lambda_{11}\,\bar{e}^5 + \lambda_{12}\,\bar{e}^6 + \lambda_{13}\,\bar{e}^7,$$

$$(\omega^{\mathrm{D}_\Lambda})^{\bar{1}}_{\bar{3}} = -(\omega^{\mathrm{D}_\Lambda})^{\bar{3}}_{\bar{1}} = (\omega^{\mathrm{D}_\Lambda})^{\bar{2}}_{\bar{4}} = -(\omega^{\mathrm{D}_\Lambda})^{\bar{4}}_{\bar{2}} = \lambda_{21}\,\bar{e}^5 + \lambda_{22}\,\bar{e}^6 + \lambda_{23}\,\bar{e}^7,$$

$$(\omega^{\mathrm{D}_\Lambda})^{\bar{1}}_{\bar{4}} = -(\omega^{\mathrm{D}_\Lambda})^{\bar{4}}_{\bar{1}} = -(\omega^{\mathrm{D}_\Lambda})^{\bar{2}}_{\bar{3}} = (\omega^{\mathrm{D}_\Lambda})^{\bar{3}}_{\bar{2}} = \lambda_{31}\,\bar{e}^5 + \lambda_{32}\,\bar{e}^6 + \lambda_{33}\,\bar{e}^7.$$

Lemma

The connection D_Λ *is a* G_2*-instanton with respect to the* G_2 *structure* $\bar{\Theta}$ *which preserves the metric iff* $\mathrm{rank}(\Lambda) \leq 1$*. In this case, the first Pontrjagin form* $p_1(\mathrm{D}_\Lambda)$ *of the* G_2*-instanton* D_Λ *is given by*

$$8\pi^2 p_1(\mathrm{D}_\Lambda) = -4\lambda^2\, e^{1234},$$

where $\lambda = |\Lambda A|$ *is the norm of the product matrix* ΛA*.*

After this preparation, we are left with solving the anomaly cancellation condition $d\bar{T} = \frac{\alpha'}{4} 8\pi^2 \Big(p_1(\nabla^-) - p_1(D_\Lambda) \Big)$, which in general is a highly overdetermined system for the dilaton function f.

In our case the anomaly becomes *the single* non-linear equation

$$\triangle e^{2f} + 2|A|^2 + \frac{\alpha'}{4} \Big[8\mathcal{F}_2[f] + 8\triangle_4 f - 3|A|^2 \triangle e^{-2f} + 4\lambda^2 \Big] = 0.$$

We remind that this is an equation on $\mathbb{R}^4$ for the dilaton function f.

An important question interesting for both string theory and nonlinear analysis is whether the above non-linear PDE admits a periodic solution.

A solution

If we assume that the function f depends on one variable, $f = f(x^1)$, and for a *negative* α' we choose $2|A|^2 + \alpha'\lambda^2 = 0$, i.e., we let $\alpha' = -\alpha^2$ so that $2|A|^2 = \alpha^2\lambda^2$. This simplifies the anomaly cancellation equation to the ordinary differential equation

$$\left(e^{2f}\right)' + \frac{3}{4}\alpha^2|A|^2\left(e^{-2f}\right)' - 2\alpha^2 f'^3 = C_0 = const.$$

The substitution $u = \alpha^{-2}e^{2f}$ turns it into

$$C_0 = \left(e^{2f}\right)' + \frac{3}{4}\alpha^2|A|^2\left(e^{-2f}\right)' - 2\alpha^2 f'^3 = \frac{\alpha^2 u'}{4u^3}\left(4u^3 - 3\frac{|A|^2}{\alpha^2}u - u'^2\right).$$

For $C_0 = 0$ we solve the following ordinary differential equation for the function $u = u(x^1) > 0$

$$u'^2 = 4u^3 - 3\frac{|A|^2}{\alpha^2}u = 4u\,(u-d)\,(u+d)\,,\ d = \sqrt{3|A|^2}/\alpha. \quad (*)$$

Replacing the real derivative with the complex derivative leads to the Weierstrass' equation $\left(\frac{d\,\mathcal{P}}{dz}\right)^2 = 4\mathcal{P}\left(\mathcal{P} - d\right)\left(\mathcal{P} + d\right)$for the Weierstrass $\mathcal{P}$ function with a pole at the origin where it has the expansion

$$\mathcal{P}(z) = \frac{1}{z^2} + \frac{d^2}{5}z^2 + \frac{d^4}{300}z^6 + \cdots ,$$

(no z^4 term and only even powers).
In addition if $\tau_\pm$ are the basic half-periods such that τ_+ is real and τ_- is purely imaginary we have that $\mathcal{P}$ is real valued on the lines $\mathfrak{Re}\, z = m\tau_+$ or $\mathfrak{Im}\, z = m\tau_-$, $m \in \mathbb{Z}$.
Furthermore, in the fundamental region centered at the origin, where $\mathcal{P}$ has a pole of order two, we have that $\mathcal{P}(z)$ decreases from $+\infty$ to d to 0 to $-d$ to $-\infty$ as z varies along the sides of the half-period rectangle from 0 to τ_+ to $\tau_+ + \tau_-$ to τ_- to 0.

Thus, $u(x^1) = \mathcal{P}(x^1)$ defines a non-negative $2\tau_+$-periodic function with singularities at the points $2n\tau_+$, $n \in \mathbb{Z}$, which solves the real equation (*). From the Laurent expansion of the Weierstrass' function it follows

$$u(x_1) = \frac{1}{(x^1)^2}\left(1 + \frac{d^2}{5}(x^1)^4 + \cdots\right).$$

By construction, $f = \frac{1}{2}\ln(\alpha^2 u)$ is a periodic function with singularities on the real line which is a solution to the anomaly cancellation equation
The G_2 structure defined by $\bar{\Theta}$ descends to the 7-dimensional nilmanifold $M^7 = \Gamma\backslash K_A$ with singularity, determined by the singularity of u, where K_A is the 2-step nilpotent Lie group with Lie algebra $\mathfrak{K}_A$ and Γ is a lattice with the same period as f, i.e., $2\tau_+$ in all variables.

Theorem

In fact, M^7 is the total space of a $\mathbb{T}^3$ bundle over the asymptotically hyperbolic manifold M^4 which is a conformally compact 4-torus with conformal boundary at infinity a flat 3-torus. Thus, we obtain the complete solution to the Strominger system in dimension seven with non-constant dilaton, non-trivial instanton and flux and with a negative α' parameter.

Solutions through contractions

A contraction of the quaternionic heisenberg algebra can be obtained considering the matrix

$$A_\varepsilon \stackrel{def}{=} \begin{pmatrix} 0 & b & 0 \\ a & 0 & -b \\ 0 & 0 & \varepsilon \end{pmatrix}.$$

Letting $\varepsilon \to 0$ into A_ε we get in the limit, the structure equations of a six dimensional two step nilpotent Lie algebra $\mathfrak{h}_5$,

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = b\,\sigma_2, \quad de^6 = a\,\sigma_1 - b\,\sigma_3. \tag{1}$$

Notice that we have $\bar{e}^7 = e^7_\varepsilon = \varepsilon\gamma^7 \to 0$ as $\varepsilon \to 0$.

With the above choice of A_ε we write the G_2-form in the usual way as

$$\bar{\Theta}_\varepsilon = \bar{F} \wedge e^7_\varepsilon + \bar{\Psi}^+, \quad \bar{F} = e^{2f}\omega_1 + e^{56}, \quad \bar{\Psi}^+ = e^{2f}(\omega_2 \wedge e^5 - \omega_3 \wedge e^6)$$

indicating with subscript ε the dependence on ε through the matrix A_ε. In addition, we let $\bar{\Psi}^- = e^{2f}(\omega_2 \wedge e^6 + \omega^3 \wedge e^5)$.
In the limit $\varepsilon \to 0$, the forms $\bar{F}$, $\bar{\Psi}^\pm$ define an $SU(3)$ structure $(\bar{F}, \bar{\Psi}^\pm)$ on a six dimensional space, obtained through the ansatz proposed by Goldstein-Prokushkin (2004) as a $\mathbb{T}^2$ bundle over $\mathbb{T}^4$ (corresponding to $f = 0$).

- Therefore, this *SU*(3) structure solves the first two Killing spinor equations.
- Furthermore, the Pontrjagin form of the ∇^- connection is given by the ”same” formula. In fact, the connection forms and the corresponding curvature 2-forms (notice that $(\Omega^-_\varepsilon)^{\bar{i}}_{\bar{7}} \to 0$ for all i) converge to those of the ∇^- connection of the *SU*(3) case.
- Similarly, the seven dimensional anomaly cancellation condition turn into the anomaly cancellation conditions for the corresponding six dimensional structure.

6-D heterotic solution

As a consequence we obtain a six-dimensional solutions with non-constant dilaton.

Theorem

The conformally compact manifold $M^6 = (\Gamma \backslash H_5, \bar{g}, J, \nabla^-, A_\Lambda)$ *is a Hermitian manifold which solves the Strominger system with non-constant dilaton* f*, non-trivial flux* $H = \bar{T}$*, non-flat instanton* A_Λ *using the first Pontrjagin form of* ∇^- *and negative* α'*. Furthermore, the heterotic equations of motion are satisfied up to first order of* α'*.*

It is remarkable that the geometric structures, the partial differential equations and their solutions found in dimension seven starting with the quaternionic Heisenbeg group as above converge trough contraction to the heterotic solutions on 6-dimensional non-Kähler space.
Finally, using suiatable contractraction it is possible to obtain non-trivial solutions to the Strominger system in dimension 5 as well.

Strominger's system in 6-D

- The gravitino equation shows that there exists a parallel spinor with respect to the $(+)$-connection. This reduces the structure group $SO(2n)$ to a subgroup of $SU(n)$ since the holonomy group of ∇^+ reduces to a subgroup of $SU(n)$, i.e., the manifold is an almost Hermitian manifold admitting a linear connection having totally skew-symmetric torsion which preserves both the almost Hermitian structure and a non-vanishing $(n, 0)$-form (complex volume form).
- The dilatino equation shows that the almost complex structure is integrable and the trace of the torsion 3-form with respect to the Kähler form is an exact 1-form. NOTE: Strominger showed the existence of a unique Hermitian connection ∇^+ with a skew-symmetric torsion on any Hermitian manifold. He also shows that the ∇^+-parallel complex volume form supplies a holomorphic complex volume form whose norm determines the dilaton.

The instanton condition in 6-D

- The instanton condition: F^A is contained in a Lie algebra of a Lie group which is a stabilizer of a non-trivial spinor. In dimension, $F^A \subset \mathfrak{su}(3)$ - the Donaldson-Uhlenbeck-Yau instanton. The $SU(3)$-instanton means that the trace of F^A with respect to the Kähler 2 form as well as the (2,0)+(0,2)-part of F^A vanish simultaneously,

$$(F^A)^i_j(JE_k, JE_l) = (F^A)^i_j(E_k, E_l), \qquad \sum_{k=1}^{6}(F^A)^i_j(E_k, JE_k) = 0.$$

- The anomally cancellation condition

$$dH = \frac{\alpha'}{4} 8\pi^2 (p_1(M^d) - p_1(E)) = \frac{\alpha'}{4}\Big(Tr(R \wedge R) - Tr(F^A \wedge F^A)\Big), \tag{2}$$

where $p_1(M)$ and $p_1(E)$ are the first Pontrjagin forms of M with respect to a metric connection ∇ with curvature R and a vector bundle E with connection A.

$SU(3)$ structures and Strominger's system

Let (M, J, g) be a Hermitian 6-manifold with Kähler form $F(\cdot,\cdot) = g(\cdot, J\cdot)$. An SU(3)-structure is determined by a non-degenerate (3,0)-form $\Psi = \Psi^+ + \sqrt{-1}\,\Psi^-$ satisfying $F \wedge \Psi^\pm = 0$, $\Psi^+ \wedge \Psi^- = \frac{2}{3} F \wedge F \wedge F$.
The Lee form θ is $\theta(\cdot) = \delta F(J\cdot)$. The flux H, i.e., the torsion of the connection ∇^+ preserving the Hermitian structure (J, g) is

$$H = T = d^c F, \qquad d^c F(X, Y, Z) \stackrel{def}{=} -dF(JX, JY, JZ).$$

The dilatino and gravitino equations $\Leftrightarrow$ the 6-manifold is complex conformally balanced, $\theta = 2d\phi$, with a non-vanishing holomorphic volume form Ψ satisfying the Strominger condition $2F \lrcorner dF + \Psi^+ \lrcorner d\Psi^+ = 0$, [Cardoso-Curio-Dall'Agata-Lüst-Manousselis-Zoupanos (2003)].
If the dilaton is constant, $\theta = 0$, then the Strominger condition becomes $dF \wedge F = d\Psi^+ = d\Psi^- = 0$. Compact examples of the latter on nilmanifolds were given by Ugarte-Villacampma '09 & '11.

Goldstein-Prokushkin' anszatz

A geometric model is given by a certain $\mathbb{T}^2$-bundle over a Calabi-Yau surface. Let Γ_i, $1 \leq i \leq 2$, be two closed 2-forms on a CY surface Z with ASD (1,1)-part, which represent integral cohomology classes. Denote by ω_1 and by $\omega_2 + \sqrt{-1}\omega_3$ the (closed) Kähler form and the holomorphic volume form on Z. Then, there is a (non-Kähler) 6-dimensional manifold M^6, which is the total space of a $\mathbb{T}^2$-bundle over Z, equipped with an $SU(3)$-structure $g = g_{CY} + \eta_1^2 + \eta_2^2$, $F = \omega_1 + \eta_1 \wedge \eta_2$, $\Psi^+ = \omega_2 \wedge \eta_1 - \omega_3 \wedge \eta_2, \quad \Psi^- = \omega_2 \wedge \eta_2 + \omega_3 \wedge \eta_1$, where η_i, $1 \leq i \leq 2$, is a 1-form on M^6 such that $d\eta_i = \Gamma_i$, $1 \leq i \leq 2$. From the construction the $SU(3)$ structure solves the first two Killing spinor equations with constant dilaton.

For any smooth function f on M^4, the $SU(3)$-structure on M^6 given by $F = e^{2f}\omega_1 + \eta_1 \wedge \eta_2$, $\Psi^+ = e^{2f}\Big[\omega_2 \wedge \eta_1 - \omega_3 \wedge \eta_2\Big]$, $\Psi^- = e^{2f}\Big[\omega_2 \wedge \eta_2 + \omega_3 \wedge \eta_1\Big]$ solves the first two Killing spinor equations with non-constant dilaton $\phi = 2f$. The metric has the form $g_f = e^{2f} g_{CY} + \eta_1^2 + \eta_2^2$.

The instanton and anomally

The Goldstein-Prokushkin's ansatz guaranties solution to the first two Killing spinor eq's. To achieve a smooth solution to the Strominger system we still have to determine an auxiliary vector bundle with an instanton and a linear connection on M^6 in order to satisfy the anomaly cancellation condition .
Taking the first Pontrjagin form of the Chern connection leads to an equation of Monge-Ampère type for the dilaton function, while it is reduced to a PDE of Laplace type for the dilaton when using the first Pontrjagin form of the $(-)$-connection, [Becker-Sethi, Becker-Bertinato-Chung-Guo (2009)].

Solutions

1. The $\mathbb{T}^2$-bundle over a K3 surface construction with connection 1-forms of ASD curvature was used by Fu-Yau to produce the first compact smooth solutions in dimension 6 solving the heterotic supersymmetry equations with non-zero flux and *non-constant dilaton* together with the anomaly cancellation with the first Pontrjagin form of the Chern connection, [Becker-Becker-Fu-Tseng-Yau '06, Li-Yau '05, Fu-Yau '08].
2. Examples with *constant dilaton* using the first Pontrjagin form of $\nabla^{\pm}$were given by Fernandez-Ivanov-Ugarte-Villacampa, Ugarte-Villacampa '09.

A complete solution with positive α'

Lemma

The $(-)$-connection of the G_2 structure $\bar{\Theta}$ is a G_2 instanton with respect to $\bar{\Theta}$ iff the dilaton satisfies $\triangle e^{2f} + 2|A|^2 = 0$, i.e., the torsion 3-form is closed, $d\bar{T} = 0$.

Let D_B be the ∇^- connection obtained by replacing A with the matrix B, but allowing B to be singular, $B \in \mathfrak{gl}_3(\mathbb{R})$. Hence, the connection D_B is a G_2-instanton with respect to the considered G_2 structure iff $\triangle e^{2f} = -2|B|^2$.

Since $8\pi^2\Big(p_1(\nabla^-) - p_1(\mathrm{D}_B)\Big) = -3\Big(|A|^2 - |B|^2\Big)\left(\triangle e^{-2f}\right)\, e^{1234}$, the anomaly cancellation condition is

$$
\begin{aligned}
& d\bar{T} - \frac{\alpha'}{4} 8\pi^2\Big(p_1(\nabla^-) - p_1(\mathrm{D}_B)\Big) \\
& = -\left[\triangle e^{2f} + 2|A|^2 - \frac{3}{4}\alpha'\Big(|A|^2 - |B|^2\Big)\left(\triangle e^{-2f}\right)\right] e^{1234} = 0
\end{aligned}
$$

coupled with $\triangle e^{2f} = -2|B|^2$.
For $B = O$ and a fixed $a \in \mathbb{R}^4$ we take $e^{2f} = \frac{3\alpha'}{4|x-a|^2}, \quad x \in \mathbb{R}^4$.
Taking the singularity at the origin, in the coordinate $t = \sqrt{3\alpha'}/2 \ln\left(4|x|^2/3\alpha'\right) = -\sqrt{3\alpha'}\, f$, we have that the dilaton and the $4-D$ metric are

$$
f = -t\sqrt{3\alpha'}, \qquad \bar{g}_H = \sum_{i=1}^{4} e^{2f}(e^i)^2 = dt^2 + 3\alpha' ds_3^2,
$$

where ds_3^2 is the metric on the unit sphere in $\mathbb{R}^4$. Hence, the horizontal metric is complete, which implies that the metric $\bar{g} = \bar{g}_H + (e^5)^2 + (e^6)^2 + (e^7)^2$ is also complete.

Thus, we proved

Theorem

The non-compact complete simply connected manifold $(K_A, \bar{\Theta}, \nabla^-, \mathrm{D}_O, f)$ described above is a complete G_2 manifold which solves the Strominger system with non-constant dilaton f, non-zero flux $H = \bar{T}$ and non-flat instanton D_O using the first Pontrjagin form of ∇^- and positive α'. Furthermore, $(K_A, \bar{\Theta}, \nabla^-, \mathrm{D}_O, f)$ also solves the heterotic equations of motion up to the first order of α'.

On the other hand, in the case $|A|^2 = |B|^2 \neq 0$ the anomaly condition is trivially satisfied for any α', provided the torsion is closed, see Lemma 5. In this case the solution is given by the solutions of $\triangle e^{2f} + 2|A|^2 = 0$. Furthermore, both ∇^- and D_B are G_2-instantons. For example, a particular solution is obtained by taking

$$e^{2f} = \frac{|A|^2}{4}(1 - |x|^2)$$

defined in the unit ball.