The quaternionic contact Yamabe problem on a 3-Sasakian manifold.

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July, 2015

The equations

- Yamabe: (M^n, g): $\bar{g} = u^{4/(n-2)}\, g$,

$$4\frac{n-1}{n-2}\triangle u - S\,u = -\overline{S}\,u^{2^*-1}.$$

- CR Yamabe: (M^{2n+1}, η, J), $\bar{\eta} = u^{4/(Q-2)}\eta$,

$$4\frac{n+1}{n}\triangle u - S\,u = -\overline{S}\,u^{2^*-1}.$$

- QC Yamabe: (M^{4n+3}, η), $\bar{\eta} = u^{4/(Q-2)}\eta$,

$$4\frac{n+2}{n+1}\,\triangle u - S\,u = -\overline{S}\,u^{2^*-1}.$$

- $\triangle f = -\lambda_1 f$.

Riemannian Obata theorems

Theorem 1. *a) (Uniqueness in Einstein class) Let $(M, \bar{g})$ be a connected compact Riemannian manifold. If $\bar{g}$ is Einstein and $g = \phi^2 \, \bar{g}$ with $\bar{S} = S = n(n-1)$, then $\phi = 1$ unless $(M, \bar{g})$ is the round unit sphere (S^n, g_{st}).*
b) (Yamabe problem on the round sphere) If g is conformal to g_{st} on S^n, $g = \phi^2 g_{st}$, with $S = n(n-1)$, then $g = \Phi^* g_{st}$ for $\Phi \in Diff(S^n)$.

Theorem 2. *Let (M, g) be an n-dimensional compact Riemannian manifold with*

$$Ric(X, X) \geq (n-1) g(X, X).$$

If $\lambda \neq 0$ is an eigenvalue, $\triangle f = -\lambda f$, then $\lambda \geq n$ (Lichnerowicz) and $\lambda = n$ iff (M, g) is isometric with S^n (Obata), in which case f is a spherical harmonic of order one.

The PDE on $\mathbb{R}^n$ - extremals of the L^2 Sobolev embedding inequality

Stereographic proj., $\mathcal{C} : S^n \setminus N \to \mathbb{R}^n$, $(\mathcal{C}^{-1})^* g_{st} = 4u^{4/(n-2)} dx^2$. The Yamabe problem on the round sphere is equivalent to:

Theorem 3. *(Aubin, Talenti) If $u \geq 0$ satisfies the Yamabe equation on $\mathbb{R}^n$*

$$\triangle u = -n(n-2)\, u^{2^*-1}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$$

then up to a translation and rescaling $u = (1+|x|^2)^{-(n-2)/2}$.

$$\text{Rescaling:} \quad u_\lambda(x) \equiv \lambda^{n/2^*}\, \delta_\lambda u \stackrel{def}{=} \lambda^{n/2^*}\, u(\lambda x), \qquad \lambda > 0.$$

Key: reduce to radial functions via symmetrization arguments. These are not (fully) available in sub-Riemannian settings (ex. groups of Iwasawa type) except for solutions with "partial" symmetry w/ Garofalo or the lowest energy solutions (extremals for Folland-Stein L^2 Sobolev type inequality): Branson & Fontana & Morpurgo and Frank & Lieb in the CR case, w/ Ivanov - Minchev in the quaternion case, Christ & Liu & Zhang in the octonian case.

Uniqueness. Recall, $g = \phi^2\, \bar{g}$ and $S = \bar{S} = n(n-1)$

Suppose $\bar{g}$ is Einstein, $0 = \overline{Ric_o} = Ric_o + \frac{n-2}{\phi}(\nabla^2\phi)_0$. The contracted Bianchi identity and S=const give $\nabla^* Ric_0 = \frac{n-2}{2n}\nabla S = 0$, hence

$$\nabla^*\left(Ric_o\,\nabla\phi\right) = (\nabla^* Ric_o)(\nabla\phi) + g(Ric_o, \nabla^2\phi) = \frac{n-2}{2n} g(\nabla S, \nabla\phi) - \frac{\phi}{n-2}|Ric_o|^2.$$

This *divergence formula* shows that g is *also an Einstein metric* and $X = \nabla\phi$ is a *gradient conformal vector field*,

$$Ric_o = (\nabla^2\phi)_0 = 0.$$

If X is a conformal vector field then we have the *infinitesimal Yamabe equation*

$$\triangle(div\, X) = -\frac{1}{n-1}(div\, X)S - \frac{n}{2(n-1)}X(S).$$

Now, for $S = n(n-1)$ it follows $f = \triangle\phi$ satisfies $\triangle f = -nf$. Thus either f =const or f is an eigenfunction with the lowest possible eigenvalue hence g is isometric to g_{st} by *Obata's eigenvalue theorem*.

The case of the sphere

Taking into account *the divergence formula*, using the stereographic projection we can reduce to a conformal map of the Euclidean space, which sends the *Euclidean metric to a conformal to it Einstein metric*. By a purely local argument *the resulting system can be integrated*, in effect proving also *Liuoville's theorem*, which gives the form of u as in Aubin and Talenti's theorem in $\mathbb{R}^n$ and then ϕ on S^n after transferring the equations back to the unit sphere.

Remark: Such argument was used in the quaternionic contact setting to classify all qc-Einstein structures on the unit $4n+3$ dimensional sphere (quaternionic Heisenberg group) conformal to the standard qc-structure.

OBATA TYPE RESULTS ON CR AND QC MANIFOLDS

Sub-Riemannian conformal infinities

On the open unit ball B in $\mathbb{C}^{n+1}$ consider the Bergman metric

$$h = \frac{4}{\rho} g_{euc} + \frac{1}{\rho^2}\left((d\rho)^2 + (Id\rho)^2\right), \qquad \rho = 1 - |x|^2.$$

As $\rho \to 0$, $\rho \cdot h$ is finite only on $H = Ker\,(I\,d\rho)$, which is the kernel of the contact form $\theta = I\,d\rho$. The conformal infinity of $\rho \cdot h$ is the conformal class of a pseudohermitian CR structure on S^{2n+1}.
In the quaternion case, consider the open unit ball B in $\mathbb{H}^{n+1}$ and the hyperbolic metric

$$h = \frac{4}{\rho} g_{euc} + \frac{1}{\rho^2}\left((d\rho)^2 + (I_1 d\rho)^2 + (I_2 d\rho)^2 + (I_3 d\rho)^2\right).$$

The conformal infinity is the conformal class of a (QC) quaternionic contact structure on S^{4n+3}. Here, $\rho \cdot h$ defines a conformal class of degenerate metrics with kernel

$$H = \cap_{j=1}^{3} Ker\,(I_j\,d\rho),$$

which carries a quaternionic structure.

CR SETTING

The solution in the Sasaki-Einstein case

Theorem 4.

a) *(Jerison & Lee '88) If θ is the contact form of a pseudo-Hermitian structure proportional to the standard contact form $\bar{\theta}$ on the unit sphere in $\mathbb{C}^{n+1}$ and the pseudohermitian scalar curvature S_θ =const, then up to a multiplicative constant $\theta = \Phi^* \bar{\theta}$ with Φ a CR automorphism of the sphere.*

b) *(X. Wang '13, Ivanov & Vassilev '14) The pseudoconformal class of a Sasaki-Einstein pseudo-Hermitian structure different from the standard Sasaki-Einstein structure on the round sphere contains a unique (up to homothety) pseudo-Hermitian form of constant CR scalar curvature.*

CR manifolds

(M, θ, J) is **strictly pseudoconvex pseudo-Hermitian manifold** if

i) θ is a contact form, $H = \ker\theta$ has a compatible Hermitian structure: $J : H \to H$, $J^2 = -id_H$, $2g(X, Y) \overset{def}{=} d\theta(X, JY)$, $X,\ Y \in H$, $\quad g(X, Y) = g(JX, JY)$;

ii) g is positive definite on H;

iii) integrability: $[JX, Y] + [X, JY] \in H$ and $[JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0$.

Reeb field ξ: $\theta(\xi) = 1$ and $\xi \lrcorner d\theta = 0$.

Tanaka-Webster connection. Unique linear connection ∇ such that (i) ξ, J, θ and g are parallel; (ii) the torsion satisfies:

- $T(X, Y) = 2\omega(X, Y)\xi$, where $\omega(X, Y) \overset{def}{=} g(JX, Y)$, $X,\ Y \in H$;
- the Webster torsion A, $A \overset{def}{=} T(\xi, .) : H \to H$, is symmetric and anti-commutes with J, $AJ = -JA$.

Note: $A = 0 \Leftrightarrow$ Sasakian structure $\Leftrightarrow$ $\mathcal{L}_\xi g = 0$.

Curvature of the Tanaka-Webster connection

Define the Riemannian metric "$h = g + \eta^2$". Let $\{\epsilon_a\}_{a=1}^{2n}$-ONB of the *horizontal* space H.

- Tanaka-Webster curvature: $R(A,B)C \overset{def}{=} [\nabla_A, \nabla_B]C - \nabla_{[A,B]}C$ and $R(A,B,C,D) \overset{def}{=} h(R(A,B)C,D)$.
- Ricci tensor: $Ric(A,B) = R(\epsilon_a, A, B, \epsilon_a) \overset{def}{=} \sum_{a=1}^{2n} R(\epsilon_a, A, B, \epsilon_a)$; scalar curvature $S = Ric(\epsilon_a, \epsilon_a)$;
- Ricci form: $\rho(A,B) = \frac{1}{2} R(A,B,\epsilon_a, J\epsilon_a)$.

Proposition 5.*We have the following type decomposition of the Ricci tensor with $B = \rho_0$, $\rho(JX,Y) = B(JX,Y) + \frac{1}{2n} g(X,Y)$,*

$$Ric(X,Y) = 2(n-1)A(JX,Y) + B(JX,Y) + \frac{S}{2n} g(X,Y).$$

A torsion-free pseudo-Einstein CR manifold is Sasaki- Einstein if $S = 4n(n+1)$.

CR divergence formula

Theorem 6 (Jerison, D. & Lee, J. '88). *Let $(M, \bar{\theta})$ be a compact Sasaki-Einstein manifold. If $\theta = 2h\bar{\theta}$ is also of constant positive pseudo-Hermitian scalar curvature $S = 4n(n+1)$, then (M, θ) is again a Sasaki-Einstein space.*

"Proof": Divergence formula: for a certain horizontal vector field X_h we have

$$\nabla^* X_h = \frac{1}{2}\left(\frac{1}{2} + h\right)\left(|D|^2 + |E|^2\right) + \frac{h}{4}|D^h + E^h|^2 + \frac{h}{2}Q(d, e, u),$$

where $D(X,Y) = -4A(X,Y)$ and $E = \frac{2}{n+2}B(X,Y)$ are up to a constant multiple the Webster torsion and the traceless J-invariant component of the Ricci tensor of the Tanaka-Webster connection. With $f = \frac{1}{2} + h + \frac{|\nabla h|^2}{4h}, \quad d = h^{-1}DJ\nabla h, \quad e = h^{-1}EJ\nabla h$, and $u = \frac{1}{n+2}\nabla^*(JD)$ we have

$$X_h = f[d + e] - dh(\xi)\left(Jd - Je + 6Ju\right).$$

Infinitesimal CR transformations: $\mathcal{L}_Q\theta = f\,\theta$ and $\mathcal{L}_Q J = 0$.

If $\bar\theta = \Phi_t^*\theta = u_t^{2/n}\,\theta$, then $\frac{4(n+1)}{n}\triangle u_t - Su_t = -(S\circ\Phi_t)\,u_t^{2^*-1}$.
Differentiating at $t=0$, the function $\phi = \frac{d}{dt}u_t|_{t=0}$ satisfies

$$\frac{4(n+1)}{n}\Delta\phi - S\,\phi = -dS(Q) - S(2^*-1)\,\phi.$$

Proposition 7. *An infinitesimal CR automorphism Q satisfies*

$$\Delta(\nabla^* Q_H) \;=\; -\;\frac{n}{2(n+1)}dS(Q)\;-\;\frac{S}{2(n+1)}\nabla^* Q_H.$$

Proof: Use $\mathcal{L}_Q g(X,Y) = \frac{1}{n}(\nabla^* Q_H)\,g(X,Y)$. Hence $\bar g = u_t^{2/n}\,g$ gives $\frac{2}{n}\phi = \frac{1}{n}\,\nabla^* Q_H$. Also $2^* = \frac{2(n+1)}{n}$.

Characterization:

- $f = d\sigma(\xi)$ and $Q = -\frac{1}{2}J\nabla\sigma - \sigma\,\xi$, where Q_H ("contact Hamiltonian field") is determined by $\theta(Q_H) = 0$ and $i_{Q_H}\,d\theta \equiv 0\ \ (mod\ \theta)$;
- $[\nabla^2\sigma]_{[-1]}(X,Y) \equiv \frac{1}{2}\left[\nabla^2\sigma(X,Y) - \nabla^2\sigma(JX,JY)\right] = -2\sigma A(JX,Y)$.

Consequences of $\Delta(\nabla^* Q_H) = -\frac{n}{2(n+1)} dS(Q) - \frac{S}{2(n+1)} \nabla^* Q_H$

When $A = 0$, Ricci's identity gives $\nabla^3 h(X, Y, \xi) = \nabla^3 h(\xi, X, Y)$ while $\theta = 2h\bar{\theta}$ gives $\left[\nabla^2 h\right]_{[-1]}(X, Y) = -2hA(X, JY) = 0$. Hence, the vector field

$$Q = -\frac{1}{2} J\nabla(\xi h) - (\xi h)\xi$$

is an infinitesimal CR vector field unless it vanishes. Since $S = 4n(n+1)$ it follows $\phi = \nabla^* Q_H$ either vanishes identically, i.e., $h = const$ or ϕ is an eigenfuction of the sublaplacian realizing the smallest possible eigenvalue on a (pseudo-Einstein) Sasakian manifold and $h \neq const$.

The CR Lichnerowicz-Obata theorem shows that (M, θ) is homothetic to the CR unit sphere.

Remark: For $f = \frac{1}{2} + h + \frac{|\nabla h|^2}{4h}$ it follows $Q = -\frac{1}{2}\nabla f - dh(\xi)\xi$ and $\phi = \triangle f$.

CR Lichnerowicz theorem

Theorem 8 (Greenleaf, A. '85 for $n \geq 3$***; Li, S.-Y., & Luk, H.-S. '04 for n=2).*** *Let M be a compact spcph manifold of dimension $2n+1$, s.t., for some $k_0 = const > 0$ we have the Lichnerowicz-type bound*

$$Ric(X,X) + 4A(X,JX) \geq k_0 g(X,X), \qquad X \in H.$$

If $n > 1$, then any eigenvalue λ of the sub-Laplacian satisfies $\lambda \geq \frac{n}{n+1} k_0$.

The standard Sasakian unit sphere has first eigenvalue equal to 2n with eigenspace spanned by the restrictions of all linear functions to the sphere.

Theorem 9 (Chiu, H.-L. '06). *If* $n = 1$ *the estimate* $\lambda \geq \frac{n}{n+1} k_0$ *holds assuming in addition that the CR-Paneitz operator is non-negative* $\int_M f \cdot Cf\, Vol_\theta \geq 0$, *where* Cf *is the CR-Paneitz operator,*

$$Cf = \nabla^4 f(e_a, e_a, e_b, e_b) + \nabla^4 f(e_a, Je_a, e_b, Je_b) \\ - 4n \nabla^* A(J\nabla f) - 4n\, g(\nabla^2 f, JA).$$

Note: Li, S.-Y., & Luk, H.-S. '04 for $n = 1$ with condition.
Given a function f we define the one form,

$$P_f(X) = \nabla^3 f(X, e_b, e_b) + \nabla^3 f(JX, e_b, Je_b) + 4nA(X, J\nabla f)$$

so we have $Cf = -\nabla^* P$.

CR Obata type theorem

Theorem 10** ($n \geq 2$**, Li, S.-Y., Wang, X. '13; n=1 w/ Ivanov '14).
Suppose (M, J, θ), $\dim M = 2n+1$, *is a compact spcph manifold which satisfies the Lichnerowicz-type bound. If* $n \geq 2$*, then* $\lambda = \frac{n}{n+1} k_0$ *is an eigenvalue iff up-to a scaling* (M, J, θ) *is the standard pseudo-Hermitian CR structure on the unit sphere in* $\mathbb{C}^{n+1}$.
If $n = 1$ *the same conclusion holds assuming in addition* $C \geq 0$.

Earlier results

- Sasakian case (enough for the CR Yamabe problem on the sphere!), Chang, S.-C., & Chiu, H.-L., for $n \geq 2$ in J. Geom. Anal. '09; for $n = 1$ in Math. Ann. '09.
- Non-Sasakian case, Chang, S.-C., & Wu, C.-T., '12, assuming: (i) for $n \geq 2$, $A_{\alpha\beta,\bar{\beta}} = 0$ and $A_{\alpha\beta,\gamma\bar{\gamma}} = 0$; (ii) for $n = 1$, $A_{11,\bar{1}} = 0$ and $P_1 f = 0$.
- w/ Ivanov '12 - assuming $\nabla^* A = 0$ and $C \geq 0$ when $n = 1$.

QUATERNIONIC CONTACT CASE

Solution of the Yamabe problem in the 3-Sasakain case

Theorem 11 (w/ Ivanov & Minchev arXiv:1504.03142). *a) Let* $(M, \bar{\eta})$ *be a compact locally 3-Sasakian qc manifold of qc-scalar curvature* $16n(n+2)$. *If* $\eta = 2h\bar{\eta}$ *is qc-conformal to* $\bar{\eta}$ *structure which is also of constant qc-scalar curvature, then up to a homothety* (M, η) *is locally 3-Sasakian manifold. Furthermore, the function h is constant unless* $(M, \bar{\eta})$ *is the unit 3-Sasakian sphere.*
b) Let $\eta = 2h\tilde{\eta}$ *with* $\tilde{\eta}$ *the standard qc-structure on a 3-Sasakian sphere of dimension* $4n+3$. *If* η *has constant qc-scalar curvature* $16n(n+2)$, *then* η *is obtained from* $\tilde{\eta}$ *by a conformal quaternionic contact automorphism.*

Remark: The 7-D case of b) was completed earlier '10.

The qc-Yamabe equation on the quaternionic Heisenberg group of homogeneous dimension $Q = 4n + 6$

Corrolary 12. *If* $0 \leq \Phi \in \mathcal{D}^{1,2}(\boldsymbol{G}(\mathbb{H}))$, S_Θ *=const,*

$$\frac{4(Q+2)}{Q-2} \triangle_{\tilde{\Theta}} \Phi = -S_\Theta \, \Phi^{2^*-1},$$

then for some fixed $(q_o, \omega_o) \in \boldsymbol{G}(\mathbb{H})$, *constants* $c_0 > 0$ *and* $\sigma > 0$ *such that* $S_\Theta = 128n(n+2)c_0\sigma$ *we have* $\Phi = (2h)^{-(Q-2)/4}$ *with*

$$h(q,\omega) \; = \; c_0 \left[\left(\sigma \; + \; |q + q_0|^2 \right)^2 \; + \; |\omega \; + \; \omega_o \; + \; 2\, Im\, q_o\, \bar{q}|^2 \right].$$

The sub-Laplacian is $\triangle_{\tilde{\Theta}} u = \sum_{a=1}^{n} \left(T_\alpha^2 u + X_\alpha^2 u + Y_\alpha^2 u + Z_\alpha^2 u \right).$

Quaternionic Heisenberg Group

$$\boldsymbol{G}\,(\mathbb{H}) = \mathbb{H}^n \times \mathrm{Im}\mathbb{H}, \quad (q,\omega) \in \boldsymbol{G}\,(\mathbb{H}),$$

$$(q_o,\omega_o)\circ(q,\omega) \;=\; (q_o \;+\; q,\omega \;+\; \omega_o \;+\; 2\,\mathrm{Im}\, q_o\,\bar{q}),$$

i) $\tilde{\Theta} \;=\; (\tilde{\Theta}_1,\ \tilde{\Theta}_2,\ \tilde{\Theta}_3) \;=\; \frac{1}{2}\,(d\omega \;-\; q\cdot d\bar{q} \;+\; dq\;\cdot\bar{q})$ or

$$\tilde{\Theta}_1 \;=\; \frac{1}{2}\,dx \;-\; x^\alpha dt^\alpha \;+\; t^\alpha dx^\alpha \;-\; z^\alpha dy^\alpha \;+\; y^\alpha dz^\alpha$$
$$\tilde{\Theta}_2 \;=\; \frac{1}{2}\,dy \;-\; y^\alpha dt^\alpha \;+\; z^\alpha dx^\alpha \;+\; t^\alpha dy^\alpha \;-\; x^\alpha dz^\alpha$$
$$\tilde{\Theta}_2 \;=\; \frac{1}{2}\,dz \;-\; z^\alpha dt^\alpha \;-\; y^\alpha dx^\alpha \;+\; x^\alpha dy^\alpha \;+\; t^\alpha dz^\alpha.$$

ii) Left-invariant horizontal vector fields

$$T_\alpha = \frac{\partial}{\partial t_\alpha} + 2x^\alpha\frac{\partial}{\partial x} + 2y^\alpha\frac{\partial}{\partial y} + 2z^\alpha\frac{\partial}{\partial z},\ X_\alpha = \frac{\partial}{\partial x_\alpha} - 2t^\alpha\frac{\partial}{\partial x} - 2z^\alpha\frac{\partial}{\partial y} + 2y^\alpha\frac{\partial}{\partial z},$$
$$Y_\alpha = \frac{\partial}{\partial y_\alpha} + 2z^\alpha\frac{\partial}{\partial x} - 2t^\alpha\frac{\partial}{\partial y} - 2x^\alpha\frac{\partial}{\partial z},\ Z_\alpha = \frac{\partial}{\partial z_\alpha} - 2y^\alpha\frac{\partial}{\partial x} + 2x^\alpha\frac{\partial}{\partial y} - 2t^\alpha\frac{\partial}{\partial z}.$$

iii) Left-invariant Reeb fields ξ_1,ξ_2,ξ_3 are $\xi_1 = 2\frac{\partial}{\partial x}, \quad \xi_2 = 2\frac{\partial}{\partial y}, \quad \xi_3 = 2\frac{\partial}{\partial z}$.

iv) On $\boldsymbol{G}\,(\mathbb{H})$, the left-invariant connection is the Biquard connection. It is flat!

- Let $\Psi \in \mathrm{End}(H)$. The $Sp(n)$-invariant parts are follows

$$\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}.$$

- The two $Sp(n)Sp(1)$-invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \qquad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}.$$

Using $\mathrm{End}(H) \overset{g}{\cong} \Lambda^{1,1}$ the $Sp(n)Sp(1)$-invariant components are the projections on the eigenspaces of $\Upsilon = I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3$.

Quaternionic Contact Structure (M^{4n+3}, η)

i) co-dim three distribution H, locally, $H = \bigcap_{s=1}^{3} Ker\, \eta_s$, $\eta_s \in T^*M$.

ii) H carries a quaternion structure: a 2-sphere bundle of ”almost complex structures” (locally) generated by $I_s : H \to H$, $I_s^2 = -1$, satisfying $I_1 I_2 = -I_2 I_1 = I_3$;

iii) a ”horizontal metric” g on H, such that for all $X, Y \in H$

$$g(I_s X, I_s Y) = g(X, Y) \qquad 2\omega_s(X, Y) \stackrel{def}{=} 2g(I_s X, Y) = d\eta_s(X, Y).$$

Reeb vector fields: $TM = H \oplus V$, for $V = span\{\xi_1, \xi_2, \xi_3\}$ where

$$\eta_s(\xi_k) = \delta_{sk}, \quad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \quad (\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}.$$

If $n = 1$, assume that the Reeb vector fields exist [**Duchemin, D.**].
The Biquard connection: There exists a unique linear connection ∇ on M with the properties: (i) V and H are parallel; (ii) g and $\Omega = \sum_{j=1}^{3} \omega_j \wedge \omega_j$ are parallel; (iii) the torsion satisfies

- $\forall X, Y \in H, \quad T(X, Y) = -[X, Y]|_V = 2\omega_i(X, Y)\xi_i \in V$
- $\forall \xi \in V,\ X \in H$, $T_\xi(X) \equiv T(\xi, X) \in H$ and
  $T_\xi \in (sp(n) + sp(1))^\perp$, $T_{\xi_j} = T^0_{\xi_j} + I_j U$, $U \in \Psi_{[3]}$.
  $T^0_{\xi_j}$-symmetric, $I_j U$-skew-symmetric..

We extend the horizontal metric g to a Riemannian metric h on M by requiring $span\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and $h(\xi_s, \xi_t) = \delta_{st}$.
N.B. h as well as the Biquard connection do not depend on the action of $SO(3)$ on V.

- qc-curvature: $\mathcal{R}(A, B)C = [\nabla_A, \nabla_B]C - \nabla_{[A,B]}C$
- qc-Ricci tensor: $Ric(A, B) = \mathcal{R}(e_a, A, B, e_a) \overset{def}{=} \sum_{a=1}^{4n} h(\mathcal{R}(e_a, A)B, e_a)$;
- qc-scalar curvature: $\mathrm{Scal} = tr_H\, Ric = Ric(e_a, e_a)$;

Theorem 13 (w/ Ivanov & Minchev '14). *If $T^0 \overset{def}{=} T^0_{\xi_i} I_i$, then $T^0 \in \Psi_{[-1]}$ and $Ric = (2n+2)T^0 + (4n+10)U + \frac{Scal}{4n} g$.*

- M is called ***qc-Einstein*** if $T^0 = U = 0$. For a qc-Einstein $\Rightarrow$ $Scal = const$ [w/ Ivanov & Minchev '10 & '1?] (non-trivial in 7-D, use W^{qc}!). M is called *qc-pseudo-Einstein* if $U = 0$.

Theorem 14 (w/ Ivanov& Minchev, '14). *Suppose $Scal > 0$. The next conditions are equivalent:*

i) *(M^{4n+3}, η) is qc-Einstein manifold.*
ii) *M is locally 3-Sasakian*

Embedded qc manifolds [w/ Ivanov & Minchev arXiv:1406.4256]

Theorem 15. *If M is a qc-manifold embedded as a hypersurface in a hyper-Kähler manifold, then M is qc-conformal to a qc-Einstein structure. In particular, the qc Yamabe problem has a solution.*

Theorem 16. *If M is a connected qc-hypersurface of $\mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$ then, up to a quaternionic affine transformation of $\mathbb{H}^{n+1}$, M is contained in one of the following three hyperquadrics:*

$$(i)\ |q_1|^2 + \cdots + |q_n|^2 + |p|^2 = 1, \quad (ii)\ |q_1|^2 + \cdots + |q_n|^2 - |p|^2 = -1,$$
$$(iii)\ |q_1|^2 + \cdots + |q_n|^2 + \mathbb{R}e(p) = 0.$$

Here $(q_1, q_2, \ldots q_n, p)$ denote the standard quaternionic coordinates of $\mathbb{H}^{n+1}$. In particular, if M is a compact qc-hypersurface of $\mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$ then, up to a quaternionic affine transformation of $\mathbb{H}^{n+1}$, M is the standard 3-Sasakian sphere.

Standard qc-structure on 3-Sasakain sphere

- Contact 3-form on the sphere $S = \{|q|^2 + |p|^2 = 1\} \subset \mathbb{H}^n \times \mathbb{H}$,

$$\tilde{\eta} \;=\; dq \cdot \bar{q} \;+\; dp \cdot \bar{p} \;-\; q \cdot d\bar{q} - \; p \cdot d\bar{p}.$$

- Identify $\boldsymbol{G}\,(\mathbb{H})$ with the boundary Σ of a Siegel domain in $\mathbb{H}^n \times \mathbb{H}$,

$$\Sigma \;=\; \{(q', p') \in \mathbb{H}^n \times \mathbb{H} \;:\; \operatorname{Re} p' \;=\; |q'|^2\},$$

by using the map $(q', \omega') \mapsto (q', |q'|^2 - \omega')$.

Proposition 17. *The Cayley transform,* $\mathcal{C} : S \setminus \{pt.\} \to \Sigma$*,*

$$(q', p') \;=\; \mathcal{C}\left((q, p)\right) = ((1+p)^{-1}\, q, (1+p)^{-1}\,(1-p)).$$

is a qc-conformal transformation

$$\mathcal{C}^*\,\tilde{\Theta} \;=\; \frac{1}{2\,|1+p\,|^2}\,\lambda\,\tilde{\eta}\,\bar{\lambda}, \qquad \lambda \text{ - } \textit{unit quaternion.}$$

QC divergence formula

Theorem 18 (w/ Ivanov & Minchev arXiv:1504.03142). *Suppose* (M^{4n+3}, η) *is a qc structure which is qc-conformal to a qc-Einstein structure* $(M^{4n+3}, \bar{\eta})$, $\tilde{\eta} = \frac{1}{2h}\eta$. *If* $Scal_\eta = Scal_{\tilde{\eta}} = 16n(n+2)$, *then* (M^{4n+3}, η) *is also qc-Einstein. In fact, with* $f = \frac{1}{2} + h + \frac{1}{4}h^{-1}|\nabla h|^2$, *we have*

$$\nabla^* \left(f(D+E) + \sum_{s=1}^{3} dh(\xi_s) \left(I_s E + F_s + 4 I_s A_s - \frac{10}{3} I_s A \right) \right)$$
$$= \left(\frac{1}{2} + h \right) \left(|T^0|^2 + 4|U|^2 \right) + 2h|\mathbb{D} + \mathbb{E}|^2 + h \langle QV, V \rangle.$$

where Q *is a positive definite matrix,* $V = (E, D_1, D_2, D_3, A_1, A_2, A_3)$, *and*

$$E = -2h^{-1} U \nabla h, \qquad D_i = -\frac{1}{2} h^{-1} (T^0 - I_i T^0 I_i) \nabla h, \qquad F_i = -\frac{1}{2} h^{-1} T^0 I_i \nabla h,$$

$$A_i = I_i[\xi_j, \xi_k], \qquad A = \sum_{i=1}^{3} A_i, \qquad D = -h^{-1} T^0 \nabla h.$$

Infinitesimal QC transformations w/ Ivanov & Minchev '14

A vector field Q on a qc manifold (M, η) is a *qc vector field* if its flow preserves the horizontal distribution $H = \ker \eta$,

$$\mathcal{L}_Q \eta = (\nu I + O) \cdot \eta,$$

where ν is a smooth function and $O \in so(3)$ with smooth entries. Thus, we also have

$$\mathcal{L}_Q g = \nu g, \qquad \mathcal{L}_Q I = O \cdot I, \qquad I = (I_1, I_2, I_3)^t.$$

The function $\nu = \frac{1}{2n} \nabla^* Q_H$ since

$$g(\nabla_X Q_H, Y) \; + \; g(\nabla_Y Q_H, X) \; + \; 2\eta_s(Q) g(T^0_{\xi_s} X, Y) = \nu \, g(X, Y).$$

The infinitesimal version of the qc Yamabe equation for a qc vector field is

Proposition 19. *Let (M^{4n+3}, η) be a qc manifold. For any qc vector field Q on M we have*

$$\Delta(\nabla^* Q_H) \; = \; -\frac{n}{2(n+2)} Q(Scal) \; - \; \frac{Scal}{4(n+2)} \nabla^* Q_H.$$

Lemma 20. *Let (M,η) and $(M,\bar{\eta})$ be qc-Einsten manifolds with equal qc-scalar curvatures $16n(n+2)$. If $\overline{\eta}=\frac{1}{2h}\eta$ for some smooth $h>0$, then*

$$Q=\frac{1}{2}\nabla f+\sum_{s=1}^{3}dh(\xi_s)\xi_s$$

is a qc vector field on M, where $f=\frac{1}{2}\ +\ h\ +\ \frac{1}{4}h^{-1}|\nabla h|^2$ is the function in the divergence formula.

It follows, $\phi=\frac{1}{2}\triangle f$ is an eigenfunction of the sub-Laplacian with eigenvalue $-4n$ unless $\Delta f\equiv 0$. In the first case, the qc version of the Lichnerowicz-Obata eigenfunction sphere theorem shows that (M,η) is the 3-Sasakain sphere. If $\Delta f=0$, then
$f=\frac{1}{2}+h+\frac{1}{4}h^{-1}|\nabla h|^2=const$ since M is compact. It follows that $h=1/2$ by considering the points where h achieves its minimum and maximum and using the qc Yamabe equation.

QC Lichnerowicz

Theorem 21 (w/ Ivanov, S., & Petkov, A. ’13 & ’14). *Let* (M,η) *be a compact QC manifold of dimension* $4n+3$*. Suppose, for* $\alpha_n = \frac{2(2n+3)}{2n+1}$, $\beta_n = \frac{4(2n-1)(n+2)}{(2n+1)(n-1)}$ *and for any* $X \in H$

$$\mathcal{L}(X,X) \stackrel{def}{=} 2Sg(X,X) + \alpha_n T^0(X,X) + \beta_n U(X,X) \geq 4g(X,X).$$

If $n=1$*, assume in addition the positivity of the P-function of any eigenfunction. Then, any eigenvalue* λ *of the sub-Laplacian* $\triangle$ *satisfies the inequality*

$$\lambda \geq 4n$$

The 3-Sasakian sphere achieves equality in the Theorem. The eigenspace of the first non-zero eigenvalue of the sub-Laplacian on the unit 3-Sasakian sphere in Euclidean space is given by the restrictions to the sphere of all linear functions.

Definition of the QC P-function

a) The $P-$form of a function f is the 1-form

$$P_f(X) = \nabla^3 f(X, e_b, e_b) + \sum_{t=1}^{3} \nabla^3 f(I_t X, e_b, I_t e_b) - 4nSdf(X) + 4nT^0(X, \nabla f) - \frac{8n(n-2)}{n-1} U(X, \nabla f).$$

b) The $P-$function of f is the function $P_f(\nabla f)$.

c) The $C-$operator is the 4-th order differential operator on M (independent of f!)

$$f \mapsto Cf = \nabla^* P_f = (\nabla_{e_a} P_f)(e_a).$$

d) The $P-$function of f is non-negative if

$$\int_M f \cdot Cf \, Vol_\eta = -\int_M P_f(\nabla f) \, Vol_\eta \geq 0.$$

If the above holds for any $f \in \mathcal{C}_o^\infty(M)$ we say that the $C-$operator is non-negative, $C \geq 0$.

Properties of the C-operator

Theorem 22 (w/ Ivanov & Petkov, '13). *a) $C \geq 0$ for $n > 1$. Furthermore $Cf = 0$ iff $(\nabla^2 f)_{[3][0]}(X, Y) = 0$. In this case the $P-$form of f vanishes as well.*
b) If $n = 1$ and M is qc-Einstein with $Scal \geq 0$, the $P-$function of an eigenfunction of the sub-Laplacian is non-negative,

$$\triangle f = \lambda f \qquad \Rightarrow \qquad -\int_M P_f(\nabla f)\, Vol_\eta \geq 0.$$

- $(\nabla_{e_a}(\nabla^2 f)_{[3][0]})(e_a, X) = \frac{n-1}{4n} P_f(X)$, hence

$$\frac{n-1}{4n}\int_M f \cdot Cf\, Vol_\eta = -\frac{n-1}{4n}\int_M P_f(\nabla f)\, Vol_\eta = \int_M |(\nabla^2 f)_{[3][0]}|^2\, Vol_\eta,$$

  after using the Ricci identities, the divergence formula and the orthogonality of the components of the horizontal Hessian.
- qc-Einstein $\Rightarrow Scal = const$, $\nabla^3 f(\xi_s, X, Y) = \nabla^3 f(X, Y, \xi_s)$, and the vertical space is integrable; $\nabla^2 f(\xi_k, \xi_j) - \nabla^2 f(\xi_j, \xi_k) = -S df(\xi_i)$
- $\int_M |P_f|^2\, Vol_\theta = -(\lambda + 4S)\int_M P_f(\nabla f)\, Vol_\theta$

The QC Obata type theorem in the compact case

Theorem 23 (w/ Ivanov & Petkov, arxiv1303.0409). *Let (M, η) be a compact QC manifold of dimension $4n + 3$ which satisfies a Lichnerowicz' type bound $\mathcal{L}(X, X) \geq 4g(X, X)$. Then, there is a function f with $\triangle f = 4nf$ if and only if M is qc-homothetic to the 3-Sasakian sphere, assuming in addition M is qc-Einstein when $n = 1$.*

Remarks:

- The 7-D case is still open in the general case.
- The results follow from another theorem where only completeness and knowledge of the horizontal Hessian are assumed.

Proof of QC eigenvalue Obata for a qc-Einstein

1. Show that $(\nabla^h)^2 f(X,Y) = -fh(X,Y)$, ($h$- Riemannian metric!).
2. Obata's result shows (M,h) is homothetic to the unit sphere in quaternion space.
3. Show qc-conformal flatness.
4. Use the qc-Liouville theorem to see $(M,g,\eta,\mathbb{Q})$ is qc-conformal to S^{4n+3}, i.e., we have $\eta = \kappa\Psi F^*\tilde{\eta}$ for some diffeomorphism $F : M \to S^{4n+3}$, $0 < \kappa \in \mathcal{C}^\infty(M)$, and $\Psi \in \mathcal{C}^\infty(M : SO(3))$

   Theorem 24 (Čap, A., & Slovák, J., '09; w/ Ivanov & Petkov arXiv:1303.0409). *Every qc-conformal transformation between open subsets of the 3-Sasakian unit sphere is the restriction of a global qc-conformal transformation.*

   Rmrk: Cowling, M., & Ottazzi, A., *Conformal maps of Carnot groups*, arXiv:1312.6423.

   Theorem 25 (w/ Ivanov & Minchev). *Let* $\Theta = \frac{1}{2h}\tilde{\Theta}$ *be a conformal deformation of the standard qc-structure* $\tilde{\Theta}$ *on the quaternionic Heisenberg group* $\boldsymbol{G}\,(\mathbb{H})$. *If* Θ *is also qc-Einstein, then*

   $$h(q,\omega) \;=\; c_0\left[\left(\sigma \;+\; |q+q_0|^2\right)^2 \;+\; |\omega \;+\; \omega_o \;+\; 2\,Im\; q_o\,\bar{q}|^2\right].$$

   with $c_0 > 0$ *and* $\sigma \in \mathbb{R}$. *Furthermore,* $S_\Theta = 128n(n+2)c_0\sigma$.
5. compare the metrics on H to see homothety.

QC Conformal Curvature tensor

- "Schouten" tensor $L(X,Y) = \frac{1}{2}T^0(X,Y) + U(X,Y) + \frac{Scal}{32n(n+2)}\, g(X,Y)$.
- Conformal curvature

$$
\begin{aligned}
W^{qc}(X,Y,Z,V) &= R(X,Y,Z,V) + (g \owedge L)(X,Y,Z,V) \\
&\qquad + \sum_{s=1}^{3} (\omega_s \owedge I_s L)(X,Y,Z,V) \\
&- \frac{1}{2}\sum_{(i,j,k)} \omega_i(X,Y)\Big[L(Z,I_iV) - L(I_iZ,V) + L(I_jZ,I_kV) - L(I_kZ,I_jV)\Big] \\
&- \sum_{s=1}^{3} \omega_s(Z,V)\Big[L(X,I_sY) - L(I_sX,Y)\Big] + \frac{1}{2n}(trL)\sum_{s=1}^{3}\omega_s(X,Y)\omega_s(Z,V),
\end{aligned}
$$

where $\sum_{(i,j,k)}$ denotes the cyclic sum.

W^{qc} is qc-conformal invariant, i.e., if $\bar\eta = \kappa\Psi\eta$ then $W^{qc}_{\bar\eta} = \phi\, W^{qc}_{\eta}$, $0 < \kappa \in \mathcal{C}^\infty(M)$, and $\Psi \in \mathcal{C}^\infty(M : SO(3))$

Theorem 26 (w/ Ivanov '10). *A qc manifold is locally qc-conformal to the quaternionic sphere S^{4n+3} or quaternion Heisenberg group iff the qc conformal curvature vanishes, $W^{qc} = 0$.*