AN OBATA TYPE RESULT FOR THE FIRST EIGENVALUE OF THE SUB-LAPLACIAN ON A CR MANIFOLD WITH A DIVERGENCE-FREE TORSION

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ABSTRACT. We prove a CR version of the Obata's result for the first eigenvalue of the sub-Laplacian in the setting of a compact strictly pseudoconvex pseudohermitian manifold which satisfies a Lichnerowicz type condition and has a divergence free pseudohermitian torsion. We show that if the first positive eigenvalue of the sub-Laplacian takes the smallest possible value then, up to a homothety of the pseudohermitian structure, the manifold is the standard Sasakian unit sphere. We also give a version of this theorem using the existence of a function with traceless horizontal Hessian on a complete, with respect to Webster's metric, pseudohermitian manifold.

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1. INTRODUCTION

The classical theorems of Lichnerowicz [29] and Obata [33] give correspondingly a lower bound for the first eigenvalue of the Laplacian on a compact manifold with a lower Ricci bound and characterize the case of equality. In [29] it was shown that for every compact Riemannian manifold (M, h) of dimension n for which the Ricci curvature is greater than or equal to that of the round unit n-dimensional sphere $S^n(1)$, i.e., $Ric(X, Y) \geq (n-1)h(X, Y)$, we have that the first positive eigenvalue λ_1 of the (positive) Laplace operator is greater than or equal to the first eigenvalue of the sphere, $\lambda_1 \geq n$.

Subsequently in [33] it was shown that the lower bound for the eigenvalue is achieved iff the Riemannian manifold is isometric to $S^n(1)$. Lichnerowicz proved his result using the classical Bochner-Weitzenböck formula. In turn, Obata showed that under these assumptions the trace-free part of the Riemannian Hessian of an eigenfunction f with eigenvalue $\lambda_1 = n$ vanishes,

$$(1.1) \qquad\qquad\qquad\qquad\qquad D^2 f = -fh,$$

after which he defined an isometry using analysis based on the geodesics and Hessian comparison of the distance function from a point. More precisely, Obata showed in [33] that if on a complete Riemannian manifold there exists a non-constant function satisfying (1.1) then the manifold is isometric to the unit sphere. Later Gallot [18] generalized these results to statements involving the higher eigenvalues and corresponding eigenfunctions of the Laplace operator.

From the sub-ellipticity of the sub-Laplacian defined in many well studied sub-Riemannian geometries it follows that its spectrum is discrete on a compact manifold. It is therefore natural to ask if there is a sub-Riemannian version of the above results. In fact, a CR analogue of the Lichnerowiecz theorem was found by Greenleaf [22] for dimensions $2n + 1 > 5$, while the corresponding results for $n = 2$ and $n = 1$ were achieved later in [32] and [16], respectively. As a continuation of this line of results in the setting of geometries modeled on the rank one symmetric spaces in [26] it was proven a quaternionic contact version of the Lichnerowicz result.

The CR Lichnerowicz type result states that on a compact $2n + 1$-dimensional strictly pseudoconvex pseudo-hermitian manifold satisfying a certain positivity condition the first eigenvalue of the sub-laplacian is grater than or equal to the first eigenvalue of the standard Sasakian sphere. For the exact statement of the CR Lichnerowicz type result we refer the reader to Theorem 8.8. We give a proof of Theorem 8.8 using the known techniques from [22], [32], [16], but use an argument based solely on the non-negativity of the CR-Paneitz operator thereby slightly simplifying the known proofs. Greenleaf [22] showed the result for $n \geq 3$, while S.-Y. Li and H.-S. Luk adapted Greenleaf's proof to cover the case $n = 2$. They also gave a version of the case $n = 1$ assuming further a condition on the covariant derivative of the pseudohermitian torsion. Part b) in Theorem 8.8 was established by H.-L. Chiu in [16]. We remark that if $n > 1$ the CR-Paneitz operator is always non-negative, cf. Lemma 8.4 while in the case $n = 1$ the vanishing of the pseudohermitian torsion implies that the CR-Paneitz operator is non-negative, see [16] and [11].

Other results in the CR case relevant for this paper have been proved in [10, 9, 8], [3] and [14] adding a corresponding inequality for $n = 1$, or characterizing the equality case in the vanishing pseudohermitian torsion case (the Sasakian case).

The problem of the existence of an Obata-type theorem in pseudohermitian manifold was considered in [8] where the following CR analogue of Obata's theorem was conjectured.

Conjecture 1.1 ([8]). *Let (M, θ) be a closed pseudohermitian (2n+1)-manifold with $n \geq 2$. In addition we assume the CR-Paneitz operator is nonnegative if $n = 1$. Suppose there is a positive constant k_0 such that the pseudohermitian Ricci curvature Ric and the pseudohermitian torsion A satisfy the inequality (1.2). If $\frac{n}{n+1}k_0$ is an eigenvalue of the sub-Laplacian then (M, θ) is the standard (Sasakian) CR structure on the unit sphere in $\mathbb{C}^{n+1}$.*

This conjecture was proved in the case of a vanishing pseudohermitian torsion (Sasakian case) in [8] for $n \geq 2$ and in [9] for $n = 1$. The non-Sasakian case was also considered in [15] where Conjecture 1.1 was established under the following assumptions on the pseudohermitian torsion (in complex coordinates):

$$\begin{aligned}
&\text{(i) for} \quad n \geq 2, \quad A_{\alpha\beta, \bar\beta} = 0, \quad \text{and} \quad A_{\alpha\beta, \gamma\bar\gamma} = 0, \quad [15, \text{Theorem 1.3}] ; \\
&\text{(ii) for} \quad n = 1, \quad A_{11, \bar 1} = 0, \quad \text{and} \quad P_1 f = 0 \quad [15, \text{Theorem 1.4}],
\end{aligned}$$

where
$$P_\alpha f = f_{\bar\beta}{}^{\bar\beta}{}_\alpha + in A_{\alpha\beta} f^\beta$$

is the operator characterizing CR-pluriharmonic functions when $n = 1$, see also the paragraph after Remark 8.3. The first condition, $A_{\alpha\beta,\,\bar\beta} = 0$, means that the (horizontal real) divergence of A with respect to the Tanaka-Webster connection ∇ vanishes,

$$(\nabla^* A)(X) = -(\nabla_{e_a} A)(e_a, X) = 0, \qquad X \in H = Ker\,\theta.$$

One purpose of this paper is to establish Conjecture 1.1 in the (non-Sasakian) case of a divergence-free pseudohermitian torsion where we prove the following result.

Theorem 1.2. *Let (M, θ) be a compact strictly pseudoconvex pseudohermitian CR manifold of dimension $2n + 1$. Suppose there is a positive constant k_0 such that the pseudohermitian Ricci curvature Ric and the pseudohermitian torsion A satisfy the inequality*

$$(1.2) \qquad Ric(X, X) + 4A(X, JX) \geq k_0 g(X, X), \qquad X \in H = Ker\,\theta.$$

Furthermore, suppose the horizontal divergence of the pseudohermitian torsion vanishes,

$$\nabla^* A = 0.$$

   a) *If $n \geq 2$ and $\lambda = \frac{n}{n+1} k_0$ is an eigenvalue of the sub-Laplacian, then up-to a scaling of θ by a positive constant (M, θ) is the standard (Sasakian) CR structure on the unit sphere in $\mathbb{C}^{n+1}$.*
   b) *If $n = 1$ and $\lambda = \frac{1}{2} k_0$ is an eigenvalue of the sub-Laplacian, the same conclusion can be reached assuming in addition that the CR-Paneitz operator is non-negative, i.e., for any smooth function f on M we have*

$$\int_M f \cdot (Cf)\,Vol_\theta \geq 0,$$

   where Cf is the CR-Paneitz operator given by (8.1).

The value of the scaling is determined, for example, by the fact that the standard psudohermitian structure on the unit sphere has first eigenvalue equal to $2n$. The corresponding eigenspace is spanned by the restrictions of all linear functions to the sphere.

Our approach is based on Lemma 3.1 where we find the explicit form of the Hessian with respect to the Tanaka-Webster connection of an extremal eigenfunction f, i.e., an eigenfunction with eigenvalue $n/(n+1)k_0$, and the formula for the pseudohermitian curvature. As mentioned earlier a proof of Greenleaf's result based on the non-negativity of the CR-Paneitz operator can be found in the Appendix. This proof shows formula (3.1) for the horizontal Hessian of f, which after a rescaling can be put in the form

$$\nabla^2 f(X, Y) = -fg(X, Y) - df(\xi)\omega(X, Y), \qquad X, Y \in H = Ker\,\theta.$$

We prove Theorem 1.2 as a consequence of Theorem 4.7 and Theorem 5.2 taking into account the already established CR Obata theorem for pseudohermitian manifold with a vanishing pseudohermitian torsion. Thus, the key new results are Theorem 4.7 and Theorem 5.2 which show correspondingly in the case $n \geq 2$ and $n = 1$ that if the pseudohermitian torsion is divergence-free and we have an eigenfunction f with a horizontal Hessian given by the above formula then the pseudohermitian torsion vanishes, i.e., we have a Sasakian structure.

The local nature of the analysis leading to the proof of Theorem 1.2 allows us to prove our second result, which is the following CR-version of Obata's Theorem [33].

Theorem 1.3. *Let (M, θ) be a strictly pseudoconvex pseudohermitian CR manifold of dimension $2n + 1 \geq 5$ with a divergence-free pseudohermitian torsion, $\nabla^* A = 0$. Assume, further, that M is complete with respect to the Riemannian metric*

$$(1.3) \qquad h = g + \theta^2.$$

If there is a smooth function $f \not\equiv 0$ whose Hessian with respect to the Tanaka-Webster connection satisfies

$$(1.4) \qquad \nabla^2 f(X, Y) = -fg(X, Y) - df(\xi)\omega(X, Y), \qquad X, Y \in H = Ker\,\theta,$$

then up to a scaling of θ by a positive constant (M, θ) is the standard (Sasakian) CR structure on the unit sphere in $\mathbb{C}^{n+1}$.

In dimension three the above result holds provided the pseudohermitian torsion vansihes, $A = 0$.

We finish the introduction by recalling that Lichnerowicz' type results (not in sharp forms) in a general sub-Riemannian setting were shown in [1], [2] (these two papers apply only to the vanishing pseudohermitian torsion CR case) and [24]. It will be interesting to consider whether the results of [1], [2], [13] and [14] can be extended to the non-Sasakian, but divergence free pseudohermitian torsion case.

Convention 1.4.

    a) We shall use X, Y, Z, U to denote horizontal vector fields, i.e. $X, Y, Z, U \in H = Ker\,\theta$.

    b) $\{e_1, \ldots, e_{2n}\}$ denotes a local orthonormal basis of the horizontal space H.

    c) The summation convention over repeated vectors from the basis $\{e_1, \ldots, e_{2n}\}$ will be used. For example, for a (0,4)-tensor P, the formula $k = P(e_b, e_a, e_a, e_b)$ means

$$k = \sum_{a,b=1}^{2n} P(e_b, e_a, e_a, e_b);$$

2. Pseudohermitian manifolds and the Tanaka-Webster connection

In this section we will briefly review the basic notions of the pseudohermitian geometry of a CR manifold. Also, we recall some results (in their real form) from [35, 36, 37, 30], see also [17, 28, 27], which we will use in this paper.

A CR manifold is a smooth manifold M of real dimension 2n+1, with a fixed n-dimensional complex sub-bundle $\mathcal{H}$ of the complexified tangent bundle $\mathbb{C}TM$ satisfying $\mathcal{H} \cap \overline{\mathcal{H}} = 0$ and $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. If we let $H = Re\,\mathcal{H} \oplus \overline{\mathcal{H}}$, the real sub-bundle H is equipped with a formally integrable almost complex structure J. We assume that M is oriented and there exists a globally defined compatible contact form θ such that the *horizontal space* is given by

$$H = Ker\,\theta.$$

In other words, the hermitian bilinear form

$$2g(X, Y) = -d\theta(JX, Y)$$

is non-degenerate. The CR structure is called strictly pseudoconvex if g is a positive definite tensor on H. The vector field ξ dual to θ with respect to g satisfying $\xi \lrcorner d\theta = 0$ is called the Reeb vector field. The almost complex structure J is formally integrable in the sense that

$$([JX, Y] + [X, JY]) \in H$$

and the Nijenhuis tensor

$$N^J(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0.$$

A CR manifold (M, θ, g) with a fixed compatible contact form θ is called *a pseudohermitian manifold*. In this case the 2-form

$$d\theta_{|_H} := 2\omega$$

is called the fundamental form. Note that the contact form is determined up to a conformal factor, i.e. $\bar{\theta} = \nu\theta$ for a positive smooth function ν, defines another pseudohermitian structure called pseudo-conformal to the original one.

2.1. **Invariant decompositions.** As usual any endomorphism Ψ of H can be decomposed with respect to the complex structure J uniquely into its $U(n)$-invariant $(2,0)+(0,2)$ and $(1,1)$ parts. In short we will denote these components correspondingly by $\Psi_{[-1]}$ and $\Psi_{[1]}$. Furthermore, we shall use the same notation for the corresponding two tensor, $\Psi(X,Y) = g(\Psi X, Y)$. Explicitly, $\Psi = \Psi = \Psi_{[1]} + \Psi_{[-1]}$, where

$$(2.1) \qquad \Psi_{[1]}(X,Y) = \frac{1}{2}\left[\Psi(X,Y) + \Psi(JX,JY)\right], \qquad \Psi_{[-1]}(X,Y) = \frac{1}{2}\left[\Psi(X,Y) - \Psi(JX,JY)\right].$$

The above notation is justified by the fact that the $(2,0)+(0,2)$ and $(1,1)$ components are the projections on the eigenspaces of the operator $\Upsilon = J \otimes J$, $(\Upsilon\Psi)(X,Y) \overset{def}{=} \Psi(JX,JY)$ corresponding, respectively, to the eigenvalues -1 and 1. Note that both the metric g and the 2-form ω belong to the $[1]$-component, since $g(X,Y) = g(JX,JY)$ and $\omega(X,Y) = \omega(JX,JY)$. Furthermore, the two components are orthogonal to each other with respect to g.

2.2. **The Tanaka-Webster connection.** The Tanaka-Webster connection [35, 36, 37] is the unique linear connection ∇ with torsion T preserving a given pseudohermitian structure, i.e., it has the properties

$$(2.2) \qquad \begin{aligned} &\nabla\xi = \nabla J = \nabla\theta = \nabla g = 0, \\ &T(X,Y) = d\theta(X,Y)\xi = 2\omega(X,Y)\xi, \quad T(\xi,X) \in H, \\ &g(T(\xi,X),Y) = g(T(\xi,Y),X) = -g(T(\xi,JX),JY). \end{aligned}$$

For a smooth function f on a pseudohermitian manifold M we denote by ∇f its horizontal gradient,

$$(2.3) \qquad g(\nabla f, X) = df(X).$$

The horizontal sub-Laplacian $\triangle f$ and the norm of the horizontal gradient $\nabla f = df(e_a)e_a$ of a smooth function f on M are defined respectively by

$$(2.4) \qquad \triangle f = -\, tr_H^g(\nabla df) = \nabla^* df = -\,\nabla df(e_a, e_a), \qquad |\nabla f|^2 = df(e_a)\,df(e_a).$$

The function $f \not\equiv 0$ is an eigenfunction of the sub-Laplacian if

$$(2.5) \qquad \triangle f = \lambda f,$$

where λ is a necessarily non-negative constant.

It is well known that the endomorphism $T(\xi,.)$ is the obstruction a pseudohermitian manifold to be Sasakian. The symmetric endomorphism $T_\xi : H \longrightarrow H$ is denoted by A,

$$A(X,Y) \overset{def}{=} T(\xi,X,Y),$$

and is called *the (Webster) torsion of the pseudohermitian manifold or pseudohermitian torsion.* It is a completely trace-free tensor of type $(2,0)+(0,2)$,

$$(2.6) \qquad A(e_a, e_a) = A(e_a, Je_a) = 0, \quad A(X,Y) = A(Y,X) = -A(JX,JY).$$

Let R be the curvature of the Tanaka-Webster connection. The pseudohermitian Ricci tensor Ric, the pseudohermitian scalar curvature S and the pseudohermitian Ricci 2-form ρ are defined by

$$Ric(A,B) = R(e_a, A, B, e_a), \quad S = Ric(e_a, e_a), \quad \rho(A,B) = \frac{1}{2}R(A,B,e_a,Ie_a).$$

We summarize below the well known properties of the curvature R of the Tanaka-Webster connection [36, 37, 30] using real expression, see also [17, 28, 27].

$$R(X,Y,JZ,JV) = R(X,Y,Z,V) = -R(X,Y,V,Z), \qquad R(X,Y,Z,\xi) = 0,$$

$$(2.7) \quad \frac{1}{2}\Big[R(X,Y,Z,V) - R(JX,JY,Z,V)\Big] = -g(X,Z)A(Y,JV) - g(Y,V)A(X,JZ) + g(Y,Z)A(X,JV)$$
$$+ g(X,V)A(Y,JZ) - \omega(X,Z)A(Y,V) - \omega(Y,V)A(X,Z) + \omega(Y,Z)A(X,V) + \omega(X,V)A(Y,Z),$$

$$(2.8) \qquad R(\xi,X,Y,Z) = (\nabla_Y A)(Z,X) - (\nabla_Z A)(Y,X),$$

$$(2.9) \qquad \begin{aligned} &Ric(X,Y) = Ric(Y,X), \\ &Ric(X,Y) - Ric(JX,JY) = 4(n-1)A(X,JY), \end{aligned}$$

(2.10) $$2\rho(X, JY) = -Ric(X, Y) - Ric(JX, JY) = R(e_a, Je_a, X, JY),$$

(2.11) $$2(\nabla_{e_a} Ric)(e_a, X) = dS(X).$$

The equalities (2.9) and (2.10) imply

(2.12) $$Ric(X, Y) = \rho(JX, Y) + 2(n-1)A(JX, Y),$$

i.e., ρ is the $(1,1)$-part of the pseudohermitian Ricci tensor, sometimes also called the Webster Ricci tensor, while the $(2,0) + (0,2)$-part is given by the pseudohermitian torsion A.

2.3. The Ricci identities for the Tanaka-Webster connection.
We shall use repeatedly the following Ricci identities of order two and three for a smooth function f, see also [27],

(2.13)
$$\nabla^2 f(X, Y) - \nabla^2 f(Y, X) = -2\omega(X, Y)df(\xi)$$
$$\nabla^2 f(X, \xi) - \nabla^2 f(\xi, X) = A(X, \nabla f)$$
$$\nabla^3 f(X, Y, Z) - \nabla^3 f(Y, X, Z) = -R(X, Y, Z, \nabla f) - 2\omega(X, Y)\nabla^2 f(\xi, Z)$$
$$\nabla^3 f(X, Y, Z) - \nabla^3 f(Z, Y, X) = -R(X, Y, Z, \nabla f) - R(Y, Z, X, \nabla f) - 2\omega(X, Y)\nabla^2 f(\xi, \ Z)$$
$$\hspace{3cm} - 2\omega(Y, Z)\nabla^2 f(\xi, X) + 2\omega(Z, X)\nabla^2 f(\xi, Y) + 2\omega(Z, X)A(Y, \nabla f)$$
$$\nabla^3 f(\xi, X, Y) - \nabla^3 f(X, \xi, Y) = (\nabla_{\nabla f} A)(Y, X) - (\nabla_Y A)(\nabla f, X) - \nabla^2 f(AX, Y)$$
$$\nabla^3 f(X, Y, \xi) - \nabla^3 f(\xi, X, Y) = \nabla^2 f(AX, Y) + \nabla^2 f(X, AY) + (\nabla_X A)(Y, \nabla f) + (\nabla_Y A)(X, \nabla f)$$
$$\hspace{8cm} - (\nabla_{\nabla f})A(X, Y).$$

We note that the above Ricci identities for the Tanaka-Webster connection follow from the general Ricci identities for a connection with torsion applying the properties of the pseudohermitian torsion listed in (2.2) and the curvature identity (2.8). For example,

$$\nabla^3 f(\xi, X, Y) - \nabla^3 f(X, \xi, Y) = -R(\xi, X, Y, \nabla f) - \nabla^2 f(T(\xi, X), Y)$$
$$= R(X, \xi, Y, \nabla f) - A(X, e_a)\nabla^2 f(e_a, Y) = (\nabla_{\nabla f} A)(Y, X) - (\nabla_Y A)(\nabla f, X) - A(X, e_a)\nabla^2 f(e_a, Y),$$

where we used (2.8).

An important consequence of the first Ricci identity is the following fundamental formula

(2.14) $$g(\nabla^2 f, \omega) = \nabla^2 f(e_a, Je_a) = -2n\, df(\xi).$$

On the other hand, by (2.4) the trace with respect to the metric is the negative sub-Laplacian

$$g(\nabla^2 f, g) = \nabla^2 f(e_a, e_a) = -\triangle f.$$

We also recall the horizontal divergence theorem [35]. Let (M, g, θ) be a pseudohermitian manifold of dimension $2n+1$. For a fixed local 1-form θ the form

$$Vol_\theta = \theta \wedge \omega^n$$

is a globally defined volume form since Vol_θ is independent on the local one form θ. We define the (horizontal) divergence of a horizontal vector field/one-form $\sigma \in \Lambda^1(H)$ defined by

$$\nabla^* \sigma = -tr|_H \nabla\sigma = -(\nabla_{e_a}\sigma)e_a.$$

The following Proposition, which allows "integration by parts", is well known [35].

Proposition 2.1. *On a compact pseudohermitian manifold M the following divergence formula holds true*

$$\int_M (\nabla^*\sigma)\, Vol_\theta = 0.$$

3. The Hessian of an extremal function in the extremal case.

Our goal is to determine the full Hessian of an "*extremal first eigenfunction*" that is an eigenfunction with the smallest possible in the sense of Theorem 8.8 eigenvalue.

Lemma 3.1. *Let M be a compact strictly pseudoconvex CR manifold of dimension $2n+1$, $n \geq 1$ satisfying*

$$Ric(X,X) + 4A(X,JX) = \rho(JX,Y) + 2(n+1)A(JX,Y) \geq k_0 \, g(X,Y)$$

while if $n = 1$ assume, further, that the CR-Paneitz operator is non-negative on f, i.e., (8.12) holds true.

If $\frac{n}{n+1}k_0$ is an eigenvalue of the sub-Laplacian, then the corresponding eigenfunctions satisfy the identity

$$(3.1) \qquad \nabla^2 f(X,Y) = -\frac{k_0}{2(n+1)} fg(X,Y) - df(\xi)\omega(X,Y).$$

Proof. The result is a simple consequence of the proof of Theorem 8.8. Under the assumptions of the Lemma, the first integral in the inequality (8.16) vanishes while the other terms are non-negative because of (8.17). Therefore,

$$(\nabla^2 f)_{[-1]} = 0$$

and (8.18) holds true. These imply the formula for the horizontal Hessian

$$\nabla^2 f(X,Y) = -\frac{1}{2n}(\triangle f) \cdot g(X,Y) + \frac{1}{2n} g(\nabla^2 f, \omega) \cdot \omega(X,Y).$$

Now, taking into account that f is an extremal first eigenfunction we obtain the coefficient in front of the metric. Finally, the skew-symmetric part of the horizontal Hessian is determined by the first Ricci identity in (2.13). $\square$

Remark 3.2. *In the extremal case, in addition to the above identities we have from (8.15) in the proof of Theorem 8.8 the following identities*

$$(3.2) \qquad Ric(\nabla f, \nabla f) + 4A(J\nabla f, \nabla f) = k_0|\nabla f|^2, \qquad \int_M P_f(\nabla f) \, Vol_\theta = 0.$$

Using a homothety we can reduce to the case $\lambda_1 = 2n$ and $k_0 = 2(n+1)$, which are the values for the standard Sasakian round sphere. Henceforth, we shall work under these assumptions. Thus, for an extremal first eigenfunction f (by definition $f \not\equiv 0$) and $n \geq 1$ we have the equalities

$$\lambda = 2n, \qquad \triangle f = 2nf, \qquad \int_M (\triangle f)^2 \, Vol_\theta = 2n \int_M |\nabla f|^2 \, Vol_\theta.$$

In addition, the horizontal Hessian of f satisfies (3.1), which with the assumed normalization takes the form given in equation (1.4).

3.1. **The divergence-free torsion and the vertical derivative of an extremal function.** Here we show one of our main observations that the vertical derivative of an extremal function is again an extremal function provided the pseudohermitian torsion is divergence-free. We begin with an identity satisfied by every extremal eigenfunction.

Lemma 3.3. *Let M be a strictly pseudoconvex pseudohermitian CR manifold of dimension $2n+1 \geq 3$. If f is an eigenfunction of the sub-Laplacian satisfying (1.4), then the following formula for the third covariant derivative holds true*

$$(3.3) \quad \nabla^3 f(X,Y,\xi) = -df(\xi)g(X,Y) - (\xi^2 f)\omega(X,Y) - 2fA(X,Y)$$
$$+ (\nabla_X A)(Y,\nabla f) + (\nabla_Y A)(X,\nabla f) - (\nabla_{\nabla f} A)(X,Y).$$

Proof. We start by substituting the horizontal Hessian of f given in (1.4) in the forth Ricci identity of (2.13).

$$\nabla^3 f(X,Y,\xi) = \nabla^3 f(\xi,X,Y) + \nabla^2 f(AX,Y) + \nabla^2 f(X,AY) + (\nabla_X A)(Y,\nabla f) + (\nabla_Y A)(X,\nabla f) - (\nabla_{\nabla f} A)(X,Y)$$
$$= \nabla^3 f(\xi,X,Y) - fA(X,Y) + df(\xi)A(X,JY) - fA(X,Y) - df(\xi)A(JX,Y)$$
$$+ (\nabla_X A)(Y,\nabla f) + (\nabla_Y A)(X,\nabla f) - (\nabla_{\nabla f} A)(X,Y)$$
$$= \nabla^3 f(\xi,X,Y) - 2fA(X,Y) + (\nabla_X A)(Y,\nabla f) + (\nabla_Y A)(X,\nabla f) - (\nabla_{\nabla f} A)(X,Y).$$

The first term, $\nabla^3 f(\xi, X, Y)$, in the left-hand side is computed by differentiating the formula for the horizontal Hessian, (1.4). A substitution of the thus obtained formula in the one above gives the desired (3.3) which completes the proof. $\qquad\square$

With the help of Lemma 3.3 we turn to our main result in this sub-section.

Lemma 3.4. *Let M be a strictly pseudoconvex pseudohermitian CR manifold of dimension $2n + 1 \geq 3$. If the pseudohermitian torsion is divergence-free with respect to the Tanaka-Webster connection, $(\nabla_{e_a} A)(e_a, X) = 0$, and f is an eigenfunction satisfying (1.4) then the function ξf is an eigenfunction with the same eigenvalue,*

$$(3.4) \qquad\qquad\qquad\qquad \triangle(\xi f) = 2n(\xi f).$$

In particular, if M is compact satisfying (1.2) then the horizontal Hessian of ξf is given by

$$(3.5) \qquad\qquad \nabla^2(\xi f)(X, Y) = \nabla^3 f(X, Y, \xi) = -df(\xi)g(X, Y) - (\xi^2 f)\omega(X, Y).$$

Proof. From the last Ricci identity in (2.13) we have

$$\triangle(\xi f) - \xi(\triangle f) = \nabla^3 f(e_a, e_a, \xi) - \nabla^3 f(\xi, e_a, e_a) = 2g(A, \nabla^2 f) - 2(\nabla^* A)(\nabla f) + (\nabla_{\nabla f} A)(e_a, e_a) = 0,$$

using that the torsion is trace- and divergence- free, and the fact that $g(A, \nabla^2 f) = 0$ by (1.4). Hence, (3.4) holds.

The second part follows from the just proved (3.4) and Lemma 3.1. $\qquad\square$

Remark 3.5. *In the compact case, the above lemma can also be seen with the help of the following "vertical Bochner formula" valid for any smooth function f*

$$(3.6) \qquad -\triangle(\xi f)^2 = 2|\nabla(\xi f)|^2 - 2df(\xi) \cdot \xi(\triangle f) + 4df(\xi) \cdot g(A, \nabla^2 f) - 4df(\xi)(\nabla^* A)(\nabla f).$$

However, the argument in Lemma 3.4 is purely local.

To prove (3.6) we use the last of the Ricci identities (2.13) and the fact that the torsion is trace free to obtain

$$
\begin{aligned}
-\frac{1}{2}\triangle(\xi f)^2 &= \nabla^3 f(e_a, e_a, \xi)df(\xi) + \nabla^2 f(e_a, \xi)\nabla^2 f(e_a, \xi) \\
&= \left[\nabla^3 f(\xi, e_a, e_a) + 2g(\nabla^2 f, A) - 2(\nabla^* A)(\nabla f)\right]df(\xi) + |\nabla(\xi f)|^2 \\
&= |\nabla(\xi f)|^2 - df(\xi) \cdot \xi(\triangle f) + 2df(\xi) \cdot g(A, \nabla^2 f) - 2df(\xi)(\nabla^* A)(\nabla f),
\end{aligned}
$$

which completes the proof of (3.6).

4. VANISHING OF THE PSEUDOHERMITIAN TORSION IN THE EXTREMAL CASE FOR $n \geq 2$

In this section we prove Theorem 4.7 which is one of our main results valid in dimension at least five, i.e., we assume $n \geq 2$. The assumptions in this section, unless noted otherwise, are that M is a strictly pseudoconvex pseudohermitian CR manifold of dimension at least five and f satisfies (1.4).

4.1. **Curvature in the extremal case.** We start with a calculation of the curvature tensor. This is achieved by using (1.4), (2.10) and the Ricci identities (2.13). After some standard calculations we obtain the following formula

$$(4.1) \quad R(Z, X, Y, \nabla f) = \left[df(Z)g(X, Y) - df(X)g(Z, Y)\right] + \nabla df(\xi, Z)\omega(X, Y)$$
$$- \nabla df(\xi, X)\omega(Z, Y) - 2\nabla df(\xi, Y)\omega(Z, X) + A(Z, \nabla f)\omega(X, Y) - A(X, \nabla f)\omega(Z, Y).$$

Taking the traces in (4.1) we obtain using (2.7)

$$
(4.2) \quad
\begin{aligned}
&Ric(Z, \nabla f) = (2n - 1)df(Z) - A(JZ, \nabla f) - 3\nabla df(\xi, JZ) \\
&Ric(JZ, J\nabla f) = R(JZ, Je_a, e_a, \nabla f) = df(Z) - (2n - 1)A(JZ, \nabla f) - (2n + 1)\nabla df(\xi, JZ).
\end{aligned}
$$

We note that the above derivation of (4.1) and (4.2) holds also when $n = 1$.

4.2. The vertical parts of the Hessian in the extremal case.

Subtracting the equations in (4.2) and using (2.9) we obtain

$$\nabla df(\xi, JZ) = -df(Z) + A(JZ, \nabla f) \tag{4.3}$$

after dividing by $n - 1$ since $n > 1$. Equation (4.3) and the Ricci identity yield

$$\nabla^2 f(\xi, Y) = df(JY) + A(Y, \nabla f), \qquad \nabla^2 f(Y, \xi) = df(JY) + 2A(Y, \nabla f). \tag{4.4}$$

At this point we have not yet determined $\xi^2 f$, but this will be achieved in Lemma 4.4.

4.3. The relation between A and $A\nabla f$.

Lemma 4.1. *Let M be a strictly pseudoconvex pseudohermitian CR manifold of dimension $2n + 1 \geq 5$. If f is a function satisfying (1.4), then we have the following identity*

$$|\nabla f|^2 |A|^2 = 2|A\nabla f|^2. \tag{4.5}$$

Proof. A substitution of (4.4) in (4.1) implies

$$R(X, Y, Z, \nabla f) = df(X)g(Y, Z) - df(Y)g(X, Z) + df(JX)\omega(Y, Z) - df(JY)\omega(X, Z) - 2df(JZ)\omega(X, Y)$$
$$- 2\omega(X, Y)A(Z, \nabla f) + 2A(X, \nabla f)\omega(Y, Z) - 2A(Y, \nabla f)\omega(X, Z).$$

Consequently, we obtain

$$R(X, Y, Z, \nabla f) - R(JX, JY, Z, \nabla f) \tag{4.6}$$
$$= 2\omega(Y, Z)A(X, \nabla f) - 2\omega(X, Z)A(Y, \nabla f) + 2g(Y, Z)A(JX, \nabla f) - 2g(X, Z)A(JY, \nabla f).$$

On the other hand, from (2.7) we have

$$R(X, Y, Z, \nabla f) - R(JX, JY, Z, \nabla f) \tag{4.7}$$
$$= -2g(X, Z)A(Y, J\nabla f) - 2g(Y, \nabla f)A(X, JZ) + 2g(Y, Z)A(X, J\nabla f) + 2g(X, \nabla f)A(Y, JZ)$$
$$- 2\omega(X, Z)A(Y, \nabla f) - 2\omega(Y, \nabla f)A(X, Z) + 2\omega(Y, Z)A(X, \nabla f) + 2\omega(X, \nabla f)A(Y, Z).$$

Comparing equations (4.6) and (4.7), and also taking into account the type of A, $A(JX, Y) = A(X, JY)$, we come to

$$0 = -2df(Y)A(X, JZ) + 2df(X)A(Y, JZ) - 2df(JY)A(X, Z) + 2df(JX)A(Y, Z). \tag{4.8}$$

For $X = \nabla f$ in (4.8) we obtain the identity

$$|\nabla f|^2 A(Y, Z) = df(Y)A(\nabla f, Z) - df(JY)A(\nabla f, JZ)$$

which proves (4.5). $\square$

4.4. The vertical derivative of an extremal eigenfunction.

Lemma 4.2. *Let M be a strictly pseudoconvex pseudohermitian CR manifold of dimension $2n + 1 \geq 5$. If f is an eigenfunction of the sub-Laplacian satisfying (1.4), then the following formula for the third covariant derivative holds true*

$$\nabla^3 f(X, Y, \xi) = -df(\xi)g(X, Y) + f\omega(X, Y) - 2fA(X, Y) - 2df(\xi)A(JX, Y) + 2(\nabla_X A)(Y, \nabla f). \tag{4.9}$$

Proof. Differentiating the last identity in (4.4) we obtain

$$\nabla^3 f(X, Y, \xi) = \nabla^2 f(X, JY) + 2(\nabla_X A)(Y, \nabla f) + 2A(Y, \nabla_X(\nabla f)).$$

Now, invoking (1.4) gives the desired formula. $\square$

Lemma 4.3. *Let M be a strictly pseudoconvex pseudohermitian CR manifold of dimension $2n + 1 \geq 5$. If f is an eigenfunction satisfying (1.4) then we have*

$$\nabla^2 f(\xi, \xi) = \xi^2 f = -f - \frac{1}{n}(\nabla_{e_a} A)(e_a, J\nabla f). \tag{4.10}$$

Proof. Comparing equations (4.9) and (3.3) we obtain the identity

$$- (\xi^2 f)\omega(X,Y) + (\nabla_X A)(Y,\nabla f) + (\nabla_Y A)(X,\nabla f) - (\nabla_{\nabla f} A)(X,Y)$$
$$= f\omega(X,Y) - 2df(\xi)A(JX,Y) + 2(\nabla_X A)(Y,\nabla f).$$

Taking a trace we obtain (4.10).   □

Lemma 4.4. *Let M be a strictly pseudoconvex pseudohermitian CR manifold of dimension $2n+1 \geq 5$. If the pseudohermitian torsion is divergence-free, $(\nabla^* A)(X) = 0$, and f is an eigenfunction satisfying (1.4) then the next formulas hold true*

$$(4.11) \qquad \nabla^2 f(\xi,\xi) = \xi^2 f = -f, \qquad (\nabla_X A)(Y,\nabla f) = fA(X,Y) + df(\xi)A(X,JY).$$

Proof. The first part follows immediately from Lemma 4.3. However, both parts can be seen as follows. Using (3.5) in (4.9) we obtain the identity

$$(4.12) \qquad (\nabla_X A)(Y,\nabla f) = fA(X,Y) + df(\xi)A(X,JY) - \frac{1}{2}\Big[\nabla^2 f(\xi,\xi) + f\Big]\omega(X,Y).$$

Equation (4.12) yields

$$(4.13) \quad (\nabla_X A)(JY,\nabla f) \; = \; (\nabla_X A)(Y,J\nabla f) \; = \; fA(X,JY) - df(\xi)A(X,Y) - \frac{1}{2}\Big[\nabla^2 f(\xi,\xi) \, + \, f\Big]g(X,Y),$$

where we used (2.6). Taking the trace of (4.13) using the fact that the pseudohermitian torsion is both divergence-free and trace-free we obtain the proof of the lemma.   □

4.5. **The elliptic eigenvalue problem.** A consequence of the above Lemma 4.4 is the following fact, which plays a crucial role in resolving Conjecture 1.1 in the vanishing torsion case by allowing the reduction to the Riemannian Obata theorem. Furthermore, the elliptic equation satisfied by an extremal eigenfunction shows that $|\nabla f| \neq 0$, hence $df \neq 0$, in a dense set since $f \neq const$.

Corollary 4.5. *Let M be a strictly pseudoconvex pseudohermitian CR manifold of dimension $2n+1 \geq 5$. If the pseudohermitian torsion of M is divergence-free, $(\nabla^* A)(X) = 0$, and f is an eigenfunction satisfying (1.4), then f is an eigenfunction of the (positive) Riemannian Laplacian $\triangle^h$ associated to the Riemannian metric (1.3) on M,*

$$(4.14) \qquad\qquad\qquad\qquad\qquad \triangle^h f = (2n+1)f.$$

Proof. We denote by D the Levi-Civita connection of h. With respect to the local orthonormal basis e_a, ξ, $a = 1, \ldots, 2n$, for any smooth function f we have

$$-\triangle^h f = h(D_{e_a}(Df), e_a) + h(D_\xi(Df), \xi).$$

Since g and θ are parallel for the Tanaka-Webster connection ∇ it follows $\nabla h = 0$, which allows us to find the relation between the two connections:

$$(4.15) \quad h(\nabla_E B, C) = h(D_E B, C) + \frac{1}{2}\Big[h(T(E,B),C) - h(T(B,C),E) + h(T(C,E),B)\Big], \qquad E, B, C \in \mathcal{T}(M).$$

Therefore we have

$$h(\nabla_{e_a} B, e_a) = h(D_{e_a} B, e_a) - h(T(B,e_a),e_a), \qquad h(\nabla_\xi B, \xi) = h(D_\xi B, \xi) - h(T(B,\xi),\xi), \qquad B \in \mathcal{T}(M).$$

With this relation in mind, taking into account the properties of the torsion, the formula for the Laplacian reduces to

$$(4.16) \qquad\qquad -\triangle^h f = -\triangle f + h(\nabla_\xi(Df),\xi) = -\triangle f + (\nabla^2 f)(\xi,\xi) = -\triangle f + (\xi^2 f).$$

In particular, if f satisfies (1.4) then we have $\triangle f = 2nf$, while Lemma 4.4 gives $\xi^2 f = -f$, hence the claimed identity.   □

4.6. **Vanishing of the pseudohermitian torsion in the case** $n > 1$.

Lemma 4.6. *Let M be a strictly pseudoconvex pseudohermitian CR manifold of dimension $2n + 1 \geq 5$. If the pseudohermitian torsion is divergence-free, $(\nabla_{e_a} A)(e_a, X) = 0$, and f is an eigenfunction satisfying* (1.4) *then*

$$(4.17) \qquad\qquad A\nabla f = 0.$$

Proof. From Lemma 3.4 it follows that ξf is also an extremal first eigenfunction. Therefore, the second equation in (4.11) of Lemma 4.4 applied to ξf gives

$$(4.18) \qquad (\nabla_X A)(Y, \nabla(\xi f)) = (\xi f)A(X, Y) + (\xi^2 f)A(X, JY).$$

A substitution of the second equality of (4.4) in (4.18) shows

$$-(\nabla_X A)(Y, J\nabla f) + 2(\nabla_X A)(Y, A\nabla f) = (\xi f)A(X, Y) + (\xi^2 f)A(X, JY).$$

Noting that $A(Y, JX) = A(JY, X)$ by the last equality of (2.6), the above identity together with Lemma 4.4 give

$$(4.19) \qquad 2(\nabla_X A)(Y, A\nabla f) = (\nabla_X A)(JY, \nabla f) + df(\xi)A(X, Y) - fA(X, JY) = 0.$$

Therefore, using the symmetry of A and (4.11) of Lemma 4.4 we have

$$(4.20) \qquad 0 = (\nabla_X A)(\nabla f, A\nabla f) = (\nabla_X A)(A\nabla f, \nabla f) = fA(X, A\nabla f) + df(\xi)A(JX, A\nabla f).$$

Replacing X with JX in (4.20) yields

$$(4.21) \qquad\qquad 0 = fA(JX, A\nabla f) - df(\xi)A(X, A\nabla f).$$

From Corollary 4.5 it follows that f is an eigenfunction for a Riemannian Laplacian, hence it cannot vanish on an open set unless $f \equiv 0$. Since by assumption f is non-trivial, equations (4.20) and (4.21), taking into account $f^2 + (df(\xi))^2 \neq 0$, a.e., imply

$$(4.22) \qquad\qquad A(X, A\nabla f) = 0, \quad \text{i.e,} \quad A\nabla f = 0.$$

$\square$

Theorem 4.7. *Let M be a strictly pseudoconvex pseudohermitian CR manifold of dimension $2n + 1 \geq 5$. If the pseudohermitian torsion is divergence-free, $(\nabla_{e_a} A)(e_a, Z) = 0$ and f is an eigenfunction satisfying* (1.4) *then the pseudohermitian torsion vanishes, $A = 0$.*

Proof. Since the pseudohermitian torsion is divergence-free we have Lemma 4.6. Now, Lemma 4.1 shows that $A = 0$. $\square$

5. VANISHING OF THE PSEUDOHERMITIAN TORSION IN THE EXTREMAL THREE DIMENSIONAL CASE

In this section we prove our first main result in dimension three. We shall assume, unless explicitly stated otherwise, that M is a compact strictly pseudoconvex pseudohermitian CR manifold of dimension three for which (1.2) holds and f is a smooth function on M satisfying (1.4). In particular, we have done the normalization, if necessary, so that (1.2) holds with $k_0 = 4$. Since the horizontal space is two dimensional we can use ∇f, $J\nabla f$ as a basis at the points where $|\nabla f| \neq 0$. In fact, similarly to the higher dimensional case, we have $|\nabla f| \neq 0$ almost everywhere. This follows from Lemma 5.1 showing that f satisfies a certain elliptic equation which implies that f cannot vanish on any open set since otherwise $f \equiv 0$ which is a contradiction.

5.1. **The elliptic value problem in dimension three.** In dimension three we have the following result.

Lemma 5.1. *Let M be a strictly pseudoconvex pseudohermitian CR manifold of dimension three and pseudohermitian scalar curvature S. If the pseudohermitian torsion of M is divergence-free, $(\nabla^* A)(X) = 0$, and f is an eigenfunction satisfying* (1.4)*, then f satisfies the following elliptic equation*

$$(5.1) \qquad\qquad \triangle^h f = \left(2 + \frac{S - 2}{6} \right) f - \frac{1}{12} g(\nabla f, \nabla S)$$

involving the (positive) Riemannian Laplacian $\triangle^h$ associated to the Riemannian metric (1.3).

Proof. We start by proving the identity

$$(5.2) \qquad 6\xi^2 f = -(S-2)f + \frac{1}{2}g(\nabla f, \nabla S).$$

On one hand from (2.14) applied to the function ξf we have

$$\nabla^2 f(Je_a, e_a, \xi) = 2\xi^2 f.$$

On the other hand we use (4.2) to find

$$3\nabla^2 f(Z, \xi) = -df(JZ) + 2A(Z, \nabla f) - \rho(Z, \nabla f)$$

taking into account (2.12). Now, differentiating the above identity, then taking a trace, after which using the formula for the horizontal Hessian (1.4) gives

$$6\xi^2 f = 3\nabla^2 f(Je_a, e_a, \xi) = \triangle f + 2(\nabla^* A)(J\nabla f) + 2g(\nabla^2 f, A) - (\nabla^* \rho)(J\nabla f) - \nabla^2 f(Je_a, \rho e_a)$$

$$= 2f - (\nabla^* \rho)(J\nabla f) - [-f\rho(e_a, Je_a) + (\xi f)\rho(e_a, e_a)] = 2f - fS + \frac{1}{2}dS(\nabla f)$$

after using (2.11) and (2.12) for the last equality. This proves (5.2).

At this point we invoke the proof of Corollary 4.5. In fact, a substituition of the above found expression for $\xi^2 f$ into (4.16), which holds also in dimension three, gives (5.1). $\qquad\square$

5.2. **Vanishing of the pseudohermitian torsion in the case** $n = 1$. Since $\nabla f, J\nabla f$ is a basis (not an orthonormal one!) of the horizontal space almost everywhere the vanishing of the pseudohermitian torsion, $A = 0$, is implied by $A(\nabla f, \nabla f) = A(J\nabla f, \nabla f) = 0$. We turn to the main result of this section.

Theorem 5.2. *Let M be a compact strictly pseudoconvex pseudohermitian CR manifold of dimension three for which the Lichnerowicz condition* (1.2) *holds. If the pseudohermitian torsion is divergence-free, $(\nabla_{e_a} A)(e_a, Z) = 0$ and f is an eigenfunction satisfying* (1.4) *then the pseudohermitian torsion vanishes, $A = 0$, and the pseudohermitian scalar curvature is a constant, $S = 8$.*

Proof. Equation (3.2) yields

$$(5.3) \qquad Ric(\nabla f, \nabla f) = \frac{S}{2}|\nabla f|^2 = 4|\nabla f|^2 - 4A(J\nabla f, \nabla f).$$

Setting $Z = \nabla f$ in (4.2) and using (5.3) we have

$$(5.4) \qquad \nabla^2 f(\xi, J\nabla f) = -|\nabla f|^2 + A(J\nabla f, \nabla f) = -\frac{1}{4}Ric(\nabla f, \nabla f) = -\frac{S}{8}|\nabla f|^2.$$

Taking $Z = J\nabla f$ in (4.2) and using that in dimension three $Ric(X, JX) = 0$ give

$$(5.5) \qquad \nabla^2 f(\xi, \nabla f) = -\frac{1}{3}A(\nabla f, \nabla f), \qquad \nabla^2 f(\nabla f, \xi) = \frac{2}{3}A(\nabla f, \nabla f).$$

By Lemma 3.4 we know that ξf is also an extremal eigenfunction, hence we have

$$(5.6) \qquad Ric(\nabla(\xi f), \nabla(\xi f)) = 4|\nabla(\xi f)|^2 - 4A(\nabla(\xi f), J\nabla(\xi f)).$$

Since ∇f and $J\nabla f$ are orthogonal we have

$$|\nabla f|^2 \nabla(\xi f) = \nabla^2 f(\nabla f, \xi)\nabla f + \nabla^2 f(J\nabla f, \xi)J\nabla f.$$

Therefore, we have the following identities

$$(5.7) \qquad |\nabla f|^4 |\nabla(\xi f)|^2 = \left[\left(\nabla^2 f(\nabla f, \xi) \right)^2 + \left(\nabla^2 f(J\nabla f, \xi) \right)^2 \right] |\nabla f|^2,$$

$$(5.8) \qquad |\nabla f|^4 \, Ric(\nabla(\xi f), \nabla(\xi f)) = \left[\left(\nabla^2 f(\nabla f, \xi) \right)^2 + \left(\nabla^2 f(J\nabla f, \xi) \right)^2 \right] Ric(\nabla f, \nabla f),$$

$$(5.9) \quad |\nabla f|^4 \, A(\nabla(\xi f), J\nabla(\xi f) = \left[\left(\nabla^2 f(\nabla f, \xi) \right)^2 - \left(\nabla^2 f(J\nabla f, \xi) \right)^2 \right] A(\nabla f, J\nabla f)$$
$$- 2\left(\nabla^2 f(\nabla f, \xi) \right) \left(\nabla^2 f(J\nabla f, \xi) \right) A(\nabla f, \nabla f).$$

Substituting (5.9), (5.8), (5.7) in (5.6) and using (5.3) we obtain

$$(5.10) \qquad |\nabla f|^4 \left\{ \left(\nabla^2 f(J\nabla f, \xi) \right)^2 A(\nabla f, J\nabla f) + \nabla^2 f(\nabla f, \xi) \nabla^2 f(J\nabla f, \xi) A(\nabla f, \nabla f) \right\} = 0.$$

Assumption (1.2) implies that in our case the pseudohermitian scalar curvature satisfies the inequality $S \geq 8$, hence (5.3) yields

$$(5.11) \qquad A(J\nabla f, \nabla f) = \left(1 - \frac{S}{8} \right) |\nabla f|^2 \leq 0.$$

Equation (5.4), the Ricci identities and (5.11) imply the inequality

$$(5.12) \qquad \nabla^2 f(J\nabla f, \xi) = \nabla^2 f(\xi, J\nabla f) + A(J\nabla f, \nabla f) = A(J\nabla f, \nabla f) - \frac{S}{8} |\nabla f|^2 \leq 0.$$

Taking into account (5.12) and (5.5) we obtain from (5.10)

$$(5.13) \qquad \nabla^2 f(J\nabla f, \xi) A(\nabla f, J\nabla f) + \frac{2}{3} \left(A(\nabla f, \nabla f) \right)^2 = 0.$$

The first term in (5.13) is nonnegative from (5.11) and (5.12). Therefore, we conclude

$$A(\nabla f, J\nabla f) = A(\nabla f, \nabla f) = 0, \quad i.e. \quad A = 0.$$

The claim for the pseudohermitian scalar curvature follows for example from (5.3).   □

An immediate corollary from Lemma 5.1 and Theorem 5.2 is the fact that on a strictly pseudoconvex pseudohermitian CR manifold of dimension three with a divergence-free pseudohermitian torsion every extremal eigenfunction is an eigenfunction of the Riemannian Laplacians (4.14). In other words, Corollary 4.5 is valid for $n \geq 1$.

6. Proof of main Theorems

In this section we give the proofs of our main results.

6.1. **Proof of Theorem 1.2.** By Lemma 3.1 we obtain the formula for the horizontal Hessian of an extremal eigenfunction. Now, we can apply Theorem 4.7 when $n > 1$ or Theorem 5.2 when $n = 1$ to conclude that the pseudohermitian torsion vanishes. The claim of the Theorem follows by applying either the known result in the Sasakian case or by applying Theorem 1.3. The first argument, essentially the one due to [8] and [9], goes as follows. When the pseudohermitian torsion vanishes, $A = 0$, we have from Corollary 4.5 and the paragraph at the end of Section 5 that for $n \geq 1$ an extremal eigenfunction f is also an eigenfunction of the Riemannian Laplacian associated to the metric (1.3), i.e.,

$$(6.1) \qquad \triangle^h f = (2n+1)f.$$

On the other hand, Theorem 1.6 in [17] gives the following relation between the curvature R^h of the Levi-Civita connection of the metric h and the curvature R of the Tanaka-Webster connection when the pseudohermitian torsin vanishes

$$(6.2) \quad R^h(E, B, C, D) = R(E, B, C, D) + g(JE, C)g(JB, D) - g(JB, C)g(JE, D) - 2g(E, JB)g(JC, D)$$
$$- \theta(D)\theta(B)g(E, C) + \theta(D)\theta(E)g(B, C) - \theta(A)\theta(C)g(B, D) + \theta(B)\theta(C)g(E, D), \quad B, C, D, E \in \mathfrak{T}(M).$$

Taking the trace in (6.2) and using (2.8) with $A = 0$ we obtain for the Ricci tensor Ric^h of the Levi-Civita connection the next expression

$$(6.3) \qquad Ric^h(X, Y) = R^h(e_a, X, Y, e_a) + R^h(\xi, X, Y, \xi) = Ric(X, Y) - 2g(X, Y), \quad Ric^h(\xi, \xi) = 2n.$$

The inequality $Ric(X, X) \geq 2(n+1)g(X, X)$ together with (6.3) yields

$$(6.4) \qquad Ric^h(B, C) \geq 2ng(B, C), \qquad B, C \in \mathfrak{T}(M).$$

An application of the Riemannian Lichnerowicz theorem gives the estimate $\lambda^h \geq 2n+1$ for the first eigenvalue of $\triangle^h$. By Obata's result equality is possible iff (M, h) is the unit round sphere. Recalling (6.1) completes the proof.

6.2. **Proof of Theorem 1.3.** We shall prove Theorem 1.3 by a reduction to the corresponding Riemannian Obata theorem on a complete Riemannian manifold. In fact, we will show that the Riemannian Hessian computed with respect to the Levi-Civita connection D of the metric h defined in (1.3) satisfies (1.1) and then apply the Obata theorem [33] to conclude that (M, h) is isometric to the unit sphere.

For $n > 1$ by Theorem 4.7 it follows that $A = 0$. Therefore, (4.4) and (4.11) imply

$$(6.5) \qquad \nabla^2 f(\xi, Y) = \nabla^2 f(Y, \xi) = df(JY), \qquad \xi^2 f = -f.$$

Next, we show that (6.5) also holds in dimension three when the pseudohermitian torsion vanishes. In the three dimensional case we have $Ric(X, Y) = \frac{S}{2} g(X, Y)$. After a substitution of this equality in (4.2), taking into account $A = 0$, we obtain

$$(6.6) \qquad \nabla^2 f(\xi, Z) = \nabla^2 f(Z, \xi) = \frac{(S-2)}{6} df(JZ).$$

Differentiating (6.6) and using (1.4) we find

$$(6.7) \qquad \nabla^3 f(Y, Z, \xi) = \frac{1}{6}\Big[dS(Y)df(JZ) + (S-2)f\omega(Y, Z) - (S-2)df(\xi)g(Y, Z) \Big].$$

On the other hand, setting $A = 0$ in (3.3), we have

$$(6.8) \qquad \nabla^3 f(Y, Z, \xi) = -df(\xi)g(Y, Z) - (\xi^2 f)\omega(Y, Z).$$

In particular, the function ξf also satisfies (1.4). Therefore using Lemma 5.1 it follows that either $\xi f \equiv 0$ or $\xi f \neq 0$ almost everywhere. In the first case it follows $\nabla f = 0$ taking into account (6.6), hence $f \equiv 0$, which is not possible by assumption. Thus, the second case holds, i.e., $\xi f \neq 0$ almost everywhere.

Continuing our calculation, we note that (6.7) and (6.8) give

$$(6.9) \qquad \frac{S-8}{6} df(\xi)g(Y, Z) - \Big(\xi^2 f + \frac{S-2}{6} \Big)\omega(Y, Z) - \frac{1}{6} dS(Y)df(JZ) = 0,$$

which implies

$$\frac{S-8}{3} df(\xi)|\nabla f|^2 = 0.$$

Thus, the pseudohermitian scalar curvature is constant, $S = 8$, invoking again Lemma 5.1. Equation (6.9) reduces then to

$$(6.10) \qquad \Big(\xi^2 f + \frac{S-2}{6} \Big)\omega(Y, Z) = 0$$

since $dS = 0$. The equality (6.10) yields

$$\xi^2 f = -f,$$

which together with (6.6) and $S = 8$ imply the validity of (6.5) also in dimension three.

Finally, we use [17, Lemma 1.3] which gives the relation between D and ∇. In the case $A = 0$ we obtain

$$(6.11) \qquad D_B C = \nabla_B C + \theta(B)JC + \theta(C)JB - \omega(B, C)\xi, \quad B, C \in \mathcal{T}(M),$$

where J is extended with $J\xi = 0$. Notice that Ω in [17] is equal to $-\omega$. Using (6.11) together with (1.4) and (6.5) we calculate that (1.1) holds. The proof of Theorem 1.3 is complete.

7. SOME EXAMPLES

In this short section we give some cases in which Theorem 4.7 can be applied.

Corollary 7.1. *Let M be a strictly pseudoconvex pseudohermitian CR manifold of dimension $2n+1 \geq 3$. If the vertical part of the Ricci tensor of the Tanaka-Webster connection vanishes, $Ric(\xi, X) = 0$, or the vertical part of the Ricci 2-form of the Tanaka-Webster connection is zero, $\rho(\xi, X) = 0$ and f is an eigenfunction satisfying (1.4) then the pseudohermitian torsion vanishes, $A = 0$.*

In particular, if the vertical curvature of the Tanaka-Webster connection is zero, and f is an eigenfunction satisfying (1.4) then the pseudohermitian torsion vanishes, $A = 0$.

Proof. Suppose that either

$$0 = Ric(\xi, X) = R(\xi, e_a, e_a, X) = R(\xi, X, e_a, Z) = 0$$

or

$$0 = \rho(\xi, X) = R(\xi, X, e_a, Je_a) = R(\xi, X, e_a, Z) = 0.$$

In both cases (2.8) yields $(\nabla_{e_a} A)(e_a, X) = 0$, since the pseudohermitian torsion A is trace-free, symmetric and of type (0,2)+(2,0) with respect to J. Hence, the pseudohermitian torsion is divergence-free and the proof follows from Theorem 4.7. □

The formula (2.8) yields that the vertical curvature of the Tanaka-Webster connection vanishes if and only if the pseudohermitian torsion is a Codazzi tensor with respect to ∇,

$$(\nabla_Y A)(Z, X) = (\nabla_Z A)(Y, X).$$

Due to Corollary 7.1, interesting examples where Theorem 1.2 and Theorem 1.3 can be applied are provided by pseudohermitian structures for which the pseudohermitian torsion A is a Codazzi tensor with respect to the Tanaka-Webster connection. This includes, of course, the case of a parallel pseudohermitian torsion (contact (κ, μ)-spaces), $(\nabla_X A)(Y, Z) = 0$, for horizontal X, Y and Z, see [6] and [34].

8. APPENDIX

8.1. **The CR-Paneitz operator.**
The purpose of this section is to record, for self-sufficiency, some of the results of [30] and [20] using real variables. For a full description and further references about conformally invariant operators see [21], [19] and, in particular, [23] for the three dimensional case.

Definition 8.1 ([30, 20]). *a) Let $B(X, Y)$ be the $(1, 1)$ component of the horizontal Hessian $(\nabla^2 f)(X, Y)$,*

$$B(X, Y) \equiv B[f](X, Y) = (\nabla^2 f)_{[1]}(X, Y) = \frac{1}{2} \left[(\nabla^2 f)(X, Y) + (\nabla^2 f)(JX, JY) \right]$$

and then define $B_0(X, Y)$ to be the completely traceless part of B,

$$B_0(X, Y) \equiv B_0[f](X, Y) = B(X, Y) + \frac{\triangle f}{2n} g(X, Y) - \frac{1}{2n} g(\nabla^2 f, \omega) \omega(X, Y).$$

b) Given a function f we define the one form,

$$(8.1) \qquad P(X) \equiv P_f(X) = \nabla^3 f(X, e_b, e_b) + \nabla^3 f(JX, e_b, Je_b) + 4nA(X, J\nabla f)$$

and also a fourth order differential operator (the so called CR-CR-Paneitz operator in [16]),

$$(8.2) \quad Cf = -\nabla^* P = (\nabla_{e_a} P)(e_a) = \nabla^4 f(e_a, e_a, e_b, e_b) + \nabla^4 f(e_a, Je_a, e_b, Je_b) - 4n\nabla^* A(J\nabla f) - 4n\, g(\nabla^2 f, JA).$$

Remark 8.2. *On a compact pseudohermitian manifold the divergence formula gives*

$$\int_M f \cdot Cf\, Vol_\theta = - \int_M P_f(\nabla f)\, Vol_\theta$$

for a smooth function f. Therefore, the non-negativity condition (8.12) can also be written as

$$- \int_M P_f(\nabla f)\, Vol_\theta \geq 0.$$

In the three dimensional case this condition is a CR invariant since it is independent of the choice of the contact form. This follows from the conformal invariance of C proven in [23]. In the case of vanishing pseudohermitian torsion, we have, up to a multiplicative constant, $C = \Box_b \bar{\Box}_b$, where $\Box_b$ is the Kohn Laplacian.

From [30], see also [5] and [4], it is known that if $n \geq 2$, a function $f \in \mathcal{C}^3(M)$ is CR-pluriharmonic, i.e, locally it is the real part of a CR holomorphic function, if and only if $B_0[f] = 0$. In fact, as shown in [20] with the help of the next Lemma only one fourth-order equation $Cf = 0$ suffices for $B_0[f] = 0$ to hold. Furthermore, the CR-Paneitz operator is non-negative. When $n = 1$ the situation is more delicate. In the three dimensional case, CR-pluriharmonic functions are characterized by the kernel of the third order operator $P[f] = 0$ [30]. However, the single equation $Cf = 0$ is enough again assuming the vanishing of the pseudohermitian torsion

[20], see also [21]. On the other hand, [11] showed that if the pseudohermitian torsion vanishes the CR-Paneitz operator is essentially positive, i.e., there is a constant $\Lambda > 0$ such that

$$\int_M f \cdot (Cf)\, Vol_\theta \geq \Lambda \int_M f^2\, Vol_\theta.$$

for all real smooth functions $f \in (Ker\, C)^\perp$, i.e., $\int_M f \cdot \phi\, Vol_\theta = 0$ if $C\phi = 0$. In addition, the non-negativity of the CR CR-Paneitz operator is relevant in the embedding problem for a three dimensional strictly pseudoconvex CR manifold. In the Sasakian case, it is known that M is embeddable, [31], and the CR CR-Paneitz operator is nonnegative, see [16], [11]. Furthermore, [12] showed that if the pseudohermitian scalar curvature of M is positive and C is non-negative, then M is embeddable in some $\mathbb{C}^n$.

Remark 8.3. *In the quaternionic contact setting the paper [25] describes differential operators and conformally invariant two forms whose kernel contains the real parts of all anti-CRF functions. In particular, there is a partial result characterizing real parts of anti-CRF functions. The "extra" assumption concerns an analog of the $\partial\bar{\partial}$ lemma in the quaternionic setting.*

We turn to one of the basic results relating the above defined operators.

Lemma 8.4 ([20]). *On a compact strictly pseudoconvex pseudohermitian CR manifold of dimension $2n + 1$, $n \geq 1$, the following identity holds true*

$$(\nabla_{e_a} B_0)(e_a, X) = \frac{n-1}{2n} P_f(X), \qquad \int_M |B_0|^2\, Vol_\theta = -\frac{n-1}{2n} \int_M P_f(\nabla f)\, Vol_\theta = \frac{n-1}{2n} \int_M f \cdot (Cf)\, Vol_\theta.$$

In particular, if $n > 1$ the CR-Paneitz operator is non-negative.

Proof. Taking into account Ricci's identity (2.13) we have

$$\nabla^3 f(e_a, e_a, X) = \nabla^3 f(X, e_a, e_a) + Ric(X, \nabla f) + 4\nabla^2 f(\xi, JX) + 2A(JX, \xi)$$
$$\nabla^3 f(e_a, Je_a, JX) = \nabla^3 f(JX, Je_a, e_a) - 2\rho(JX, \nabla f) + Ric(JX, J\nabla f) - 4n\nabla^2 f(\xi, \ JX)$$
$$- 2A(JX, \nabla f).$$

Therefore, using (2.9) and (2.14), which implies $\nabla^2 f(JX, \xi) = \frac{1}{2n}\nabla^3 f(JX, e_c, Je_c)$, we have

$$(8.3) \quad 2\nabla_{e_a}(\nabla^2 f)_{[1]}(e_a, X) = \nabla^3 f(X, e_a, e_a) - \nabla^3 f(JX, e_a, Je_a) + \frac{2(n-1)}{n}\nabla^3 f(JX, e_a, Je_a)$$
$$+ 4(n-1)A(X, J\nabla f).$$

The trace part of B is computed as follows

$$(8.4) \quad \nabla_{e_a}\left(\frac{1}{2n}\nabla^2 f(e_c, e_c)g - df(\xi)\omega\right)(e_a, X) = \frac{1}{2n}\nabla^3 f(X, e_c, e_c) + \nabla^2 f(JX, \xi)$$
$$= \frac{1}{2n}\nabla^3 f(X, e_a, e_a) - \frac{1}{2n}\nabla^3 f(JX, e_a, Je_a).$$

The above identities (8.3) and (8.4) imply

$$(\nabla B_0)(e_a, e_a, X) \quad = \quad \frac{n-1}{2n} P(X),$$

which completes the proof of the first formula. The second identity follows by an integration by parts. $\square$

As was observed earlier in [16] and [11], see also [7], if the pseudohermitian torsion vanishes then the CR-Paneitz operator is non-negative also in dimension three. For completeness, since it suffices for the proofs of Theorem 8.8 and Theorem 1.2 in the case of vanishing pseudohermitian torsion, we note that (8.12) holds for an eigenfunction of the sub-Laplacian, $\triangle f = \lambda f$, using an idea of [20, Proposition 3.2]. Indeed, assuming $A = 0$ we have by the last Ricci identity (2.13) and (8.1) the identities

$$[\xi, \triangle]f = 0, \qquad P_f(X) = \nabla^3 f(X, e_b, e_b) + \nabla^3 f(JX, e_b, Je_b).$$

Therefore, with the help of (2.14) we have (recall (8.2))

$$\int_M |P_f|^2 \, Vol_\theta = \int_M \triangle f \cdot (Cf) \, Vol_\theta + \int_M \nabla^3 f(Je_a, e_b, Je_b)\Big(\nabla^3(e_a, e_c, e_c) + \nabla^3 f(Je_a, e_c, Je_c)\Big) \, Vol_\theta$$

$$= \lambda \int_M f \cdot (Cf) \, Vol_\theta - 2n \int_M \nabla^2 f(e_b, Je_b) \, [\xi, \triangle] f \, Vol_\theta = \lambda \int_M f \cdot (Cf) \, Vol_\theta.$$

Thus, since $\triangle f$ is a non-negative operator, if the pseudohermitian torsion vanishes and f is an eigenfunction, then the CR-Paneitz operator is non-negative for f, i.e., (8.12) holds for every eigenfunction. When dealing with a divergence-free pseudohermitian torsion it is worth noting that the CR-Paneitz operator is non-negative for any function satisfying (1.4). Indeed, in the case of a divergence-free pseudohermitian torsion the above identities hold taking into account the last Ricci identity (2.13), the fact that A belongs to the $(2,0) + (0,2)$ space, and definition (8.2).

8.2. Greenleaf's Bochner formula for the sub-Laplacian.
The first eigenvalue of the sub-Laplacian is the smallest positive constant in (2.5) which we denote by λ_1.

Theorem 8.5 ([22]). *On a strictly pseudoconvex pseudohermitian manifold of dimension $2n + 1$, $n \geq 1$, the following Bochner-type identity holds*

$$(8.5) \qquad \frac{1}{2}\triangle |\nabla f|^2 = -g(\nabla(\triangle f), \nabla f) + Ric(\nabla f, \nabla f) + 2A(J\nabla f, \nabla f) + |\nabla df|^2 + 4\nabla df(\xi, J\nabla f).$$

Proof. By definition we have

$$(8.6) \qquad -\frac{1}{2}\triangle |\nabla f|^2 = \nabla^3 f(e_a, e_a, e_b) df(e_b) + \nabla^2 f(e_a, e_b)\nabla^2 f(e_a, e_b) = \nabla^3 f(e_a, e_a, e_b) df(e_b) + |\nabla^2 f|^2.$$

To evaluate the first term in the right hand side of (8.6) we use the Ricci identities (2.13). Taking into account $(\nabla_X T)(Y, Z) = 0$, applying successively the Ricci identities (2.13), we obtain

$$(8.7) \qquad \nabla^3 f(e_a, e_a, e_b) df(e_b) = -g(\nabla(\triangle f), \nabla f) + Ric(\nabla f, \nabla f) + 2A(J\nabla f, \nabla f) + 4\nabla^2 f(\xi, J\nabla f).$$

A substitution of (8.7) in (8.6) completes the proof of (8.5). □

We have to evaluate the last term of (8.5). We recall the notation (2.1) of the two components of the $U(n)$-invariant decomposition of the horizontal Hessian $\nabla^2 f$. The first integral formula for the last term in (8.5) originally proved in [22] follows.

Lemma 8.6 ([22]). *On a compact strictly pseudoconvex pseudohermitian CR manifold of dimension $2n + 1$, $n \geq 1$, we have the identity*

$$(8.8) \qquad \int_M \nabla^2 f(\xi, J\nabla f) \, Vol_\theta = -\int_M \Big[\frac{1}{2n} g(\nabla^2 f, \omega)^2 + A(J\nabla f, \nabla f)\Big] \, Vol_\theta.$$

Proof. Integrating (2.14) we compute

$$(8.9) \qquad 4n^2 \int_M (\xi f)^2 \, Vol_\theta = -2n \int_M g(\nabla^2 f, \omega) \cdot df(\xi) \, Vol_\theta.$$

Let us consider the horizontal 1-form defined by

$$D_2(X) = df(JX) df(\xi)$$

whose divergence is, taking into account the second formula of (2.13),

$$(8.10) \qquad \nabla^* D_2 = g(\nabla^2 f, \omega) \cdot df(\xi) - \nabla^2 f(\xi, J\nabla f) - A(J\nabla f, \nabla f).$$

Integrating (8.10) over M and using (8.9) implies (8.8) which completes the proof of the lemma. □

We shall need one more representation of the last term in (8.5).

Lemma 8.7. *On a compact strictly pseudoconvex pseudohermitian CR manifold of dimension $2n + 1$, $n \geq 1$, we have the identity*

$$\nabla^2 f(\xi, Z) = \frac{1}{2n} \nabla^3 f(Z, Je_a, e_a) - A(Z, \nabla f).$$

In addition,

$$\int_M \nabla^2 f(\xi, J\nabla f) Vol_\theta = \int_M \left[-\frac{1}{2n} (\triangle f)^2 + A(J\nabla f, \nabla f) - \frac{1}{2n} P_f(\nabla f) \right] Vol_\theta.$$

Proof. We compute using the first two Ricci identity in (2.13)

$$2\nabla^3 f(Z, Je_a, e_a) = \nabla^3 f(Z, Je_a, e_a) - \nabla^3 f(Z, e_a, Je_a) = -2\omega(Je_a, e_a)\nabla^2 f(Z, \xi)$$
$$= 4n \left(\nabla^2 f(\xi, Z) + A(Z, \nabla f) \right),$$

which proves the first formula. The second identity follows from the above formula, the definition (8.1) of P_f and an integration by parts. $\qquad \square$

8.3. **The CR Lichneorwicz type theorem.** At this point we are ready to give in details, using real notation as in [28, 27], the known version of the Lichnerowicz' result on a compact strictly pseudo-convex CR manifold [22], [32], [16].

Theorem 8.8 ([22], [32], [16]). *Let (M, θ) be a compact strictly pseudoconvex pseudohermitian CR manifold of dimension $2n + 1$. Suppose there is a positive constant k_0 such that the pseudohermitian Ricci curvature Ric and the pseudohermitian torsion A satisfy the inequality*

$$(8.11) \qquad\qquad Ric(X, X) + 4A(X, JX) \geq k_0 g(X, X).$$

a) If $n > 1$, then any eigenvalue λ of the sub-Laplacian $\triangle$ satisfies the inequality

$$\lambda \geq \frac{n}{n+1} k_0.$$

b) If $n = 1$ the same estimate holds,

$$\lambda \geq \frac{1}{2} k_0,$$

assuming in addition that the CR-Paneitz operator is non-negative, i.e., for any smooth function f we have

$$(8.12) \qquad\qquad \int_M f \cdot Cf \, Vol_\theta \geq 0,$$

where Cf is defined in (8.2).

As well known the standard CR structure on the sphere achieves equality in this inequality. We note that the assumption on the pseudohermitian Ricci curvature of the Tanaka-Webster connection and the pseudohermitian torsion can be put in the equivalent form

$$Ric(X, X) + 4A(X, JX) = \rho(JX, X) + 2(n+1)A(JX, X)$$

using the pseudohermitian Ricci 2-form ρ of the Tanaka-Webster connection. We turn to the proof of Theorem 8.8.

Proof. Integrating the Bochner type formula (8.5) we obtain

$$0 = \int_M \left[-(\triangle f)^2 + \left| (\nabla^2 f)_{[1]} \right|^2 + \left| (\nabla^2 f)_{[-1]} \right|^2 + Ric(\nabla f, \nabla f) + 2A(J\nabla f, \nabla f) + 4\nabla^2 f(\xi, J\nabla f) \right] Vol_\theta.$$

We use Lemma 8.7 to represent the last term, which turns the above identity in the following form

$$(8.13) \quad 0 = \int_M \left[-(\triangle f)^2 + \left| (\nabla^2 f)_{[1]} \right|^2 + \left| (\nabla^2 f)_{[-1]} \right|^2 + Ric(\nabla f, \nabla f) + 6A(J\nabla f, \nabla f) \right.$$
$$\left. - \frac{2}{n}(\triangle f)^2 - \frac{2}{n} P(\nabla f) \right] Vol_\theta.$$

Lemma 8.6 and Lemma 8.7 give the following identity

$$(8.14) \qquad 2\int_M A(J\nabla f, \nabla f) Vol_\theta = \int_M \left[-\frac{1}{2n} g(\nabla^2 f, \omega)^2 + \frac{1}{2n}(\triangle f)^2 + \frac{1}{2n} P(\nabla f) \right] Vol_\theta.$$

After a substitution of (8.14) in (8.13) we come to the key identity

$$(8.15) \quad 0 = \int_M \left[Ric(\nabla f, \nabla f) + 4A(J\nabla f, \nabla f) - \frac{n+1}{n}(\triangle f)^2 \right] Vol_\theta$$
$$+ \int_M \left[\left| (\nabla^2 f)_{[1]} \right|^2 - \frac{1}{2n}(\triangle f)^2 - \frac{1}{2n}g(\nabla^2 f, \omega)^2 + \left| (\nabla^2 f)_{[-1]} \right|^2 - \frac{3}{2n}P(\nabla f) \right] Vol_\theta.$$

If we assume (1.2) and use an eigenfunction $\triangle f = \lambda f$ in (8.15) we obtain the inequality

$$(8.16) \quad 0 \geq \int_M \left(k_0 - \frac{n+1}{n}\lambda \right) |\nabla f|^2 \, Vol_\theta + \int_M \left[\left| (\nabla^2 f)_{[1]} \right|^2 - \frac{1}{2n}(\triangle f)^2 - \frac{1}{2n}g(\nabla^2 f, \omega)^2 \right] Vol_\theta$$
$$+ \int_M \left[\left| (\nabla^2 f)_{[-1]} \right|^2 - \frac{3}{2n}P(\nabla f) \right] Vol_\theta.$$

A projection on the span of the orthonormal set $\left\{ \frac{1}{\sqrt{2n}}g, \ \frac{1}{\sqrt{2n}}w \right\}$ in the $(1,1)$ space gives the inequality

$$(8.17) \qquad\qquad |(\nabla^2 f)_{[1]}|^2 \geq \frac{1}{2n}(\triangle f)^2 + \frac{1}{2n}\left(g(\nabla^2 f, \omega) \right)^2$$

with equality holding if and only if

$$(8.18) \qquad\qquad (\nabla^2 f)_{[1]} = \frac{1}{2n}(\triangle f) \cdot g + \frac{1}{2n}g(\nabla^2 f, \omega) \cdot \omega.$$

We obtain from (8.16) and (8.17) the inequality

$$0 \geq \int_M \left[\left(-\frac{n+1}{n}\lambda + k_0 \right) |\nabla f|^2 + \left| (\nabla^2 f)_{[-1]} \right|^2 - \frac{3}{2n}P(\nabla f) \right] Vol_\theta,$$

which, taking into account Lemma 8.4 and the assumptions of the Theorem, implies Greenleaf's inequality

$$\lambda \geq \frac{n}{n+1}k_0.$$

The proof of Theorem 8.8 is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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