

THE SHARP LOWER BOUND OF THE FIRST EIGENVALUE OF THE SUB-LAPLACIAN ON A QUATERNIONIC CONTACT MANIFOLD

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ABSTRACT. The main technical result of the paper is a Bochner type formula for the sub-laplacian on a quaternionic contact manifold. With the help of this formula we establish a version of Lichnerowicz' theorem giving a lower bound of the eigenvalues of the sub-Laplacian under a lower bound on the $Sp(n)Sp(1)$ components of the qc-Ricci curvature. It is shown that in the case of a 3-Sasakian manifold the lower bound is reached iff the quaternionic contact manifold is a round 3-Sasakian sphere. Another goal of the paper is to establish a-priori estimates for square integrals of horizontal derivatives of smooth compactly supported functions. As an application, we prove a sharp inequality bounding the horizontal Hessian of a function by its sub-Laplacian on the quaternionic Heisenberg group.

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1. INTRODUCTION

The first circle of results of this paper are motivated by the classical theorems of Lichnerowicz [32] and Obata [36] giving correspondingly a lower bound of the first eigenvalue of the Laplacian on a compact manifold with a Ricci bound and characterizing the case of equality. In fact, in [32] it was shown that for every compact Riemannian manifold of dimension n for which the Ricci curvature is greater than or equal to that of the round unit n -dimensional sphere $S^n(1)$, i.e.,

$$Ric(X, Y) \geq (n - 1)g(X, Y)$$

we have that the first positive eigenvalue λ_1 of the (positive) Laplace operator is greater than or equal to the first eigenvalue of the sphere,

$$\lambda_1 \geq n.$$

Date: December 7, 2011.

2010 Mathematics Subject Classification. 53C26, 53C25, 58J60.

Key words and phrases. quaternionic contact structures, qc conformal flatness, qc conformal curvature, Einstein metrics.

This work has been rationally funded by.

Subsequently in [36] it was shown that the lower bound for the eigenvalue is achieved iff the Riemannian manifold is isometric to $S^n(1)$. Lichnerowicz proved his result using the classical Bochner-Weitzenböck formula. In turn, Obata showed that under these assumptions the eigenfunction ϕ satisfies the system $\nabla^2\phi = -\phi g$, after which he defines an isometry using analysis based on the geodesics and Hessian comparison of the distance function from a point. Later Gallot [20] generalized these results to statements involving the higher eigenvalues and corresponding eigenfunctions of the Laplace operator.

It is natural to ask if there is a sub-Riemannian version of the above results. Greenleaf [23] gave a version of Lichnerowicz' result on a compact strongly pseudo-convex CR manifold. Suppose M is $2n + 1$, $n \geq 3$ dimensional strongly pseudo-convex CR manifold. If

$$Ric(X, Y) + 4A(X, JY) \geq (n + 1)g(X, X)$$

for all horizontal vectors X , where Ric and A are, correspondingly, the Ricci curvature and the Webster torsion of the Tanaka-Webster connection (in the notation from [30, 29]), then the first positive eigenvalue λ_1 of the sub-Laplacian satisfies the inequality $\lambda_1 \geq n$. The standard CR structure on the sphere achieves equality in this inequality. Further results in the CR case have been proved in [33], [14], [11, 10, 9], [3] and [12] adding a corresponding inequality for $n = 1$, or characterizing the equality case in the vanishing torsion case (the Sasakian case).

One purpose of this paper is to consider these questions in the setting of a closed compact quaternionic contact manifold. The Lichnerowicz type result is as follows.

Theorem 1.1. *Let (M, g, \mathbb{Q}) be a compact quaternionic contact manifold of dimension $4n + 3 > 7$. If the Ricci tensor and torsion of the Biquard connection satisfy the inequality*

$$(1.1) \quad Ric(X, X) + \frac{2(4n + 5)}{2n + 1}T^0(X, X) + \frac{3(2n^2 + 5n - 1)}{(n - 1)(2n + 1)}U(X, X) \geq k_0g(X, X)$$

for some positive constant k_0 then any positive eigenvalue λ of the sub-Laplacian Δ satisfies the inequality

$$\lambda \geq \frac{n}{n + 2}k_0.$$

The second goal then is to investigate the case of equality in Theorem 1.1. We restrict our considerations to the case when the torsion of the Biquard connection vanishes, $T^0 = U = 0$. In this case it is known [24] that the qc manifold is qc-Einstein, $Ric = k.g$, the qc-scalar curvature is constant ($n > 1$) and if it is positive then the qc manifold is locally qc equivalent to a 3-Sasakian space. The corresponding result in the negative scalar curvature case can be found in [28] and [29]. We prove the following result.

Theorem 1.2. *Let (M, g, \mathbb{Q}) be a compact qc-Einstein manifold of dimension $4n + 3 > 7$ of qc scalar curvature $Scal = 16n(n + 2)$,*

$$Ric(X, Y) = \frac{1}{4n}Scalg(X, Y) = 4(n + 2)g(X, Y).$$

The first positive eigenvalue λ_1 of the sub-Laplacian equals $4n$ if and only if (M, g, \mathbb{Q}) is qc equivalent to the 3-Sasakian sphere of dimension $4n + 3$.

In particular, on a 3-Sasakian manifold of dimension $(4n + 3)$, $n > 1$, the first positive eigenvalue of the sub-laplacian is equal to $4n$ if and only if the 3-Sasakian manifold is qc-equivalent to the 3-Sasakian sphere.

We note that in [26] is given an explicit formula for the eigenfunctions of the above eigenvalue, see also [1].

The second main theme of the paper is the derivation of an a-priori inequality between the (horizontal) Hessian and sub-Laplacian. For the Heisenberg group a corresponding sharp estimate was found in [17]. Equipped with our estimate we precise the scope of use of [16] for the quaternionic Heisenberg group. We recall that the main application is the establishment of the $\mathcal{C}^{1,\alpha}$ regularity for the p sub-Laplacian with p close to 2. The exact interval for p around 2 is determined by the constant found in this paper. Using Bochner's identity we will find an integral identity. Such integral identities have been exploited earlier in [21] in the setting of Carnot groups. A similar method based on Greenleaf's formula was employed in [13], but

due to the different quaternionic linear algebra our proof proceeds in a way particular to the quaternionic case. Here we prove the following result.

Theorem 1.3. *Let (M, η) be a $(4n+3)$ -dimensional qc manifold, $n > 1$. For any $f \in \mathcal{C}_o^\infty(M)$ the following inequality holds true*

$$(1.2) \quad \int_M |\Delta f|^2 Vol_\eta \geq \frac{n}{n+1} \int_M |\nabla^2 f|^2 Vol_\eta + \frac{n^2}{n^2-1} \int_M Ric(\nabla f, \nabla f) Vol_\eta \\ + \frac{n^2}{n^2-1} \int_M -\frac{4}{n} T^0(\nabla f, \nabla f) - 6U(\nabla f, \nabla f) - 6S|\nabla f|^2 Vol_\eta \\ = \frac{n}{n+1} \int_M |\nabla^2 f|^2 Vol_\eta + \int_M \frac{2n(n+2)}{n+1} T^0(\nabla f, \nabla f) + \frac{4n^2}{n-1} U(\nabla f, \nabla f) + \frac{2n^2}{n+1} S|\nabla f|^2 Vol_\eta.$$

The proofs of the last Theorem is presented in Section 6.

Convention 1.4.

- a) We shall use X, Y, Z, U to denote horizontal vector fields, i.e. $X, Y, Z, U \in H$.
- b) $\{e_1, \dots, e_{4n}\}$ denotes a local orthonormal basis of the horizontal space H .
- c) The summation convention over repeated vectors from the basis $\{e_1, \dots, e_{4n}\}$ will be used. For example, for a $(0,4)$ -tensor P , the formula $k = P(e_b, e_a, e_a, e_b)$ means

$$k = \sum_{a,b=1}^{4n} P(e_b, e_a, e_a, e_b);$$

- d) The triple (i, j, k) denotes any cyclic permutation of $(1, 2, 3)$.
- e) s will be any number from the set $\{1, 2, 3\}$, $s \in \{1, 2, 3\}$.

Acknowledgments The research is partially supported by the Contract 181/2011 with the University of Sofia ‘St.Kl.Ohridski’ and Contract ‘Idei’, DID 02-39/21.12.2009. . S.I and D.V. are partially supported by Contract ‘Idei’, DO 02-257/18.12.2008.

2. QUATERNIONIC CONTACT MANIFOLDS

In this section we will briefly review the basic notions of quaternionic contact geometry and recall some results from [4], [24] and [27] which we will use in this paper.

2.1. Quaternionic contact structures and the Biquard connection. A quaternionic contact (qc) manifold (M, g, \mathbb{Q}) is a $4n+3$ -dimensional manifold M with a codimension three distribution H locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in \mathbb{R}^3 . In addition H has an $Sp(n)Sp(1)$ structure, that is, it is equipped with a Riemannian metric g and a rank-three bundle \mathbb{Q} consisting of endomorphisms of H locally generated by three almost complex structures I_1, I_2, I_3 on H satisfying the identities of the imaginary unit quaternions, $I_1 I_2 = -I_2 I_1 = I_3$, $I_1 I_2 I_3 = -id|_H$ which are hermitian compatible with the metric $g(I_s \cdot, I_s \cdot) = g(\cdot, \cdot)$ and the following compatibility condition holds $2g(I_s X, Y) = d\eta_s(X, Y)$, $X, Y \in H$.

A special phenomena, noted in [4], is that the contact form η determines the quaternionic structure and the metric on the horizontal distribution in a unique way.

If the first Pontryagin class of M vanishes then the 2-sphere bundle of \mathbb{R}^3 -valued 1-forms is trivial [2], i.e. there is a globally defined form η that annihilates H , we denote the corresponding qc manifold (M, η) . In this case the 2-sphere of associated almost complex structures is also globally defined on H .

On a qc manifold with a fixed metric g on H there exists a canonical connection defined in [4] when the dimension $(4n+3) > 7$, and in [18] for the 7-dimensional case.

Theorem 2.1. [4] *Let (M, g, \mathbb{Q}) be a qc manifold of dimension $4n+3 > 7$ and a fixed metric g on H in the conformal class $[g]$. Then there exists a unique connection ∇ with torsion T on M^{4n+3} and a unique supplementary subspace V to H in TM , such that:*

- i) ∇ preserves the decomposition $H \oplus V$ and the $Sp(n)Sp(1)$ structure on H , i.e. $\nabla g = 0, \nabla \sigma \in \Gamma(\mathbb{Q})$ for a section $\sigma \in \Gamma(\mathbb{Q})$, and its torsion on H is given by $T(X, Y) = -[X, Y]|_V$;

- ii) for $\xi \in V$, the endomorphism $T(\xi, \cdot)|_H$ of H lies in $(sp(n) \oplus sp(1))^\perp \subset gl(4n)$;
- iii) the connection on V is induced by the natural identification φ of V with the subspace $sp(1)$ of the endomorphisms of H , i.e. $\nabla\varphi = 0$.

In ii), the inner product \langle, \rangle of $End(H)$ is given by $\langle A, B \rangle = \sum_{i=1}^{4n} g(A(e_i), B(e_i))$, for $A, B \in End(H)$. We shall call the above connection *the Biquard connection*. When the dimension of M is at least eleven [4] also described the supplementary distribution V , which is (locally) generated by the so called Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ determined by

$$(2.1) \quad \begin{aligned} \eta_s(\xi_k) &= \delta_{sk}, & (\xi_s \lrcorner d\eta_s)|_H &= 0, \\ (\xi_s \lrcorner d\eta_k)|_H &= -(\xi_k \lrcorner d\eta_s)|_H, \end{aligned}$$

where \lrcorner denotes the interior multiplication. If the dimension of M is seven Duchemin shows in [18] that if we assume, in addition, the existence of Reeb vector fields as in (2.1), then Theorem 2.1 holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2.1).

The qc conformal curvature tensor W^{qc} , introduced in [27], is the obstruction for a qc structure to be locally qc conformal to the flat structure on the quaternionic Heisenberg group $\mathbf{G}(\mathbb{H})$. A qc conformally flat structure is also locally qc conformal to the standard 3-Sasaki sphere due to the local qc conformal equivalence of the standard 3-Sasakian structure on the $(4n+3)$ -dimensional sphere and the quaternionic Heisenberg group [24, 27].

Notice that equations (2.1) are invariant under the natural $SO(3)$ action. Using the triple of Reeb vector fields we extend g to a metric on M by requiring $span\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and $g(\xi_s, \xi_k) = \delta_{sk}$. The extended metric does not depend on the action of $SO(3)$ on V , but it changes in an obvious manner if η is multiplied by a conformal factor. Clearly, the Biquard connection preserves the extended metric on TM , $\nabla g = 0$. Since the Biquard connection is metric it is connected with the Levi-Civita connection ∇^g of the metric g by the general formula

$$(2.2) \quad g(\nabla_A B, C) = g(\nabla_A^g B, C) + \frac{1}{2} [g(T(A, B), C) - g(T(B, C), A) + g(T(C, A), B)].$$

The covariant derivative of the qc structure with respect to the Biquard connection and the covariant derivative of the distribution V are given by

$$(2.3) \quad \nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j, \quad \nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j.$$

The $sp(1)$ -connection 1-forms α_s on H are expressed in [4] by

$$(2.4) \quad \alpha_i(X) = d\eta_k(\xi_j, X) = -d\eta_j(\xi_k, X), \quad X \in H, \quad \xi_i \in V,$$

while the $sp(1)$ -connection 1-forms α_s on the vertical space V are calculated in [24]

$$(2.5) \quad \alpha_i(\xi_s) = d\eta_s(\xi_j, \xi_k) - \delta_{is} \left(\frac{S}{2} + \frac{1}{2} (d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_3, \xi_1) + d\eta_3(\xi_1, \xi_2)) \right),$$

where S is the *normalized* qc scalar curvature defined below in (2.11). The vanishing of the $sp(1)$ -connection 1-forms on H implies the vanishing of the torsion endomorphism of the Biquard connection (see [24]).

The fundamental 2-forms ω_s of the quaternionic structure Q are defined by

$$(2.6) \quad 2\omega_{s|H} = d\eta_{s|H}, \quad \xi \lrcorner \omega_s = 0, \quad \xi \in V.$$

Due to (2.6), the torsion restricted to H has the form

$$(2.7) \quad T(X, Y) = -[X, Y]|_V = 2\omega_1(X, Y)\xi_1 + 2\omega_2(X, Y)\xi_2 + 2\omega_3(X, Y)\xi_3.$$

2.2. Invariant decompositions. Any endomorphism Ψ of H can be decomposed with respect to the quaternionic structure (\mathbb{Q}, g) uniquely into four $Sp(n)$ -invariant parts $\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$, where Ψ^{+++} commutes with all three I_i , Ψ^{+--} commutes with I_1 and anti-commutes with the others two and etc. Explicitly,

$$(2.8) \quad \begin{aligned} 4\Psi^{+++} &= \Psi - I_1\Psi I_1 - I_2\Psi I_2 - I_3\Psi I_3, & 4\Psi^{+--} &= \Psi - I_1\Psi I_1 + I_2\Psi I_2 + I_3\Psi I_3, \\ 4\Psi^{-+-} &= \Psi + I_1\Psi I_1 - I_2\Psi I_2 + I_3\Psi I_3, & 4\Psi^{--+} &= \Psi + I_1\Psi I_1 + I_2\Psi I_2 - I_3\Psi I_3. \end{aligned}$$

The two $Sp(n)Sp(1)$ -invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \quad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$$

with the following characterizing equations

$$(2.9) \quad \begin{aligned} \Psi = \Psi_{[3]} &\iff 3\Psi + I_1\Psi I_1 + I_2\Psi I_2 + I_3\Psi I_3 = 0, \\ \Psi = \Psi_{[-1]} &\iff \Psi - I_1\Psi I_1 - I_2\Psi I_2 - I_3\Psi I_3 = 0. \end{aligned}$$

With a short calculation one sees that the $Sp(n)Sp(1)$ -invariant components are the projections on the eigenspaces of the Casimir operator

$$(2.10) \quad \Upsilon = I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3$$

corresponding, respectively, to the eigenvalues 3 and -1 , see [8]. If $n = 1$ then the space of symmetric endomorphisms commuting with all I_s is 1-dimensional, i.e. the $[3]$ -component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_3 = \frac{|\Psi|^2}{4} Id|_H$. Note here that each of the three 2-forms ω_s belongs to its $[-1]$ -component, $\omega_s = \omega_{s[-1]}$ and constitute a basis of the lie algebra $sp(1)$.

2.3. The torsion tensor. The properties of the Biquard connection are encoded in the properties of the torsion endomorphism $T_\xi = T(\xi, \cdot) : H \rightarrow H$, $\xi \in V$. Decomposing the endomorphism $T_\xi \in (sp(n) + sp(1))^\perp$ into its symmetric part T_ξ^0 and skew-symmetric part b_ξ , $T_\xi = T_\xi^0 + b_\xi$. Biquard shows in [4] that the torsion T_ξ is completely trace-free, $tr T_\xi = tr T_\xi \circ I_s = 0$, its symmetric part has the properties $T_{\xi_i}^0 I_i = -I_i T_{\xi_i}^0$, $I_2(T_{\xi_2}^0)^{+--} = I_1(T_{\xi_1}^0)^{-+-}$, $I_3(T_{\xi_3}^0)^{-+-} = I_2(T_{\xi_2}^0)^{--+}$, $I_1(T_{\xi_1}^0)^{--+} = I_3(T_{\xi_3}^0)^{+--}$, where the upperscript $+++$ means commuting with all three I_i , $+-$ indicates commuting with I_1 and anti-commuting with the other two and etc. The skew-symmetric part can be represented as $b_{\xi_i} = I_i u$, where u is a traceless symmetric $(1,1)$ -tensor on H which commutes with I_1, I_2, I_3 . If $n = 1$ then the tensor u vanishes identically, $u = 0$ and the torsion is a symmetric tensor, $T_\xi = T_\xi^0$.

Any 3-Sasakian manifold has zero torsion endomorphism, and the converse is true if in addition the qc scalar curvature (see (2.11)) is a positive constant [24]. We remind that a $(4n+3)$ -dimensional Riemannian manifold (M, g) is called 3-Sasakian if the cone metric $g_c = t^2 g + dt^2$ on $C = M \times \mathbb{R}^+$ is a hyper Kähler metric, namely, it has holonomy contained in $Sp(n+1)$ [6]. A 3-Sasakian manifold of dimension $(4n+3)$ is Einstein with positive Riemannian scalar curvature $(4n+2)(4n+3)$ [31] and if complete it is compact with a finite fundamental group, (see [5] for a nice overview of 3-Sasakian spaces).

2.4. Torsion and curvature. Let $R = [\nabla, \nabla] - \nabla[\cdot, \cdot]$ be the curvature tensor of ∇ and the dimension is $4n+3$. We denote the curvature tensor of type $(0,4)$ and the torsion tensor of type $(0,3)$ by the same letter, $R(A, B, C, D) := g(R(A, B)C, D)$, $T(A, B, C) := g(T(A, B), C)$, $A, B, C, D \in \Gamma(TM)$. The Ricci tensor, the normalized scalar curvature and the Ricci 2-forms of the Biquard connection, called *qc-Ricci tensor* Ric , *normalized qc-scalar curvature* S and *qc-Ricci forms* ρ_s, τ_s , respectively, are defined by

$$(2.11) \quad \begin{aligned} Ric(A, B) &= R(e_b, A, B, e_b), \quad 8n(n+2)S = R(e_b, e_a, e_a, e_b), \\ \rho_s(A, B) &= \frac{1}{4n} R(A, B, e_a, I_s e_a), \quad \tau_s(A, B) = \frac{1}{4n} R(e_a, I_s e_a, A, B). \end{aligned}$$

The $sp(1)$ -part of R is determined by the Ricci 2-forms and the connection 1-forms by

$$(2.12) \quad R(A, B, \xi_i, \xi_j) = 2\rho_k(A, B) = (d\alpha_k + \alpha_i \wedge \alpha_j)(A, B), \quad A, B \in \Gamma(TM).$$

The two $Sp(n)Sp(1)$ -invariant trace-free symmetric 2-tensors $T^0(X, Y) = g((T_{\xi_1}^0 I_1 + T_{\xi_2}^0 I_2 + T_{\xi_3}^0 I_3)X, Y)$, $U(X, Y) = g(uX, Y)$ on H , introduced in [24], have the properties:

$$(2.13) \quad \begin{aligned} T^0(X, Y) + T^0(I_1 X, I_1 Y) + T^0(I_2 X, I_2 Y) + T^0(I_3 X, I_3 Y) &= 0, \\ U(X, Y) = U(I_1 X, I_1 Y) = U(I_2 X, I_2 Y) = U(I_3 X, I_3 Y). \end{aligned}$$

In dimension seven ($n = 1$), the tensor U vanishes identically, $U = 0$.

We shall need the following identity taken from [27, Proposition 2.3]

$$(2.14) \quad 4T^0(\xi_s, I_s X, Y) = T^0(X, Y) - T^0(I_s X, I_s Y).$$

Thus, taking into account (2.14) we have the formula

$$(2.15) \quad T(\xi_s, I_s X, Y) = T^0(\xi_s, I_s X, Y) + g(I_s u I_s X, Y) = \frac{1}{4} \left[T^0(X, Y) - T^0(I_s X, I_s Y) \right] - U(X, Y).$$

Definition 2.2. A qc structure is said to be qc Einstein if the horizontal qc-Ricci tensor is a scalar multiple of the metric,

$$Ric(X, Y) = 2(n+2)Sg(X, Y).$$

The horizontal Ricci tensor and the horizontal Ricci 2-forms can be expressed in terms of the torsion of the Biquard connection [24] (see also [25, 27]). We collect the necessary facts from [24, Theorem 1.3, Theorem 3.12, Corollary 3.14, Proposition 4.3 and Proposition 4.4] with slight modification presented in [27]

Theorem 2.3 ([24]). *On a $(4n+3)$ -dimensional qc manifold (M, η, \mathbb{Q}) with a normalized scalar curvature S we have the following relations*

$$(2.16) \quad \begin{aligned} Ric(X, Y) &= (2n+2)T^0(X, Y) + (4n+10)U(X, Y) + 2(n+2)Sg(X, Y), \\ \rho_s(X, I_s Y) &= -\frac{1}{2} \left[T^0(X, Y) + T^0(I_s X, I_s Y) \right] - 2U(X, Y) - Sg(X, Y), \\ \tau_s(X, I_s Y) &= -\frac{n+2}{2n} \left[T^0(X, Y) + T^0(I_s X, I_s Y) \right] - Sg(X, Y), \\ T(\xi_i, \xi_j) &= -S\xi_k - [\xi_i, \xi_j]_H, \quad S = -g(T(\xi_1, \xi_2), \xi_3), \\ g(T(\xi_i, \xi_j), X) &= -\rho_k(I_i X, \xi_i) = -\rho_k(I_j X, \xi_j) = -g([\xi_i, \xi_j], X), \\ \frac{1}{2}\xi_j(S) &= \rho_i(\xi_i, \xi_j) + \rho_k(\xi_k, \xi_j), \\ \rho_i(\xi_i, X) &= \frac{X(S)}{4} + \frac{1}{2} (\rho_i(\xi_j, I_k X) - \rho_j(\xi_k, I_i X) - \rho_k(\xi_i, I_j X)). \end{aligned}$$

For $n=1$ the above formulas hold with $U=0$.

The qc Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection. In this case S is constant and the vertical distribution is integrable provided $n > 1$.

2.5. The Ricci identities. We shall use repeatedly the following Ricci identities of order two and three, see also [27]. Let ξ_i , $i=1,2,3$ be the Reeb vector fields, $X, Y \in H$ and f a smooth function on the qc manifold M with ∇f its horizontal gradient of f , $g(\nabla f, X) = df(X)$. We have:

$$(2.17) \quad \begin{aligned} \nabla^2 f(X, Y) - \nabla^2 f(Y, X) &= -2 \sum_{s=1}^3 \omega_s(X, Y) df(\xi_s) \\ \nabla^2 f(X, \xi_s) - \nabla^2 f(\xi_s, X) &= T(\xi_s, X, \nabla f) \\ \nabla^3 f(X, Y, Z) - \nabla^3 f(Y, X, Z) &= -R(X, Y, Z, \nabla f) - 2 \sum_{s=1}^3 \omega_s(X, Y) \nabla^2 f(\xi_s, Z) \end{aligned}$$

where we used (2.7) in the last equalities in the first and the fourth lines.

2.6. The horizontal divergence theorem. Let (M, g, \mathbb{Q}) be a qc manifold of dimension $4n+3 > 7$ For a fixed local 1-form η and a fix $s \in \{1, 2, 3\}$ the form

$$(2.18) \quad Vol_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \omega_s^{2n}$$

is a locally defined volume form. Note that Vol_η is independent on s as well as it is independent on the local one forms η_1, η_2, η_3 . Hence it is globally defined volume form denoted with Vol_η .

We consider the (horizontal) divergence of a horizontal vector field/one-form $\sigma \in \Lambda^1(H)$ defined by

$$(2.19) \quad \nabla^* \sigma = -tr|_H \nabla \sigma = -\nabla \sigma(e_a, e_a).$$

We need the following Proposition from [24], see also [37], which allows "integration by parts".

Proposition 2.4 ([24]). *On a compact quaternionic contact manifold (M, η) the following divergence formula holds true*

$$\int_M (\nabla^* \sigma) \text{Vol}_\eta = 0.$$

3. THE BOCHNER FORMULA FOR THE SUB-LAPLACIAN

The horizontal sub-Laplacian Δf and the norm of the horizontal gradient ∇f of a smooth function f on M are defined respectively by

$$(3.1) \quad \Delta f = -\text{tr}_H^g(\nabla^2 f) = \nabla^* df = -\nabla^2 f(e_a, e_a), \quad |\nabla f|^2 = df(e_a)df(e_a).$$

The function f is an eigenfunction with eigenvalue λ of the sub-Laplacian if

$$(3.2) \quad \Delta f = \lambda f,$$

for some constant λ . The divergence formula implies that on a compact qc manifolds all eigenvalues of the sub-Laplacian are non-negative. Our main result Theorem 1.1 gives a lower bound on the positive eigenvalues. Therefore, Theorem 1.1 can also be interpreted as giving a bound from below on the first eigenvalue λ_1 , i.e., of the smallest positive eigenvalue for which (3.2) holds.

We start with the proof of the following Bochner-type formula.

Theorem 3.1. *On a qc manifold of dimension $4n + 3$ the next formula holds true*

$$(3.3) \quad \frac{1}{2} \Delta |\nabla f|^2 = |\nabla^2 f|^2 - g(\nabla(\Delta f), \nabla f) + \text{Ric}(\nabla f, \nabla f) + 2 \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) + 4 \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s \nabla f).$$

Proof. By definition we have

$$(3.4) \quad -\frac{1}{2} \Delta |\nabla f|^2 = \nabla^3 f(e_a, e_a, e_b) df(e_b) + \nabla^2 f(e_a, e_b) \nabla^2 f(e_a, e_b) = \nabla^3 f(e_a, e_a, e_b) df(e_b) + |\nabla^2 f|^2.$$

To evaluate the first term in the right hand side of (3.4) we use the Ricci identities (2.17). An application of (2.3) to (2.7) gives

$$(3.5) \quad (\nabla_X T)(Y, Z) = 0.$$

Applying successively the Ricci identities (2.17) and also (3.5) we obtain the next sequence of equalities

$$(3.6) \quad \begin{aligned} \nabla^3 f(e_a, e_a, e_b) df(e_b) &= \nabla^3 f(e_a, e_b, e_a) df(e_b) - 2 \sum_{s=1}^3 \omega_s(e_a, e_b) df(e_b) \nabla^2 f(e_a, \xi_s) \\ &= \nabla^3 f(e_b, e_a, e_a) df(e_b) - R(e_a, e_b, e_a, e_c) df(e_c) df(e_b) - 2 \sum_{s=1}^3 \omega_s(e_a, e_b) df(e_b) \left[\nabla^2 f(\xi_s, e_a) + \nabla^2 f(e_a, \xi_s) \right] \\ &= -d(\Delta f)(e_b) df(e_b) + \text{Ric}(\nabla f, \nabla f) + 4 \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s \nabla f) + 2 \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) \end{aligned}$$

A substitution of (3.6) in (3.4) completes the proof of (3.3). \square

Corollary 3.2. *On a qc manifold of dimension $4n + 3$ the next formula holds*

$$(3.7) \quad \begin{aligned} \frac{1}{2} \Delta |\nabla f|^2 &= -d(\Delta f)(e_a) df(e_a) + \text{Ric}(\nabla f, \nabla f) + 2T^0(\nabla f, \nabla f) - 6U(\nabla f, \nabla f) + |\nabla^2 f|^2 \\ &\quad + 4 \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s \nabla f). \end{aligned}$$

Proof. Using (2.15) together with (2.13), we calculate

$$(3.8) \quad 2 \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) = 2T^0(\nabla f, \nabla f) - 6U(\nabla f, \nabla f),$$

which when combined with (3.8) and (3.3) give (3.7). \square

Our next goal is to evaluate in two ways the last term of (3.7). First using the $Sp(n)Sp(1)$ -invariant orthogonal decomposition $\Psi_{[3]} \oplus \Psi_{[-1]}$ of all linear maps on H , we obtain the $Sp(n)Sp(1)$ -invariant decomposition of the horizontal Hessian $\nabla^2 f$ (after the usual identification of tensors through the metric), namely

$$(3.9) \quad \begin{aligned} (\nabla^2 f)_{[3]}(X, Y) &= \frac{1}{4} \left[\nabla^2 f(X, Y) + \sum_{s=1}^3 \nabla^2 f(I_s X, I_s Y) \right] \\ (\nabla^2 f)_{[-1]}(X, Y) &= \frac{1}{4} \left[3\nabla^2 f(X, Y) - \sum_{s=1}^3 \nabla^2 f(I_s X, I_s Y) \right]. \end{aligned}$$

We continue with the next lemma where we give the first formula for the last term of (3.7).

Lemma 3.3. *On a compact qc manifold of dimension $4n + 3$ the next integral formula holds*

$$(3.10) \quad \int_M \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s \nabla f) Vol_\eta = \int_M \left[\frac{3}{4n} |(\nabla^2 f)_{[3]}|^2 - \frac{1}{4n} |(\nabla^2 f)_{[-1]}|^2 - \frac{1}{2} \sum_{s=1}^3 \tau_s(I_s \nabla f, \nabla f) \right] Vol_\eta.$$

Proof. We recall that an orthonormal frame

$$\{e_1, e_2 = I_1 e_1, e_3 = I_2 e_1, e_4 = I_3 e_1, \dots, e_{4n} = I_3 e_{4n-3}, \xi_1, \xi_2, \xi_3\}$$

is a qc-normal frame (at a point) if the connection 1-forms of the Biquard connection vanish (at that point). Lemma 4.5 in [24] asserts that a qc-normal frame exists at each point of a qc manifold.

Using the identification of the 3-dimensional vector spaces spanned by $\{\xi_1, \xi_2, \xi_3\}$ and $\{I_1, I_2, I_3\}$ with \mathbb{R}^3 , the restriction of the action of $Sp(n)Sp(1)$ to this spaces can be identified with the action of the group $SO(3)$, i.e., $\xi_i = \sum_{t=1}^3 \Psi_{it} \bar{\xi}_t$ and $I_i = \sum_{t=1}^3 \Psi_{it} \bar{I}_t$, $i = 1, 2, 3$ with $\Psi \in SO(3)$. One verifies easily that the horizontal 1-form

$$B(X) = \sum_{s=1}^3 \nabla^2 f(I_s X, I_s e_a) df(e_a)$$

is $Sp(n)Sp(1)$ invariant on \mathbb{H} , for example $\bar{B}(X) = (\det \Psi) B(X) = B(X)$. Thus, it is sufficient to compute the divergence of B in a qc-normal frame. To avoid the introduction of new variables we shall assume that $\{e_1, \dots, e_{4n}, \xi_1, \xi_2, \xi_3\}$ is a qc-normal frame.

Using that the Biquard connection preserves the splitting of TM , the Ricci identities (2.17), the definition of τ_s and (2.7), we find

$$(3.11) \quad \begin{aligned} \nabla^* B &= \sum_{s=1}^3 \left[\nabla^3 f(e_b, I_s e_b, I_s e_a) df(e_a) + \nabla^2 f(I_s e_b, I_s e_a) \nabla^2 f(e_b, e_a) \right] \\ &= \frac{1}{2} \sum_{s=1}^3 \left[\nabla^3 f(e_b, I_s e_b, I_s e_a) - \nabla^3 f(I_s e_b, e_b, I_s e_a) \right] df(e_a) + \sum_{s=1}^3 \nabla^2 f(I_s e_b, I_s e_a) \nabla^2 f(e_b, e_a) \\ &= -\frac{1}{2} R(e_b, I_s e_b, I_s e_a, e_c) df(e_c) df(e_a) - \sum_{s=1}^3 \omega_s(e_b, I_s e_b) \nabla^2 f(\xi_s, I_s e_a) df(e_a) + \sum_{s=1}^3 \nabla^2 f(I_s e_b, I_s e_a) \nabla^2 f(e_b, e_a) \\ &= -2n \sum_{s=1}^3 \tau_s(I_s \nabla f, \nabla f) - 4n \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s \nabla f) + g(\Upsilon \nabla^2 f, \nabla^2 f), \end{aligned}$$

where we used (2.10) in the last term and the convention $I_s \alpha(X) = -\alpha(I_s X)$ for a horizontal 1-form α . Using the orthogonality of the spaces $\Psi_{[3]}$ and $\Psi_{[-1]}$ we have

$$g(\Upsilon \nabla^2 f, \nabla^2 f) = 3|(\nabla^2 f)_{[3]}|^2 - |(\nabla^2 f)_{[-1]}|^2.$$

A substitution of the last equality in (3.11) and the divergence formula give (3.10). This completes the proof of the Lemma. \square

The second integral formula for the last term in (3.7) follows.

Lemma 3.4. *On a compact qc manifold of dimension $4n + 3$ the following integral formula holds*

$$(3.12) \quad \int_M \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s \nabla f) \text{Vol}_\eta = - \int_M \left[4n \sum_{s=1}^3 (df(\xi_s))^2 + \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) \right] \text{Vol}_\eta.$$

Proof. Note, that by definition we have

$$\left[g(\nabla^2 f, \omega_s) \right]^2 = \left[\nabla^2 f(e_a, I_s e_a) \right]^2.$$

From the Ricci identities we have

$$(3.13) \quad g(\nabla^2 f, \omega_s) = \nabla^2 f(e_a, I_s e_a) = -4n df(\xi_s)$$

which implies

$$(3.14) \quad 16n^2 \int_M \sum_{s=1}^3 (df(\xi_s))^2 \text{Vol}_\eta = \int_M \sum_{s=1}^3 \left[g(\nabla^2 f, \omega_s) \right]^2 \text{Vol}_\eta = -4n \int_M \sum_{s=1}^3 g(\nabla^2 f, \omega_s) df(\xi_s) \text{Vol}_\eta.$$

Let us consider the $Sp(n)Sp(1)$ invariant horizontal 1-form defined by

$$C(X) = \sum_{s=1}^3 df(I_s X) df(\xi_s)$$

whose divergence is (computing as usual in a qc normal frame)

$$(3.15) \quad \begin{aligned} \nabla^* C &= \sum_{s=1}^3 \left[\nabla^2 f(e_a, I_s e_a) df(\xi_s) + \nabla^2 f(e_a, \xi_s) df(I_s e_a) \right] \\ &= \sum_{s=1}^3 \left[g(\nabla^2 f, \omega_s) df(\xi_s) - \nabla^2 f(\xi_s, I_s \nabla f) - T(\xi_s, I_s \nabla f, \nabla f) \right]. \end{aligned}$$

In the above calculation we used the second formula of (2.17) to obtain the second equality of (3.15). Integrate (3.15) over M and use (3.14) to get (3.12) which completes the proof of the lemma. \square

4. PROOF OF THEOREM 1.1

Proof. We begin by integrating the Bochner type formula (3.3) over the compact qc manifold M of dimension $4n + 3$. Using the divergence formula we come to

$$(4.1) \quad \begin{aligned} 0 &= \int_M \left[-(\Delta f)^2 + |(\nabla^2 f)_{[3]}|^2 + |(\nabla^2 f)_{[-1]}|^2 + Ric(\nabla f, \nabla f) + 2 \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) \right] \text{Vol}_\eta \\ &\quad + 4 \int_M \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s \nabla f) \text{Vol}_\eta. \end{aligned}$$

Following Greenleaf [23], we represent the last term in (4.1) as follows

$$\int_M \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s \nabla f) \text{Vol}_\eta = (1 - c) \int_M \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s \nabla f) \text{Vol}_\eta + c \int_M \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s \nabla f) \text{Vol}_\eta,$$

where c is a constant. Then we apply Lemma 3.3 and Lemma 3.4, correspondingly, to the first and the second terms in the obtained identity after which the above equality (4.1) takes the form

$$(4.2) \quad 0 = \int_M \left[-(\Delta f)^2 + |(\nabla^2 f)_{[3]}|^2 + |(\nabla^2 f)_{[-1]}|^2 + Ric(\nabla f, \nabla f) + 2 \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) \right] Vol_\eta \\ + 4(1-c) \int_M \left[\frac{3}{4n} |(\nabla^2 f)_{[3]}|^2 - \frac{1}{4n} |(\nabla^2 f)_{[-1]}|^2 - \frac{1}{2} \sum_{s=1}^3 \tau_s(I_s \nabla f, \nabla f) \right] Vol_\eta \\ - 4c \int_M \left[4n \sum_{s=1}^3 (df(\xi_s))^2 + \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) \right] Vol_\eta.$$

Equation (4.2) can be simplified as follows

$$(4.3) \quad 0 = \int_M \left[-(\Delta f)^2 + \left(1 + \frac{3(1-c)}{n}\right) |(\nabla^2 f)_{[3]}|^2 + \left(1 - \frac{(1-c)}{n}\right) |(\nabla^2 f)_{[-1]}|^2 + Ric(\nabla f, \nabla f) \right] Vol_\eta \\ + \int_M \left[-16nc \sum_{s=1}^3 (df(\xi_s))^2 - 2(1-c) \sum_{s=1}^3 \tau_s(I_s \nabla f, \nabla f) + (2-4c) \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) \right] Vol_\eta.$$

Using that $\left\{ \frac{1}{2\sqrt{n}} \omega_s \right\}$ is an orthonormal set in $\Psi_{[-1]}$ we have

$$(4.4) \quad |(\nabla^2 f)_{[-1]}|^2 \geq \frac{1}{4n} \sum_{s=1}^3 [g(\nabla^2 f, \omega_s)]^2 = 4n \sum_{s=1}^3 (df(\xi_s))^2,$$

while a projection on $\left\{ \frac{1}{2\sqrt{n}} g \right\}$ gives

$$(4.5) \quad |(\nabla^2 f)_{[3]}|^2 \geq \frac{1}{4n} (\Delta f)^2.$$

We obtain from (4.3) taking into account (4.5) and (4.4) that for any constant c such that

$$(4.6) \quad 1 + \frac{3(1-c)}{n} \geq 0, \quad 1 - \frac{(1-c)}{n} \geq 0$$

we have the following inequality

$$(4.7) \quad 0 \geq \int_M \left[\left(\frac{1}{4n} + \frac{3(1-c)}{4n^2} - 1 \right) (\Delta f)^2 + 16n^2 \left(\frac{1}{4n} - \frac{(1-c)}{4n^2} - \frac{c}{n} \right) \sum_{s=1}^3 (df(\xi_s))^2 \right] Vol_\eta \\ + \int_M \left[Ric(\nabla f, \nabla f) - 2(1-c) \sum_{s=1}^3 \tau_s(I_s \nabla f, \nabla f) + (2-4c) \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) \right] Vol_\eta.$$

The coefficient in front of $\sum_{s=1}^3 (df(\xi_s))^2$ is non-negative provided $c \leq \frac{n-1}{4n-1}$, so in order to cancel the $\sum_{s=1}^3 (df(\xi_s))^2$ term we take

$$(4.8) \quad c = \frac{n-1}{4n-1}.$$

Note that the inequalities (4.6) are satisfied and with this choice of c (4.7) yields

$$(4.9) \quad 0 \geq \int_M \left(\frac{2(1-n)(2n+1)}{n(4n-1)} \right) (\Delta f)^2 Vol_\eta \\ + \int_M \left[Ric(\nabla f, \nabla f) - \frac{6n}{4n-1} \sum_{s=1}^3 \tau_s(I_s \nabla f, \nabla f) + \frac{4n+2}{4n-1} \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) \right] Vol_\eta.$$

Applying the identities from Theorem 2.3 and (2.13) we calculate

$$(4.10) \quad \sum_{s=1}^3 \tau_s(I_s X, Y) = \frac{n+2}{n} T^0(X, Y) + 3Sg(X, Y).$$

Using the first equality in Theorem 2.3, (3.8) and (4.10), we express the second line in (4.9) in terms of Ric, T^0 and U as follows

$$(4.11) \quad Ric(\nabla f, \nabla f) - \frac{6n}{4n-1} \sum_{s=1}^3 \tau_s(I_s \nabla f, \nabla f) + \frac{4n+2}{4n-1} \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) \\ = \frac{2(n-1)(2n+1)}{(4n-1)(n+2)} \left[Ric(\nabla f, \nabla f) + \alpha_n T^0(\nabla f, \nabla f) + \beta_n U(\nabla f, \nabla f) \right],$$

where

$$(4.12) \quad \alpha_n = \frac{2(4n+5)}{2n+1}, \quad \beta_n = 3 \frac{2n^2+5n-1}{(2n+1)(n-1)}.$$

At this point we let f be an eigenfunction of the sub-Laplacian with eigenvalue λ , i.e., (3.2) holds. An integration by parts yields

$$(4.13) \quad \int_M (\Delta f)^2 Vol_\eta = \lambda \int_M f \Delta f Vol_\eta = \lambda \int_M |\nabla f|^2 Vol_\eta.$$

Let us assume $n \geq 2$. A substitution of (4.13) and (4.11) in (4.9) gives

$$(4.14) \quad 0 \geq \int_M -\lambda |\nabla f|^2 + \frac{n}{n+2} \left[Ric(\nabla f, \nabla f) + \alpha_n T^0(\nabla f, \nabla f) + \beta_n U(\nabla f, \nabla f) \right] Vol_\eta.$$

The conditions of the theorem together with (4.14) yield the inequality

$$(4.15) \quad 0 \geq \int_M \left(-\lambda + \frac{n}{n+2} k_0 \right) |\nabla f|^2 Vol_\eta,$$

which implies the desired inequality

$$\lambda \geq \frac{n}{n+2} k_0.$$

This completes the proof of Theorem 1.1. □

Remark 4.1. Suppose we have the case of equality in Theorem 1.1, i.e., we have

$$\lambda = \frac{n}{n+2} k_0, \quad \Delta f = \frac{n}{n+2} k_0 f$$

For c given by (4.8) equalities in (4.4) and (4.5) must hold which implies that the horizontal Hessian of the eigenfunction f is given by the next equation

$$(4.16) \quad \nabla^2 f(X, Y) = -\frac{k_0}{4(n+2)} fg(X, Y) - \sum_{s=1}^3 df(\xi_s) \omega_s(X, Y).$$

5. PROOF OF THEOREM 1.2

We proof Theorem 1.2 using the Lichnerowicz' estimate for the first positive eigenvalue of Riemannian Laplacian and the Obata' theorem [36] which says that the equality in the Lichnerowicz' estimate is achieved only on the round sphere.

5.1. Relation between the Laplacian and the sub-Laplacian. We start with the next lemma relating the Riemannian Laplacian and the sub-Laplacian.

Lemma 5.1. Let M be a $(4n+3)$ -dimensional qc manifold. Then the sub-Laplacian Δ and the Riemannian Laplacian Δ^g , corresponding to the Levi-Civita connection ∇^g of the extended metric g , are connected by

$$(5.1) \quad \Delta^g f = \Delta f - \sum_{s=1}^3 \xi_s^2 f + df \left(\sum_{s=1}^3 \nabla_{\xi_s} \xi_s \right).$$

Proof. By definition, $\Delta^g f = -\sum_{a=1}^{4n} \nabla^g df(e_a, e_a) - \sum_{s=1}^3 \nabla^g df(\xi_s, \xi_s)$, where $\{e_1, \dots, e_{4n}, \xi_1, \xi_2, \xi_3\}$ is an orthonormal basis of $H \oplus V$. Write df for the gradient of f , the last equality can be write in the form

$$(5.2) \quad \Delta^g f = -g(\nabla_{e_a}^g \tilde{d}f, e_a) - \sum_{s=1}^3 g(\nabla_{\xi_s}^g \tilde{d}f, \xi_s) = -g(\nabla_{e_a} \tilde{d}f, e_a) - \sum_{s=1}^3 g(\nabla_{\xi_s} \tilde{d}f, \xi_s),$$

where we used (2.2) and the identities

$$(5.3) \quad T(e_a, A, e_a) = T(\xi_s, A, \xi_s) = 0$$

following from the properties of the torsion tensor T of ∇ listed in (2.16). Now, we get (5.1) from (5.2). \square

Next we give an estimate between the first eigenvalues of the Riemannian Laplacian and the sub-Laplacian.

Proposition 5.2. *Let M be a $(4n+3)$ -dimensional closed compact qc manifold. The first positive eigenvalue μ of the Riemannian Laplacian and the first positive eigenvalue λ of the sub-Laplacian satisfy the following inequality*

$$(5.4) \quad \mu \leq \lambda + \int_M \sum_{s=1}^3 (df(\xi_s))^2 Vol_\eta$$

for any smooth function f with $\int_M f^2 Vol_\eta = 1$.

Proof. From the variational characterization of the first eigenvalue and (5.1) we have the estimate

$$(5.5) \quad \mu \leq \int_M (\Delta^g f) f Vol_\eta = \int_M (\Delta f) f Vol_\eta - \int_M \left[\sum_{s=1}^3 (\xi_s^2 f) f - df \left(\sum_{s=1}^3 \nabla_{\xi_s} \xi_s \right) f \right] Vol_\eta.$$

For the term $df(\sum_{s=1}^3 \nabla_{\xi_s} \xi_s)$, we obtain consecutively

$$(5.6) \quad df \left(\sum_{s=1}^3 \nabla_{\xi_s} \xi_s \right) = g(\tilde{d}f, \sum_{s=1}^3 \nabla_{\xi_s} \xi_s) = \sum_{t=1}^3 df(\xi_t) g(\xi_t, \sum_{s=1}^3 \nabla_{\xi_s} \xi_s) = \sum_{s,t=1}^3 df(\xi_t) g(\nabla_{\xi_s} \xi_s, \xi_t),$$

where we used for the third equality that the Biquard connection is metric.

Consider the vector field $f df(\xi_s) \xi_s$. We calculate its Riemannian divergence $div[f df(\xi_s) \xi_s]$ as follows

$$(5.7) \quad \begin{aligned} div[f(\xi_s f) \xi_s] &= (df(\xi_s))^2 + (\xi_s^2 f) f + f df(\xi_s) \left[g(\nabla_{e_a}^g \xi_s, e_a) + \sum_{t=1}^3 g(\nabla_{\xi_t}^g \xi_s, \xi_t) \right] \\ &= (df(\xi_s))^2 + (\xi_s^2 f) f + f df(\xi_s) \left[g(\nabla_{e_a} \xi_s, e_a) + \sum_{t=1}^3 g(\nabla_{\xi_t} \xi_s, \xi_t) \right] \\ &= (df(\xi_s))^2 + (\xi_s^2 f) f - f df(\xi_s) \sum_{t=1}^3 g(\nabla_{\xi_t} \xi_t, \xi_s) \end{aligned}$$

where we used (2.2), (5.3) and the fact that the Biquard connection preserves the splitting $H \oplus V$ to established the second and the third equality. A substitution of (5.7) and (5.6) in (5.5) followed by an application of the Riemannian divergence formula give inequality (5.4). \square

Proof of Theorem 1.2. Suppose that M is a qc-Einstein structure of dimension at least eleven with a normalized qc scalar $S = 2$, hence the qc Ricci tensor given by the first equality in (2.16) satisfies $Ric = 4(n+2)g$. Suppose the equality case of Theorem 1.2 holds, i.e., $\lambda = 4n$ and let $\Delta f = \lambda f$. After a possible rescaling of f and using the divergence formula we have then the following identities

$$(5.8) \quad \begin{aligned} \lambda &= 4n, \quad \Delta f = 4nf, \quad \int_M f^2 Vol_\eta = 1, \\ \int_M |\nabla f|^2 Vol_\eta &= \lambda = \frac{1}{\lambda} \int_M |\Delta f|^2 Vol_\eta. \end{aligned}$$

In this case Lemmas 3.3 and 3.4 together with equation (4.10) yield

$$(5.9) \quad \int_M \sum_{s=1}^3 (df(\xi_s))^2 Vol_\eta = 3.$$

Therefore, from (5.9) we have the inequality

$$(5.10) \quad \mu \leq 4n + 3.$$

On the other hand, any qc-Einstein manifold with a positive qc scalar curvature is locally 3-Sasakian [24] and it is well known that a 3-Sasakian manifold is Einstein (with respect to the extended metric) with Riemannian scalar curvature $(4n+2)$ [31], i.e., the Riemannian Ricci tensor Ric^g is given by

$$(5.11) \quad Ric^g(A, A) = (4n + 2)g(A, A).$$

By Lichnerowicz' theorem and (5.11) we have

$$(5.12) \quad \mu \geq 4n + 3.$$

The inequalities (5.10) and (5.12) yield the equality

$$(5.13) \quad \mu = 4n + 3.$$

Therefore, by Obata's result we conclude that the manifold (M, g) is isometric to the sphere $S^{4n+3}(1)$ and hence the manifold (M, g, \mathbb{Q}) is qc equivalent to the 3-Sasakian sphere of dimension $4n + 3$. This completes the proof of Theorem 1.2. \square

6. SHARP ESTIMATES FOR SQUARE INTEGRALS OF DERIVATIVES

In this section we prove Theorem 1.3.

Proof of Theorem 1.3. Notice that we are using a function which vanishes outside some compact so the integrals are well defined. The proof is similar to the proof of Theorem 1.1 except we have to express $|(\nabla^2 f)_{[3]}|^2$ in two different ways. This is the place where the qc case differs from the CR case. We start with the identity (4.1), in which we first move the integral of the square of the sub-Laplacian to the left-hand side of the equality. Then we write

$$|(\nabla^2 f)_{[3]}|^2 = (1 - c)|(\nabla^2 f)_{[3]}|^2 + c|(\nabla^2 f)_{[3]}|^2$$

and use (4.5) to obtain

$$|(\nabla^2 f)_{[3]}|^2 \geq \frac{1-c}{4n} |\Delta f|^2 + c|(\nabla^2 f)_{[3]}|^2$$

when $1 - c \geq 0$. Finally, we use (3.10) for the last term in the thus obtained form of (4.1). The result is the following inequality (valid for $1 - c \geq 0$)

$$(6.1) \quad \left(1 - \frac{1-c}{4n}\right) \int_M |\Delta f|^2 Vol_\eta \geq \int_M \left[\left(c + \frac{3}{n}\right) |(\nabla^2 f)_{[3]}|^2 + \left(1 - \frac{1}{n}\right) |(\nabla^2 f)_{[-1]}|^2 \right] Vol_\eta \\ + \int_M \left[Ric(\nabla f, \nabla f) - 2 \sum_{s=1}^3 \tau_s(I_s \nabla f, \nabla f) + 2 \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) \right] Vol_\eta.$$

In order to obtain the norm of horizontal Hessian we solve for c the equation

$$c + \frac{3}{n} = 1 - \frac{1}{n},$$

which gives $c = (n - 4)/n$. Since $1 - c = 4/n > 0$ we let $c = (n - 4)/n$ in the above inequality (6.1) which becomes

$$(6.2) \quad \frac{n^2 - 1}{n^2} \int_M |\Delta f|^2 Vol_\eta \geq \frac{n-1}{n} \int_M |\nabla^2 f|^2 Vol_\eta \\ + \int_M \left[Ric(\nabla f, \nabla f) - 2 \sum_{s=1}^3 \tau_s(I_s \nabla f, \nabla f) + 2 \sum_{s=1}^3 T(\xi_s, I_s \nabla f, \nabla f) \right] Vol_\eta.$$

Recalling the formula for the Ricci tensor in Theorem 2.3, (3.8) and (4.10) after a short simplification (using that $n > 1$) we obtain the desired inequality, which completes the proof. \square

For a qc-Einstein manifold, where $T^0 = U = 0$, Theorem 1.3 gives the next corollary taking into account that a qc-Einstein manifold of dimension eleven and higher is of constant scalar curvature, see [24].

Corollary 6.1. *Let (M, η) be a $(4n + 3)$ -dimensional qc-Einstein manifold, $n > 1$. For any $f \in \mathcal{C}_o^\infty(M)$ we have*

$$(6.3) \quad \int_M |\Delta f|^2 \text{Vol}_\eta \geq \frac{n}{n+1} \int_M |\nabla^2 f|^2 \text{Vol}_\eta + \frac{2n^2 S}{n+1} \int_M |\nabla f|^2 \text{Vol}_\eta.$$

For the quaternionic Heisenberg group with its standard qc structure, see [24] and [29], the above Corollary gives the following result. The point here is the precise value of the constant c_n since even the more general Calderón-Zygmund L^p version is well known to hold on nilpotent Lie groups, see [19] for an excellent overview.

Corollary 6.2. *Let $(\mathbf{G}(\mathbb{H}), \tilde{\Theta})$ be the $(4n + 3)$ -dimensional Heisenberg group equipped with its standard qc structure. For any $f \in \mathcal{C}_o^\infty(\mathbf{G}(\mathbb{H}))$ we have*

$$(6.4) \quad \|\nabla^2 f\|_{L^2(\mathbf{G}(\mathbb{H}))} \leq c_n \|\Delta f\|_{L^2(\mathbf{G}(\mathbb{H}))}, \quad c_n = \sqrt{1 + \frac{1}{n}}.$$

As a consequence of the above estimate, [17] and [16] which generalize Cordes' results to the sub-Riemannian setting it follows that for

$$(6.5) \quad 2 \leq p < 2 + \frac{n + n\sqrt{16n^2 + 8n - 3}}{4n^2 + 2n - 1}$$

a p -harmonic function on an open set $\Omega \subset \mathbf{G}(\mathbb{H})$ on the quaternionic Heisenberg group of dimension $4n + 3$, $f \in S^{1,p}(\mathbf{G}(\mathbb{H}))$, has in fact additional regularity $f \in S_{loc}^{2,2}(\mathbf{G}(\mathbb{H}))$. Here $S^{k,p}(\Omega)$ denote the usual non-isotropic Sobolev spaces, see for example [19]. Similarly to [17] and [16] one can then obtain a $\mathcal{C}^{1,\alpha}$ under suitable restrictions on p . Obtaining the $\mathcal{C}^{1,\alpha}$ property of the solution is in general still an open problem except in some cases, see [16], [34] and [22] and references therein. The first $\mathcal{C}^{1,\alpha}$ estimate was obtained for the-Laplacian operator on the Heisenberg group [7].

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