The quaternionic contact Yamabe problem on a 3-Sasakian manifold.

Dimiter Vassilev, University of New Mexico

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The equations

- **Yamabe**: \((M^n, g)\): \(\bar{g} = u^{4/(n-2)} g\),
  \[
  4 \frac{n-1}{n-2} \triangle u - Su = -\bar{S} u^{2*-1}.
  \]

- **CR Yamabe**: \((M^{2n+1}, \eta, J)\), \(\bar{\eta} = u^{4/(Q-2)} \eta\),
  \[
  4 \frac{n+1}{n} \triangle u - Su = -\bar{S} u^{2*-1}.
  \]

- **QC Yamabe**: \((M^{4n+3}, \eta)\), \(\bar{\eta} = u^{4/(Q-2)} \eta\),
  \[
  4 \frac{n+2}{n+1} \triangle u - Su = -\bar{S} u^{2*-1}.
  \]

- \(\triangle f = -\lambda_1 f\).
Riemannian Obata theorems

**Theorem 1.**  
*a* (Uniqueness in Einstein class) Let \((M, \bar{g})\) be a connected compact Riemannian manifold. If \(\bar{g}\) is Einstein and 
\[ g = \phi^2 \bar{g} \text{ with } \bar{S} = S = n(n - 1), \]
then \(\phi = 1\) unless \((M, \bar{g})\) is the round unit sphere \((S^n, g_{st})\).

*b* (Yamabe problem on the round sphere) If \(g\) is conformal to \(g_{st}\) on \(S^n\), 
\[ g = \phi^2 g_{st}, \text{ with } S = n(n - 1), \]
then \(g = \Phi^* g_{st}\) for \(\Phi \in \text{Diff} (S^n)\).

**Theorem 2.** Let \((M, g)\) be an \(n\)-dimensional compact Riemannian manifold with 
\[ \text{Ric}(X, X) \geq (n - 1)g(X, X). \]
If \(\lambda \neq 0\) is an eigenvalue, \(\triangle f = -\lambda f\), then \(\lambda \geq n\) (Lichnerowicz) and 
\(\lambda = n\) iff \((M, g)\) is isometric with \(S^n\) (Obata), in which case \(f\) is a spherical harmonic of order one.
The PDE on $\mathbb{R}^n$ - extremals of the $L^2$ Sobolev embedding inequality

Stereographic proj., $\mathcal{C} : S^n \setminus N \to \mathbb{R}^n$, $(\mathcal{C}^{-1})^* g_{st} = 4u^{4/(n-2)} dx^2$. The Yamabe problem on the round sphere is equivalent to:

**Theorem 3.** (Aubin, Talenti) If $u \geq 0$ satisfies the Yamabe equation on $\mathbb{R}^n$

$$\triangle u = -n(n-2)u^{2*-1}, \quad u \in D^{1,2}(\mathbb{R}^n)$$

then up to a translation and rescaling $u = (1 + |x|^2)^-(n-2)/2$.

Rescaling: $u_\lambda(x) \equiv \lambda^{n/2*} \delta_\lambda u \overset{\text{def}}{=} \lambda^{n/2*} u(\lambda x), \quad \lambda > 0$.

Key: reduce to radial functions via symmetrization arguments. These are not (fully) available in sub-Riemannian settings (ex. groups of Iwasawa type) except for solutions with "partial" symmetry w/ Garofalo or the lowest energy solutions (extremals for Folland-Stein $L^2$ Sobolev type inequality): Branson & Fontana & Morpurgo and Frank & Lieb in the CR case, w/ Ivanov - Minchev in the quaternion case, Christ & Liu & Zhang in the octonian case.
Uniqueness. Recall, \( g = \phi^2 \bar{g} \) and \( S = \bar{S} = n(n-1) \)

Suppose \( \bar{g} \) is Einstein, \( 0 = \overline{\text{Ric}_o} = \text{Ric}_o + \frac{n-2}{\phi} (\nabla^2 \phi)_0 \). The contracted Bianchi identity and \( S = \text{const} \) give \( \nabla^* \text{Ric}_0 = \frac{n-2}{2n} \nabla S = 0 \), hence

\[
\nabla^* (\text{Ric}_o \nabla \phi) = (\nabla^* \text{Ric}_o)(\nabla \phi) + g(\text{Ric}_o, \nabla^2 \phi) = \frac{n-2}{2n} g(\nabla S, \nabla \phi) - \frac{\phi}{n-2} |\text{Ric}_o|^2.
\]

This divergence formula shows that \( g \) is also an Einstein metric and \( X = \nabla \phi \) is a gradient conformal vector field,

\[
\text{Ric}_o = (\nabla^2 \phi)_0 = 0.
\]

If \( X \) is a conformal vector field then we have the infinitesimal Yamabe equation

\[
\triangle (\text{div} \, X) = -\frac{1}{n-1} (\text{div} \, X) S - \frac{n}{2(n-1)} X(S).
\]

Now, for \( S = n(n-1) \) it follows \( f = \triangle \phi \) satisfies \( \triangle f = -nf \). Thus either \( f \) = const or \( f \) is an eigenfunction with the lowest possible eigenvalue hence \( g \) is isometric to \( g_{st} \) by Obata’s eigenvalue theorem.
The case of the sphere

Taking into account the divergence formula, using the stereographic projection we can reduce to a conformal map of the Euclidean space, which sends the Euclidean metric to a conformal to it Einstein metric. By a purely local argument the resulting system can be integrated, in effect proving also Liuoville’s theorem, which gives the form of $u$ as in Aubin and Talenti’s theorem in $\mathbb{R}^n$ and then $\phi$ on $S^n$ after transferring the equations back to the unit sphere.

Remark: Such argument was used in the quaternionic contact setting to classify all qc-Einstein structures on the unit $4n + 3$ dimensional sphere (quaternionic Heisenberg group) conformal to the standard qc-structure.
Obata type results on CR and QC manifolds
Sub-Riemannian conformal infinities

On the open unit ball $B$ in $\mathbb{C}^{n+1}$ consider the Bergman metric

$$h = \frac{4}{\rho} g_{\text{euc}} + \frac{1}{\rho^2} \left( (d\rho)^2 + (Id\rho)^2 \right), \quad \rho = 1 - |x|^2.$$ 

As $\rho \to 0$, $\rho \cdot h$ is finite only on $H = \text{Ker} \ (I d\rho)$, which is the kernel of the contact form $\theta = Id\rho$. The conformal infinity of $\rho \cdot h$ is the conformal class of a pseudohermitian CR structure on $S^{2n+1}$.

In the quaternion case, consider the open unit ball $B$ in $\mathbb{H}^{n+1}$ and the hyperbolic metric

$$h = \frac{4}{\rho} g_{\text{euc}} + \frac{1}{\rho^2} \left( (d\rho)^2 + (I_1d\rho)^2 + (I_2d\rho)^2 + (I_3d\rho)^2 \right).$$

The conformal infinity is the conformal class of a (QC) quaternionic contact structure on $S^{4n+3}$. Here, $\rho \cdot h$ defines a conformal class of degenerate metrics with kernel

$$H = \cap_{j=1}^3 \text{Ker} \ (I_j d\rho),$$

which carries a quaternionic structure.
CR Setting
The solution in the Sasaki-Einstein case

**Theorem 4.**

a) (Jerison & Lee ’88) If $\theta$ is the contact form of a pseudo-Hermitian structure proportional to the standard contact form $\bar{\theta}$ on the unit sphere in $\mathbb{C}^{n+1}$ and the pseudohermitian scalar curvature $S_{\theta} = \text{const}$, then up to a multiplicative constant $\theta = \Phi^* \bar{\theta}$ with $\Phi$ a CR automorphism of the sphere.

b) (X. Wang ’13, Ivanov & Vassilev ’14) The pseudoconformal class of a Sasaki-Einstein pseudo-Hermitian structure different from the standard Sasaki-Einstein structure on the round sphere contains a unique (up to homothety) pseudo-Hermitian form of constant CR scalar curvature.
CR manifolds

$(M, \theta, J)$ is strictly pseudoconvex pseudo-Hermitian manifold if

i) $\theta$ is a contact form, $H = \text{ker} \theta$ has a compatible Hermitian structure: $J : H \to H, J^2 = -id_H$, $2g(X, Y) \overset{\text{def}}{=} d\theta(X, JY)$, $X, Y \in H$, $g(X, Y) = g(JX, JY)$;

ii) $g$ is positive definite on $H$;


Reeb field $\xi$: $\theta(\xi) = 1$ and $\xi \bot d\theta = 0$.

Tanaka-Webster connection. Unique linear connection $\nabla$ such that

(i) $\xi, J, \theta$ and $g$ are parallel; (ii) the torsion satisfies:

- $T(X, Y) = 2\omega(X, Y)\xi$, where $\omega(X, Y) \overset{\text{def}}{=} g(JX, Y)$, $X, Y \in H$;

- the Webster torsion $A, A \overset{\text{def}}{=} T(\xi, .) : H \to H$, is symmetric and anti-commutes with $J$, $AJ = -JA$.

Note: $A = 0 \iff$ Sasakian structure $\iff \mathcal{L}_\xi g = 0$. 
Curvature of the Tanaka-Webster connection

Define the Riemannian metric \( h = g + \eta^2 \). Let \( \{\epsilon_a\}_{a=1}^{2n} \)-ONB of the horizontal space \( H \).

- Tanaka-Webster curvature: \( R(A, B)C \overset{\text{def}}{=} [\nabla_A, \nabla_B]C - \nabla_{[A,B]}C \) and \( R(A, B, C, D) \overset{\text{def}}{=} h(R(A, B)C, D) \).

- Ricci tensor: \( Ric(A, B) = R(\epsilon_a, A, B, \epsilon_a) \overset{\text{def}}{=} \sum_{a=1}^{2n} R(\epsilon_a, A, B, \epsilon_a) \); scalar curvature \( S = Ric(\epsilon_a, \epsilon_a) \);

- Ricci form: \( \rho(A, B) = \frac{1}{2} R(A, B, \epsilon_a, J\epsilon_a) \).

**Proposition 5.** We have the following type decomposition of the Ricci tensor with \( B = \rho_0 \), \( \rho(JX, Y) = B(JX, Y) + \frac{1}{2n} g(X, Y) \),

\[
Ric(X, Y) = 2(n - 1)A(JX, Y) + B(JX, Y) + S \frac{2}{2n} g(X, Y).
\]

A torsion-free pseudo-Einstein CR manifold is Sasaki- Einstein if \( S = 4n(n + 1) \).
**Theorem 6 (Jerison, D. & Lee, J. ’88).** Let \((M, \bar{\theta})\) be a compact Sasaki-Einstein manifold. If \(\theta = 2h\bar{\theta}\) is also of constant positive pseudo-Hermitian scalar curvature \(S = 4n(n + 1)\), then \((M, \theta)\) is again a Sasaki-Einstein space.

”Proof”: Divergence formula: for a certain horizontal vector field \(X_h\) we have

\[
\nabla^* X_h = \frac{1}{2} \left( \frac{1}{2} + h \right) \left( |D|^2 + |E|^2 \right) + \frac{h}{4} |D^h + E^h|^2 + \frac{h}{2} Q(d, e, u),
\]

where \(D(X, Y) = -4A(X, Y)\) and \(E = \frac{2}{n+2}B(X, Y)\) are up to a constant multiple the Webster torsion and the traceless \(J\)-invariant component of the Ricci tensor of the Tanaka-Webster connection. With \(f = \frac{1}{2} + h + \frac{|\nabla h|^2}{4h}\), \(d = h^{-1}DJ\nabla h\), \(e = h^{-1}EJ\nabla h\), and \(u = \frac{1}{n+2} \nabla^*(JD)\) we have

\[
X_h = f[d + e] - dh(\xi) (Jd - Je + 6Ju).
\]
Infinitesimal CR transformations: $\mathcal{L}_Q \theta = f \theta$ and $\mathcal{L}_Q J = 0$.

If $\bar{\theta} = \Phi_t^* \theta = u_t^{2/n} \theta$, then $\frac{4(n+1)}{n} \Delta u_t - S u_t = -(S \circ \Phi_t) u_t^{2* - 1}$.

Differentiating at $t = 0$, the function $\phi = \frac{d}{dt} u_t |_{t=0}$ satisfies

$$\frac{4(n+1)}{n} \Delta \phi - S \phi = -dS(Q) - S(2^* - 1) \phi.$$ 

**Proposition 7.** An infinitesimal CR automorphism $Q$ satisfies

$$\Delta(\nabla^* Q_H) = -\frac{n}{2(n+1)} dS(Q) - \frac{S}{2(n+1)} \nabla^* Q_H.$$ 

Proof: Use $\mathcal{L}_Q g(X, Y) = \frac{1}{n} (\nabla^* Q_H) g(X, Y)$. Hence $\bar{g} = u_t^{2/n} g$ gives $\frac{2}{n} \phi = \frac{1}{n} \nabla^* Q_H$. Also $2^* = \frac{2(n+1)}{n}$.

**Characterization:**

- $f = d\sigma(\xi)$ and $Q = -\frac{1}{2} J \nabla \sigma - \sigma \xi$, where $Q_H$ ("contact Hamiltonian field") is determined by $\theta(Q_H) = 0$ and $i_{Q_H} d\theta \equiv 0 \ (\mod \theta)$;
- $[\nabla^2 \sigma]_{[-1]}(X, Y) \equiv \frac{1}{2} [\nabla^2 \sigma(X, Y) - \nabla^2 \sigma(JX, JY)] = -2\sigma A(JX, Y)$. 

Consequences of \( \Delta(\nabla^* Q_H) = -\frac{n}{2(n+1)}dS(Q) - \frac{S}{2(n+1)} \nabla^* Q_H \)

When \( A = 0 \), Ricci’s identity gives \( \nabla^3 h(X, Y, \xi) = \nabla^3 h(\xi, X, Y) \)
while \( \theta = 2h\bar{\theta} \) gives \( [\nabla^2 h]_{[-1]}(X, Y) = -2hA(X, JY) = 0 \). Hence, the vector field
\[
Q = -\frac{1}{2}J\nabla (\xi h) - (\xi h)\xi
\]
is an infinitesimal CR vector field unless it vanishes. Since \( S = 4n(n+1) \) it follows \( \phi = \nabla^* Q_H \) either vanishes identically, i.e., \( h = const \) or \( \phi \) is an eigenfunction of the sublaplacian realizing the smallest possible eigenvalue on a (pseudo-Einstein) Sasakian manifold and \( h \neq const \).

The CR Lichnerowicz-Obata theorem shows that \( (M, \theta) \) is homothetic to the CR unit sphere.

Remark: For \( f = \frac{1}{2} + h + \frac{|\nabla h|^2}{4h} \) it follows \( Q = -\frac{1}{2}\nabla f - dh(\xi)\xi \) and \( \phi = \Delta f \).
Theorem 8 (Greenleaf, A. ’85 for $n \geq 3$; Li, S.-Y., & Luk, H.-S. ’04 for $n=2$). Let $M$ be a compact spcph manifold of dimension $2n + 1$, s.t., for some $k_0 = \text{const} > 0$ we have the Lichnerowicz-type bound

$$\text{Ric}(X, X) + 4A(X, JX) \geq k_0 g(X, X), \quad X \in H.$$ 

If $n > 1$, then any eigenvalue $\lambda$ of the sub-Laplacian satisfies

$$\lambda \geq \frac{n}{n+1} k_0.$$ 

The standard Sasakian unit sphere has first eigenvalue equal to $2n$ with eigenspace spanned by the restrictions of all linear functions to the sphere.
Theorem 9 (Chiu, H.-L. ’06). If \( n = 1 \) the estimate \( \lambda \geq \frac{n}{n+1} k_0 \) holds assuming in addition that the CR-Paneitz operator is non-negative
\[
\int_M f \cdot Cf \, \text{Vol}_\theta \geq 0,
\]
where \( Cf \) is the CR-Paneitz operator,
\[
Cf = \nabla^4 f(e_a, e_a, e_b, e_b) + \nabla^4 f(e_a, Je_a, e_b, Je_b)
- 4n\nabla^* A(J\nabla f) - 4n g(\nabla^2 f, JA).
\]

Note: Li, S.-Y., & Luk, H.-S. ’04 for \( n = 1 \) with condition.

Given a function \( f \) we define the one form,
\[
P_f(X) = \nabla^3 f(X, e_b, e_b) + \nabla^3 f(JX, e_b, Je_b) + 4nA(X, J\nabla f)
\]
so we have \( Cf = -\nabla^* P \).
**Theorem 10** \((n \geq 2, \text{Li, S.-Y., Wang, X. ’13; } n=1 \text{ w/ Ivanov ’14}).\)

Suppose \((M, J, \theta)\), \(\dim M = 2n + 1\), is a compact spcph manifold which satisfies the Lichnerowicz-type bound. If \(n \geq 2\), then \(\lambda = \frac{n}{n+1} k_0\) is an eigenvalue iff up-to a scaling \((M, J, \theta)\) is the standard pseudo-Hermitian CR structure on the unit sphere in \(\mathbb{C}^{n+1}\).

If \(n = 1\) the same conclusion holds assuming in addition \(C \geq 0\).

**Earlier results**

- Sasakian case (enough for the CR Yamabe problem on the sphere!), Chang, S.-C., & Chiu, H.-L., for \(n \geq 2\) in J. Geom. Anal. ’09; for \(n = 1\) in Math. Ann. ’09.
- Non-Sasakian case, Chang, S.-C., & Wu, C.-T., ’12, assuming:
  1. for \(n \geq 2\), \(A_{\alpha\beta}, \bar{\beta} = 0\) and \(A_{\alpha\beta}, \gamma\bar{\gamma} = 0\);
  2. for \(n = 1\), \(A_{11, \bar{1}} = 0\) and \(P_1 f = 0\).
- w/ Ivanov ’12 - assuming \(\nabla^* A = 0\) and \(C \geq 0\) when \(n = 1\).
QUATERNIONIC CONTACT CASE
Theorem 11 (w/ Ivanov & Minchev arXiv:1504.03142). a) Let $(M, \bar{\eta})$ be a compact locally 3-Sasakian qc manifold of qc-scalar curvature $16n(n + 2)$. If $\eta = 2h\bar{\eta}$ is qc-conformal to $\bar{\eta}$ structure which is also of constant qc-scalar curvature, then up to a homothety $(M, \eta)$ is locally 3-Sasakian manifold. Furthermore, the function $h$ is constant unless $(M, \bar{\eta})$ is the unit 3-Sasakian sphere.

b) Let $\eta = 2h\tilde{\eta}$ with $\tilde{\eta}$ the standard qc-structure on a 3-Sasakian sphere of dimension $4n + 3$. If $\eta$ has constant qc-scalar curvature $16n(n + 2)$, then $\eta$ is obtained from $\tilde{\eta}$ by a conformal quaternionic contact automorphism.

Remark: The 7-D case of b) was completed earlier ’10.
The qc-Yamabe equation on the quaternionic Heisenberg group of homogeneous dimension \( Q = 4n + 6 \)

**Corollary 12.** If \( 0 \leq \Phi \in \mathcal{D}^{1,2} (G(\mathbb{H})) \), \( S_{\Theta} = \text{const} \),

\[
\frac{4(Q + 2)}{Q - 2} \triangle \bar{\Theta} \Phi = -S_{\Theta} \Phi^{2*-1},
\]

then for some fixed \((q_o, \omega_o) \in G(\mathbb{H})\), constants \( c_0 > 0 \) and \( \sigma > 0 \) such that \( S_{\Theta} = 128n(n + 2)c_0\sigma \) we have \( \Phi = (2h)^{-(Q-2)/4} \) with

\[
h(q, \omega) = c_0 \left[ (\sigma + |q + q_0|^2)^2 + |\omega + \omega_o + 2 \text{Im} q_o \bar{q}|^2 \right].
\]

The sub-Laplacian is \( \triangle \bar{\Theta} u = \sum_{a=1}^{n} (T_{\bar{\alpha}}^2 u + X_{\bar{\alpha}}^2 u + Y_{\bar{\alpha}}^2 u + Z_{\bar{\alpha}}^2 u) \).
Quaternionic Heisenberg Group

\[ G(\mathbb{H}) = \mathbb{H}^n \times \text{Im} \mathbb{H}, \quad (q, \omega) \in G(\mathbb{H}), \]

\[(q_0, \omega_0) \circ (q, \omega) = (q_0 + q, \omega + \omega_0 + 2 \text{Im } q_0 \bar{q}), \]

i) \[ \tilde{\Theta} = (\tilde{\Theta}_1, \tilde{\Theta}_2, \tilde{\Theta}_3) = \frac{1}{2} \left( d\omega - q \cdot d\bar{q} + dq \cdot \bar{q} \right) \]

\begin{align*}
\tilde{\Theta}_1 & = \frac{1}{2} \left( dx - x^\alpha dt^\alpha + t^\alpha dx^\alpha - z^\alpha dy^\alpha + y^\alpha dz^\alpha \right) \\
\tilde{\Theta}_2 & = \frac{1}{2} \left( dy - y^\alpha dt^\alpha + z^\alpha dx^\alpha + t^\alpha dy^\alpha - x^\alpha dz^\alpha \right) \\
\tilde{\Theta}_3 & = \frac{1}{2} \left( dz - z^\alpha dt^\alpha - y^\alpha dx^\alpha + x^\alpha dy^\alpha + t^\alpha dz^\alpha \right).
\end{align*}

ii) Left-invariant horizontal vector fields

\begin{align*}
T_\alpha & = \frac{\partial}{\partial t_\alpha} + 2x^\alpha \frac{\partial}{\partial x} + 2y^\alpha \frac{\partial}{\partial y} + 2z^\alpha \frac{\partial}{\partial z}, \\
X_\alpha & = \frac{\partial}{\partial x_\alpha} - 2t^\alpha \frac{\partial}{\partial x} - 2z^\alpha \frac{\partial}{\partial y} + 2y^\alpha \frac{\partial}{\partial z}, \\
Y_\alpha & = \frac{\partial}{\partial y_\alpha} + 2z^\alpha \frac{\partial}{\partial x} - 2t^\alpha \frac{\partial}{\partial y} - 2x^\alpha \frac{\partial}{\partial z}, \\
Z_\alpha & = \frac{\partial}{\partial z_\alpha} - 2y^\alpha \frac{\partial}{\partial x} + 2x^\alpha \frac{\partial}{\partial y} - 2t^\alpha \frac{\partial}{\partial z}.
\end{align*}

iii) Left-invariant Reeb fields \( \xi_1, \xi_2, \xi_3 \) are \( \xi_1 = 2 \frac{\partial}{\partial x}, \quad \xi_2 = 2 \frac{\partial}{\partial y}, \quad \xi_3 = 2 \frac{\partial}{\partial z}. \)

iv) On \( G(\mathbb{H}) \), the left-invariant connection is the Biquard connection. It is flat!
Let $\Psi \in \text{End}(H)$. The $Sp(n)$-invariant parts are follows

$$\Psi = \Psi^{+++} + \Psi^{+-} + \Psi^{-+} + \Psi^{--}.$$

The two $Sp(n)Sp(1)$-invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \quad \Psi_{[-1]} = \Psi^{++-} + \Psi^{-+-} + \Psi^{--+}.$$

Using $\text{End}(H) \cong \Lambda^{1,1}$ the $Sp(n)Sp(1)$-invariant components are the projections on the eigenspaces of $\Upsilon = I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3$. 
Quaternionic Contact Structure \((M^{4n+3}, \eta)\)

i) co-dim three distribution \(H\), locally, \(H = \bigcap_{s=1}^{3} \text{Ker} \, \eta_s\), \(\eta_s \in T^*M\).

ii) \(H\) carries a quaternion structure: a 2-sphere bundle of ”almost complex structures” (locally) generated by \(I_s : H \to H\),
\(I_s^2 = -1\), satisfying \(I_1I_2 = -I_2I_1 = I_3\);

iii) a ”horizontal metric” \(g\) on \(H\), such that for all \(X, Y \in H\)

\[
g(I_sX, I_sY) = g(X, Y) \quad 2\omega_s(X, Y) \overset{\text{def}}{=} 2g(I_sX, Y) = d\eta_s(X, Y).
\]

Reeb vector fields: \(TM = H \oplus V\), for \(V = \text{span}\{\xi_1, \xi_2, \xi_3\}\) where
\[
\eta_s(\xi_k) = \delta_{sk}, \quad (\xi_s \lrcorner d\eta_s)|_H = 0, \quad (\xi_s \lrcorner d\eta_k)|_H = -(\xi_k \lrcorner d\eta_s)|_H.
\]

If \(n = 1\), assume that the Reeb vector fields exist \([\text{Duchemin, D.}]\).

The Biquard connection: There exists a unique linear connection \(\nabla\) on \(M\) with the properties: (i) \(V\) and \(H\) are parallel;  
(ii) \(g\) and \(\Omega = \sum_{j=1}^{3} \omega_j \wedge \omega_j\) are parallel;  
(iii) the torsion satisfies

\[
\begin{align*}
\forall X, Y \in H, \quad T(X, Y) &= -[X, Y]|_V = 2\omega_i(X, Y)\xi_i \in V \\
\forall \xi \in V, \ X \in H, \ T\xi(X) &\equiv T(\xi, X) \in H \text{ and} \\
T\xi &\in (\text{sp}(n) + \text{sp}(1))\perp, \ T_{\xi_j} = T^0_{\xi_j} + I_jU, \ U \in \Psi[3]. \quad T^0_{\xi_j}\text{-symmetric, } I_jU\text{-skew-symmetric}.
\end{align*}
\]
We extend the horizontal metric $g$ to a Riemannian metric $h$ on $M$ by requiring $\text{span}\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and $h(\xi_s, \xi_t) = \delta_{st}$.

**N.B.** $h$ as well as the Biquard connection do not depend on the action of $SO(3)$ on $V$.

- **qc-curvature:** $\mathcal{R}(A, B)C = [\nabla_A, \nabla_B]C - \nabla_{[A,B]}C$
- **qc-Ricci tensor:** $\text{Ric}(A, B) = (\mathcal{R}(e_a, A, B, e_a) \overset{\text{def}}{=} \sum_{a=1}^{4n} h(\mathcal{R}(e_a, A)B, e_a)$;
- **qc-scalar curvature:** $\text{Scal} = tr_H \text{Ric} = \text{Ric}(e_a, e_a)$;

**Theorem 13 (w/ Ivanov & Minchev ’14).** If $T^0 \overset{\text{def}}{=} T^0_{\xi_i} I_i$, then $T^0 \in \Psi_{[-1]}$ and $\text{Ric} = (2n + 2)T^0 + (4n + 10)U + \frac{\text{Scal}}{4n} g$.

- $M$ is called **qc-Einstein** if $T^0 = U = 0$. For a qc-Einstein $\Rightarrow \text{Scal} = \text{const}$ [w/ Ivanov & Minchev ’10 & ’1?](non-trivial in 7-D, use $\mathcal{W}^{qc}$!). $M$ is called **qc-pseudo-Einstein** if $U = 0$.

**Theorem 14 (w/ Ivanov & Minchev, ’14).** Suppose $\text{Scal} > 0$. The next conditions are equivalent:

i) $(M^{4n+3}, \eta)$ is qc-Einstein manifold.

ii) $M$ is locally 3-Sasakian
Theorem 15. If $M$ is a qc-manifold embedded as a hypersurface in a hyper-Kähler manifold, then $M$ is qc-conformal to a qc-Einstein structure. In particular, the qc Yamabe problem has a solution.

Theorem 16. If $M$ is a connected qc-hypersurface of $\mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$ then, up to a quaternionic affine transformation of $\mathbb{H}^{n+1}$, $M$ is contained in one of the following three hyperquadrics:

(i) $|q_1|^2 + \cdots + |q_n|^2 + |p|^2 = 1$,  
(ii) $|q_1|^2 + \cdots + |q_n|^2 - |p|^2 = -1$,  
(iii) $|q_1|^2 + \cdots + |q_n|^2 + \Re(p) = 0$.

Here $(q_1, q_2, \ldots q_n, p)$ denote the standard quaternionic coordinates of $\mathbb{H}^{n+1}$. In particular, if $M$ is a compact qc-hypersurface of $\mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$ then, up to a quaternionic affine transformation of $\mathbb{H}^{n+1}$, $M$ is the standard 3-Sasakian sphere.
Standard qc-structure on 3-Sasakain sphere

- Contact 3-form on the sphere $S = \{|q|^2 + |p|^2 = 1\} \subset \mathbb{H}^n \times \mathbb{H}$,

$$\tilde{\eta} = dq \cdot \bar{q} + dp \cdot \bar{p} - q \cdot d\bar{q} - p \cdot d\bar{p}.$$ 

- Identify $G(\mathbb{H})$ with the boundary $\Sigma$ of a Siegel domain in $\mathbb{H}^n \times \mathbb{H}$,

$$\Sigma = \{(q', p') \in \mathbb{H}^n \times \mathbb{H} : \text{Re } p' = |q'|^2\},$$

by using the map $(q', \omega') \mapsto (q', |q'|^2 - \omega').$

**Proposition 17.** The Cayley transform, $\mathcal{C} : S \setminus \{\text{pt.}\} \to \Sigma,$

$$(q', p') = \mathcal{C} \left( (q, p) \right) = ((1 + p)^{-1} q, (1 + p)^{-1} (1 - p)).$$

is a qc-conformal transformation

$$\mathcal{C}^* \tilde{\Theta} = \frac{1}{2|1 + p|^2} \lambda \tilde{\eta} \bar{\lambda}, \quad \lambda - \text{unit quaternion}.$$
QC divergence formula

**Theorem 18 (w/ Ivanov & Minchev arXiv:1504.03142).** Suppose 
\((M^{4n+3}, \eta)\) is a qc structure which is qc-conformal to a qc-Einstein 
structure \((M^{4n+3}, \tilde{\eta})\), \(\tilde{\eta} = \frac{1}{2h} \eta\). If \(\text{Scal}_\eta = \text{Scal}_{\tilde{\eta}} = 16n(n+2)\), then 
\((M^{4n+3}, \eta)\) is also qc-Einstein. In fact, with 
\(f = \frac{1}{2} + h + \frac{1}{4} h^{-1} |\nabla h|^2\), we have

\[
\nabla^* \left( f(D + E) + \sum_{s=1}^{3} dh(\xi_s) \left( I_s E + F_s + 4I_s A_s - \frac{10}{3} I_s A \right) \right) 
= \left( \frac{1}{2} + h \right) \left( |T^0|^2 + 4|U|^2 \right) + 2h |D + E|^2 + h \langle QV, V \rangle.
\]

where \(Q\) is a positive definite matrix, \(V = (E, D_1, D_2, D_3, A_1, A_2, A_3)\), and 

\[
E = -2h^{-1} U \nabla h, \quad D_i = -\frac{1}{2} h^{-1} (T^0 - I_i T^0 I_i) \nabla h, \quad F_i = -\frac{1}{2} h^{-1} T^0 I_i \nabla h,
\]

\[
A_i = I_i [\xi_j, \xi_k], \quad A = \sum_{i=1}^{3} A_i, \quad D = -h^{-1} T^0 \nabla h.
\]
A vector field $Q$ on a qc manifold $(M, \eta)$ is a *qc vector field* if its flow preserves the horizontal distribution $H = \ker \eta$,

$$\mathcal{L}_Q \eta = (\nu I + O) \cdot \eta,$$

where $\nu$ is a smooth function and $O \in so(3)$ with smooth entries. Thus, we also have

$$\mathcal{L}_Q g = \nu g, \quad \mathcal{L}_Q I = O \cdot I, \quad I = (I_1, I_2, I_3)^t.$$

The function $\nu = \frac{1}{2n} \nabla^* Q_H$ since

$$g(\nabla_X Q_H, Y) + g(\nabla_Y Q_H, X) + 2\eta_s(Q) g(T^0_{\xi,s}X, Y) = \nu g(X, Y).$$

The infinitesimal version of the qc Yamabe equation for a qc vector field is

**Proposition 19.** Let $(M^{4n+3}, \eta)$ be a qc manifold. For any qc vector field $Q$ on $M$ we have

$$\Delta(\nabla^* Q_H) = -\frac{n}{2(n+2)} Q(\text{Scal}) - \frac{\text{Scal}}{4(n+2)} \nabla^* Q_H.$$
Lemma 20. Let \((M, \eta)\) and \((M, \bar{\eta})\) be qc-Einstein manifolds with equal qc-scalar curvatures \(16n(n + 2)\). If \(\bar{\eta} = \frac{1}{2h} \eta\) for some smooth \(h > 0\), then

\[
Q = \frac{1}{2} \nabla f + \sum_{s=1}^{3} dh(\xi_s)\xi_s
\]

is a qc vector field on \(M\), where \(f = \frac{1}{2} + h + \frac{1}{4}h^{-1}|\nabla h|^2\) is the function in the divergence formula.

It follows, \(\phi = \frac{1}{2} \Delta f\) is an eigenfunction of the sub-Laplacian with eigenvalue \(-4n\) unless \(\Delta f \equiv 0\). In the first case, the qc version of the Lichnerowicz-Obata eigenfunction sphere theorem shows that \((M, \eta)\) is the 3-Sasakain sphere. If \(\Delta f = 0\), then \(f = \frac{1}{2} + h + \frac{1}{4}h^{-1}|\nabla h|^2 = \text{const}\) since \(M\) is compact. It follows that \(h = 1/2\) by considering the points where \(h\) achieves its minimum and maximum and using the qc Yamabe equation.
**Theorem 21 (w/ Ivanov, S., & Petkov, A. ’13 & ’14).** Let \((M, \eta)\) be a compact QC manifold of dimension \(4n + 3\). Suppose, for 
\[
\alpha_n = \frac{2(2n+3)}{2n+1}, \quad \beta_n = \frac{4(2n-1)(n+2)}{(2n+1)(n-1)}
\]
and for any \(X \in H\)

\[
\mathcal{L}(X, X) \overset{\text{def}}{=} 2Sg(X, X) + \alpha_n T^0(X, X) + \beta_n U(X, X) \geq 4g(X, X).
\]

If \(n = 1\), assume in addition the positivity of the \(P\)-function of any eigenfunction. Then, any eigenvalue \(\lambda\) of the sub-Laplacian \(\Delta\) satisfies the inequality

\[
\lambda \geq 4n
\]

The 3-Sasakian sphere achieves equality in the Theorem. The eigenspace of the first non-zero eigenvalue of the sub-Laplacian on the unit 3-Sasakian sphere in Euclidean space is given by the restrictions to the sphere of all linear functions.
Definition of the QC P-function

a) The $P-$form of a function $f$ is the 1-form

$$P_f(X) = \nabla^3 f(X, e_b, e_b) + \sum_{t=1}^{3} \nabla^3 f(I_t X, e_b, I_t e_b) - 4nSdf(X) + 4nT^0(X, \nabla f) - \frac{8n(n-2)}{n-1} U(X, \nabla f).$$

b) The $P-$function of $f$ is the function $P_f(\nabla f)$.

c) The $C-$operator is the 4-th order differential operator on $M$

(independent of $f$!)

$$f \mapsto Cf = \nabla^* P_f = (\nabla_{e_a} P_f) (e_a).$$

d) The $P-$function of $f$ is non-negative if

$$\int_M f \cdot Cf Vol_\eta = -\int_M P_f(\nabla f) Vol_\eta \geq 0.$$  

If the above holds for any $f \in C_\infty^0 (M)$ we say that the $C-$operator is non-negative, $C \geq 0.$
Properties of the C-operator

Theorem 22 (w/ Ivanov & Petkov, ’13). a) \( C \geq 0 \) for \( n > 1 \).
Furthermore \( Cf = 0 \) iff \((\nabla^2 f)_{[3][0]}(X, Y) = 0\). In this case the \( P \)-form of \( f \) vanishes as well.

b) If \( n = 1 \) and \( M \) is qc-Einstein with \( \text{Scal} \geq 0 \), the \( P \)-function of an eigenfunction of the sub-Laplacian is non-negative,

\[
\triangle f = \lambda f \quad \Rightarrow \quad -\int_M Pf(\nabla f) \text{Vol}_\eta \geq 0.
\]

\( (\nabla e_a (\nabla^2 f)_{[3][0]})(e_a, X) = \frac{n-1}{4n} Pf(X) \), hence

\[
\frac{n-1}{4n} \int_M f \cdot Cf \text{Vol}_\eta = -\frac{n-1}{4n} \int_M Pf(\nabla f) \text{Vol}_\eta = \int_M |(\nabla^2 f)_{[3][0]}|^2 \text{Vol}_\eta,
\]

after using the Ricci identities, the divergence formula and the orthogonality of the components of the horizontal Hessian.

qc-Einstein \( \Rightarrow \) \( \text{Scal} = \text{const} \), \( \nabla^3 f(\xi_s, X, Y) = \nabla^3 f(X, Y, \xi_s) \), and the vertical space is integrable; \( \nabla^2 f(\xi_k, \xi_j) - \nabla^2 f(\xi_j, \xi_k) = -Sdf(\xi_i) \)

\[
\int_M |Pf|^2 \text{Vol}_\theta = -(\lambda + 4S) \int_M Pf(\nabla f) \text{Vol}_\theta
\]
The QC Obata type theorem in the compact case

**Theorem 23 (w/ Ivanov & Petkov, arxiv1303.0409).** Let $(M, \eta)$ be a compact QC manifold of dimension $4n + 3$ which satisfies a Lichnerowicz’ type bound $\mathcal{L}(X, X) \geq 4g(X, X)$. Then, there is a function $f$ with $\Delta f = 4nf$ if and only if $M$ is qc-homothetic to the 3-Sasakian sphere, assuming in addition $M$ is qc-Einstein when $n = 1$.

Remarks:

- The 7-D case is still open in the general case.
- The results follow from another theorem where only completeness and knowledge of the horizontal Hessian are assumed.
Proof of QC eigenvalue Obata for a qc-Einstein

1. Show that \((\nabla^h)^2 f (X, Y) = -fh(X, Y)\), \((h\)- Riemannian metric!).

2. Obata’s result shows \((M, h)\) is homothetic to the unit sphere in quaternion space.

3. Show qc-conformal flatness.

4. Use the qc-Liouville theorem to see \((M, g, \eta, \mathcal{Q})\) is qc-conformal to \(S^{4n+3}\), i.e., we have \(\eta = \kappa \Psi F^* \tilde{\eta}\) for some diffeomorphism \(F : M \to S^{4n+3}\), \(0 < \kappa \in C^\infty (M)\), and \(\Psi \in C^\infty (M : SO(3))\)


Theorem 25 (w/ Ivanov & Minchev). Let \(\Theta = \frac{1}{2h} \tilde{\Theta}\) be a conformal deformation of the standard qc-structure \(\tilde{\Theta}\) on the quaternionic Heisenberg group \(G (\mathbb{H})\). If \(\Theta\) is also qc-Einstein, then

\[
h(q, \omega) = c_0 \left[ (\sigma + |q + q_0|^2)^2 + |\omega + \omega_0 + 2 \text{Im} q_0 \bar{q}|^2 \right].
\]

with \(c_0 > 0\) and \(\sigma \in \mathbb{R}\). Furthermore, \(S_{\Theta} = 128n(n + 2)c_0\sigma\).

5. compare the metrics on \(H\) to see homothety.
QC Conformal Curvature tensor

- "Schouten" tensor $L(X, Y) = \frac{1}{2} T^0(X, Y) + U(X, Y) + \frac{Scal}{32n(n+2)} g(X, Y)$.

- Conformal curvature

$$W^{qc}(X, Y, Z, V) = R(X, Y, Z, V) + (g \otimes L)(X, Y, Z, V) + \sum_{s=1}^{3} (\omega_s \otimes I_s L)(X, Y, Z, V)$$

$$- \frac{1}{2} \sum_{(i,j,k)} \omega_i(X, Y) \left[ L(Z, I_i V) - L(I_i Z, V) + L(I_j Z, I_k V) - L(I_k Z, I_j V) \right]$$

$$- \sum_{s=1}^{3} \omega_s(Z, V) \left[ L(X, I_s Y) - L(I_s X, Y) \right] + \frac{1}{2n} (trL) \sum_{s=1}^{3} \omega_s(X, Y) \omega_s(Z, V),$$

where $\sum_{(i,j,k)}$ denotes the cyclic sum.

$W^{qc}$ is qc-conformal invariant, i.e., if $\tilde{\eta} = \kappa \Psi \eta$ then $W^{qc}_{\tilde{\eta}} = \phi W^{qc}_{\eta}$, $0 < \kappa \in C^\infty(M)$, and $\Psi \in C^\infty(M : SO(3))$

**Theorem 26 (w/ Ivanov ’10).** A qc manifold is locally qc-conformal to the quaternionic sphere $S^{4n+3}$ or quaternion Heisenberg group iff the qc conformal curvature vanishes, $W^{qc} = 0$. 