COMPLEX ANALYSIS PROBLEMS

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Part 1. Homework Problems, MATH562, W2011

1.1. Homework

Problem 1.1.1. *a) Let f_n be a sequence of functions that are holomorphic on the punctured unit disc $\mathbb{D}^\times$ and suppose that each f_n has a pole at $z = 0$. If the sequence $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{D}^\times$, then does the limit function f necessarily have a pole at $z = 0$?*

b) Answer the same question with "pole" replaced by "removable singularity" or "essential singularity."

Problem 1.1.2. *Compute the following residues $Res_f(z_0)$ for the given function and point.*

$$z_0 = 2i, \quad f(z) = \frac{z^2}{(z-2i)(z+3)}.$$

Problem 1.1.3. *a) Find the Laurent series of the function $f(z) = \frac{9z-z^2}{(z^2-9)(z+1)}$ on the annulus $A = \{z \in \mathbb{C} : 3 < |z| < \infty\}$.*

b) Compute the integral $\int_{|z|=4} f(z)\,dz$.

Problem 1.1.4. *Determine the number of zeros of $p(z) = z^5 + \frac{1}{3}z^3 + \frac{1}{4}z^2 + \frac{1}{3}$ inside the annulus $A = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 1\}$.*

Problem 1.1.5. *Let f be a function holomorphic on the punctured unit disk $\mathbb{D}^\times$ and f' has a pole of order k at $z = 0$. Show that $k \geq 2$ and f has a pole of order $k-1$ at $z = 0$.*

Problem 1.1.6. *Let f be holomorphic function on the closed unit disk and $f^{(k)}(0) = 0$ for $k = 0, 1, \ldots, n$ and $\max_{|z|=1} |f(z)| = M$. Show that $|f(z)| \leq M|z|^{n+1}$ for $|z| < 1$.*

Problem 1.1.7. *Let $f(z)$ denote a function which is holomorphic in $\mathbb{C} \setminus \{0\}$ and has the Laurent expansion $f(z) = \sum_{j=-\infty}^{\infty} a_j z^j$. Assuimg that $f(z)$ is real for all real z, does it follow that all coefficients a_j are real? Give a proof or counterexample.*

1.2. Homework

Problem 1.2.1. *Let z_1 and z_2 be two distinct points in the complex plane and l the line determined by them. Find the formula for the orthogonal reflection with respect to l, i.e., find the function $S(z)$, called the Schwarz function of l, such that given any point $z \in \mathbb{C}$ the reflected image is $z^* = \overline{S(z)}$. Notice that $S(z) = \bar{z}$ for $z \in l$. (Have a look at Problems (2.3.2) and (2.3.3) from last semester.)*

Problem 1.2.2. *Find the Schwarz function S of the circle of radius R centered at the point $z_0 \in \mathbb{C}$. This will be a holomorphic function S defined in a neighborhood of the circle and satisfying $S(z) = \bar{z}$ for points on the circle. First consider the case $R = 1$ and $z_0 = 0$.*

Problem 1.2.3. *a) Let $f \in \mathcal{A}(\mathbb{D}) \cap \mathcal{C}(\bar{\mathbb{D}})$ and $|f(z)| = 1$, when $|z| = 1$. Show that f can be extended by reflection to a meromorphic function on $\bar{\mathbb{C}}$ by the rule*

$$F(z) = \begin{cases} f(z), & |z| \leq 1; \\ \\ 1\Big/\overline{f(z^*)}, & |z| > 1,\ z^* = 1/\bar{z}. \end{cases}$$

b) Use the Schwarz function of the unit circle $\mathbb{T} = \partial\mathbb{D}$ to formulate (and/ or prove) the above version of the Schwarz reflection principle for reflection in the unit circle.

Problem 1.2.4. *Show that if f is an entire function which is real valued on some open interval on the real axis and purely imaginary on some open interval on the imaginary axis, then f is an odd function, $f(-z) = -f(z)$ for all $z \in C$.*

Problem 1.2.5. *Let $h \in \mathcal{A}(\mathbb{D}) \cap \mathcal{C}(\bar{\mathbb{D}})$ and $|h(z)| = 1$, when $|z| = 1$. Prove that h is a rational function. Show that up to a constant multiple of unit modulus h is a finite product of Blaschke factors $B_a(z) = \frac{z-a}{1-\bar{a}z}$ for some (finitely many - some could be the same) a's of modulus $|a| < 1$.*

Problem 1.2.6. *Let $|a| < 1$ and $L(z) = \frac{z-a}{1-\bar{a}z}$. Let $L_n = L \circ \cdots \circ L$ n-times. Prove that the sequence of holomorphic functions $\{L_n\}$ converges locally uniformly on $\mathbb{D}$ and find its limit.*

Problem 1.2.7. *Let $\mathcal{F}$ be a normal family of holomorphic functions on U. Show that the family $\mathcal{F}'$ of all complex derivatives of the functions in $\mathcal{F}$ is also a normal family. Hint: Review Problem (2.7.2).*

Problem 1.2.8. *Suppose that $\{f_n\}$ is a uniformly bounded family of holomorphic functions on a domain Ω. Let $\{z_k\}$ be a sequence of points in Ω converging to a point $z_0 \in \Omega$. Show that if for every fixed k the sequence $\{f_n(z_k)\}$ is convergent, then the $\{f_n\}$ converges locally uniformly on compact subsets of Ω.*

Problem 1.2.9. *Let Ω be a bounded domain of the complex plane $\mathbb{C}$ and $\{f_n\}$ a sequence of holomorphic function on Ω. Assume that there is a constant $C < \infty$ such that*

$$\int_\Omega |f_n(z)|^2\, dxdy < C$$

for all n. Prove that $\{f_n\}$ is a normal family. Hint: Use that $|f(z)|^2$ can be bounded by the mean value of $|f|^2$ on a small disc centered at z. This inequality was also used in Problem (2.10.6) b). See also Problem (1.2.10)

Problem 1.2.10. *Let Ω be an open set and $K \Subset L \Subset \Omega$. Show that there exists a constant M (depending on K and L) such that for any holomorphic function f on Ω we have*

$$\sup_K |f| \leq M \left(\int_L |f|^2 dxdy \right)^{1/2}.$$

Hint: First show the inequality for $K = D(z_0, r)$ and $L = D(z_0, R)$ using the Cauchy formula applied to the holomorphic function f^2. See Problem (2.6.7) and (2.7.10).

1.3. Homework

Note: Here *conformal* means injective and surjective holomorphic map, hence a biholomorphism.

Problem 1.3.1. *Determine* $Aut(\mathbb{C})$.

Problem 1.3.2. *a) Describe all holomorphic maps from* $\mathbb{C}$ *to* $\mathbb{D}$*?*
b) Can there be a holomorphic map of $\mathbb{D}$ onto $\mathbb{C}$*?*

Problem 1.3.3. *a) Prove that if* $f : \mathbb{D} \to \mathbb{D}$ *is holomorphic and has two distinct fixed points then* $f(z) = z$*.*
b) Is it true that if $f : \mathbb{D} \to \mathbb{D}$ *is holomorphic then it must have a fixed point?*

Problem 1.3.4. *Let* Ω *be a holomorphically simply connected domain in* $\mathbb{C}$ *and let* P *and* Q *be distinct points of* Ω*. Let* F_1 *and* F_2 *be conformal self-maps of* Ω*, i.e.,* $F_1, F_2 \in Aut(\Omega)$*. If* $F_1(P) = F_2(P)$ *and* $F_1(Q) = F_2(Q)$ *then prove that* $F_1 \equiv F_2$*. Note: Be careful to distinguish the case of* $\mathbb{C}$ *from that of* Ω *a proper subset of* $\mathbb{C}$*.*

Problem 1.3.5. *Let* Ω *be a bounded domain and let* ϕ *be a conformal mapping of* Ω *to itself. Let* $P \in \Omega$ *and suppose that* $\phi(P) = P$*. a) Show that if* $\phi'(P) = 1$*, then* ϕ *must be the identity.*
b) Show that in the general case (i.e. without the assumption a)) $\phi(z) = \lambda z$ *for some constant* $\lambda \in \mathbb{C}$*,* $|\lambda| = 1$*.*

Problem 1.3.6. *Show that* $f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right)$ *is a conformal map of the "upper" half disc* $\mathbb{D} \cap \{z : \Im z > 0\}$ *to the upper half-plane.*

Problem 1.3.7. *Let* Ω *be a holomorphically simply connected domain in* $\mathbb{C}$ *and let* ϕ *be a conformal mapping of* Ω *to* $\mathbb{D}$*. Set* $P = \phi^{-1}(0)$*. Let* $f : \Omega \to \mathbb{D}$ *be any holomorphic function such that* $f(P) = 0$*. Prove that* $|f'(P)| \le |\phi'(P)|$*.*

Problem 1.3.8. *Let* U *be an open subset of* $\mathbb{C}$*. If there is a continuous function* $f : U \to \mathbb{C}$ *such that* $e^{f(z)} = z$ *for* $z \in U$*, show that then* f *is necessarily holomorphic and hence a branch of* $\log z$ *on* U*.*

Problem 1.3.9. *Let* $n \in \mathbb{N}$*. Find a conformal map of the sector* $0 < arg\, z < \frac{\pi}{n}$ *onto* $\mathbb{D}$*.*

1.4. Homework

Note: Harmonic function means real-valued unless explicitly stated otherwise.

Problem 1.4.1. *Let $u : U \to \mathbb{R}$ be a continuous function. Show the equivalence of the next two properties.*

(a) $u(z_0) = \frac{1}{2\pi r}\int_{\partial D} u(z)\, ds(z)$ *for any* $D = D(z_0, r) \subset U$, *where* ds *is arc length measure on* ∂D.

(b) $u(z_0) = \frac{1}{\pi r^2}\int_{D} u(z)\, dA(z)$ *for any* $D = D(z_0, r) \subset U$, *where* $dA = dxdy$.

Problem 1.4.2. *Prove that there is no nonconstant harmonic function* $u : U \to \mathbb{R}$ *such that* $u(z) \le 0$ *for all* $z \in \mathbb{C}$.

Problem 1.4.3. *Let the function* $v(z) = \Im \exp[(1+z)/(1-z)]$.

(a) Calculate $v(z)$ *explicitly.*

(b) What limiting value does v *have as* $z \to 1$, $z \in \mathbb{D}$*?*

Problem 1.4.4. *Give two distinct harmonic functions on* $\mathbb{C}$ *that vanish on the entire real axis. Why is this not possible for holomorphic functions?*

Problem 1.4.5. *Use the open mapping principle for holomorphic functions to prove an open mapping principle for harmonic functions.*

Problem 1.4.6. *(a) Let* u *be a harmonic on a connected open set* $U \subset \mathbb{C}$. *Show that the following are equivalent:*

i) $u \equiv 0$ *on* U;

ii) u *vanishes identically on some disc* $D(z_0, r) \subset U$;

iii) there is a point $z_0 \in U$ *where all partial derivatives of* u *vanish,* $\frac{\partial^n \partial^m u}{\partial x^n \partial y^m}(z_0) = 0$.

(b) State and prove a unique continuation property for harmonic functions defined on a connected domain $U \subset \mathbb{C}$.

Problem 1.4.7. *Prove that a nonconstant harmonic function on a connected domain* $U \subset \mathbb{C}$ *cannot reach a local maximum (minimum).*

Problem 1.4.8. *Using Poisson's formula find a harmonic function on* $\mathbb{D}$ *which takes continuously the value* $+1$ *on the semi-circle* $\partial\mathbb{D} \cap \{\Im z > 0\}$ *and the value* 0 *on the semi-circle* $\partial\mathbb{D} \cap \{\Im z < 0\}$.

1.5. Homework

Problem 1.5.1. *Find a formula for the upper half-plane $\Omega = \{z \mid \Im z > 0\}$ analogous to the Poisson integral formula by mapping Ω conformally to the unit disc.*

Problem 1.5.2. *Let $u \in \mathcal{C}(\Omega)$, Ω-open subset of $\mathbb{C}$, and u harmonic in $\Omega \setminus \{z_0\}$. Prove that u is harmonic on Ω.*

Problem 1.5.3. *Let Ω be an open subset of $\mathbb{C}$, u a harmonic function on Ω with harmonic conjugate v. Show that for any $D(z_0, R) \subset \Omega$ we have the expansions*

$$u(z) = u(z_0 + re^{i\theta}) = u_0 + \sum_{n=1}^{\infty} (u_n \cos n\theta - v_n \sin n\theta)\, r^n$$

$$v(z) = v(z_0 + re^{i\theta}) = v_0 + \sum_{n=1}^{\infty} (v_n \cos n\theta + u_n \sin n\theta)\, r^n$$

for $z = z_0 + re^{i\theta} \in D(z_0, R)$ and some numbers u_n, v_n.

Problem 1.5.4. *Let $z = re^{it} \in D(0, \rho)$. Show that*

$$\frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos t + r^2} = 1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{\rho}\right)^n \cos nt$$

$$\frac{2\rho r \sin t}{\rho^2 - 2\rho r \cos t + r^2} = 2 \sum_{n=1}^{\infty} \left(\frac{r}{\rho}\right)^n \sin nt$$

converge uniformly on every compact subset of $D(0, \rho)$. Hint: use the holomorphic function

$$f(z) = \frac{\rho + z}{\rho - z}.$$

Do also the computation for $f(z) = \frac{\rho e^{i\theta} + re^{i\phi}}{\rho e^{i\theta} - re^{i\phi}}$

Problem 1.5.5. *Is there a version of Harnack's principle for a decreasing sequence of harmonic functions? If so, formulate and prove it. If not, give a counterexample.*

1.6. Homework

Problem 1.6.1. *Recall the Poisson formula for the unit disc* $\mathbb{D}$,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2} f(e^{i\theta})\, d\theta, \qquad |z|<1,$$

where f *is a piece-wise continuous function on* $\partial\mathbb{D}$. *Show that*

$$v(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{|z|^2-1}{|z-e^{i\theta}|^2} f(e^{i\theta})\, d\theta, \qquad |z|>1$$

defines a bounded harmonic function on the exterior of the unit disc, which is continuous at every point of $\partial\mathbb{D}$ *where* f *is continuous.*

Problem 1.6.2. *Show that both* $u(z)=1$ *and* $v(z)=\ln|z|$ *are solution of the Dirichlet problem on* $\mathbb{C}\setminus\mathbb{D}$ *with boundary data on* $\partial\mathbb{D}$ *given by* $f\equiv 1$. *What solution does the Poisson formula of Problem 1.6.1 give?*

Problem 1.6.3. *Show that the Dirichlet problem on* $\mathbb{C}\setminus\mathbb{D}$ *with continuous boundary data on* $\partial\mathbb{D}$ *given by a function* f *has a unique* bounded *solution. Hint: Problem 1.5.2 and the map* $w=\frac{1}{z}$ *can be helpful.*

Problem 1.6.4. *Let* $\phi:\Omega\to\mathbb{D}$ *be a conformal map,* $\phi(z_0)=0$ *and* $\phi'(z_0)>0$ *for some given point* $z_0\in\Omega$ *and* u *a harmonic function on* Ω. *Let* $\Omega_r=\phi^{-1}(D(0,r))$, $0<r<1$, *be the pre-image of the disc* $D(0,r)$.

a) Show that

$$u(z) = \frac{1}{2\pi}\int_{\partial\Omega_r} u(\zeta)\frac{|\phi(\zeta)|^2-|\phi(z)|^2}{|\phi(\zeta)-\phi(z)|^2}\frac{|\phi'(\zeta)|}{|\phi(\zeta)|}\,|d\zeta|, \qquad z\in\Omega_r.$$

b) Show that at z_0 *the above formula can be written in the form*

$$u(z_0) = \frac{1}{2\pi}\int_{\partial\Omega_r} u(\zeta)\frac{|\phi'(\zeta)|}{|\phi(\zeta)|}\,|d\zeta| \;=\; -\frac{1}{2\pi}\int_{\partial\Omega_r} u(\zeta)\frac{\partial}{\partial\nu}\ln|\phi(\zeta)|\,|d\zeta|,$$

where ν *is the inner unit normal to* Ω_r.

Problem 1.6.5. *Let* Ω *be a simply connected domain,* $\Omega\neq\mathbb{C}$. *For any* $w\in\Omega$ *let* $\phi_w:\Omega\to\mathbb{D}$ *be the conformal map such that* $\phi_w(z)=0$ *and* $\phi_w'(z)>0$. *Define the function*

$$G(z,w) = -\ln|\phi_w(z)|, \qquad w,z\in\Omega,\ z\neq w.$$

Show that for any $w\in\Omega$ *the function* $z\to G(z,w)$ *is positive and harmonic in* $\Omega\setminus\{w\}$. *Furthermore, it has the properties* $\lim_{z\to w}G(z,w)=+\infty$ *and* $\lim_{z\to\zeta}G(z,w)=0$ *for any* $\zeta\in\partial\Omega$.

Problem 1.6.6. *Show that a sub-harmonic function on a connected open set cannot reach a* local *maximum unless it is identically constant. Show that unlike harmonic functions, sub-harmonic functions can reach a local minimum without being constant. Hint: Think of a sub-harmonic function on the real line that is constant on* $(-\infty,0)$.

Problem 1.6.7. *Show that if* $h:\Omega\to\mathbb{R}$ *is harmonic and* $\phi:\mathbb{R}\to\mathbb{R}$ *is a convex function , then* $\phi\circ h$ *is subharmonic. Hint: Use Jensen's inequality: if* ϕ *is a convex function on* $\mathbb{R}$, *then* $\phi\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\leq \frac{1}{b-a}\int_a^b(\phi\circ f)(t)dt$ *valid for any* $a<b$ *and* f *a continuous function on* $[a,b]$.

Problem 1.6.8. *a) Show that if* f *is holomorphic on an open set* Ω *and* $p>0$, *then* $|f|^p$ *is subharmonic.*

b) Now suppose that f *is merely harmonic. Prove that* $|f|^p$ *is subharmonic when* $p\geq 1$, *but that in general it fails to be subharmonic for* $p<1$.

1.7. Homework

Problem 1.7.1. *Prove that in the definition of a sub-harmonic function it is enough to consider harmonic functions that are continuous on the closed disc and harmonic inside.*

Problem 1.7.2. *Let $u : U \to \mathbb{R}$ be a continuous function. Show the equivalence of the next properties.*
(a) $u(z_0) \leq \frac{1}{\pi r^2} \int_D u(z)\, dA(z)$ for any disc $D = D(z_0, r) \subset U$ and $r < r_0 = r_0(z_0)$, where $dA = dxdy$.
(b) u is sub-harmonic.

Problem 1.7.3. *Prove that if is $f \in \mathcal{C}^2(U)$ on an open set $U \subset \mathbb{C}$ and f is subharmonic, then $\triangle f \geq 0$ on U. Hint: To check the inequality at $0 \in U$ (this is enough!) use Taylor's formula to obtain*

$$\frac{1}{\pi r^2} \int_{D(0,r)} u(z) - u(0)\, dxdy = \frac{r^2}{8} \triangle u(0) \; + \; o(r^2).$$

For the converse see the next problem.

Problem 1.7.4. *Let $f : U \to \mathbb{R}$ be a $\mathcal{C}^2$ function on an open set in $U \subset \mathbb{C}$.*
(a) Recall that if $\Delta f > 0$ at a point P, then f cannot have a local maximum at P. Use this observation to deduce that if $\Delta f > 0$ everywhere on U, then f is subharmonic.
(b) If $\Delta f \geq 0$ everywhere, then, for each $\epsilon > 0$, $\Delta(f + \epsilon|z|^2) > 0$ everywhere. Use a limiting argument and part (a) to deduce that if $\Delta f \geq 0$ everywhere, then f is subharmonic.

Problem 1.7.5. *Show that if the bounded open set U has the outer segment property at the boundary point $z_0 \in \partial U$, i.e, there is a point $z_1 \in \mathbb{C}$ so that the closed segment I connecting z_0 to z_1 has the property $I \cap \overline{U} = \{z_0\}$ then z_0 is a regular point.*

1.8. Homework

Problem 1.8.1. *Determine the convergence of the product*

$$\prod_{k=1}^{\infty}\left(1+\frac{(-1)^k}{\sqrt{k}}\right).$$

Problem 1.8.2. *Let $a_n \in \mathbb{C}$, $n \in \mathbb{N}$, and set $\Pi_n = \prod_{k=1}^{n}(1+a_k)$ and $P_n = \prod_{k=1}^{n}(1+|a_k|)$. Show that*

$$P_n \leq \exp(|a_1|+\cdots+|a_n|) \qquad \textit{and} \qquad |\Pi_n - 1| \leq P_n - 1.$$

Problem 1.8.3. *Show that if $0 \leq a_n < 1$ and $\sum_{n=1}^{\infty} a_n$ is a divergent series, then*

$$\lim_{n\to\infty}\prod_{k=1}^{n}(1-a_k) = 0.$$

Problem 1.8.4. *Let $\Pi = \prod_{k=1}^{\infty}(1+a_k)$ be an absolutely convergent product. Show that for any reordering function $Q : \mathbb{N} \to \mathbb{N}$ a bijection of $\mathbb{N}$ we have*

$$\Pi = \prod_{k=1}^{\infty}(1+a_{Q(k)}).$$

Problem 1.8.5. *Determine the regions of convergence and absolute convergence of the product*

$$\prod_{k=1}^{\infty}(1+z^k).$$

Problem 1.8.6. *Determine the regions of convergence and absolute convergence of the product*

$$\prod_{k=1}^{\infty}(1-\frac{z^k}{k}).$$

Problem 1.8.7. *Determine the region of convergence of the product*

$$\prod_{k=1}^{\infty}\left|1-\frac{z}{k}\right|.$$

Problem 1.8.8. *Determine the region of convergence of the product*

$$\prod_{k=1}^{\infty}\left(1+z^{3^k}\right).$$

Problem 1.8.9. *Suppose $\sum |a_n - b_n| < \infty$. Determine the largest open set Ω for which the product*

$$\prod_{k=1}^{\infty}\frac{z-a_n}{z-b_n}$$

converges normally on Ω.

1.9. Homework

Problem 1.9.1. *Find an entire function with zeros of multiplicity one at the points* $a_n = n^2$.

Problem 1.9.2. *Show that* $\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$.

Problem 1.9.3. *Show that* $f(z) = \prod_{n=1}^{\infty}\left[\left(1 - \frac{z}{n}\right) e^{z/n}\right]$ *and* $g(z) = \prod_{n=1}^{\infty}\left[\left(1 + \frac{z}{n}\right) e^{-z/n}\right]$ *define entire functions.*

Problem 1.9.4. *Show that for some entire function* $h(z)$ *we have*

$$\sin \pi z = z\, f(z)\, g(z)\, e^{h(z)},$$

where f *and* g *are the functions from the Problem 1.9.3. (Actually* $e^h \equiv \pi$*.)*

Problem 1.9.5. *Show that*

$$\pi \cot \pi z = h'(z) + \frac{1}{z} + \sum_{n=-\infty}^{\infty}\left(\frac{1}{z-n} + \frac{1}{n}\right),$$

where h *is the function from Problem 1.9.4. Here the* $\sum$ *is over* $n \neq 0$.

Problem 1.9.6. *Let* a_n *be a sequence of non-zero complex numbers without an accumulation point in the complex plane. Let* k *be the largest non-negative integer for which the series*

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^k}$$

diverges. The the function

$$f(z) = \prod_{n=1}^{\infty}\left\{\left(1 - \frac{z}{a_n}\right)\exp\left(\frac{z}{a_n} + \cdots + \frac{1}{k}\frac{z^k}{a_n^k}\right)\right\}$$

is an entire function with zeros given by the sequence $\{a_n\}$. *In particular, when* $k = 0$ *we obtain*

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right)$$

is an entire function with zeros precisely given by the sequence a_n. *Note: Equivalently,* k *is the smallest integer for which the series*

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{k+1}}$$

converges.

Problem 1.9.7. *Suppose that*a_n *is a sequence of distinct complex numbers,* $\lim |a_n| = inft$ *and that the* A_n *are arbitrary complex numbers. Show that there exists an entire function* f *such that* $f(a_n) = A_n$. *Hint: Let* g *be an entire function with simple zeros at* a_n. *Show that*

$$\sum \frac{g(z)}{z - a_n} \cdot \frac{A_n}{g'(a_n)}\, e^{\gamma_n (z - a_n)}$$

converges for some choice of the numbers γ_n.

Problem 1.9.8. *Show that the field generated by all entire functions is the field of all meromorphic functions on* $\mathbb{C}$. *In other words, if* $q(z)$ *is a meromorphic function on* $\mathbb{C}$, *then* $q(z) = f(z)/g(z)$ *for some entire functions* f *and* g.

1.10. Homework

Problem 1.10.1. *a) Suppose that f, $g \in A(\Omega)$ have no common zeros in the open set Ω. Show that there are u, $v \in A(\Omega)$ such that*

$$fu + gv = 1.$$

b) Prove the result of part a) for finitely many functions $f_1, \dots, f_n \in A(\Omega)$ without a common zero.

Problem 1.10.2. *Show that every finitely generated maximal ideal $\mathfrak{m}$ of the ring $A(\Omega)$ is of the form $\mathfrak{m}_\lambda = \{f \in A(\Omega) \mid f(\lambda) = 0\}$ for some $\lambda \in \Omega$.*

Problem 1.10.3. *Let Z be an infinite discrete subset of Ω.*

a) Show that the set $\mathfrak{I}$ of all functions vanishing at all but finitely many points of Z is an ideal of $A(\Omega)$.

b) Show that there is no point of Z at which all functions in $\mathfrak{I}$ have a zero.

Problem 1.10.4. *Show that the ring $A(\Omega)$ has maximal ideals which are not of the form $\mathfrak{m}_\lambda$. Hint: Consider the maximal ideal $\mathfrak{m}$ containing $\mathfrak{I}$ from Problem 1.10.3.*

Problem 1.10.5. *Find a meromorphic function with simple poles at every integer number and residue one.*

Problem 1.10.6. *Show that*

$$f(z) = \pi \cot \pi z - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n} \right) - \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right)$$

is an entire function which is periodic $f(z) = f(z+1)$. Hint: Show that $f(z+1) - f(z) \equiv 0$ by grouping terms (carefully).

Problem 1.10.7. *Show that the function f defined in Problem 1.10.6 vanishes identically. For this you can proceed by following either of the following steps(do both but turn-in one solution).*

a) Let Γ_n be the square with vertices $\left(n + \frac{1}{2}\right) + i\left(n + \frac{1}{2}\right)$ and its reflections across the coordinate axes and the origin. Show that

$$\frac{1}{2\pi i} \oint_{\Gamma_n} \frac{\cot \pi \zeta}{\zeta(\zeta - z)} \, d\zeta = \frac{\cot(\pi z)}{z} - \frac{1}{\pi z^2} + \frac{1}{\pi} \sum_{j=1}^{n} \frac{1}{j(j-z)} + \frac{1}{\pi} \sum_{j=1}^{n} \frac{1}{j(j+z)}.$$

Hint: See Example 4.6.6. or your notes from last semester.

b.1) Show that

$$f'(z) = -\frac{\pi^2}{\sin^2 \pi z} + \sum_{n=-\infty}^{+\infty} \frac{1}{(z-n)^2}.$$

b.2) Show that f' is bounded on $\left\{ z \mid |\Re(z)| \le \frac{1}{2} \right\}$.

b.3) Show that

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{+\infty} \frac{1}{(z-n)^2}.$$

b.4) Show that $f \equiv 0$ using that f is an odd function.

Problem 1.10.8. *Show that*

$$\sin \pi z = \pi z \, f(z) \, g(z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right),$$

where f and g are the functions from the Problem 1.9.3.

1.11. Homework

Problem 1.11.1. *Let f be holomorphic in a neighborhood of the closure of the disk $D(0,R)$, $f(0) \neq 0$, and $n(r)$ be the number of zeros (counted with their multiplicities) inside the circle $|z| = r$, $r < R$. Show that*

$$\int_0^R \frac{n(r)}{r}\, dr \;=\; \sum_{|a_k|<R} \ln\left|\frac{R}{a_k}\right|,$$

where a_k are the zeros repeated with their multiplicities.

Problem 1.11.2. *Let f be holomorphic in a neighborhood of the closure of the disk $D(0,R)$, $f(0) \neq 0$, and $n(r)$ be the number of zeros (counted with their multiplicities) inside the circle $|z| = r$, $r < R$. Show that*

$$\int_0^R \frac{n(r)}{r}\, dr \;=\; \frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\theta})|\, d\theta \;-\; ln|f(0)|.$$

Problem 1.11.3. *Let f be holomorphic in a neighborhood of the closure of the disk $D(0,R)$, $f(0) \neq 0$, and $n(r)$ be the number of zeros (counted with their multiplicities) inside the circle $|z| = r$, $r < R$. If*

$$M(R) \;=\; \frac{1}{2\pi}\int_0^{2\pi} |f(Re^{i\theta})|\, d\theta$$

on $|z| = R$ show that

$$n(R/2) \;\leq\; \frac{1}{\ln 2}\ln\frac{M(R)}{|f(0)|}.$$

Problem 1.11.4. *Let f be holomorphic in a neighborhood of the closure of the unit disk $\mathbb{D}$, $|f(0)| = 1$ and $|f(z)| \leq 17$ on $|z| = 1$. Give an estimate of the number of roots inside $D(0, 1/2)$.*

Problem 1.11.5. *Suppose $a_n \in \mathbb{D}$ and $\sum(1 - |a_n|) = \infty$. Show that if f, $g \in H^\infty(\mathbb{D})$ and $f(a_n) = g(a_n)$ then $f \equiv g$ on $\mathbb{D}$.*

Problem 1.11.6. *Prove the Jensen-Poisson formula: if f is holomorphic in a neighborhood of the closure of the disk $D(0,r)$, $z_o \in D(0,r)$ and $f(z_o) \neq 0$ then*

$$\ln|f(z_o)| + \sum_{k=1}^{n} \ln\left|\frac{r^2 - \bar{a}_k z_o}{r(z_o - a_k)}\right| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \Re\left\{\frac{re^{i\theta} + z_o}{re^{i\theta} - z_o}\right\} \ln|f(re^{i\theta})|\, d\theta$$

where a_k are the zeros of f each repeated as many times as its multiplicity.

Problem 1.11.7. *Let f is holomorphic in a neighborhood of the closure of the unit disk $\mathbb{D}$ and $f(z_o) \neq 0$. Show:*

a)

$$|f(0)| \;\leq\; |a_1 \ldots a_k| \cdot \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|\, d\theta.$$

b) if $|f(z)| \leq M$ on $|z| = 1$ then

$$|f(0)| \;\leq\; |a_1 \ldots a_k| \cdot M.$$

1.12. Homework

Problem 1.12.1. *Show that $f(z) = 1/z$ cannot be approximated uniformly on the unit circle by holomorphic polynomials.*

Problem 1.12.2. *Show that if the Taylor series of an entire function f converges uniformly on $\mathbb{C}$ to f, then f is a polynomial.*

Problem 1.12.3. *Let $K = \overline{D(4,1)} \cup \overline{D(-4,1)}$ and $L = \overline{D(4i,1)} \cup \overline{D(-4i,1)}$. Construct a sequence $\{f_j\}$ of entire functions which converges uniformly to 1 on K and uniformly to -1 on L.*

Problem 1.12.4. *Show that if $\Omega \subset \mathbb{C}$ is an open set and $f \in A(\Omega)$, then there exists a sequence $\{r_n\}$ of rational functions with poles in $\hat{\mathbb{C}} \setminus \Omega$ such that r_n converges normally to f.*

Problem 1.12.5. *Let $u \in \mathcal{C}([0,1] \times K)$ where K is a compact subset of $\mathbb{C}$. Show that*

$$S_n(z) = \frac{1}{n}\sum_{j=1}^{n} u(j/n, z)$$

converges uniformly and find its limit.

Problem 1.12.6. *Let K be a compact for which $\mathbb{C} \setminus K$ is not connected. Show that there is a function f holomorphic in a neighborhood of K which cannot be approximated uniformly on K by polynomials. Hint: Take a z_o which belongs to a bounded connected component of K^c. Take a polynomial p so that*

$$|(z - z_o)p(z) - 1| < 1.$$

Use the maximum principle to see that this inequality holds true at z_o as well (in fact in the component containing z_o).

Problem 1.12.7. *Using Runge's theorem construct a function $f \in A(\mathbb{D})$ which has no radial limit at any boundary point, i.e.,*

$$\lim_{r \to 1-} f(re^{i\theta})$$

does not exist for all θ.

Problem 1.12.8. * *Construct an entire function with the following property: given any bounded open set U with a connected complement $\hat{\mathbb{C}} \setminus U$ and $g \in A(U)$ there is a sequence of positive integers n_k such that $F_k(z) = F(z + n_k)$ converges uniformly to g on U.*

1.13. Homework

Problem 1.13.1. *Show that for two lattices we have* $\Lambda(\omega_1, \omega_2) = \Lambda(\omega'_1, \omega'_2)$ *iff there is a matrix* $A \in GL(2, \mathbb{Z})$ *with* $\det A = \pm 1$ *such that*

$$\begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Problem 1.13.2. *a) Consider* $SL(2, \mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbb{R}, \ \det A = 1 \right\}$. *Using your knowledge of* $Aut(\mathbb{D})$ *show that the group of automorphisms of the upper half-plane* $\mathbb{H} = \{z \in \mathbb{C} : \ \Im z > 0\}$ *is the group*

$$Aut(\mathbb{H}) \ = \ SL(2, \mathbb{R})/\{\pm I\} \overset{def}{=} PSL(2, \mathbb{R}).$$

Write explicitly the isomorphism.

b) Show that

$$Aut(\hat{\mathbb{C}}) \ = \ SL(2, \mathbb{C})/\{\pm I\} \overset{def}{=} PSL(2, \mathbb{C}).$$

Write explicitly the isomorphism.

Problem 1.13.3. *Show that* Γ*, the congruence group mod* 2 *(*$\subset Aut(\mathbb{H})$*), is generated by*

$$\mu(z) = \frac{z}{2z + 1} \ \text{ and } \omega(z) = z + 2.$$

Definition. Let $z_2, \ z_3, \ z_4$ be three distinct points of $\hat{\mathbb{C}}$. Consider the function

$$z \mapsto S_{z_2, z_3, z_4}(z) = \frac{z - z_3}{z - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}$$

which is called the cross ratio of $z, \ z_2, \ z_3, \ z_4$. If z_2, z_3 or z_4 is ∞ the formula is defined by taking the limit of the above formula when the corresponding point approaches ∞. It is customary to use the notation (z, z_2, z_3, z_4) for $S_{z_2, z_3, z_4}(z)$.

Problem 1.13.4. *a) Show that the cross ratio* (z, z_1, z_2, z_3) *is the unique linear fractional transformation which sends* $z_1, \ z_2, \ z_3$ *to* $1, 0, \infty$ *respectively.*

b) Show that the cross ratio is invariant under linear fractional transformations, i.e., if $T \in Aut(\hat{\mathbb{C}})$ *then* $(z, z_1, z_2, z_3) = (Tz, Tz_1, Tz_2, Tz_3)$.

c) The cross ratio $(z, z_1, z_2, z_3) \in \mathbb{R}$ *iff the four points* z, z_1, z_2, z_3 *lie on a circle or on a straight line.*

Problem 1.13.5. *Show that the right half* W' *of the fundamental domain* W *of* Γ *can be mapped homeomorphically onto* $\overline{\mathbb{H}}$ *so that* $0, 1, \infty$ *stay fixed. Hint: Use Carathéodory's theorem.*

Problem 1.13.6. *Suppose* f *is a doubly periodic meromorphic function on* $\mathbb{C}$ *with periods 1 and i so that* f *is holomorphic on* $\mathbb{C} \setminus \{m + ni \mid m, n \in \mathbb{Z}\}$. *Prove that the residue of* f *at each of the poles is zero.*

Problem 1.13.7. *Let* $\Lambda = \Lambda(\omega_1, \omega_2)$ *with* $\omega_2/\omega_1 \notin \mathbb{R}$ *be a lattice. For* $a \in \mathbb{C}$ *let*

$$\Phi = \Phi_a = \{a + t_1\omega_1 + t_2\omega_2 \mid 0 \le t_1, t_2 < 1\}$$

be the so called fundamental parallelogram at a.

a) Show that if $\delta \ge diam(\Phi)$ *then*

$$n - 1 - \delta \le |z| \le n + \delta, \ z \in A_n.$$

b) Show that there exists a constant M *depending on the area and diameter of* Φ *such that for* $n \in \mathbb{N}$ *the annulus* $A_n = \{z \in \mathbb{C} \mid n - 1 \le |z| < n\}$ *contains at most* Mn *lattice points. Hint: Consider the fundamental parallelograms* $\Phi_a, \ a \in \Lambda$ *that intersect* A_n.

c) Show that if $k > 2$ *then* $\sum_{\omega \in \Lambda'} \frac{1}{|\omega|^k}$*, where* $\Lambda' = \Lambda \setminus \{0\}$*, converges. Hint: Either use part b) or show that* $\inf |m\omega_1 + n\omega_2|/(|m| + |n|) \ > 0$ *and then write the sum as a suitable iterated sum.*

Problem 1.13.8. *Let* $\Lambda = \Lambda(\omega_1, \omega_2)$ *with* $\omega_2/\omega_1 \notin \mathbb{R}$ *be a lattice. Construct a function which has simple zeros at all points of* Λ *(the Weirstrass sigma function). Hint: Use Problem 1.13.7 and Problem 1.9.6*

Problem 1.13.9. *Let $\phi \in A(\mathbb{D})$ be one-to-one on $\mathbb{D}$. Show that ϕ is a proper map, i.e., for every compact $K \subset G = f(\mathbb{D})$ the pre-image $\phi^{-1}(K)$ is a compact of $\mathbb{D}$. In particular for every sequence of points $z_n \in \mathbb{D}$ which converges to a boundary point $z_o \in \partial D$ the sequence $w_n = \phi(z_n)$ converges to ∂G in the sense that w_n leaves every compact contained in G.*

Problem 1.13.10. *Let $\phi \in A(\mathbb{D}) \cap \mathcal{C}(\bar{\mathbb{D}})$ be one-to-one on $\mathbb{D}$. Show that if ϕ is injective on $\mathbb{D}$ then ϕ is injective on $\overline{D}$. Hint: Show that we only need to check injectivity on ∂D. For the latter argue by contradiction: if $\phi(\zeta) = \phi(\zeta')$ then ϕ has to be constant on one of the arcs between ζ and ζ' which can be seen by considering the image of the sector defined by 0, ζ and ζ'.*

Part 2. Homework Problems, MATH561, F2010

2.1. Homework

Problem 2.1.1. *Prove the following identity for any $\bar{z} \cdot w \neq 1$,*

$$1 - \left| \frac{z - w}{1 - z\bar{w}} \right|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2}.$$

Problem 2.1.2. *Prove that the function $\phi(z) = i\frac{1-z}{1+z}$ maps the open unit disc one-to-one onto the open upper half-plane.*

Problem 2.1.3. *Let $|z_0| < 1$. Prove that the function $\psi(z) = \frac{z_0 - z}{1 - \bar{z}_0 z}$ has the following properties:*

(1) ψ is holomorphic.
(2) $\psi(0) = z_0$ and $\psi(z_0) = 0$;
(3) the unit circle is mapped onto the unit circle;
(4) maps the open unit disc $\mathbb{D}$ one-to-one and onto itself.

Problem 2.1.4. *Show that a real-valued holomorphic polynomial must be identically constant.*

Problem 2.1.5. *Prove that $\triangle(|f|^2) = 4\left|\frac{\partial f}{\partial z}\right|^2$ if f is a holomorphic function.*

Problem 2.1.6. *Prove that if $f \in \mathcal{C}^2$ is holomorphic and non-vanishing, then $\ln|f|$ is harmonic.*

Problem 2.1.7. *Show that the function $f(z) = 1/z$ is holomorphic on the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$, but does not have a holomorphic antiderivative on A.*

Problem 2.1.8. *Write the Cauchy-Riemann equations in polar coordinates.*

Problem 2.1.9. *Define at every point $z \in \mathbb{R}^2$ a linear map $J_z : T_z\,\mathbb{R}^2 \to T_z\,\mathbb{R}^2$ given by*

$$J_z\left(\frac{\partial}{\partial x}\Big|_z\right) = \frac{\partial}{\partial y}\Big|_z, \quad J_z\left(\frac{\partial}{\partial y}\Big|_z\right) = -\frac{\partial}{\partial x}\Big|_z,$$

i.e., J_z is a c.c.w. rotation by 90^o. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(z) = (u(z), v(z))$, is a smooth map.

We showed that $f_* \circ J_z = J_{f(z)} \circ f_*$ at every $z \in \mathbb{R}^2$ iff f is a holomorphic function in the sense that it satisfies the Cauchy-Riemann equations.

(a) Derive the system satisfied by the derivatives of $f_* \circ J_z = -J_{f(z)} \circ f_*$.

(b) What can you say if J_z was defined to be a c.c.w rotation by angle ϕ instead of $\pi/2$ and should we have a continuum of complex analysis qualifying exams?

2.2. Homework

Problem 2.2.1. *Let $U_1 \subset U_2 \subset \dots$ be a sequence of connected open subsets of $\mathbb{C}$ and U be their union. Show that if $f \in A(U)$ and for every j, the restriction of f to U_j has a holomorphic antiderivative, then f has a holomorphic antiderivative on U.*

Problem 2.2.2. *Find all solutions of $z^{1+i} = 9$.*

Problem 2.2.3. *(a) Show that the function $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ is holomorphic.*

(b) Describe the holomorphic map $z \mapsto \cos z$ by showing images of some suitable curves in the complex plane, for example, lines parallel to the coordinate axes. In particular, show that the vertical strip $0 < x < \pi$ is mapped one-to-one and onto the region $\mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$.

(c) Is $\cos z$, $z \in \mathbb{C}$, a bounded function? Show that $|\cos(x + iy)| \le e^y$ for $y \ge 0$.

Problem 2.2.4. *(a) Let $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$. Find $\frac{\partial \sin z}{\partial z}$ and $\frac{\partial \sin z}{\partial \bar{z}}$.*

(b) Show that $(\sin z)' = \cos z$.

Problem 2.2.5. *(a) Show that the function $\cosh z = \frac{e^{z}+e^{-z}}{2}$ and $\sinh z = \frac{e^{z}-e^{-z}}{2}$ are holomorphic. Find their complex derivatives.*

(b) Describe the holomorphic map $z \mapsto \cosh z$ by showing images of some suitable curves in the complex plane, for example, lines parallel to the coordinate axes.

Problem 2.2.6. *(a) Write the complex polynomial $P(z, \bar{z}) = z^3 - \bar{z}^2 z$ in real notation, i.e., in terms of x and y, $z = x + iy$.*

(b) Write the polynomial $F(x, y) = (x - y^2) + i(x + y^2)$ as a polynomial in z and $\bar{z}$.

Problem 2.2.7. *If possible, find a holomorphic function $f = u + iv$ with the given real part.*

(a) $u = x^2 - y^2$.

(b) $u = \cosh x \cos y$.

(c) $u = x^3$.

Problem 2.2.8. *Find all holomorphic functions $w = f(z)$ defined on $\mathbb{C}$ whose image is contained in the coordinate axes $uv = 0$, $w = u + iv$.*

2.3. Homework

Problem 2.3.1. *Show that if h is a complex valued harmonic polynomial, then h can be written in the form $h(z) = p(z) - \overline{q(z)}$, where p and q are holomorphic polynomials*

Problem 2.3.2. *a) Let C be the unit circle in $\mathbb{C}$. Find a holomorphic function $S(z)$ defined in a neighborhood of C (all point in an annulus around C) so that $S(z) = \bar{z}$ for every $z \in C$.*

b) Show that $|S'(z)| = 1$ for $z \in C$.

Problem 2.3.3. *Suppose γ is a smooth arc in $\mathbb{C}$ for which there is a holomorphic function $S(z)$ defined near γ and such that $S(z) = \bar{z}$ for every $z \in \gamma$. Let $z_o \in \gamma$ and $k = S'(z_o) \in \mathbb{C}$.*

a) Show that $|S'(z)| = 1$ for $z \in \gamma$.

b) Show that the equation of the tangent line to γ at z_o can be written as

$$\bar{z} = S'(z_o) \cdot (z - z_o) + \bar{z}_o.$$

c) Show that $\frac{1}{2} \arg S'(z_o)$ is the angle the tangent to γ at z_o makes with the real axis (so the slope of the tangent is determined).

Problem 2.3.4. *Suppose f and g are $\mathcal{C}^1$ complex valued functions (not necessarily holomorphic!) and $\frac{\partial f}{\partial z} = \frac{\partial g}{\partial z}$ for all $z \in \mathbb{C}$. How are f and g related?*

Problem 2.3.5. *Compute the following line integrals.*

a) $\int_\gamma \frac{z^2}{z-1}\, dz$, where γ is the circle of radius 3 centered at the origin and c.c.w. orientation.

b) $\int_\gamma \frac{z}{(z+4)(z-1+i)}\, dz$, where γ is the circle of radius 1 centered at the origin and c.c.w. orientation.

c) $\int_\gamma \bar{z} + z^2\bar{z}\, dz$ where γ is the unit square of side of length two, centered at (0,0) with clockwise orientation.

Problem 2.3.6. *Show that if F is holomorphic and $F \in \mathcal{C}^2$, then $f(z) = F'(z)$ is also holomorphic.*

Problem 2.3.7. *Let $f : \mathbb{C} \to \mathbb{C}$ be a $\mathcal{C}^1(\mathbb{C})$ function (so f is not given to be holomorphic!). Suppose that for any piecewise linear curve $\gamma : [a, b] \to \mathbb{C}$ satisfying $\gamma(a) = \gamma(b)$ it holds that $\int_\gamma f(z)dz = 0$.*

(a) Prove that f has a $\mathcal{C}^2$ holomorphic antiderivative F, i.e., there is $F \in \mathcal{C}^2(\mathbb{C})$, $\frac{\partial F}{\partial \bar{z}} = 0$ and $F' = f$. You might want to fix a point $z_o \in \mathbb{C}$ and consider $F(z) = \int_\gamma f(\zeta)\, d\zeta$ where γ is a suitable curve connecting z_o to z.

(b) Show that f is holomorphic.

Problem 2.3.8. *Let φ be a real valued $\mathcal{C}^1$ function, $\varphi : \mathbb{C} \to \mathbb{R}$, for which the integral $\int\int_{\mathbb{R}^2} |\nabla\varphi|^2 dA$ exists, where dA is the area element in the plane. Let $z = f(w)$ be any holomorphic function on $\mathbb{C}$ and $\psi(w) = \varphi(f(w))$. Show that*

$$\int\int_{\mathbb{R}^2} |\nabla\varphi|^2 dA = \int\int_{\mathbb{R}^2} |\nabla\psi|^2 dA.$$

You might want to recall that $|\nabla\varphi|^2 = 4|\frac{\partial\varphi}{\partial z}|^2$ and $dz \wedge d\bar{z} = -2i dx \wedge dy$ in order to do the verification in one or two lines.

2.4. Homework

Problem 2.4.1. *Prove that the Cauchy integral formula*

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(z)}{\zeta - z}\, d\zeta, \qquad z \in \mathbb{D}$$

holds for any function $f \in \mathcal{C}(\mathbb{D}) \cap A(\mathbb{D})$*, where* $\mathbb{D}$ *is the unit disc. The point is that* f *is not a* $\mathcal{C}^1(\bar{\mathbb{D}})$ *function. Notice that you can use the proven formula on every disc* $D(0,r)$*,* $0 < r < 1$*, and its boundary.*

Problem 2.4.2. *Let* $D(z_o, r)$ *denote the open disc centered at* z_o *and radius* r*. Compute the integrals:*

a)

$$\int_{\partial D(8i,2)} z^3\, dz;$$

b)

$$\int_{\partial D(6+i,2)} (\bar{z} - i)^2\, dz;$$

c)

$$\frac{1}{2\pi i} \int_{\partial D(0,1)} \frac{1}{z+2}\, dz;$$

d)

$$\frac{1}{2\pi i} \int_{\partial D(0,2)} \frac{1}{z+1}\, dz.$$

Problem 2.4.3. *Let* f *be a continuous function on* $\partial\mathbb{D}$*– the boundary of the unit circle oriented c.c.w. Let*

$$F(z) = \begin{cases} f(z), & |z| = 1; \\ \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(\zeta)}{\zeta - z}\, d\zeta, & |z| < 1. \end{cases}$$

Is it true that $F \in \mathcal{C}(\bar{\mathbb{D}})$*? You might want to consider* $f(z) = \bar{z}$*.*

Problem 2.4.4. *Let* $D \subset \mathbb{C}$ *be an open disc with center* 0*. Suppose that both* f *and* g *are holomorphic functions on* $D \setminus 0$*. Show that if* $\frac{\partial f}{\partial z} = \frac{\partial g}{\partial z}$ *on* $D \setminus 0$*, then* f *and* g *differ by a constant.*

Problem 2.4.5. *Calculate* $\int_\gamma \frac{d\zeta}{(\zeta-1)(\zeta-2i)}$*, where* γ *is the circle with center 0, radius 4, and counterclockwise orientation.*

Problem 2.4.6. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be* $\mathcal{C}^1$ *function. Suppose that for any piecewise linear curve* $\gamma : [a,b] \to \mathbb{C}$ *satisfying* $\gamma(a) = \gamma(b)$ *it holds that* $\int_\gamma f(z)\, dz = 0$*.*

(a) Prove that there is a holomorphic function F *on* $\mathbb{C}$ *such that* $\frac{\partial F}{\partial z} = f$*.*

(b) Show that f *is holomorphic.*

2.5. Homework

Problem 2.5.1. *(a) Prove that if $U \subset \mathbb{C}$ is open and connected and if $p, q \in U$, then there is a piecewise $\mathcal{C}^1$ curve from p to q consisting of horizontal and vertical line segments. [Hint: Show that, with $p \in U$ fixed, the set of points $q \in U$ that are reachable from p by curves of the required type is both open and closed in U.]*

(b) Let $f \in \mathcal{C}(\mathbb{C})$ and holomorphic on the complement of the coordinate axes. Prove that f is actually holomorphic on $\mathbb{C}$.

Problem 2.5.2. *Show that the conclusion of Morera's theorem still holds if it is only assumed that the integral of f around the boundary of every rectangle in U or around every triangle in U is 0.*

Problem 2.5.3. *Let $\gamma : [0, 1] \to \mathbb{C}$ be any $\mathcal{C}^1$ curve. Define*

$$f(z) = \int_\gamma \frac{1}{\zeta - z}\, d\zeta.$$

(a) Prove that f is holomorphic on $\mathbb{C} \setminus T_\gamma$, where T_γ is the trace (image) of γ.

(b) In case $\gamma(t) = t$, show that there is no way to extend f to a continuous function on all of $\mathbb{C}$.

Problem 2.5.4. *Let γ be the unit circle $|z| = 1$ oriented c.c.w. and*

$$f(z) = \int_\gamma \frac{1}{\zeta - z}\, d\zeta.$$

Find $f''(0)$.

Problem 2.5.5. *Explain why the following string of equalities is incorrect:*

$$\frac{d^2}{dx^2} \int_{-1}^{1} \ln|x - t|\, dt \;=\; \int_{-1}^{1} \frac{d^2}{dx^2} \ln|x - t|\, dt \;=\; \int_{-1}^{1} \frac{-1}{(x - t)^2}\, dt.$$

Problem 2.5.6. *Show that if f is an entire function such that for some real number $M > 0$ and positive integer k we have*

$$|f(z) \leq M|z|^k$$

for all $|z| \geq 1$, then f is a polynomial of degree at most k.

Problem 2.5.7. *Evaluate the following integrals:*

(a) $\int_{|z|=2} \frac{2z}{(z-3)^2}\, dz$;

(b) $\int_{|z|=1} \frac{z}{z(z-3)}\, dz$;

(c) $\int_{|z+2|=1} \frac{z}{4z-z^2}\, dz$;

(d) $\int_{|z|=2} \frac{2z}{z(z-3)}\, dz$;

2.6. Homework

Problem 2.6.1. *Show that if the sequence $f_n \in \mathcal{C}(U : \mathbb{C})$, $U \subset \mathbb{C}$ converges locally uniformly, then the limit function $f(z) = lim_{n\to\infty} f_n(z)$, $z \in U$ is a continuous function on U, i.e., $f \in \mathcal{C}(U : \mathbb{C})$.*

Problem 2.6.2. *Show the mean value property for a holomorphic function, i.e., if f is holomorphic in some open set U then for any disc $D = D(z_0, R) \subset U$ we have*

$$f(0) = \frac{1}{2\pi R} \int_{\partial D} f(\zeta)\, |d\zeta|.$$

Note: $|d\zeta|$ is the arc-length parameter on the circle.

Problem 2.6.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ be the holomorphic function defined on the disc of convergence around z_0 of the power series (suppose the disc is of non-zero radius). Show that*

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Problem 2.6.4. *Determine the disc of convergence of each of the following series.*

(a) $\sum_{k=3}^{\infty} k z^k$;

(b) $\sum_{k=2}^{\infty} k^{\ln k} (z+1)^k$;

(c) $\sum_{k=2}^{\infty} (\ln k)^{\ln k} (z-3)^k$;

(d) $\sum_{k=0}^{\infty} p(k) z^k$ *where p is some fixed polynomial;*

(e) $\sum_{k=1}^{\infty} 3^k (z+2i)^k$;

(f) $\sum_{k=2}^{\infty} \frac{k}{k^2+4} z^k$

(g) $\sum_{k=0}^{\infty} k e^{-k} z^k$;

(h) $\sum_{k=1}^{\infty} \frac{1}{k!} (z-5)^k$;

(i) $\sum_{k=1}^{\infty} k^{-k} z^k$.

Problem 2.6.5. *The functions $f_k(x) = \sin(kx)$ are $\mathcal{C}^\infty$ and bounded by 1 on the interval $[-1, 1]$ yet their derivatives at 0 are unbounded. Contrast this situation with the functions $f_k(z) = \sin(kz)$ on the unit disc. The Cauchy estimates provide bounds for $|\frac{\partial f_k}{\partial z}(0)|$. Why are these two examples not contradictory?*

Problem 2.6.6. *Suppose that $f : D(0,1) \to \mathbb{C}$ is holomorphic and that $|f(z)| \le 2$, $z \in D(O,1)$. Derive an estimate for $\left|\frac{d^3 f}{dz^3}(\frac{i}{3})\right|$.*

Problem 2.6.7. *Show that if $f : D(0,R) \to \mathbb{C}$ is holomorphic, then*

$$|f(0)|^2 \le \frac{1}{\pi R^2} \int_{D(0,R)} |f(z)|^2\, dxdy.$$

Hint: The function f^2 is holomorphic too. Use the Cauchy integral formula to obtain on any circle $|\zeta| = r < R$ to obtain a bound from above of $|f(0)|^2$ by a constant multiple of $\int_{|\zeta|=r} |f(\zeta)|^2\, |d\zeta|$. Then use that in polar coordinates $dxdy = r d\phi\, dr$ with $r d\phi = |d\zeta|$ – the arc-length parameter on the circle of radius r.

2.7. Homework

Problem 2.7.1. *Show that a uniformly convergent sequence of continuous functions defined on some subset $A \subset \mathbb{C}$ have a limit which is a continuous function.*

Problem 2.7.2. *Let $K_1 \Subset K_2 \Subset \Omega$, where Ω is an open subset of $\mathbb{C}$. Show that there exists a constant $C = C(K_1, K_2, \Omega)$ such that for any $f \in A(\Omega)$ we have*

$$\max_{z \in K_1} |f'(z)| \leq C \max_{z \in K_2} |f(z)|.$$

Problem 2.7.3. *Show that if $f \in A(\mathbb{C})$ and the Taylor series of f centered at 0 converges uniformly to f on $\mathbb{C}$, then f is a polynomial.*

Find the power series expansion for each of the following holomorphic functions about the given point. Determine the radius of convergence of each series.

Problem 2.7.4. $f(z) = 1/(1 + 2z),\ P = 0.$

Problem 2.7.5. $f(z) = z^2/(4 - z),\ P = i.$

Problem 2.7.6. $f(z) = 1/z,\ P = 2 - i.$

Problem 2.7.7. $f(z) = (z - \frac{1}{2})/(1 - \frac{z}{2}),\ P = 0.$

Problem 2.7.8. $f(z) = \frac{\sin z}{1+z^2},\ P = 0.$

Problem 2.7.9. *Suppose $f \in A(D)$, $D = D(0, 1)$. Is it possible to have $|f^{(n)}(0)| \geq e^n n!$?*

Problem 2.7.10. *(a) For $n, m \in \mathbb{N}$ compute $\int_{|z|=r} z^n \bar{z}^m \, |dz|$.*

(b) Let $f \in A(D)$, $D = D(0, 1)$, and $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Prove Parseval's equality

$$\frac{1}{2\pi} \int_{|z|=r} |f(z)|^2 \, |dz| \; = \; \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

valid for any $0 < r < 1$.

2.8. Homework

Problem 2.8.1. *For a natural number q let $z^{1/q}$ be the q-th root function, which is real and positive for z-real and positive, i.e., if $z = re^{i\phi}$, $-\pi \le \phi < \pi$, then $z^{1/q} = \sqrt[q]{r}\, e^{-i\frac{\phi}{q}}$.*

(a) Let $\alpha = p/q$–rational number. Show that

$$\frac{d^n}{dz^n}(z^\alpha) = \alpha(\alpha-1)\dots(\alpha-n+1)z^{\alpha-n}, \qquad z \in \mathbb{C} \setminus \{(-\infty, 0]\}.$$

In particular $z \mapsto z^{p/q}$ is a holomorphic function on $\mathbb{C} \setminus \{(-\infty, 0]\}$.

(b) Let $\alpha = 1/2$. Show that

$$z^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}(z-1)^n, \qquad |z-1| < 1.$$

This is the Taylor series of $z^{1/2}$ centered at $z = 1$.

(c) Let $\alpha = 1/2$. Show that

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} z^n, \qquad |z| < 1.$$

This is the Taylor series of $(1+z)^{1/2}$ centered at $z = 0$.

Problem 2.8.2. *Let $f \in A(\bar{\Omega})$, Ω–open bounded subset of $\mathbb{C}$. Show that f has at most finitely many zeros in $\bar{\Omega}$.*

Problem 2.8.3. *Using the open mapping theorem, show that if $f \in A(\Omega)$, Ω–open subset of $\mathbb{C}$, then we have the following* Maximum Principle.

If $|f(z)|$ has a local maximum M at $z_0 \in \Omega$, then $f \equiv M$. As usual, z_0 is a point of local maximum of the modulus of f if $|f(z)| \le |f(z_0)|$ for all z in some neighborhood D of z_0. Hint: Note that $w \in f(D)$ must satisfy $|w| \le |f(z_0)|$.

Problem 2.8.4. *Show that if $f \in A(\Omega)$, Ω–open subset of $\mathbb{C}$, then we have the following* Minimum Principle. *If $|f(z)|$ has a local minimum m at $z_o \in \Omega$, then $f \equiv m$ or $m = 0$. Thus, the modulus of a non-constant holomorphic function cannot achieve a strictly positive local minimum. (Minimum principle)*

Problem 2.8.5. *Let $f \in A(\Omega) \cap \mathcal{C}(\bar{\Omega})$, Ω–open bounded subset of $\mathbb{C}$. Show the following versions of the maximum/ minimum principles.*

(a) The $\max_{z \in \bar{\Omega}} |f(z)|$ is achieved on $\partial\Omega$, thus

$$|f(z)| \le \max_{\zeta \in \partial\Omega} |f(\zeta)|, \qquad z \in \bar{\Omega}.$$

(b) The $\min_{z \in \bar{\Omega}} |f(z)|$ is achieved on $\partial\Omega$ if f has no zeros in Ω.

Problem 2.8.6. *Show that the claim of Problem 2.8.5(b) is not true if we drop the condition $f(z) \neq 0$, $z \in \Omega$.*

Problem 2.8.7. *Show that if $f \in A(\mathbb{D})$, $\mathbb{D} = D(0,1)$ with $|f(z)| = 1$, z when $|z| = 1$, then $\mathbb{D} \subset f(\mathbb{D})$.*

Problem 2.8.8. *(a) Show that there is no universal $r > 0$ such that for all $f \in A(\mathbb{D})$, $\mathbb{D} = D(0,1)$ satisfying $f(0) = 0$ we have that $D(0,r) \subset f(D)$. Hint: $f_n(z) = \frac{z}{2n}$ might be helpful.*

(b) Show that $f_n(z) = n(e^{z/n} - 1)$ is a better example since $f_n'(0) = 1$.

2.9. Homework

Problem 2.9.1. *Find the maximum of* $|f(z)|$ *on* $\bar{\mathbb{D}}$ *for* $f(z) = \frac{z^3}{z-5}$.

Problem 2.9.2. *Show that if we have a sequence of functions* $f_n \in \mathcal{C}(\bar{\Omega}) \cap A(\Omega)$ *which converges uniformly on* $\partial\Omega$ *then* f_n *converges uniformly to a function* f *on* $\bar{\Omega}$ *and* $f \in \mathcal{C}(\bar{\Omega}) \cap A(\Omega)$. *Hint: Use the maximum principle.*

Problem 2.9.3. *Show that if* $f \in A(\Omega)$, *then either* f *is constant or* $\Re f$ *cannot have a local maximum in* Ω.

Problem 2.9.4. *Show that if* $f_n \in A(\Omega)$ *is a sequence of nowhere vanishing functions,* $f_n(z) \neq 0$ *for* $z \in \Omega$, *which converges locally uniformly to a function* f, *then either* $f \equiv 0$ *or* $f(z) \neq 0$ *for* $z \in \Omega$.

Problem 2.9.5. *Show that if* $f_n \in A(\Omega)$ *is a sequence of one-to-one functions, which converge locally uniformly to a function* f, *then either* $f \equiv const$ *or* f *is also one-to-one.*

Problem 2.9.6. *Show that the linear fractional transformations of the form*

$$f(z) = \lambda \frac{z - c}{1 - \bar{c}z}$$

where $|\lambda| = 1$ *and* $c \in \mathbb{D}$ *form a group of biholomorphisms of the unit disc* $\mathbb{D}$.

Problem 2.9.7. *Show that the group defined in the last problem is Aut*$(\mathbb{D})$. *Hint: Recall the proof for the group* $SU(1,1)/\{\pm I\}$ *or see directly that the group in this problem is* $SU(1,1)/\{\pm I\}$.

Problem 2.9.8. *Show that if* $f \in$*Aut*$(\mathbb{D})$, *then we have equalities in the Schwarz-Pick inequalities.*

Problem 2.9.9. *For* $z_1, z_2 \in \mathbb{D}$ *let*

$$\rho(z_1, z_2) = \tanh^{-1}\left(\left|\frac{z_1 - z_2}{1 - z_1\bar{z}_2}\right|\right).$$

a) Show that if $f \in$*Aut*$(\mathbb{D})$, *then* $\rho\left(f(z_1), f(z_2)\right) = \rho(z_1, z_2)$ *for* $z_1, z_2 \in \mathbb{D}$. *Hint: Use Problem 2.9.8.*

b) Show that ρ *defines a distance function on* $\mathbb{D}$. *This is the* Poincare distance *of* $\mathbb{D}$. *Hint: For the triangle inequality,reduce to the case* $\rho(z_1, z_2) \leq \rho(z_1, 0) + \rho(0, z_2) = \rho(|z_1|, 0) + \rho(0, |z_2|)$ *using that Aut*$(\mathbb{D})$ *is transitive and (a). Use also that* $\tanh^{-1}\left(\frac{r_1+r_2}{1+r_1r_2}\right) = \tanh^{-1}(r_1) + \tanh^{-1}(r_2)$ *for* $r_1, r_2 \in [0, 1]$.

Problem 2.9.10. *a) Show that if* $f : \mathbb{D} \to \mathbb{D}$ *is holomorphic then* f *is a contraction on* $\mathbb{D}$ *considered with the Poincare distance. Hint: Use Schwarz-Pick.*

b) Show that if $f : \mathbb{D} \to \mathbb{D}$ *is holomorphic then* $f \in$*Aut*$(\mathbb{D})$ *iff* f *is an isometry of* $\mathbb{D}$ *considered with the Poincare distance.*

2.10. Homework

Problem 2.10.1. *Determine the type of singularity of each of the following function at* $z = 0$.
a) $f(z) = \frac{1}{z}$*; b)* $f(z) = \sin\frac{1}{z}$*; c)* $f(z) = \frac{1}{z^3}$.

Problem 2.10.2. *Determine the type of singularity of each of the following function at* $z = 0$.
a) $f(z) = \frac{\sin z}{z}$*; b)* $f(z) = \frac{\cos z}{z}$*;*

Problem 2.10.3. *Determine the type of singularity of each of the following function at* $z = 0$*. a)* $f(z) = z \cdot e^{\frac{1}{z}} \cdot e^{-\frac{1}{z^2}}$*; b)* $\frac{\sum_k = 2^\infty 2^k z^k}{z^3}$.

Problem 2.10.4. *Let* f *be holomorphic on* $U \setminus \{z_0\}$*,* $z_0 \in U$*,* U*- open subset of* $\mathbb{C}$*. If* f *has an essential singularity at* z_0*, then what type of singularity does* $1/f$ *have at* z_0*? What about when* f *has a removable singularity or a pole at* z_0*?*

Problem 2.10.5. *Let* $z_0 \in U$*-open subset of* $\mathbb{C}$*. Let*

$$A_j = \{f \text{ - holomorphic on } U \setminus \{z_0\} : \ f \text{ has a singularity of type (j) at } z_0\},$$

where (j) refers to singularities of types (i) removable, (ii) pole, or (iii) essential singularity. Is A_j *closed under addition? Multiplication? Division?*

Problem 2.10.6. *Let* $\mathbb{D}^\times = \mathbb{D} \setminus \{0\}$*. Prove the following two refined versions of Riemann's theorem.*

a) If f *is holomorphic on* $\mathbb{D}^\times$ *and* $\lim z f(z) = 0$ *when* $z \to 0$ *then* 0 *is a removable singularity.*
b) If f *is holomorphic on* $\mathbb{D}^\times$ *and if* $\int_{\mathbb{D}} |f(z)|^2 \, dxdy < \infty$*, then* 0 *is a removable singularity.*

Problem 2.10.7. *Suppose that* $z = 0$ *for some integer* $N \geq 0$ *we have that* $z^N f(z)$ *is bounded near* $z = 0$*. Show that* $f(z) = \frac{g(z)}{z^n}$ *for some function g holomorphic on the unit disc* $\mathbb{D}$*. If we take N to be the smallest such integer, then the corresponding g will satisfy the condition* $g(0) \neq 0$*.*

Problem 2.10.8. *Suppose* $z = 0$ *is not a removable singularity of the holomorphic function* f *on* $\mathbb{D}^\times = D(0,1) \setminus \{0\}$*. Show that* $z = 0$ *is a pole singularity iff for some integer* $N \geq 1$ *we have that* $z^N f(z)$ *is bounded near* $z = 0$*.*

Problem 2.10.9. *In this problem you are asked to do what we did for* $SU(1,1)/\{\pm I\}$ *starting with another subgroup of* $GL(2,\mathbb{C})$*. Consider* $SL(2,\mathbb{R}) = \left\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a,b,c,d \in \mathbb{R}, \ \det A = 1\right\}$.

a) Show that $SL(2,\mathbb{R})$ *is a subgroup of* $GL(2,\mathbb{C})$ *(even* $GL(2:\mathbb{R})$*).*
b) Using Problem 2.1.2 and your knowledge of $\mathrm{Aut}(\mathbb{D})$ *show that the group of automorphisms of the upper half-plane* $\mathbb{H} = \{z \in \mathbb{C} : \ \Im z > 0\}$ *is the the group* $SL(2,\mathbb{R})/\{\pm I\}$*.*

2.11. Homework

Problem 2.11.1. *Find the Laurent series on the annulus $1 < |z| < 5$ for the function*

$$f(z) = \frac{z+1}{(z-1)(z-5)}.$$

Problem 2.11.2. *Calculate the annulus of convergence for each of the following Laurent series: a) $\sum_{n=-\infty}^{\infty} 2^{-n} z^n$; b) $\sum_{n=0}^{\infty} 4^{-n} z^n + \sum_{n=-\infty}^{-1} 3^n z^n$.*

Problem 2.11.3. *Calculate the annulus of convergence (including possible boundary points) for each of the following Laurent series: a) $\sum_{n=-\infty}^{\infty} z^n/n^2$; b) $\sum_{n=-\infty}^{\infty} z^n/n^n$.*

Problem 2.11.4. *Calculate the annulus of convergence (including possible boundary points) for each of the following Laurent series: a) $\sum_{n=-\infty}^{10} z^n/|n|!$; b) $\sum_{n=-20}^{\infty} n^2 z^n$.*

Problem 2.11.5. *Give an example of a (formal) doubly infinite series such that*

$$\lim_{n\to\infty} \sum_{k=-n}^{n} a_k z^k$$

exists for some $z \neq 0$ but such that

$$\sum_{k=-\infty}^{\infty} a_k z^k$$

fails to converge for that same z.

Problem 2.11.6. *a) Let f_n be a sequence of functions that are holomorphic on the punctured unit disc $\mathbb{D}^{\times}$ and suppose that each f_n has a pole at $z = 0$. If the sequence $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{D}^{\times}$, then does the limit function f necessarily have a pole at $z = 0$?*

b) Answer the same question with "pole" replaced by "removable singularity" or "essential singularity."

Problem 2.11.7. *Prove that f has an essential singularity on the punctured unit disc $\mathbb{D}^{\times} = D(0,1) \setminus \{0\}$ iff for each positive integer N there is a sequence $z_n \in \mathbb{D}^{\times}$ with $\lim_{z_n\to 0} = 0$ and $|z_n^N f(z_n)| \geq N$. Thus, f "blows up" faster than any positive power of $1/z$ along some sequence converging to 0.*

Problem 2.11.8. *Let f be holomorphic on the punctured disc $D^x = D(P,r) \setminus \{z_0\}$ and suppose that f has a pole of order k at z_0. Then the Laurent series coefficients a_n of f expanded about the point z_0, for $n = -k, -k+1, -k+2, \ldots$, are given by the formula*

$$a_n = \frac{1}{(k+n)!} \left(\frac{\partial}{\partial z}\right)^{k+n} \left((z - z_0) \cdot f(z)\right)_{|z=z_0}.$$

Problem 2.11.9. *Prove that if f has a non-removable singularity at z_0, then e^f has an essential singularity at z_0.*

Problem 2.11.10. *Calculate the first four terms of the Laurent expansion of the given function f about the given point z_0. In each case, specify the annulus of convergence of the expansion and the residue of f at z_0.*

a) $f(z) = \sin(1/z)$ *at* $z_0 = 0$*;*
b) $f(z) = \frac{z}{(z+1)^3}$ *at* $z_0 = -1$*;*
c) $f(z) = e^z/z^3$ *at* $z_0 = 0$*;*

2.12. Homework

Problem 2.12.1. *Suppose f, $g \in A(D)$, $D = D(z_0, R_0)$, are two holomorphic functions such that f has a simple zero at z_0 while $g(z_0) \neq 0$. Show that $Res\,(\frac{g}{f}, z_0) = \frac{g(z_0)}{f'(z_0)}$.*

Problem 2.12.2. *Compute each of the following residues $Res_f(z_0)$ for the given function and point.*

a) $z_0 = 2i, \quad f(z) = \frac{z^2}{(z-2i)(z+3)}$;
b) $z_0 = -3, \quad f(z) = \frac{z^2+1}{z(z+3)^2}$;
c) $z_0 = 2, \quad f(z) = \frac{z}{(z+1)(z-2)}$.

Problem 2.12.3. *Compute each of the following residues $Res_f(z_0)$ for the given function and point.*

a) $z_0 = i+1, \quad f(z) = \frac{e^z}{(z-i-1)^3}$;
b) $z_0 = \pi, \quad f(z) = \frac{\cot z}{z^2(z+i)^2}$;
c) $z_0 = 0, \quad f(z) = \frac{\sin z}{z^3(z-2)(z+1)}$.

Problem 2.12.4. *Use the calculus of residues to compute the integral $\frac{1}{2\pi i}\oint f(z)\,dz$, where $f(z) = e^{iz}/\,[\sin z \cos z]$ and γ is the quadrilateral with vertices $\pm 5i$, ± 10.*

Problem 2.12.5. *Use the calculus of residues to compute the following integrals, where the boundary is positively oriented unless said explicitly otherwise.*

a) $\frac{1}{2\pi i}\oint_{\partial D(0,5)} f(z)\,dz$, *where* $f(z) = z/\,[(z+1)(z+2i)]$;
b) $\frac{1}{2\pi i}\oint_{\partial D(0,5)} f(z)\,dz$, *where* $f(z) = e^z/\,[(z+1)\sin z]$.

Problem 2.12.6. *Use the calculus of residues to compute the following integrals, where the boundary is positively oriented unless said explicitly otherwise.*

a) $\frac{1}{2\pi i}\oint_{\partial D(0,8)} f(z)\,dz$, *where* $f(z) = \cot z/\left[(z-6i)^2+64\right]$;
b) $\frac{1}{2\pi i}\oint_{\gamma} f(z)\,dz$, *where* $f(z) = f(z) = e^z/\,[z(z+1)(z+2)]$ *and γ is the negatively oriented triangle with vertices $1 \pm i$ and -3.*

Problem 2.12.7. *Compute the integral $\int_{-\infty}^{\infty} \frac{x^2}{x^4-4x^2+5}\,dx$.*

Problem 2.12.8. *Compute the integral $\int_{-\infty}^{\infty} \frac{x\sin x}{x^4+1}\,dx$.*

Problem 2.12.9. *Compute the integral $\int_0^{2\pi} \frac{d\theta}{(2-\sin\theta)^2}$ by using $z = e^{i\theta}$.*

Problem 2.12.10. *Compute the integral $\int_{-\infty}^{\infty} \frac{\cos x}{e^x+e^{-x}}\,dx$ by using $f(z) = e^{iz}/(e^z+e^{-z})$.*

2.13. Homework

Problem 2.13.1. *Suppose f is a holomorphic function on a set of the form $\mathbb{C} \setminus D(0, R)$ for some $R > 0$ and $F(z) = f(1/z)$, $|z| < 1/R$.*

a) If f has a pole of order k at ∞, what property does $1/f$ have at ∞?
b) What property does $1/F(z)$ have at 0?

Problem 2.13.2. *Suppose $f : \hat{\mathbb{C}} \to \mathbb{C}$ is holomorphic. Show that $f = const$.*

Problem 2.13.3. *Calculate the residue of the given function at ∞.*
a) $f(z) = z^3 - 7z^2 + 8$; b) $f(z) = z^2 e^z$.

Problem 2.13.4. *Calculate the residue of the given function at ∞.*
a) $f(z) = e^z / p(z)$, $p(z)$ a polynomial; b) $f(z) = p(z)e^z$, $p(z)$ a polynomial.

Problem 2.13.5. *Calculate the residue of the given function at ∞. a) $f(z) = \sin z$; b) $f(z) = \cot z$. Note: Be careful what you answer here!*

Problem 2.13.6. *Determine all poles of the function $f(z) = 1/\sin(1/(1-z))$ in the unit disc $\mathbb{D}$.*

Problem 2.13.7. *Let f be a non-constant holomorphic function on a neighborhood of the closed disc $\bar{D} = \overline{D(z_o, r_o)}$. Show that f has at most finitely many zeros on $\bar{D}$.*

Problem 2.13.8. *Classify the singularities of the following functions on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and find the principal (singular) part at each of them. a) $f(z) = \frac{1-z^3}{1-z^2}$; b) $f(z) = \frac{\sin z}{z^3}$.*

Problem 2.13.9. *Show that if Ω is an open subset of $\mathbb{C}$ and S is a discrete subset of Ω then S is relatively closed in Ω.*

Problem 2.13.10. *Let Ω be an open connected subset of $\mathbb{C}$ and S a discrete subset of Ω. Show that S is at most countable. Hint: If K is a compact, $K \subset \Omega$, what can you say about $S \cap K$? Sets of the form $K_n = \bar{B}(0, n) \cap \{z \in \Omega : dist(z, \partial\Omega) \geq 1/n\}$, $n \in \mathbb{N}$ might be useful.*

2.14. Homework

Problem 2.14.1. *Let f be holomorphic on the punctured unit disc $\mathbb{D}^\times$. What kind of singularity is $z = 0$ if we know that f is an injective map, i.e, $z \neq z'$ implies $f(z) \neq f(z')$?*

Problem 2.14.2. *Show that if $f : \mathbb{C} \to \mathbb{C}$ is a one to one holomorphic map which is conformal, then $f(z) = az + b$, i.e., f is a linear function.*

Hint: What kind of singularity is ∞? You can use without a proof that a polynomial of degree n has n complex roots.

Problem 2.14.3. *Let $f(z) = -2/(z^2 - z - 2)$.*

a) Find the Laurent expansion of f in the region $|z + 1| > 3$.
b) Classify the singularities of f including the ∞ point and find the residues at the singular points.

Problem 2.14.4. *Evaluate the integral*

$$\oint_\gamma \frac{e^z}{(z+1)(z-2i+1)}\,dz,$$

where γ is the ellipse $\frac{x^2}{4} + y^2 = 1$ with positive orientation (c.c.w.).

Problem 2.14.5. *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and $M(r) = \max_{|z|=r} |f(z)|$.*

a) Show that $M(r)$ is a continuous and non-decreasing function.
b) Show that $M(r)$ is increasing if $f \neq const$, i.e., if $M(r) \geq M(R)$ for some $r < R$, then $f = const$.

Problem 2.14.6. *Let f be a function holomorphic on the unit disc $\mathbb{D}$ and $f(0) = 0$. Show that*

$$F(z) = \sum_{k=1}^{\infty} f_k(z)$$

defines a holomorphic function on $\mathbb{D}$, where $f_k(z) = f(z^k)$, $k = 0, 1, 2, \ldots$.

Hint: Show that $F_n = \sum_{k=1}^{n} f_k(z)$ converges locally uniformly on $\mathbb{D}$ by using Schwarz's lemma.

2.15. Homework

Problem 2.15.1. *Estimate the number of zeros of the given function in the given region U. a) $f(z) = z^8 + 5z^7 - 20$, $U = D(0,6)$; b) $f(z) = z^3 - 3z^2 + 2$, $U = D(0,1)$; c) $f(z) = z^2 e^z - z$, $U = D(0,2)$.*

Problem 2.15.2. *Show that the polynomial $p(z) = z^4 + 2z^3 + 3z^2 + z + 2$ has exactly two zeros in the right half-plane.*

Problem 2.15.3. *Let p be a polynomial on $\mathbb{C}$.*

a) *Show that if the zeros are contained in a given half-plane $V \subset \mathbb{C}$, then the same is true for the zeros of p'.*
b) *Show that the zeros of p' are contained in the closed convex hull of the zeros of p. The closed convex hull of a set S is the intersection of all closed convex sets that contain S.*

Problem 2.15.4. *Let $P_t(z) = \sum_{k=0}^{n} a_k(t) z^k$, where $a_k \in \mathcal{C}([0,1])$. In other words P_t is a one-parameter family of polynomials of the* same degree *depending continuously on a parameter.*

a) *Let $Z = \{(z,t) : P_t(z) = 0\}$. If $(z_0, t_0) \in Z$ and $\left.\frac{\partial P_{t_0}}{\partial z}\right|_{z=z_0} \neq 0$, then show, using the argument principle, that there is an $\epsilon > 0$ such that for all t sufficiently close to t_0 there is a unique $z \in D(z_0, \epsilon)$ with $P_t(z) = 0$.*
b) *Prove that if the roots of P_{t_0} are distinct (no multiple roots) then the same is true for P_t for all t sufficiently close to t_0.*

Problem 2.15.5. *Let $g \in A(\Omega)$, Ω–open subset of $\mathbb{C} \setminus \{(-\infty, 0]\}$. Show that if g does not have a zero in Ω then we can find $h \in A(\Omega)$ such that $h^n = g$, i.e., we can define a holomorphic n-th root of g.*

Problem 2.15.6. *Give a proof of Hurwitz' preservation of zeros theorem using the Argument Principle.*

Problem 2.15.7. *Let a_n be a strictly decreasing sequence of positive real numbers.*

a) *Show that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function on $\mathbb{D}$.*
b) *Show that every partial sum of the above series has no zeros in $\mathbb{D}$.*
c) *Show that f has no zeros in $\mathbb{D}$.*

Problem 2.15.8. *Show that $\int_0^{2\pi} \frac{d\theta}{a + \sin\theta} = \frac{2\pi}{\sqrt{a^2-1}}$ for $a > 1$.*

Problem 2.15.9. *Show that $\int_o^{\infty} \frac{x^{\lambda-1}}{x + e^{i\theta}}\, dx = \frac{\pi}{\sin \pi\lambda} e^{i(\lambda-1)\theta}$, where $0 < \lambda < 1$ and $-\pi < \theta < \pi$.*

Problem 2.15.10. *Use the calculus of residues to sum the series $\sum_{k=0}^{\infty} \frac{1}{k^4+1}$. Hint: Use either the tangent or cotangent function to introduce infinitely many poles that are located at the integer values that you want to study.*

Problem 2.15.11. *(Integral formula for the inverse function). Suppose f is holomorphic on $D(0,R)$, $f(0) = 0$, $f'(z) \neq 0$ on $D^{\times}(0,R)$. Let $0 < r < R$. Show that*

$$g(w) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{\zeta f'(\zeta)}{f(\zeta) - w}\, d\zeta$$

is the inverse function of f. More precisely, if $m = \min_{|\zeta|=r} |f(\zeta)|$, then g is holomorphic on $|w| < m$ and $f(g(w)) = w$ there, while $g(f(z)) = z$ for $z \in f^{-1}(|w| < m) \cap D(0,r)$. Hint: Use Rouche and the residue theorem by noting that g is the residue of a certain function.