THE NON-LINEAR DIRICHLET PROBLEM AND THE CR YAMABE PROBLEM

NICOLA GAROFALO AND DIMITER VASSILEV

1. Introduction

To introduce the questions addressed in this paper we recall that a Carnot group $\boldsymbol{G}$ is a simply connected nilpotent Lie group such that its Lie algebra $\mathfrak{g}$ admits a stratification $\mathfrak{g} = \overset{r}{\underset{j=1}{\oplus}} V_j$, with $[V_1, V_j] = V_{j+1}$ for $1 \leq j < r$, $[V_1, V_r] = \{0\}$. We assume that a scalar product $< \cdot, \cdot >$ is given on $\mathfrak{g}$ for which the $V_j's$ are mutually orthogonal. Every Carnot group is naturally equipped with a family of non-isotropic dilations def ined by

$$\delta_\lambda(g) \ = \ exp \circ \Delta_\lambda \circ exp^{-1}(g), \quad g \in \boldsymbol{G},$$

where $exp : \mathfrak{g} \to \boldsymbol{G}$ is the exponential map and $\Delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ is defined by $\Delta_\lambda(X_1 + ... + X_r) = \lambda X_1 + ... + \lambda^r X_r$. The topological dimension of $\boldsymbol{G}$ is $N \ = \ \sum_{j=1}^{r} dim\, V_j$, whereas the *homogeneous dimension* of $\boldsymbol{G}$, attached to the group of dilations $\{\delta_\lambda\}_{\lambda>0}$, is given by $Q \ = \ \sum_{j=1}^{r} j \ dim\, V_j$. We denote by $dH = dH(g)$ a fixed Haar measure on $\boldsymbol{G}$. One has $dH(\delta_\lambda(g)) = \lambda^Q dH(g)$, so that the number Q plays the role of a dimension with respect to the group dilations.

The Euclidean distance to the origin $|\cdot|$ on $\mathfrak{g}$ induces a homogeneous norm $|\cdot|_{\mathfrak{g}}$ on $\mathfrak{g}$ and (via the exponential map) one on the group $\boldsymbol{G}$ in the following way (see also [15]). For $\xi \in \mathfrak{g}$, with $\xi = \xi_1 + ... + \xi_r$, $\xi_i \in V_i$, we let

$$|\xi|_{\mathfrak{g}} \ = \ \left(\sum_{i=1}^{r} |\xi_i|^{2r!/i} \right)^{2r!}, \tag{1.1}$$

and then define $|g|_{\boldsymbol{G}} \ = \ |\xi|_{\mathfrak{g}}$ if $g = exp\ \xi$. Such homogeneous norm on $\boldsymbol{G}$ can be used to define a pseudo-distance on $\boldsymbol{G}$:

$$\rho(g, h) \ = \ |h^{-1} g|_{\boldsymbol{G}}. \tag{1.2}$$

Let $X = \{X_1, \ \dots \ ,X_m\}$ be a basis of V_1 and continue to denote by X the corresponding system of sections on $\boldsymbol{G}$. The pseudo-distance (1.2) is equivalent to the Carnot-Carathéodory distance $d(\cdot, \cdot)$ generated by the system X, i.e., there exists a constant $C = C(\boldsymbol{G}) > 0$ such that

$$C \ \rho(g, h) \ \leq d(g, h) \ \leq \ C^{-1} \ \rho(g, h), \qquad g, h \in \boldsymbol{G}, \tag{1.3}$$

Date: June 20, 1999.

The first author was supported in part by NSF Grant No. DMS-9706892.

The second author was supported in part by a grant of the Purdue Research Foundation and also by the first author's NSF Grant No. DMS-9706892.

see [36]. If $B(x,R) = \{y \in \boldsymbol{G} \mid d(x,y) < R\}$, then by left-translation and dilation it is easy to see that the Haar measure of $B(x,R)$ is proportional to R^Q, where $Q = \sum_{i=1}^{r} i \dim V_i$ is the homogeneous dimension of $\boldsymbol{G}$. One has for every $f,g,h \in \boldsymbol{G}$ and for any $\lambda > 0$

$$d(gf,gh) = d(f,h), \qquad d(\delta_\lambda(g), \delta_\lambda(h)) = \lambda\, d(g,h).$$

The *sub-Laplacian* associated to X is the second-order partial differential operator on $\boldsymbol{G}$ given by

$$\mathcal{L} \;=\; -\sum_{j=1}^{m} X_j^* X_j \;=\; \sum_{j=1}^{m} X_j^2$$

(we recall that in a Carnot group one has $X_j^* = -X_j$, see [15]). By the assumption on the Lie algebra one immediately sees that the system X satisfies the well-known finite rank condition, therefore thanks to Hörmander's theorem [22] the operator $\mathcal{L}$ is hypoelliptic. However, it fails to be elliptic, and the loss of regularity is measured by the step r of the stratification of $\mathfrak{g}$. For a function u on $\boldsymbol{G}$ we let $|Xu| = (\sum_{j=1}^m (X_j u)^2)^{1/2}$. For $1 \le p < Q$ we set

$$\overset{o}{\mathcal{D}}{}^{1,p}(\Omega) = \overline{C_o^\infty(\Omega)}^{||\cdot||_{\mathcal{D}^{1,p}(\Omega)}},$$

where $\mathcal{D}^{1,p}(\Omega)$ indicates the space of functions $u \in L^{p^*}(\Omega)$ having distributional horizontal gradient $Xu = (X_1 u, ..., X_m u) \in L^p(\Omega)$. The space $\mathcal{D}^{1,p}(\Omega)$ is endowed with the obvious norm

$$||u||_{\mathcal{D}^{1,p}(\Omega)} \;=\; ||u||_{L^{p^*}(\Omega)} + ||Xu||_{L^p(\Omega)}.$$

Here, $p^* = \frac{pQ}{Q-p}$ is the Sobolev exponent relative to p. The relevance of such number is emphasized by the following important embedding due to Folland and Stein [15], [16].

Theorem (Folland and Stein). *Let $\Omega \subset \boldsymbol{G}$ be an open set. For any $1 < p < Q$ there exists $S_p = S_p(\boldsymbol{G}) > 0$ such that for $u \in C_o^\infty(\Omega)$*

$$\left(\int_\Omega |u|^{p^*} dH\right)^{1/p^*} \le S_p \left(\int_\Omega |Xu|^p dH\right)^{1/p}. \tag{1.4}$$

We are interested in the following non-linear Dirichlet problem

$$\begin{cases} \mathcal{L}u \;=\; -\; u^{\frac{Q+2}{Q-2}} \\ u \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega), \quad u \ge 0. \end{cases} \tag{1.5}$$

When Ω coincides with the whole group $\boldsymbol{G}$ we will talk of an *entire solution* to the problem (1.5). We are interested in questions of existence and non-existence of weak solutions when:

- [i] Ω is bounded;
- [ii] Ω is unbounded, yet it is not the whole group;
- [iii] Ω coincides with the whole group.

The exponent $\frac{Q+2}{Q-2} = 2^* - 1$ is critical for the case $p = 2$ of the embedding (1.4). To motivate our results we recall that in the classical Riemannian setting the equation $\Delta u = -u^{(n+2)/(n-2)}$ is connected to the compact Yamab e problem [40], [3], [38], see also the book [4] and the survey article [32]. There exists an analogue of such problem in CR geometry, namely: *Given a compact, strictly pseudo-convex CR manifold, find a choice of contact form for which the Webster-Tanaka pseudo-hermitian scalar curvature is constant.* The pde associated to the CR Yamabe problem is the one that appears in (1.5). Although on the formal level this problem has many similarities with its Riemann ian predecessor, the analysis is considerably harder since, as we mentioned, the sub-Laplacian $\mathcal{L}$ fails to be elliptic everywhere. In 1984-88 D. Jerison and J. Lee in a series of important papers [24], [25], [26], [**?**] gave a complete solution to the CR Yamabe problem when the CR manifold M has dimension ≥ 5 and M is not locally CR equivalent to the sphere in $\mathbb{C}^{n+1}$. They proved first that the CR Yamabe problem can be solved on any compact CR mani fold M provided that the CR Yamabe invariant of M is strictly less than that of the sphere in $\mathbb{C}^{n+1}$. Similarly to Aubin's approach in the Riemannian case, in order to determine when the problem can be solved they then proved that the Yama be functional is minimized by the standard Levi form on the sphere and its images under CR automorphisms. A crucial step in this analysis is the explicit computation of the extremal functions in the special case when $p = 2$ and $\boldsymbol{G}$ is the Heisenberg grou p in the above stated Folland-Stein embedding . Jerison and Lee made the deep discovery that, up to group translations and dilations, a suitable multiple of the function

$$u(z,t) \ = \ ((1+|z|^2)^2 \ + \ t^2)^{-(Q-2)/4}, \tag{1.6}$$

is the only positive solution of (1.5) when $\Omega = \mathbb{H}^n$. Here, we have denoted with (z,t), $z \in \mathbb{C}^n, t \in \mathbb{R}$, the variable point in $\mathbb{H}^n$.

In 1980 A. Kaplan [28] introduced a class of Carnot groups of step two in connection with hypoellipticity questions. Such groups, which are called of *Heisenberg type*, constitute a direct and important generalization of the Heisenberg group, a s they include, in particular, the nilpotent component in the Iwasawa decomposition of simple groups of rank one. In his first work on the subject Kaplan [28] constructed an explicit fundamental solution for the sub-Laplacian, thus extending Follan d's result for the Heisenberg group [14], see (1.8). In [6] Capogna, Danielli and one of us found explicit formulas for the fundamental solution of the p-sub-Laplacian in any group of Heisenberg type, and for the horizontal p-capacity of rings.

Some years ago we discovered that when $\boldsymbol{G}$ is a group of Heisenberg type, then problem (1.5) possesses the following remarkable family of entire solutions.

Theorem 1.1. *Let $\boldsymbol{G}$ be a group of Heisenberg type. For every $\epsilon > 0$ the function*

$$K_\epsilon(g) \ = \ C_\epsilon \ \left((\epsilon^2 + |x(g)|^2)^2 + 16|y(g)|^2\right)^{-(Q-2)/4}, \qquad g \in \boldsymbol{G}, \tag{1.7}$$

where $C_\epsilon = [m(Q-2)\epsilon^2]^{(Q-2)/4}$, is a positive, entire solution of the Yamabe equation (1.5).

The symbols $x(g), y(g)$ in (1.7) respectively denote the projection of the exponential coordinates of the point $g \in \boldsymbol{G}$ onto the first and second layer of the Lie algebra $\mathfrak{g}$, whereas m indicates the dimension of the first layer. One s hould compare (1.7) with the Jerison-Lee minimizer (1.6). To give a glimpse of the complexity of the present situation with respect to the classical one we recall Folland's mentioned fundamental solution for the Kohn sub-Laplacian on $\mathbb{H}^n$

$$\Gamma(z,t) = C_Q(|z|^4 + t^2)^{-(Q-2)/4}, \tag{1.8}$$

where C_Q is a suitable constant. Whereas Γ is (remarkably) a function of the natural homogeneous gauge $\rho = \rho(z,t) = (|z|^4 + t^2)^{1/4}$, the Jerison-Lee minimizer in (1.6) is not. This is in strong contrast with the famous results of Aubin [1], [2] and Talenti [39] who proved that for every value of p the minimizers in the Sobolev embedding are functions with spherical symmetry. After discovering the entire solutions K_ϵ we formulated the following

Conjecture: *In a group of Heisenberg type, up to group translations the functions K_ϵ in* (1.7) *are the only positive entire solutions to (1.5).*

If true, such conjecture would generalize Jerison and Lee's cited result to groups of Heisenberg type. This problem turns out to be considerably harder than its already difficult Heisenberg group predecessor. In a forthcoming work we plan to come back to it and prove the full conjecture. However, in section four we announce some partial progress toward it.

In closing we mention that the results described in this paper are contained in the two papers [19], [20].

2. **Bounded domains**

We next describe the main results, starting with the case of bounded domains. In the following definition the notion of starlikeness is expressed by means of the infinitesimal generator Z of the group dilations $\{\delta_\lambda\}_{\lambda>0}$.

Definition 2.1. *Let $D \subset \boldsymbol{G}$ be a connected open set of class C^1 containing the group identity e. We say that D is* starlike with respect to the identity e *(or simply starlike)* along a subset $M \subset \partial D$*, if*

$$< Z, \eta > (g) \ \geq \ 0$$

at every $g \in M$. D is called starlike with respect to the identity e *if it is starlike along $M = \partial D$. We say that D is* uniformly starlike with respect to e along M *if there exists a constant $\alpha = \alpha_D > 0$ such that for every $g \in M$*

$$< Z, \eta > (g) \ \geq \ \alpha.$$

A domain as above is called starlike (uniformly starlike) with respect to one of its points g along $M \subset \partial D$*, if $g^{-1}D$ is starlike (uniformly starlike) along $g^{-1}M$ with respect to e.*

Theorem 2.2. *Let $D \subset \boldsymbol{G}$ be C^∞, bounded and starlike with respect to $g_o \in D$. Suppose that $u \in \Gamma^{0,\alpha}(\overline{D})$ is a non-negative solution of*

$$\begin{cases} \mathcal{L}u \ = \ - \ f(u) \\ u \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega), \quad u \geq 0, \end{cases} \tag{2.1}$$

with $f \in C^\infty(\mathbb{R})$. Assume in addition that $Xu \in L^\infty(D)$ and $Zu \in L^\infty(D)$. If

$$2QF(u) \ - \ (Q-2)uf(u) \ \leq \ 0, \tag{2.2}$$

then $u \equiv 0$. In particular, (2.1) *has no non-trivial such solution when $f(u) = u^q$, if $q \geq \frac{Q+2}{Q-2}$.*

Remark 2.3. *The inequality (2.2) is the analogue of the famous Pohožaev condition for Laplace equation, see* [37]*. We mention that the first non-existence result for the Heisenberg group $\mathbb{H}^n$ was obtained via an integral identity of Rellich-Pohoza ev type in* [17]*. In that paper however the relevant solutions were a priori assumed to be globally smooth and the delicate question of regularity at characteristic points was not addressed.*

It is important to remark that the vector field Z is neither left-invariant, nor it is *sub-unitary* according to C. Fefferman and D.H. Phong [13]. One easily sees that, in exponential coordinates, the vector field Z involves commutators up to maximum length. In the classical case the boundary regularity of the relevant solution which is necessary to apply the Rellich-Pohožaev identity is guaranteed, via standard elliptic theory, by suitable smoothness assumptions on the ground domain Ω, see, e.g., [37]. The situation is drastically different in the sub-elliptic setting even if the domain Ω is C^∞, due to the presence of characteristic points on the boundary of Ω. We recall that the characteristic set of a smooth domain $\Omega \subset \boldsymbol{G}$ with respect to the system X is

$$\Sigma = \Sigma_{\Omega,X} = \{g \in \partial\Omega \mid X_j(g) \in T_g(\partial\Omega), j = 1, ..., m\}.$$

A bounded domain with trivial topology in a group of Heisenberg type always has a non-empty characteristic set. Theorem 2.2 constitutes the main motivation for our study of the regularity near the characteristic set. Due to the well-know n counterexample of Jerison [23] to the boundary regularity in a neighborhood of a characteristic point it is not clear *a priori* that Theorem 2.2 has any content at all. The next two results prove that it does indeed, at lea st if the ground domain Ω satisfy some very natural and easily verifiable geometric conditions.

Henceforth, we consider a C^∞, connected, bounded open set $\Omega \subset \boldsymbol{G}$. We suppose that Ω satisfies the following natural condition: There exist $A, r_o > 0$ such that for every $Q \in \partial\Omega$ and every $0 < r < r_0$

$$|(\boldsymbol{G} \setminus \Omega) \cap B(Q,r)| \geq A|B(Q,r)|. \tag{2.3}$$

Such geometric assumption is fulfilled if, e.g., Ω satisfies the uniform corkscrew condition, see [7], [9]. These papers contain an extensive study of examples of domains which, in particular, satisfy (2.3). For us it is impo rtant that (2.3) allows to adapt to the present setting the classical arguments that lead, via Moser's iteration, to obtain $u \in \Gamma^{0,\alpha}(\bar\Omega)$ for some $0 < \alpha < 1$, see, e.g., [21], Section 8.10. Extending u with zero outside Ω, we can assume henceforth that

$$u \in \Gamma^{0,\alpha}(\boldsymbol{G}). \tag{2.4}$$

If we suppose further that Ω is a C^∞ domain, and denote by $\Sigma = \Sigma_{\Omega,X}$ the characteristic set of Ω, then thanks to the results of Kohn and Nirenberg [29], for every $Q \in \partial\Omega \setminus \Sigma$ there exists a neig hborhood U of Q such that $u \in C^\infty(\bar\Omega \cap U)$. From these considerations it is clear that the main new obstacle to overcome is the regularity of a weak solution to (1.5) near the characteristic set Σ.

Since our assumptions on Ω are of a local nature, and they involve the geometry of the domain near its characteristic set Σ, there is no restriction in assuming the existence of $\rho \in C^\infty(\boldsymbol{G})$ and of $\gamma_\Omega > 0$ such that for so me $R \in \mathbb{R}$

$$\Omega = \{g \in \boldsymbol{G} \mid \rho(g) < R\}, \tag{2.5}$$

and for which one has $|D\rho(g)| \geq \gamma_\Omega > 0$, for every g in some relatively compact neighborhood K of ∂D. The outward pointing unit normal to $\partial\Omega$ is $\eta = \frac{D\rho}{|D\rho|}$.

We now assume that Ω be uniformly starlike along Σ, see Definition 2.1, with respect to one of its points, which by performing a left-translation we can take to be the group identity e. We explicitly remark that when this is the case, then by the compactness of Σ we can find a bounded open set U and a constant $\delta > 0$ such that $\Sigma \subset U$ and for which, setting $\Delta = \partial\Omega \cap U$, one has

$$Z\rho(g_o) \geq \delta \ > \ 0, \qquad \text{for } g_o \in \Delta. \tag{2.6}$$

We note that the uniform transversality condition (2.6) implies that the trajectories of Z starting from points of Δ fill a full open set ω interior to Ω. By possibly shrinking the set U we can assume that $\omega = \Omega \cap U$. To fix the notation we suppose that there exists $\lambda_o > 0$ such that

$$\delta_\lambda g_o \in \omega \qquad \text{for} \quad \lambda_o < \lambda < 1.$$

Hereafter, given a point $g \in \boldsymbol{G}$ we respectively denote with

$$x(g) \;=\; (x_1(g), ..., x_m(g)), \qquad\qquad y(g) \;=\; (y_1(g), ..., y_k(g))$$

the projection of the exponential coordinates of g on the first and second layer of the Lie algebra $\mathfrak{g}$. We define $\psi(g) = |x(g)|^2$.

In addition to (2.6) we assume that there exists $C_o > 0$ such that the definining function ρ of Ω satisfies the following *convexity* condition

$$\mathcal{L}\rho \;\geq\; C_o \;< X\rho, X\psi > \qquad \text{in} \quad \omega. \tag{2.7}$$

We emphasize that a sufficient condition for (2.7) to hold is the strict $\mathcal{L}$-sub-harmonicity of the defining function ρ of Ω near the characteristic set Σ. On the other hand, since on Σ we evidently have $< X\rho, Xsi >= 0$, the $\mathcal{L}$-sub-harmonicity of ρ on Σ (but not the strict one) is also necessary.

The following two theorems constitute the main regularity results of this section.

Theorem 2.4. *Consider a C^∞ domain Ω in a Carnot group $\boldsymbol{G}$ satisfying* (2.3)*,* (2.6) *and* (2.7)*. Let u be a weak solution of* (1.5)*, then*

$$Xu \;\in\; L^\infty(\Omega).$$

In the next result we establish the boundedness of the $Z-$derivative of the solution of (1.5) near the characteristic set. We stress once again that such derivative involves commutators of the vectors X_j up to maximum order.

Theorem 2.5. *Let $\boldsymbol{G}$ be a Carnot group of step two. Consider a C^∞ connected, bounded open set $\Omega \subset \boldsymbol{G}$ satisfying* (2.3)*,* (2.6) *and* (2.7)*. Under these assumptions, if u is a weak solution of* (1.5) *one has*

$$Zu \;\in\; L^\infty(\Omega). \tag{2.8}$$

Remark 2.6. *Unlike Theorem 2.4, in Theorem 2.5 we have assumed that the group $\boldsymbol{G}$ be of step two. We do not presently know whether Theorem 2.5 continues to hold for groups of arbitrary step.*

We next provide an important class of domains to which Theorems 2.4, 2.5, and therefore Theorem 2.2 apply. Let $\boldsymbol{G}$ be a Carnot group of step two. We define the function

$$f_\epsilon(g) \;=\; \left((\epsilon^2 \;+\; |x(g)|^2)^2 \;+\; 16|y(g)|^2\right)^{1/4}, \qquad\qquad \epsilon \in \mathbb{R}. \tag{2.9}$$

For $R > 0$ and $\epsilon \in \mathbb{R}$, with $\epsilon^2 < R^2$, consider the C^∞ bounded open set

$$\Omega_{R,\epsilon} \;=\; \{g \in \boldsymbol{G} \mid f_\epsilon(g) < R\}. \tag{2.10}$$

When $\epsilon = 0$ it is clear that $\Omega_{R,\epsilon}$ is nothing but a gauge pseudo-ball centered at the group identity e, except that the natural gauge is defined in (1.1) without the factor 16. Here we have introduced such (immaterial) facto r for the purpose of keeping a consistent definition with the case of groups of Heisenberg type discussed in the next section. For $g \in \boldsymbol{G}$, we let

$$\Omega_{R,\epsilon}(g) \;=\; \{h \in \boldsymbol{G} \mid f_\epsilon(g^{-1}h) < R\} \;=\; g\;\Omega_{R,\epsilon}.$$

It is very easy to verify that the domains $\Omega_{R,\epsilon}(g)$ fulfill the geometric assumptions (2.3), (2.6) and (2.7), so that Theorems 2.4, 2.5 and 2.2 can be applied. This proves the fo llowing basic result.

Theorem 2.7. *Let $\boldsymbol{G}$ be a Carnot group of step two. Given any $g \in \boldsymbol{G}, R \in \mathbb{R}$ and $\epsilon \in \mathbb{R}$ with $\epsilon^2 < R^2$, the function $u \equiv 0$ is the only non-negative weak solution of* (1.5) *in $\Omega_{R,\epsilon}(g)$.*

Besides having an interest in its own right, Theorem 2.7 also plays an important role in the analysis of unbounded domains, to which task we now turn.

3. **Unbounded domains different from the whole group**

Our first objective is to introduce an appropriate notion of cones and half-spaces in a Carnot group. This can be done in a natural way by means of the exponential map, or instead working directly on the group by exploiting its homogeneous structure. This latter approach was fully developed in [7]. Below, we will use the former approach. We recall that for a point $g \in \boldsymbol{G}$ we denote with $x(g) = (x_1(g), ..., x_m(g))$ and $y(g) = (y_1(g), ..., y_k(g))$ the projection of the exponential coordinates of g on the first and second layer of the Lie algebra $\mathfrak{g}$. We indicate with $\mathbb{R}^k_+$ the cone $\{(y_1, ..., y_k) \in \mathbb{R}^k \mid y_i \geq 0, i = 1, ..., k\}$.

Definition 3.1. *Let $\boldsymbol{G}$ be a Carnot group of step two. Given $M, b \in \mathbb{R}$, and $\boldsymbol{a} \in \mathbb{R}^k \setminus \{0\}$, we call the open sets*

$$C^+_{M,b,\boldsymbol{a}} \; = \; \{g \in \boldsymbol{G} \mid < y(g), \boldsymbol{a} > \; > \; M|x(g)|^2 + b\}$$

and

$$C^-_{M,b,\boldsymbol{a}} \; = \; \{g \in \boldsymbol{G} \mid < y(g), \boldsymbol{a} > \; < \; -M|x(g)|^2 + b\}$$

characteristic cones. *In the case in which $\boldsymbol{a} \in \mathbb{R}^k_+ \setminus \{0\}$, then we call the cone* convex *if $M \geq 0$,* concave *if $M < 0$. When $M = 0$ we use the notation $H^\pm_{b,\boldsymbol{a}}$ to indicate the* ch aracteristic half-spaces

$$C^+_{0,b,\boldsymbol{a}} \; = \; \{g \in \boldsymbol{G} \mid < y(g), \boldsymbol{a} > \; > \; b\}, \qquad C^-_{0,b,\boldsymbol{a}} \; = \; \{g \in \boldsymbol{G} \mid < y(g), \boldsymbol{a} > \; < \; b\}.$$

The boundaries of such half-spaces are called characteristic hyperplanes.

The relevance of these domains becomes especially evident in the framework of groups of Heisenberg type due to a remarkable notion of inversion and Kelvin transform which for the Heisenberg group $\mathbb{H}^n$ were first developed by Korányi in [30]. Subs equently, such inversion formula, as well as the Kelvin transform, were generalized in [11] and [10] to all groups of Heisenberg type. We begin with the formal definition of group of Heisenberg type. Let $\boldsymbol{G}$ be a group of step two with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ and consider the map $J : V_2 \to End(V_1)$ defined by

$$< J(\xi_2)\xi_1', \xi_1" > = < [\xi_1', \xi_1"], \xi_2 >, \qquad \xi_1', \xi_1" \in V_1, \xi_2 \in V_2. \tag{3.1}$$

From the definition it is immediately obvious that

$$< J(\xi_2)\xi_1, \xi_1 > \; = \; 0, \qquad \xi_1 \in V_1, \xi_2 \in V_2.$$

Definition 3.2. *A Carnot group of step two, $\boldsymbol{G}$, is called of* Heisenberg type *if for every vector $\xi_2 \in V_2$ the map $J(\xi_2) : V_1 \to V_1$ defined by* (3.1) *is orthogonal, i.e.,*

$$|J(\xi_2)\xi_1| \; = \; |\xi_2| \; |\xi_1|. \tag{3.2}$$

Definition 3.2 is due to A. Kaplan [28]. We are now ready to introduce the CR inversion and Kelvin transform in a group of Heisenberg type, see [30], [11] and [10].

Definition 3.3. *Let $\boldsymbol{G}$ be a group of Heisenberg type with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$. For $g = exp(\xi) \in \boldsymbol{G}$, with $\xi = \xi_1 + \xi_2$, the* inversion $\sigma : \boldsymbol{G}^* \to \boldsymbol{G}^*$, *where $\boldsymbol{G}^* = \boldsymbol{G} \setminus \{e\}$ is defined by*

$$\sigma(g) = \Big(- \ \big(|x|^2 + 4J(\xi_2)\big)^{-1}\xi_1, - \ \frac{\xi_2}{|x|^4 + 16|y|^2}\Big),$$

where the map J is as in (3.1). *One easily verifies that*

$$\sigma^2(g) \ = \ g, \qquad\qquad g \in \boldsymbol{G}^*.$$

As in definition (2.9) in the sequel we will use the renormalized gauge

$$N(g) = \big(|x(g)|^4 + 16|y(g)|^2\big)^{1/4}, \tag{3.3}$$

since the latter is better suited than (1.1) to the structure of a group of Heisenberg type. This fact is witnessed by the following remarkable fact which was discovered by Kaplan [28]. In a group of Heisenberg type the fundamental solution Γ of the sub-Laplacian $\mathcal{L}$ is given by the formula

$$\Gamma(g,h) \ = \ C(\boldsymbol{G}) \ \ N(h^{-1}g)^{-(Q-2)}, \qquad\qquad g,h \in \boldsymbol{G}, g \neq h, \tag{3.4}$$

where $C(\boldsymbol{G}) > 0$ is a suitable constant. Equation (3.4) will play an important role in Definition 3.6 below. Writing $\sigma(g) = \exp(\eta)$, with $\eta = \eta_1 + \eta_2$, for the image of g we see easily from Definition 3.3 and (3.2) that

$$|\eta_1| \ = \ \frac{|\xi_1|}{N(g)^2}, \qquad \text{and} \qquad |\eta_2| \ = \ \frac{|\xi_2|}{N(g)^4} \tag{3.5}$$

An immediate consequence of (3.5) is that

$$N(\sigma(g)) \ = \ N(g)^{-1}, \qquad\qquad g \in \boldsymbol{G}^*. \tag{3.6}$$

A direct verification shows that the inversion anticommutes with the group dilations, i.e.,

$$\sigma(\delta_\lambda(g)) = \delta_{\lambda^{-1}}(\sigma(g)), \qquad\qquad g \in \boldsymbol{G}^*. \tag{3.7}$$

A corollary of (3.7) is that starlikeness behaves well under inversion. This is contained in the following result.

Proposition 3.4. *Let $\rho \in C^\infty(\boldsymbol{G})$. The following formula holds*

$$Z(\rho \circ \sigma) \ = \ - \ (Z\rho) \circ \sigma.$$

In connection with Proposition 3.4 we mention that the starlikeness of the level sets of positive $\mathcal{L}-$harmonic functions in unbounded domains in Carnot groups was first obtained in [12].

The next proposition underlines the remarkable connection between the convex cones in groups of Heisenberg type and the bounded domains introduced in (2.10).

Proposition 3.5. *Let $\boldsymbol{G}$ be a group of Heisenberg type with the inversion as in Definition 3.3. For every $M \geq 0$, $b > 0$, $\boldsymbol{a} \in \mathbb{R}^k_+ \setminus \{0\}$, define $\epsilon = \sqrt{M/2b}$, $R^2 = \sqrt{16M^2 + |\boldsymbol{a}|^2/8b}$, and co nsider the set*

$$\Omega_{R,\epsilon} \;=\; \{g \in \boldsymbol{G} \mid (|x(g)|^2 + \epsilon^2)^2 + 16|y(g)|^2 \;<\; R^4\}.$$

One has

$$\sigma(C^+_{M,b,\boldsymbol{a}}) \;= (0, -\frac{\boldsymbol{a}}{32b})\; \Omega_{R,\epsilon} \;=\; \left\{g \in \boldsymbol{G} \mid \;\; (|x(g)|^2 + \epsilon^2)^2 + 16|y(g) + \frac{\boldsymbol{a}}{32b}|^2 \;<\; R^4\right\}.$$

In particular, the image through the inversion of the characteristic half-space $H^+_{b,\boldsymbol{a}} \;=\; \{g \in \boldsymbol{G} \mid < y(g), \boldsymbol{a} > \; > \; b\}$ is the gauge ball $B((0, -\frac{\boldsymbol{a}}{32b}), R) \;=\; \{g \in \boldsymbol{G} \mid |x(g)|^4 + \left|y(g) + \frac{textbfa}{32b}\right|^2 \;<\; R^4\}$.

In the sequel we denote by Ω^* the image of a generic domain Ω under the inversion σ. We stress that, since we have chosen not to define the inversion of the point at infinity, in the case in which Ω is a neighborhood of ∞, by which we mean that there exists a ball $B(e, R)$ such that $(\boldsymbol{G} \setminus \overline{B}(e, R)) \subset \Omega$, then Ω^* is a punctured neighborhood of the identity, i.e., $\Omega^* = D \setminus \{e\}$, for an open set D such that $e \in D$. The reader shoul d keep this point in mind for the statement of the next results.

Definition 3.6. *Let $\boldsymbol{G}$ be a group of Heisenberg type, and consider a function u on $\boldsymbol{G}$. The* CR Kelvin transform *of u is defined by the equation*

$$u^*(g) \;=\; N(g)^{-(Q-2)}\; u(\sigma(g)), \qquad\qquad g \in \boldsymbol{G}^*.$$

An important subset of that of groups of Heisenberg type is the class of groups of Iwasawa type. Such groups arise as the nilpotent component N in the Iwasawa decomposition KAN of a simple group of rank one. When $\boldsymbol{G}$ is a group of Iwasawa type, then it was proved in [10] that the inversion and the Kelvin transform possess some remarkable properties. In the following theorem we collect the two which are most important in the sequel.

Theorem 3.7 (see [10])**.** *Let $\boldsymbol{G}$ be a group of Iwasawa type. The Jacobian of the inversion is given by*

$$d(H \circ \sigma)(g) \;=\; N(g)^{-2Q}\; dH(g), \qquad\qquad g \in \boldsymbol{G}^*.$$

The Kelvin transform u^* of a function satisfies the equation

$$\mathcal{L}u^*(g) \;=\; N(g)^{-(Q+2)}(\mathcal{L}u)(\sigma(g)), \qquad\qquad g \in \boldsymbol{G}^*.$$

The following theorem is an important consequence of the conformal properties of the inversion and of the Kelvin transform expressed by Theorem 3.7.

Theorem 3.8. *The Kelvin transform is an isometry between $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega)$ and $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega^*)$.*

Such result is used in combination with the conformal invariance of the Yamabe type equation expressed by the following proposition.

Proposition 3.9. *Let u be a solution of*

$$\begin{cases}\mathcal{L}u \;=\; -\;u^p \\ u\in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega), \quad u\geq 0,\end{cases} \tag{3.8}$$

and denote by u^* its Kelvin transform. Then u^* satisfies

$$\mathcal{L}u^*(g) \;=\; -\;N(g)^{p(Q-2)-(Q+2)}\;u^*(g)^p \qquad\qquad g\in\Omega^*. \tag{3.9}$$

In particular, when $p=\frac{Q+2}{Q-2}$ we conclude that if u satisfies problem (1.5)*, then u^* is a solution of the same problem in Ω^*.*

The following theorem asserts that if u^* is a solution to (1.5) in a neighborhood of infinity, then the Kelvin transform of u^* has a removable singularity at the group identity e. It plays a crucial role in converting the Yamabe prob lem (1.5) on an unbounded domain to the same problem on a bounded one, via the CR inversion.

Theorem 3.10. *Let $\boldsymbol{G}$ be an Iwasawa group. Suppose that u^* is a solution of* (1.5) *in Ω^*, with Ω^* a neighborhood of infinity. Let u be the Kelvin transform of u^* defined in Ω, then the group identity e is a removable singulari ty, i.e., u can be extended as a smooth function in a neighborhood of e where the equation is satisfied.*

Using Theorem 3.8, Proposition 3.9, and Theorems 3.10, 2.4, 2.5 and 2.2, we obtain the main non-existence result for unbounded domains (which do not coincide with the whole group) .

Theorem 3.11. *Let $\boldsymbol{G}$ be a group of Iwasawa type. Consider a C^∞ unbounded open set $\Omega^*\subset\boldsymbol{G}$ and denote by Ω its image through the inversion. Suppose that $\Omega=D\setminus\{e\}$, where D is a bounded open set, containing the identity, which satisfies all the hypothesis in Theorem 2.4. In this situation there exists no solution to problem* (1.5) *in Ω^*, other than $u^*\equiv 0$.*

Here is a basic consequence of Theorem 3.11.

Corollary 3.12. *Let $\boldsymbol{G}$ be a group of Iwasawa type and consider the unbounded domain $\Omega^*=\{g\in\boldsymbol{G}\mid N(gg_o^{-1})>R\}$, where N is the gauge in (3.3), $g_o\in\boldsymbol{G}$ and $R>0$ are fixed. There exist no non-trivial solution to* (1.5) *in Ω^*.*

Proof. By left-translation and rescaling we can suppose that $g_o=e, R=1$. In this situation, it is easy to verify Ω^* is mapped by the inversion in $D=\Omega\setminus\{e\}$, where $\Omega=\{g\in\boldsymbol{G}\mid N(g)<1\}$. To complete the proof it is enough to observe that, as it was proved in Theorem 2.7 (case $\epsilon=0$), the domain Ω fulfills the assumptions in Theorem 2.4.

□

We finally consider a notable class of unbounded domains with non-compact boundary, the convex characteristic cones, and prove that these sets do not support non-trivial solutions to the Yamabe problem (1.5).

Theorem 3.13. *Consider a group of Iwasawa type $\boldsymbol{G}$. Let $C^{\pm}_{M,b,\boldsymbol{a}}\subset\boldsymbol{G}$ be a convex characteristic cone as in Definition 3.1. There exists no solution to* (1.5) *in $\Omega^*=C^{+}_{M,b,\boldsymbol{a}}$, other than $u\equiv 0$. In particular, there exist no non-trivial solutions for the characteristic half-spaces $H^{\pm}_{b,\boldsymbol{a}}$.*

Proof. Suppose u^* is a non-trivial solution to (1.5) in $C^+_{M,b,\mathbf{a}}$ and denote by u its Kelvin transform. In view of Proposition 3.5, u is defined in $(0, -\frac{\mathbf{a}}{32b})\ \Omega_{R,\epsilon}$, where $\Omega_{R,\epsilon}$ is the domain in (2.10), with R and ϵ specified as in Proposition 3.5. By left-translation we obtain a new non-trivial function, which for simplicit y we continue to denote with u, defined in the bounded open set $\Omega_{R,\epsilon}$. From Theorem 3.8 we infer that $u \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega_{R,\epsilon})$. Thanks to Proposition 3.9 we know that u is a non-trivial solution to problem (1.5) in $\Omega_{R,\epsilon}$. At this point we invoke Theorem 2.7 to reach a contradiction. The proof is thus completed.

□

In connection with Theorem 3.13 we mention that Lanconelli and Uguzzoni [31] have recently obtained in the special case of the Heisenberg group $\mathbb{H}^n$ an interesting non-existence result for the non-characteristic hyperplanes, i.e., those hyperplanes which are parallel to the grou p center (the t-axis). Their analysis is essentially different from ours since, given the absence of characteristic points on the boundary, their focus is on the asymptotic behavior of a solution to (1.5) at infinity. In a note added in p roof in [31] it is said that in the forthcoming article [41] Uguzzoni has obtained, for the characteristic hyperplanes H_a in the Heisenberg group, a uniqueness result similar to the second part of our Theorem 3.13.

4. **Entire solutions**

We finally describe some progress toward the proof of the main Conjecture in section one. Before proceeding we note that an important consequence of a suitable adaptation of the method of concentration of compactness due to P. L. Lions [33], [34] allows to prove that in any Carnot group (1.5) always admits at least one entire solution, see [42]. In this regard an elementary, yet crucial observation, is that if u is an entire solution to (1.5), then such are also the two functions

$$\tau_h u \overset{def}{=} u \circ \tau_h, \qquad h \in \boldsymbol{G}, \tag{4.1}$$

where $\tau_h : \boldsymbol{G} \to \boldsymbol{G}$ is the operator of left-translation $\tau_h(g) = hg$, and

$$u_\lambda \overset{def}{=} \lambda^{(Q-2)/2}\, u \circ \delta_\lambda, \qquad \lambda > 0. \tag{4.2}$$

We need the following definition.

Definition 4.1. *Let $\boldsymbol{G}$ be a Carnot group of step two with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$. We say that a function $U : \boldsymbol{G} \to \mathbb{R}$ has* partial symmetry *if there exist an element $g_o \in \boldsymbol{G}$ such that for every $g = exp(x(g) + y(g)) \in \boldsymbol{G}$ one has*

$$U(g_o g) \ = \ u(|x(g)|, y(g)),$$

for some function $u : [0,\infty) \times V_2 \to \mathbb{R}$.

A function U is said to have cylindrical symmetry *if there exist $g_o \in \boldsymbol{G}$ and $\phi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ for which*

$$U(g_o g) \ = \ \phi(|x(g)|, |y(g)|),$$

for every $g \in \boldsymbol{G}$.

Our main result is the following.

Theorem 4.2. *Let $\boldsymbol{G}$ be a group of Iwasawa type. If $U \not\equiv 0$ is an entire solution to (1.5) having partial symmetry, then up to group translations and dilations we must have $U = K_\epsilon$, where K_ϵ is the function in Theorem* **??**.

Theorem 4.2 is a direct consequence of the following two results.

Theorem 4.3. *Let $\boldsymbol{G}$ be an Iwasawa group. Suppose $U \not\equiv 0$ is an entire solution of (1.5). If U has partial symmetry, then U has cylindrical symmetry.*

Theorem 4.4. *Let $U \not\equiv 0$ be an entire solution to (1.5) in a Iwasawa group $\boldsymbol{G}$ and suppose that U has cylindrical symmetry. There exists $\epsilon > 0$ such that*

$$U(g) \;=\; [m(Q-2)\epsilon^2]^{(Q-2)/4}\;((\epsilon^2 + |x(g)|^2)^2 \;+\; 16|y(g)|^2)^{-(Q-2)/4}.$$

All other cylindrically symmetric solutions are obtained from this one by the left-translations (4.1).

One should notice that, unlike the Euclidean case, in the Folland-Stein embedding there exists no spherical symmetrization, and therefore the search of minimizers cannot be reduced to an ordinary differential equation, as in the famous results of Aubin ¸teA1, [2] and Talenti [39]. Therefore, after Theorem 4.3 is in force one still needs to confront the non-trivial problem of the uniqueness of positive solutions of a certain non-linear pde in the Poincaré half-plane. This aspect is taken care of by Theorem 4.4.

In closing we mention that some interesting existence and non-existence results for positive entire solutions of the equation $\mathcal{L}u = -K(x)u^p$ in Carnot groups were announced by G. Lu and J. Wei in [35]. These authors also study the asympt otic behavior at infinity of the relevant solutions. We also mention that we have recently received a preprint by I. Birindelli and J. Prajapat [5] in which the authors prove in the context of the Heisenberg group $\mathbb{H}^n$ an interesting non-existence theorem for positive entire solutions having cylindrical symmetry of the equation $\mathcal{L}u = -u^p$, with sub-critical exponent $p < (Q+2)/(Q-2)$.

References

[1] T. Aubin, *Problèms isopérimétriques et espaces de Sobolev*, J. Diff. Geometry, **11** (1976), 573-598.

[2] ———, *Espaces de Sobolev sur les variétés Riemanniennes*, Bull. Sc. Math., **100** (1976), 149-173.

[3] ———, *Equations différentielles non linéaires et Problème de Yamabe concernant la courbure scalaire*, J. Math. Pures et appl., **55** (1976), 269-296.

[4] ———, *Nonlinear analysis on manifolds. Monge-Ampère equations*, Springer-Verlag, New York, 1982.

[5] I. Birindelli & J. Prajapat, *Nonlinear Liouville type theorems in the Heisenberg group via the moving plane method*, preprint.

[6] L. Capogna, D. Danielli & N. Garofalo, *Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations*, Amer. J. of Math., **118** (1997), 1153-1196.

[7] L. Capogna & N. Garofalo, *NTA domains for Carnot-Carathéodory metrics and a Fatou type theorem*, J. Fourier Anal. Appl., 4, **4-5** (1998), 403-432.

[8] L. Capogna, N. Garofalo & Duy-Minh Nhieu, A version of a theorem of Dahlberg for the sub-elliptic Dirichlet problem, Math. Res. Letters, **5** (1998), 541-549.

[9] ———, *The Dirichlet problem for sub-Laplacians*, preprint.

[10] M. Cowling, A. H. Dooley, A. Korányi & F. Ricci, *H-type groups and Iwasawa decompositions*, Adv. Math., **87** (1991), 1-41.

[11] M. Cowling & A. Korányi, *Harmonic analysis on Heisenberg type groups from a geometric viewpoint*, in "Lie Group Representation III", pp.60-100, Lec. Notes in Math., **1077** (1984), Springer-Verlag.

[12] D. Danielli & N. Garofalo, *Geometric properties of solutions to subelliptic equations in nilpotent Lie groups*, Lect. Notes in Pure and Appl. Math., "Reaction Diffusion Systems", Trieste, October 1995, Ed. G. Caristi Invernizzi, E. Mitidieri, Ma rcel Dekker, **194** (1998).

[13] C. Fefferman & D.H. Phong, *Subelliptic eigenvalue problems*, Proceedings of the Conference in Harmonic Analysis in Honor of A. Zygmund, Wadsworth Math. Ser., Belmont, CA, (1981), 530-606.

[14] G. Folland, *A fundamental solution for a subelliptic operator*, Bull. Amer. Math. Soc., **79** (1973), 373-376.

[15] G. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Math., **13** (1975), 161-207.

[16] G. B. Folland & E. M. Stein, *Estimates for the $\bar{\partial}_b$ Complex and Analysis on the Heisenberg Group*, Comm. Pure Appl. Math., **27** (1974), 429-522.

[17] N. Garofalo & E. Lanconelli, *Existence and non-existence results for semilinear elliptic problems in unbounded domains*, Indiana Univ. Math. J., **41** (1992) 71-98.

[18] N. Garofalo & D. M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math., **49** (1996), 1081-1144.

[19] N. Garofalo, D. N. Vassilev, *Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians*, preprint.

[20] ———, *Symmetry properties of positive entire solutions of Yamabe type equations on groups of Heisenberg type*, preprint.

[21] D. Gilbarg & N. S. Trudinger, *Elliptic partial differential equations of second order*, Second ed., Revised third printing, Vol. 244, Springer-Verlag, Grundlehren der mathematischen Wissenschaften, 1998.

[22] H. Hörmander, *Hypoelliptic second-order differential equations*, Acta Math., **119** (1967), 147-171.

[23] ———, *The Dirichlet problem for the Kohn Laplacian on the Heisenberg group, II*, J. of Funct. Anal. **43** (1981), 224-257.

[24] D. Jerison & J. Lee, *A subelliptic, nonlinear eigenvalue problem and scalar curvature on CR manifolds*, Contemporary Math., **27** (1984), 57-63.

[25] ———, *The Yamabe problem on CR manifolds*, J. Diff. Eq., **25** (1987), 167-197.

[26] ———, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. of the Amer. Math. Soc., 1, **1** (1988), 1-13.

[27] ———, *Intrinsic CR normal coordinates and the CR Yamabe problem*, J. Differential Geom., **29** (1989), no. 2, 303–343.

[28] A. Kaplan, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Amer. Math. Soc., **258** (1980), 147-153.

[29] J. J. Kohn & L. Nirenberg, *Non-coercive boundary value problems*, Comm. Pure and Appl. Math., 18, **18** (1965), 443-492.

[30] A. Korányi, *Kelvin transform and harmonic polynomials on the Heisenberg group*, Adv.Math. **56** (1985), 28-38.

[31] E. Lanconelli & F. Uguzzoni, *Asymptotic behavior and non-existence theorems for semilinear Dirichlet problems involving critical exponent on unbounded domains of the Heisenberg group*, Boll. Un. Mat. Ital., (8) **1-B** (1998), 139-168.

[32] J. M. Lee & T. H. Parker, *The Yamabe problem*, Bull. Am. Math. Soc., 1, **17** (1987), 37-91.

[33] ———, *The concentration compactness principle in the calculus of variations. The limit case. Part 1*, Rev. Mat. Iberoamericana **1.1** (1985), 145-201.

[34] ———, *The concentration compactness principle in the calculus of variations. The limit case. Part 2*, Rev. Mat. Iberoamericana **1.2** (1985), 45-121.

[35] G. Lu & J. Wei, *On positive entire solutions to the Yamabe-type problem on the Heisenberg and stratified groups*, ERA Amer. Math. Soc., **3** (1997), 83-89.

[36] A. Nagel, E.M. Stein & S. Wainger, *Balls and metrics defined by vector fields I: Basic properties*, Acta Math. **155** (1985), 103-147.

[37] S. Pohožaev, *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl., **6** (1965), 1408-1411.

[38] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1984), no. 2, 479–495.

[39] G. Talenti, *Best constant in the Sobolev inequality*, Ann. Mat. Pura e Appl., **110** (1976), 353-372.

[40] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa, **22** (1968), 265-274.

[41] F. Uguzzoni, *A non-existence theorem for a semilinear Dirichlet problem involving critical exponent on half spaces of the Heisenberg group*, preprint.

[42] D. Vassilev, Ph. D. Dissertation, Purdue University.

(Nicola Garofalo) Dipartimento di Metodi e Modelli Matematici, Via Belzoni, 7, 35100 Padova, and, Department of Mathematics, Purdue University, West Lafayette IN 47907-1935

E-mail address: garofalo@ulam.dmsa.unipd.it and garofalo@math.purdue.edu

(Dimiter Vassilev) Department of Mathematics, Purdue University, West Lafayette IN 47907-1935

E-mail address: dvassil@math.purdue.edu