On sub-Riemannian and Riemannian spaces associated to a Lorentzian manifold

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Abstract We present a certain construction of a sub-Riemannian and Riemannian spaces naturally associated to a Lorentzian manifold. Some additional structures and relations between geometric properties of the corresponding spaces will be explored. The emphasis will be on keeping the text as self-sufficient as possible while linking various well developed fields.

1 Some notations and conventions

Throughout the text we let $d \geq 1$ be a positive integer, which equals the space dimension of the Lorentzian manifold M. Latin indices will vary from 1 to d, while Greek indices will vary from 0 to d, and we will assume a summation on repeated indices. We will consider a Lorentzian metric g of signature $(1, d)$ on M. In local coordinates, $g = (g_{\alpha\beta})$. We shall raise and lower indices using the metric and its inverse $g^{-1} = (g^{\alpha\beta})$. In particular, for a tangent vector $y = y^\alpha \frac{\partial}{\partial x^\alpha} \in T_x M$ we have $y_\alpha = g_{\alpha\beta} y^\beta$ and $y^\alpha = g^{\alpha\beta} y_\beta$ and its length is given by

$$|y|^2 \overset{def}{=} g_x(y, y) = g_{\alpha\beta}(x)\, y^\alpha y^\beta = y^\alpha y_\alpha \in \mathbb{R}.$$

The simplest example is the case of Minkowski space where $M = \mathbb{R}^{1,d}$ and

$$|y|^2 = (y^0)^2 - (y^1)^2 - \cdots - (y^d)^2.$$

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In the general case, given a point and a normal coordinate system centered at it the metric takes the above form at the given point.

As usual, the Lorentzian metric induces a decomposition of the tangent vectors at each point, which with our agreement of the signature of the metric means that a tangent vector $v \in TM$ is timelike if $|v| > 0$, lightlike if $|v| = 0$, and spacelike if $|v| < 0$.

We shall assume throughout that the considered spacetime is non-compact and time oriented, i.e., there exists a continuous timelike vector field. The future cone at every point is the part of the timelike double cone that contains the fixed timelike global vector field.

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2 Causal Set Theory

The causal order of space-time has a long history. One of the more recent developments lead to a candidate of a framework on which a theory of quantum gravity can be based, see [7], [11] and references therein. Causal set theory provides a way of discretization that avoids preferred frame, [3], while preserving Lorentz invariance as a fundamental property. Both of these underlining notions are present in our constructions as well. Let (M, g) be a noncompact time-oriented Lorentzian manifold. We define a point p to be in the past of a point q, $p \prec q$, if there is a smooth, future-directed timelike curve from p to q. The future $I^+(p)$ and past $I^-(p)$ of p are defined, respectively, by

$$I^+(p) = \{q \mid p \prec q\} \quad \text{and} \quad I^-(p) = \{q \mid q \prec p\}.$$

(M, g) is called future (past) distinguishing if $I^+(p) = I^+(q)$ $(I^-(p) = I^-(q))$ implies $p = q$.

To a certain extent the Lorentzian geometry of space-time can be recovered from its causal order.

Theorem 1 ([8], [9]) *Assume M is both future and past distinguishing. Then, the causal structure determines the metric up to a conformal factor.*

The modern version of Causal Set Theory began with the paper of Bombelli, Lee, Meyer and Sorkin [4].

Definition 1 A causal set is a partially ordered set $(C, \prec)$ where $\prec$ is (i) acyclic; (ii) transitive and (iii) locally finite, i.e, for all $x, y \in C$ the set $A(x, y) = \{z \mid x \prec z \prec y\}$ is finite.

The absence of cycles in the above definition can be replaced with irreflexivity of the partial order. The general idea is that a causal set replaces the continuum manifold, while the latter is regarded as an approximation of the causal set.

A causal set $(C, \prec)$ with elements given through an injection in a spacetime (M, g) and order induced from the causal structure of the spacetime is said to be an embedding. The approximation is of density ρ_c if there exists an order preserving injection $\Phi : C \to M$ such that $\Phi(C)$ is uniformly distributed with density ρ_c. Here, every spacetime region of volume V contains approximately $\rho_c V$ elements of C. This approach leads to several problems, including symmetry breaking.

The adopted approach to handle this uniformity issue has been to introduce the concept of *sprinkling*, where we begin with a spacetime (M, g) and then randomly "sprinkle" elements onto M via a Poisson process. Thus, the probability of finding n elements in a spacetime region of volume V is $P_V(n) = \frac{(\rho_c V)^n}{n!} e^{-\rho_c V}$. The set of events within a proper time τ_0 in the future of a point $p = 0$ is the region between the light cone and the hyperboloid $|y|^2 = (y^0)^2 - (y^1)^2 - \cdots - (y^d)^2 \approx \tau_0^2$. This is a non-compact region and almost surely contains infinite number of points q directly in the future of p, i.e, $p \prec q$ and there is no r such that $p \prec r \prec q$. For a causal set that is approximated, for example, by Minkowski spacetime, every element therefore has an infinite number of nearest neighbours in its future and past light-cones. This "non-locality" complicates the definitions of discrete version of continuum quantities, including D'Alembertian, leading to non-convergent infinite sums.

The sub-Riemannian and Riemannian spaces introduced in the subsequent sections arose in our goal to remove the non-localities by modeling and discretizing a future directed timelike sector of the tangent bundle to Lorentzian space as a (sub-)Riemannian manifold.

3 Sub-Riemannian space

In this section we associate a sub-Riemannian space to the considered spacetime. For more details on the relevant definitions and results in sub-Riemannian geometry we refer to [1], [2], and [10]. The space we shall define is a sub-bundle of the tangent bundle. The latter and the cotangent bundle have appeared in theories of a maximal proper acceleration seeking geometric formulation of quantum mechanics whose early ideas can be found in [5, 6].

Definition 2 Let M be a time-oriented Lorentzian manifold. Define the *phase space manifold* $\mathcal{M}$ to be the elements of TM consisting of all points whose tangent component is a timelike future-oriented vector.

Definition 3 Let $\pi : TM \to M$ be the natural projection. For a point $\xi \in \mathcal{M}$ we shall say that (x, y) is a *normal coordinate system centered* at ξ if x is a normal coordinate system centered at $\pi(\xi) \in M$ and y is the fiber coordinate.

Thus, locally, using normal coordinate systems centered at the corresponding point ξ, we have

$$\mathcal{M} = \{\xi \in TM \mid y^0 > 0, \, |y|^2 > 0\}.$$

Definition 4 Let (x, y) be a normal coordinate system at the point $\xi \in \mathcal{M}$. Consider the following forms on $\mathcal{M}$ defined near ξ,

$$\theta^k = y^k dx^0 - y^0 dx^k.$$

The *horizontal space at* ξ is the joint kernel of the forms θ^k at ξ,

$$\mathcal{H}_\xi = \cap_{k=1}^d Ker\, \theta^k|_\xi \subset T_\xi \mathcal{M}.$$

Let us observe that we obtain a (well defined) sub-bundle $\mathcal{H}$ of the tangent bundle of the phase space $\mathcal{M}$ due to the following proposition, which also exhibits the Lorentz invariance of the horizontal space.

Proposition 1 *We have* $span\{\theta^1, \ldots, \theta^d\} = span\{\eta^0, \eta^1, \ldots, \eta^d\}$, *where*

$$\eta^\alpha \stackrel{def}{=} dx^\alpha - \frac{y^\alpha}{|y|^2} y_\beta dx^\beta.$$

Furthermore, the horizontal space $\mathcal{H}$ *is Lorentz invariant under Lorentz transformations (of the spacetime coordinates).*

Proof The first claim follows from the following easily verifiable identities

$$\eta^\alpha(\frac{\partial}{\partial y^\beta}) = 0, \; \eta^\alpha(y^\beta \frac{\partial}{\partial x^\beta}) = y^\alpha - \frac{y^\alpha y_\beta y^\beta}{|y|^2} = 0, \text{ and } \theta^k = y^k \eta^0 - y^0 \eta^k.$$

For the proof of the second part let x and $\tilde{x}$ are normal coordinates centered at the same point p of M, hence $x = \Lambda \tilde{x}$ where Λ is a Lorentzian matrix. Using vector notation we have at p the identities $dx = \Lambda d\tilde{x}$, $y = \Lambda \tilde{y}$. The invariance follows from $\eta = dx - \frac{\langle y, dx \rangle}{|y|^2} y$, where we used the notation $\langle y, dx \rangle = y_\alpha dx^\alpha$.$\square$

In fact, we are in the realm of sub-Riemannian geometry since the horizontal vector fields and their commutators span the whole tangent space, i.e., the horizontal space is *bracket generating* (completely non-holonomic).

Proposition 2 *For* $\xi \in \mathcal{M}$ *and* (x, y) *a normal coordinate system of* $\mathcal{M}$ *centered at* ξ *we have that*

$$\mathcal{H}_\xi = span\,\{V = y^\beta \frac{\partial}{\partial x^\beta}, \; \frac{\partial}{\partial y^\alpha}\}.$$

The horizontal space $\mathcal{H}$ *is a rank* $d+2$ *bracket generating sub-bundle (distribution) of* $T\mathcal{M}$ *which satisfies Hörmander's condition of step one.*

Proof It is obvious that the right-hand side involves $d+2$ linearly independent vectors that are annihilated by the 1-forms defining the horizontal space. Furthermore, since we have

$$[V, \frac{\partial}{\partial y^\alpha}] = \frac{\partial}{\partial x^\alpha} \quad \text{and} \quad \mathcal{H}^{[1]} \stackrel{def}{=} \mathcal{H} + [\mathcal{H}, \mathcal{H}] = T\mathcal{M},$$

the horizontal bundle $\mathcal{H}$ together with its commutator span the tangent space.

Physically, the vector V should be thought as a vector defining the future direction. It can be used to define a causal structure on the sub-Riemannian and Riemannian spaces we define.

By Chow - Rashevskii' theorem, see [1], [2] and [10], the bracket generating condition is sufficient for any two points of $\mathcal{M}$ to be connected by a horizontal curve, i.e., a curve whose velocity lies in the horizontal direction. We note explicitly that in our case $\mathcal{H}$ is not strong bracket generating due to $\mathcal{H}_\xi + [\frac{\partial}{\partial y^\alpha}, \mathcal{H}]_\xi \neq T_\xi\mathcal{M}$. Recall, [1] and [10], that *strong bracket generating* or *fat distribution* means either of the following equivalent conditions, where $\xi \in \mathcal{M}$ and w, $w' \in \mathcal{H}_\xi$, $w \neq 0$, with horizontal extensions W and W': (i) $\mathcal{H} + [W, \mathcal{H}]_\xi = T_\xi\mathcal{M}$; (ii) the curvature (Levi) form $\mathcal{L} : \mathcal{H} \times \mathcal{H} \to T\mathcal{M}/\mathcal{H}$,

$$\mathcal{L}(w, w') = [W, W']_\xi \; mod \; \mathcal{H}_\xi$$

defines a surjective map $\mathcal{L}(w, .)$, i.e., the *dual curvature is symplectic*. In general, the strong bracket generating property (which does not hold here) excludes the existence of abnormal (sub-Riemannian) geodesics, see 3.1.1.

3.1 Sub-Riemannian metrics

Let f and h be smooth positive functions on $\mathcal{M}$. For $0 < b < 1$, $\xi = (x, y) \in \mathcal{M}$ and $y = y^\alpha \frac{\partial}{\partial x^\alpha}$, recalling that $y_\alpha = g_{\alpha\beta} y^\beta$, define

$$G = G_b(\xi) \stackrel{def}{=} f(|y|) \frac{y_\alpha y_\beta}{|y|^2}\, dx^\alpha \otimes dx^\beta \; + \; h(|y|) \left(\frac{y_\alpha y_\beta}{|y|^2} \; - \; b\, g_{\alpha\beta} \right) dy^\alpha \otimes dy^\beta.$$

Theorem 2 *The above formula for G defines a positive definite and Lorentz invariant symmetric tensor on $\mathcal{H}$, i.e., G is invariant under the transformations*

$$(x, y) \mapsto (\Lambda x, \Lambda y),$$

where Λ is a Lorentz transformation with respect to the given Lorentzian metric g on M.

Proof The Lorentz invariance is obvious. We sketch the proof of the positivity. For $W = u^\alpha \frac{\partial}{\partial y^\alpha} + aV \in \mathcal{H}$, where V was defined in Proposition 2, we have

$$G(W, W) = a^2 f(|y|)\, |y|^2 \; + \; h(|y|)\, \frac{g(y, u)^2 - b|u|^2 |y|^2}{|y|^2},$$

where $u = u^\alpha \frac{\partial}{\partial x^\alpha}$. Consider the two cases (recall $|y|^2 > 0$), depending on u being timelike or non-timelike. If u is timelike we have the reverse Cauchy-Schwarz inequality $(y^\alpha u_\alpha)^2 \geq |y|^2 |u|^2$, hence

$$G(W,W) \geq a^2 f(|y|)\, |y|^2 \; + \; h(|y|)\, (1-b)|u|^2.$$

In the second case, where $|u|^2 \leq 0$, we use that $b|u|^2|y|^2 \leq 0$. □

When $M = \mathbb{R}^{1,d}$ is a Lorentz spacetime, then G is also invariant under translations in x.

3.1.1 The Sub-Riemannian distance and geodesics

For the sake of giving some context and the possible difficulties we may encounter we recall some known results.

As usual, we define the Carnot-Carathédory (sub-Riemannian) distance using the arclength of "admissible" curves. For sufficiently smooth (e.g. locally rectifiable) $\gamma : [0,1] \to \mathcal{M}$ which is horizontal, $z'(\tau) \in \mathcal{H}_{\gamma(\tau)}$, we define the Carnot-Carathédory (sub-Riemannian) length of γ by the formula

$$l(\gamma) = \int_0^1 \sqrt{G_b(z', z')} d\tau.$$

For z_1, $z_2 \in \mathcal{M}$ the Carnot-Carathédory (sub-Riemannian) distance(abbr. CC-distance) between the two points is

$$d_{CC}(z_0, z_1) = \inf\{l(\gamma) \mid \gamma(0) = z_0, \gamma(1) = z_1, \gamma'(\tau) \in \mathcal{H}_{\gamma(\tau)}\}.$$

As well known, the CC-distance defines a topology equivalent to the manifold topology.A (horizontal) curve is called a *minimizing CC-geodesic* if it achieves the CC-distance between its endpoints. It is called a CC-geodesic if it is locally a minimizing CC-geodesic.

Let $W_0, \ldots, W_{d+2}$ be a local orthonormal frame for the horizontal distribution $\mathcal{H}$. For $\theta \in T^*_\xi \mathcal{M}$ define the Hamiltonian function

$$H(\theta) = \frac{1}{2} \left(\langle \theta, W_0(\xi) \rangle^2 + \cdots + \langle \theta, W_{d+2}(\xi) \rangle^2 \right).$$

H is a fiber-quadratic positive semi-definite form on $T^*\mathcal{M}$ of rank $d+3 = \dim \mathcal{H}$. Recall that $T^*\mathcal{M}$ has the canonical symplectic form.

The projection to $\mathcal{M}$ of an integral curve for the Hamiltonian vector field with Hamiltonian H is a CC-geodesic.

In the Riemannian case this characterizes the geodesics. In the sub-Riemannian case there could be CC-geodesics which are not the projections of integral curves for the Hamiltonian vector field of H. Let $\mathcal{H}^\perp \subset T^*\mathcal{M}$ be the annihilator of the distribution $\mathcal{H}$. Let ω be the restriction to $\mathcal{H}^\perp$ of the canonical symplectic form of $T^*\mathcal{M} \subset T(TM)$. A *characteristic curve* for $\mathcal{H}^\perp$ is an absolutely continuous nowhere vanishing curve $\eta : [0,1] \to \mathcal{H}^\perp$ whose derivative lies in the kernel of ω whenever it exists, $\omega(\theta'(t), \Theta) = 0$ for all $\Theta \in T_{\theta(t)}\mathcal{H}^\perp$. An admissible curve γ on $\mathcal{M}$ is *singular* if and only if it is the projection of a characteristic curve, which depends only on the distribution, not on the sub-Riemannian metric. We note that there is another (equivalent) definition of singular curves as the critical points of the end-point map. It is also known that if ω is "symplectic", i.e., has trivial kernel then there are no characteristics ("strong bracket generating case"). However, every CC-geodesic is a singular curve or a normal geodesics.

A sub-Riemannian metric space is complete if and only if the closed metric balls (or all sufficiently small balls) are compact. In this case, there exists a minimizing CC-geodesics between any two given point. This is the case when the sub-Riemannian metric G is the restriction of a complete Riemannian metric on $\mathcal{M}$.

4 The Riemannian space

Below we shall use the notation set at the beginning of Section 3.1. The following proposition is an easy corollary of the previous constructions, see in particular Theorem 2.

Theorem 3 *For $a > 0$, $\hat{G}_{a,b} \stackrel{def}{=} G_b - af\, g_{\alpha\beta}\eta^\alpha \otimes \eta^\beta$ defines a "Lorentz invariant" Riemannian metric on $\mathcal{M}$. Explicitly, dropping a and b in the notation,*

$$\hat{G} = f \cdot \left(\frac{(1+a) y_\alpha y_\beta}{|y|^2} - a g_{\alpha\beta} \right) dx^\alpha \otimes dx^\beta + h \cdot \left(\frac{y_\alpha y_\beta}{|y|^2} - b g_{\alpha\beta} \right) dy^\alpha \otimes dy^\beta .$$

The Riemannian metrics $\hat{G}_{a,b}$ are a Riemannian approximation of the sub-Riemannian metric G_b, which in the limit $a \to \infty$ converge to the sub-Riemannian space.

5 Some remarks on the Minkowski space case

Assume that g is the Minkowski metric on $\mathbb{R}^{1,d}$. We note that, the horizontal space $\mathcal{H}$ is invariant under Lorentz transformations and in the case of Minkowski space also under translations in the space time variable x. The exhibited Lorentz invariances imply

Proposition 3 *Let $\gamma(\tau) = (x(\tau), y(\tau))$ be a smooth phase space curve, Λ a Lorentz transformation of $\mathbb{R}^{1,d}$, and $\gamma_\Lambda(\tau) \stackrel{def}{=} (\Lambda x(\tau), \Lambda y(\tau))$. If γ is a*

Riemannian geodesic (for $\hat{G}$) then γ_Λ is also a geodesic. The same is true in the setting of CC-geodesics.

An interesting case comes from letting $f = k = const$, $h = l/|y|^2$, $l = const$. With these assumptions, the Riemannian metric $\hat{G}$ is complete, hence, the sub-Riemannian metric is complete as well. Furthermore, using vector notation and letting $\langle y, x' \rangle = y_\alpha dx^\alpha / d\tau$ etc., with prime denoting derivative with respect to the parameter τ, the geodesic equations of the Riemannian metric $\hat{G}$ are

$$x'' = (a+1)\left(\frac{a-1}{a}\frac{\langle y, x'\rangle\langle y, y'\rangle}{|y|^2} - \langle x', y'\rangle\right)\frac{y}{|y|^2} + \frac{1+a}{a}\frac{\langle y, x'\rangle}{|y|^2}y'$$

$$y'' = \left(\frac{k(a+1)}{la}\langle y, x'\rangle^2 - |y'|^2\right)\frac{y}{|y|^2} - \frac{k(a+1)}{la}\langle y, x'\rangle x' + \frac{2}{|y|}|y|'y'.$$

Proposition 4 *If $\gamma(\tau) = (x(\tau), y(\tau))$ is a geodesic of $(\mathcal{M}, \hat{G})$ such that $\gamma(0) = (0, v)$ and $\gamma'(0) = (u, w)$ with v, v' and w parallel vectors in $\mathbb{R}^{1,d}$, then the same condition holds throughout the definition of γ*

As a corollary, we have that every timelike line in Minkowski space is locally the projection of some geodesic of $(\mathcal{M}, \hat{G})$.

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