STRONG UNIQUE CONTINUATION FOR GENERALIZED BAOUENDI-GRUSHIN OPERATORS

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1. Introduction and statement of the results

We say that a given partial differential operator $\mathcal{L}$ in $\mathbb{R}^N$ has the strong unique continuation property (SUCP) if every weak solution u of the equation $\mathcal{L}u = 0$, which vanishes to infinite order at some $z_o \in \mathbb{R}^N$, i.e.,

$$\lim_{r\to 0} \frac{1}{r^k} \int_{B_r(z_o)} |u(z)|^2 \, dV = 0, \quad \text{for all} \quad k > 0,$$

must vanish identically in some neighborhood of z_o. In other words non-trivial solutions can have at most finite order of vanishing.

In this paper we study the strong unique continuation property for a class of "variable coefficient" operators whose "constant coefficient" model at one point is the so called Baouendi-Grushin operator [B], [Gr1], [Gr2]. We recall that the latter is the following operator on $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$, $N = n + m$,

$$\mathcal{L}_o = \sum_{i=1}^{N} X_i X_i u, \tag{1.1}$$

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where the vector fields are given by

$$X_k = \frac{\partial}{\partial x_k}, \quad k = 1, \dots, n, \qquad X_{n+j} = |x|^\alpha \frac{\partial}{\partial y_j}, \quad j = 1, \dots, m. \quad (1.2)$$

Here $\alpha > 0$ is a fixed parameter, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_m) \in \mathbb{R}^m$. When $\alpha = 0$, $\mathcal{L}_o$ is just the standard Laplacian in $\mathbb{R}^N$. For $\alpha > 0$ the ellipticity of the operator $\mathcal{L}_o$ becomes degenerate on the *characteristic submanifold* $M = \mathbb{R}^n \times \{0\}$ of $\mathbb{R}^N$.

The SUCP for the operator $\mathcal{L}_o$ was proved in [G2]. In the same paper this is also proved for the operator $\mathcal{L}u + < \vec{V_1}, Du > + V_o u = 0$ with suitable assumptions on $\vec{V_1}$ and V_o. To give an idea, for example

$$|V_o| \ \leq \ \frac{C}{\rho}\psi \ \text{ and } \ | < \vec{V_1}, Du > | \ \leq \ C|Xu|\, \psi^{1/2}$$

is enough (note the use of an apriori estimate on the gradient Du in the above conditions). Here Du is the gradient of u, Xu is the horizontal gradient (1.7) of u, and ρ and ψ are defined correspondingly in (1.5) and (1.6). With a completely different method, based on a subtle two-weighted Carleman estimate, the sucp was established in [GS1] for zero order perturbations $\mathcal{L}_o - V_o$ when $\alpha = 1$ and $y \in \mathbb{R}$ (i.e. $m = 1$), where the potential V_o is allowed to belong to some appropriate L^p spaces,

$$V_o \in L^p_{loc}$$

$p > Q - 2$ if $n-$even, $p > \frac{2n^2}{n+1}$ if $n-$odd.

The operator for which we prove the SUCP is the following

$$\mathcal{L}u = \sum_{i,j=1}^{N} X_j(a_{ij}(x,y) X_i u) = 0. \quad (1.3)$$

We assume that $A = \big(a_{ij}(x,y)\big), i,j = 1, ..., N$, is a $N \times N$ matrix-valued function on $\mathbb{R}^N$ which, for simplicity, we take such that $A(0) =$Id. Furthermore, we assume A is symmetric and uniformly elliptic matrix. Thus $a_{ij} = a_{ji}$ and there exists $\lambda > 0$ such that for any $\eta \in \mathbb{R}^N$

$$\lambda|\eta|^2 \ \leq < A(x,y)\eta, \eta > \leq \ \lambda^{-1}|\eta|^2. \quad (1.4)$$

Our main concern is whether, under suitable assumptions on the matrix A, the sucp continues to hold for the operator $\mathcal{L}$. To put our result in perspective we mention that when $\alpha = 0$ in (1.2), so that $\mathcal{L}_o$ is the standard Laplacian, we have the following well-known SUCP due to Aronszaijn, Krzywicki and Szarski [AKS]

Theorem 1.1 *Suppose Ω is a connected open set in $\mathbb{R}^n$. Let $u \in W^{1,2}_{loc}(\Omega)$ be a weak solution of $Lu = \sum \partial_i(a_{ij}(z)\partial_j u) = 0$, where $A(z) = \big(a_{ij}(z)\big)$ is a symmetric, positive definite matrix with Lipschitz continuous entries. If there is point $z_o \in \Omega$ at which u has a zero of infinite order in, then $u = 0$ a.e..*

Furthermore, it was shown in [M] that such assumption is optimal. The case $n = 2$ is exceptional since, according to a result of Bers and Nirenberg, the SUCP holds for bounded measurable coefficients. Our results, Theorems 1.3 and 1.4 can be seen as a generalization of Theorem 1.1, in the sense that, in the limit as $\alpha \to 0$ we recapture both the assumptions and the conclusion of the elliptic case, see Remark 1.3. The approach, however, is different from that in [AKS], which is based on Carleman inequalities along with results from Riemannian geometry that do not seem to be adaptable to our context due to the lack of ellipticity. Instead, we have used the ideas developed in [GL1], [GL2], [G2], and simplified in [K]. Our main result is Theorem 1.3, which gives a quantitative control of the order of zero of a weak solution to (1.3). Such result is proved under some hypothesis on the matrix A which are listed as assumptions (H) below. The latter should be interpreted as a sort of Lipschitz continuity with respect to a suitable pseudo-distance associated to the system of vector fields (1.2).

In what follows we let $\xi = (x, y)$. To state our main result, we recall the following *gauge* from [G2] associated to the operator $\mathcal{L}_o$

$$\rho = \rho(\xi) \overset{def}{=} (|x|^{2(\alpha+1)} + (\alpha+1)^2|y|^2)^{\frac{1}{2(\alpha+1)}}. \tag{1.5}$$

Let $B_r = \{\rho < r\}$ be the pseudo-balls with respect to ρ centered at the origin in $\mathbb{R}^N$ with radius r. It is worth stressing that if α is an even positive integer, then the Carnot-Carathéodory distance associated to the system of vector fields in (1.2) is comparable to $\rho(\xi)$. We will also need the angle function ψ defined as follows [G2]

$$\psi = \psi(\xi) \overset{def}{=} |X\rho|^2(\xi) = \frac{|x|^{2\alpha}}{\rho^{2\alpha}}, \qquad \xi \neq 0. \tag{1.6}$$

Hereafter, given a function f, we denote the gradient along the system of vector fields in (1.2) (called also *horizontal gradient*)

$$Xf \; = \; (X_1 f, ..., X_N f) \tag{1.7}$$

and let $|Xf|^2 = \sum_{j=1}^{N}(X_j f)^2$. The function ψ vanishes at every point of the characteristic manifold M, and clearly satisfies $0 \le \psi \le 1$.

Definition 1.2 *A weak solution to $\mathcal{L}u = 0$ in an open set Ω is a function $u \in C(\Omega)$ such that the (distributional) horizontal gradient $Xu \in L^2_{loc}(\Omega)$, and the equation $\mathcal{L}u = 0$ is satisfied in the variational sense in Ω, i.e.,*

$$\int_\Omega < AXu,\, X\phi > dV \; = \; 0$$

for every $\phi \in C^\infty_o(\Omega)$.

For convenience, we have required that a weak solution be a continuous function since we will take traces on hypersurfaces. We note however that such assumption could be considerably relaxed if one assumes the existence of sub-unit curve joining any two points. Under this additional hypothesis, the assumption $u, Xu \in L^2_{loc}(\Omega)$ would suffice to apply the results in [FL], [FS], and conclude that a weak solution u is (after modification on a set of measure zero) Hölder continuous with respect to the Carnot-Carathéodory distance, and therefore (with a different exponent) also with respect to the Euclidean distance. Of course, when α is an even positive integer the system of vector fields is smooth and satisfies the Hörmander finite rank condition. In this case, the existence of a sub-unit curve joining any two points follows from the theorem of Chow-Rashevsky.

Theorem 1.3 *Let A be a symmetric matrix satisfying* (1.4) *and the hypothesis (H) below with relative constant Λ. Suppose u is a weak solution of* (1.3) *in a neighborhood of the origin Ω. Under these assumptions, there exist positive constants $C = C(u, \alpha, \lambda, \Lambda, N)$ and $r_o = r_o(u, \alpha, \lambda, \Lambda, N)$, such that, for any $2r \le r_o$, we have*

$$\int_{B_{2r}} u^2 \psi \, dV \le C \int_{B_r} u^2 \psi \, dV.$$

The dependence of the constant C on u is quite explicit. As it is well known [GL1], Theorem 1.3 implies the following sucp.

Theorem 1.4 *With the assumptions of Theorem 1.3, the operator $\mathcal{L}$ has the* SUCP.

We have stated the above theorem when the point of vanishing is the origin. Obviously the result is true for any other point with the appropriate modification of the hypothesis (H). In order to state our main assumptions (H) on the matrix A it will be useful to think of the latter in the following block form,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Here the entries are respectively $n \times n$, $n \times m$, $m \times n$ and $m \times m$ matrices, and we assume that $A_{12}^t = A_{21}$.

The proof of Theorem 1.3 relies crucially on the following assumptions on the matrix A. These will be our main hypothesis and will be assumed to hold throughout the paper.

HYPOTHESIS 1.5 *There exists a positive constant Λ such that one has in B_ϵ for some $\epsilon > 0$ the following estimates*

$$|a_{ij} - \delta_{ij}| \ \leq \ \begin{cases} \Lambda\rho, & \text{for } \ 1 \leq i,\, j \leq n \\ \\ \Lambda\psi^{\frac{1}{2}+\frac{1}{2\alpha}}\rho \ = \ \Lambda\frac{|x|^{\alpha+1}}{\rho^\alpha}, & \text{otherwise} \end{cases}$$

$$\text{(H)}$$

$$|X_k a_{ij}| \ \leq \ \begin{cases} \Lambda, & \text{for } \ 1 \leq k \leq n, \ \text{ and } \ 1 \leq i,\, j \leq n \\ \\ \Lambda\psi^{\frac{1}{2}} \ = \ \Lambda\frac{|x|^\alpha}{\rho^\alpha}, & \text{otherwise .} \end{cases}$$

A simple, yet interesting example of a matrix satisfying the conditions (H) is

$$A = \begin{pmatrix} 1+\rho f(x,y) & |x|^{\alpha+1}g(x,y) \\ |x|^{\alpha+1}g(x,y) & 1+|x|^{\alpha+1}h(x,y) \end{pmatrix},$$

where f, g and h are functions which are Lipschitz continuous at the origin of $\mathbb{R}^2$ with respect to the Euclidean metric. Here, $n = m = 1$.

2. Monotonicity of the generalized frequency

We begin by introducing the relevant quantities. Since our results are local in nature, from now on, we focus our attention on a pseudo-ball B_{R_0} centered at the origin and such that u is a weak solution of $\mathcal{L}u = 0$ in B_{R_0} in which (1.4) and the hypothesis (H) hold. For $0 < r < R_o$ we define correspondingly the height $H(r)$ and the Dirichlet integral $D(r)$ of u on the pseudo-ball B_r

$$H(r) \;= \int_{\partial B_r} u^2 < AX\rho, X\rho > \frac{d\sigma}{|D\rho|}$$

$$D(r) \;= \int_{B_r} < AXu, Xu > dV.$$

Consider further the frequency function

$$N(r) \overset{def}{=} \begin{cases} \frac{rD(r)}{H(r)}, \text{ if } H \neq 0 \\ 0, \text{ if } H = 0 \end{cases}$$

The following two lemmas are the key to proving the monotonicity of the modified frequency

$$\tilde{N}(r) = N(r)\, e^{2Mr},$$

where $M > 0$ will be suitably chosen. In the sequel we shall briefly sketch the main steps in their proofs, referring to [GV] for full details.

Lemma 2.1 *a) There exists a positive constant* $C_1 = C_1(\alpha, \lambda, \Lambda, N)$ *such that for a.e.* $r \in (0, R_o)$ *one has*

$$\Big| H^{'}(r) - \frac{Q-1}{r} H(r) - 2D(r) \Big| \leq C_1 H(r).$$

b) There exists a positive number $r_o = r_o(\alpha, \lambda, \Lambda, N) \leq R_o$ *such that either* $H(r) = 0$ *on* $(0, r_o)$ *or* $H(r) > 0$ *on* $(0, r_o)$.

The proof involves a lengthy computation with the use of integration by parts, the co-area formula and the fact that, up to a constant, ρ^{2-Q} is the fundamental solution of $\mathcal{L}_o$ with singularity at $(0, 0)$. Here, $Q = n + (\alpha + 1)m$ is the homogeneous dimension corresponding to dilations with infinitesimal generator (radial vector field)

$$Z = \sum_{1 \leq i \leq n} x_i \frac{\partial}{\partial x_i} \; + \; (\alpha + 1) \sum_{1 \leq j \leq m} y_i \frac{\partial}{\partial y_i}.$$

Lemma 2.2 *There exists a constant $C_2 = C_2(\alpha, \lambda, \Lambda, N) > 0$ such that for a.e. $r \in (0, R_o)$ one has*

$$D'(r) \geq 2 \int_{\partial B_r} \frac{< AXu, X\rho >^2}{\mu} \frac{d\sigma}{|D\rho|} \; + \; \frac{Q-2}{r} D(r) \; - \; C_2 D(r),$$

where $\mu \overset{def}{=} < AX\rho, \; X\rho >$.

To prove this lemma we use the co-area formula and Rellich's identity, which imply

$$D'(r) = \; 2 \int_{\partial B_r} \frac{< AXu, X\rho >^2}{\mu} \frac{d\sigma}{|D\rho|} + \frac{1}{r} \int_{B_r} (div\, F) < AXu, Xu >$$

$$- \frac{2}{r} \int_{B_r} < AXu, [X, F]u > + \frac{1}{r} \int_{B_r} < (FA)Xu, Xu >,$$

with

$$F \; = \; \rho \sum_{i,j=1}^{N} \frac{a_{ij} X_j \rho}{\mu} X_i, \qquad x \neq 0.$$

We show that F can be extended continuously to a vector field near the origin. In fact, such extension is sufficiently smooth. The vector field F should be thought of as a small perturbation of the radial vector field. A computation shows that we can estimate the terms in the right-hand side of the above identity as required.

The above two lemmas imply the following monotonicity theorem

Theorem 2.3 *Suppose u is a solution of $\mathcal{L}u = 0$ in a neighborhood of the origin in which (1.4) and the hypothesis (H) hold true. Under these conditions there exist positive constants $r_o = r_o(u, \alpha, \lambda, \Lambda, N)$ and $M = M(\alpha, \lambda, \Lambda, N)$ such that*

$$\tilde{N}(r) \; = \; N(r)\, exp(2Mr)$$

is a continuous monotone nondecreasing function for $r \in (0, r_o)$.

Proof The proof of Theorem 2.3 follows from lemmas 2.1 and 2.2. Let $M = \max\{C_1, C_2\}$, where C_1 and C_2 are the constants from Lemmas

2.1 and 2.2. With r_o as defined in Lemma 2.1 we have that either $u \equiv 0$ in B_{r_o} or $H(r) > 0$ for $0 < r < r_o$. In the first case the frequency is identically zero on $(0, r_o)$ so let us consider the second case for which $H(r) > 0$. The continuity of $\tilde{N}(r)$ follows from the continuity of each of the functions involved in its definition. Furthermore, for a.e. $r \in (0, r_o)$ we have

$$\begin{aligned}\big(\ln \frac{rD(r)}{H(r)} e^{2Mr}\big)' &= \frac{1}{r} + \frac{D'}{D} - \frac{H'}{H} + 2M \\ &\geq \frac{1}{r} + \frac{Q-2}{r} + \frac{2}{D}\int_{\partial B_r} \frac{<AXu, X\rho>^2}{<AX\rho, X\rho>}\frac{d\sigma}{|D\rho|} \\ &\quad - \frac{Q-1}{r} - 2\frac{D}{H} \geq 0\ ,\end{aligned}$$

where we have applied first lemmas 2.1 and 2.2, and then the Cauchy-Schwarz inequality. The reader should keep in mind the following formula

$$D(r) = \int_{\partial B_r} u {<AXu, X\rho>} \frac{d\sigma}{|D\rho|}\ .$$

3. Proof of the doubling property

In this section we shall prove Theorem 1.3. If the solution vanishes in some neighborhood of the origin then the doubling for all sufficiently small balls is trivially satisfied. Let us consider next the case of a non-trivial solution. Let r_o be the number defined in Lemma 2.1 and $2r \leq r_o$. By the co-area formula

$$\int_0^R \int_{\partial B_r} u^2 \psi \frac{d\sigma}{|D\rho|} dr = \int_{B_R} u^2 \psi\, dV.$$

From the ellipticity of A we have

$$\int_0^R H(r)\, dr \approx \int_{B_R} u^2 \psi\, dV,$$

which shows it is enough to prove the doubling property for the height function H. Now we compute

$$\begin{aligned}\ln \frac{H(2r)}{2^{Q-1}H(r)} &= \ln \frac{H(2r)}{2^{Q-1}r^{Q-1}} - \frac{H(2r)}{r^{Q-1}} = \int_r^{2r} \left\{ \frac{H'}{H} - \frac{Q-1}{t} \right\} dt \\ &\leq \int_r^{2r} \left\{ 2\frac{D(t)}{H(t)} + M \right\} dt = \int_r^{2r} 2\tilde{N}(t)\frac{e^{-2Mt}}{t} dt + Mr \\ &\leq 2\tilde{N}(r_o)\int_r^{2r} \frac{1}{t} dt + M = 2\tilde{N}(r_o)\ln 2 + M.\end{aligned}$$

In the above inequalities we have used the monotonicity of the modified frequency. Hence

$$H(2r) \;\leq\; 2^{Q-1} e^{2\tilde{N}(r_o)\ln 2 \,+\, M} H(r),$$

i.e., the doubling property holds.

Remark 3.1 *We observe that for non-trivial solution we have the doubling property for all balls $B_{2r} \subset \Omega$ and $2r \leq R$, where $R > 0$ is a fixed number, since for "big" balls, i.e., $2r \geq r_o$ we have*

$$\frac{\int_{B_{2r}} u^2 \psi \, dV}{\int_{B_r} u^2 \psi \, dV} \leq \frac{\int_{B_R} u^2 \psi \, dV}{\int_{B_{r_o/2}} u^2 \psi \, dV}.$$

Of course, in this case the constant C in the doubling property depends on $\tilde{N}(R)$.

4. Proof of the SUCP

In this section we shall prove Theorem 1.4. Suppose u is a solution which vanishes to infinite order at the origin. Let $|B_r| = \omega_o r^Q$. Fix a number $\kappa > 0$ such that $C_o 2^{-Q\kappa} = 1$. For any r sufficiently small and $p \in \mathbb{N}$ the doubling property applied p times gives

$$\begin{aligned}
\int_{B_r} u^2 \psi \, dV &\leq C_o^p \int_{B_{r/2^p}} u^2 \psi \, dV \\
&\leq \omega_o^\kappa C_o^p \frac{r^{Q\kappa}}{2^{Qp\kappa}} \frac{1}{|B_{r/2^p}|^\kappa} \int_{B_{r/2^p}} u^2 \psi \, dV \\
&\leq \omega_o^\kappa r^{Q\kappa} \frac{1}{|B_{r/2^p}|^\kappa} \int_{B_{r/2^p}} u^2 \psi \, dV \to 0
\end{aligned}$$

when $p \to \infty$. This ends the proof.

References

[AKS] N. Aronszajn, A. Krzywicki, J. Szarski, *A unique continuation theorem for exterior differential forms on Riemannian manifolds*, Arkiv för Matematik, **4** (1962), 417-435.

[B] M. Baouendi, *Sur une classe d'opérateurs elliptiques degénérés*, Bull. Soc. Math. France, **95** (1967), 45-87.

[Ba] H. Bahouri, *Non prolongement unique des solutions d'opérateurs "Somme de carrés"*, Ann. Inst. Fourier, Grenoble, **36**-4 (1986), 137-155.

[Bo] J. M. Bony, *Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptique dégénérés*, Ann. Inst. Fourier, Grenoble, **19**-1 (1969), 277-304.

[FL] B. Franchi, E. Lanconelli, *Hölder regularity theorem for a class of linear non uniformly elliptic operators with measurable coefficients*, Ann. Sc. Norm. Sup. Pisa, **4** (1983), 523-541.

[FS] B. Franchi, R. Serapioni, *Pointwise estimates for a class of strongly degenerate elliptic operators:A geometrical approach*, Ann. Sc. Norm. Sup. Pisa, **14** (1987), 527-568.

[G1] N. Garofalo, *Lecture notes on unique continuation*, Summer School on Elliptic Equations, Cortona, 1991.

[G2] N. Garofalo, *Unique continuation for a class of ellipic operators which degenerate on a manifold of arbitrary codimension*, J. Diff. Eq., **104**-1 (1993), 117-146.

[GLa] N. Garofalo, E. Lanconelli, *Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation*, Ann. Inst. Fourier, Grenoble, **44**-2 (1990), 313-356.

[GL1] N. Garofalo, F. H. Lin, *Monotonicity properties of variational integrals, A_p weights and unique continuation*, Indiana Univ. Math J., **35**-2 (1986), 245-268.

[GL2] N. Garofalo, F. H. Lin, *Unique continuation for elliptic operators:a geometric-variational approach*, Comm. Pure Appl. Math., **40** (1987), 347-366.

[GS1] N. Garofalo, Z. W. Shen, *A Carleman estimate for a subelliptic operator and unique continuation*, Ann. Inst. Fourier, Grenoble, **44** (1994), 129-166.

[GS2] N. Garofalo, Z. W. Shen, *Absence of positive eigenvalues for a class of subelliptic operators*, Math. Annalen, **304** (1996), 701-715.

[Gr1] V. Grushin, *On a class of hypoelliptic operators*, Math. USSR Sbornik, **12**-3 (1970), 458-476.

[Gr2] V. Grushin, *On a class of hypoelliptic pseudodifferential operators degenerate on a submanifold*, Math. USSR Sbornik, **13**-2 (1971), 155-186.

[GV] N. Garofalo, D. Vassilev, *Strong unique continuation of solutions of equations involving generalized Baouendi-Grushin operators*, preprint 2003

[K] I. Kukavica, *Quantitative uniqueness for second order elliptic operators*, Duke Math. J. **91** (1998), no. 2, 225–240

[M] K. Miller, *Nonunique continuation for uniformly parabolic and elliptic equations in self-adjoint divergence form with Hölder continuous coefficients*, Arch. Rational Mech. Anal., **54** (1974), 105-117.

[RS] L. P. Rothschild, E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math., **37** (1976), 247-320.