HOMEWORK PROBLEMS, MATH 431-536

D. VASSILEV, FALL 2013

The Due Homework Problems are due at the beginning of class the Monday following the week of assignment. Please check again the homework problems after class as advanced postings of homework could change.

1. Homework 1

Problem 1.1. Let $U$, $V$, $W$, and $U_\alpha$, $\alpha \in A$-some index set, be subsets of some set $X$. Prove the following identities:

a) $(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$;   

b) $(U \cap V) \setminus W = (U \setminus W) \cap (V \setminus W)$;   

c) Does $U \setminus (V \setminus W) = (U \setminus V) \setminus W$?   

d) $X \cap \alpha \in A U = \cup \alpha \in A (X \setminus U_\alpha)$;

Problem 1.2. Describe the relation between the closed ball $C(x,r)$ and the closure $B(x,r)$ of the open ball $B(x,r)$. In particular, determine if $C(x,r) = B(x,r)$.

Due Problem 1.1. Show that two metrics on the set $X$ equivalent if and only if they define the same convergent sequences.

Problem 1.3. Show that every metric on a set $X$ is equivalent to a bounded metric, i.e., if $d$ is a metric on $X$, then there is a metric $\rho$ on $X$ equivalent to $d$ and such that $\rho(x,y) \leq 1$, $x,y \in X$.

Due Problem 1.2. Find all the limit points of the following subsets of the real line $\mathbb{R}$:

(a) $\{\frac{1}{n} \sin n \mid n \in \mathbb{N}\}$;

(b) $\{\frac{1}{m} + \frac{1}{n} \mid m,n \in \mathbb{N}\}$.

Problem 1.4. Let $B$ be the set of binary sequences $\{x = (x_1, x_2, \ldots, x_n, \text{dots}) \mid n \in \mathbb{N}, x_n = 0 \text{ or } 1\}$. For two points $x, y \in B$ define

$$d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}.$$  

a) Show that $d$ is a metric on $B$.

b) Let $F$ be the subset of $B$ containing all binary sequences which have only finitely many non-zero elements. Show that $F$ is dense in $B$.

Problem 1.5. Suppose $(X,d)$ is a complete metric space. Show that a subspace $E$ of $X$ is closed iff $E$ is a complete metric space.

Problem 1.6. Prove that any countable union of sets of the first category in $X$ is again of the first category in $X$. Here $X$ is a complete metric space.

Problem 1.7. Show that if $f$ is an infinitely many times differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for every $x \in \mathbb{R}$ there some derivative $f^{(n_x)}$ vanishing at $x$, $f^{(n_x)}(x) = 0$, then $f$ is a polynomial function on some interval.

Remark: Actually $f$ is a polynomial on $\mathbb{R}$, but this requires another use of Baire’s theorem.

Date: November 25, 2013.
2. Homework 2

Problem 2.1. Show that a subset $A$ of the Euclidean space $\mathbb{R}^n$ is totally bounded iff $A$ is bounded.

Problem 2.2. Consider $M_2$-the metric space of all $2 \times 2$ matrices equipped with the Euclidean metric, i.e.,
\[
d(A, B) = \left( \sum_{i=1}^{4} (a_i - b_i)^2 \right)^{1/2}, \quad A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.
\]
a) Determine if $\text{SL}(2)$ - the set of all matrices in $M_2$ of determinant equal to one is a compact.
b) Determine if $\text{O}(2)$ - the set of all orthogonal matrices in $M_2$ is a compact.

Problem 2.3. Let $E$ be the subspace of $\mathbb{R}^2$ obtained from the circle centered at $(0, 1/2)$ of radius $1/2$ by deleting the point $(0, 1)$. Define a function $h$ from $\mathbb{R}$ to $E$ so that $h(s)$ is the point at which the line segment from $(s, 0)$ to $(0, 1)$ meets $E$.

(a) Show that $h$ is a homeomorphism from $\mathbb{R}$ to $E$.
(b) Show that $\rho(s, t) = ||h(s) - h(t)||$, $s, t \in \mathbb{R}$ defines a metric on $\mathbb{R}$ which is equivalent to the usual metric on $\mathbb{R}$. Here, $||h(s) - h(t)||$ is the Euclidean distance.
(c) Show that $(\mathbb{R}, \rho)$ is totally bounded, but not complete.
(d) Identify the completion of $(\mathbb{R}, \rho)$.

Problem 2.4. Show that the space of continuous real-valued functions on the unit interval $[0, 1]$ considered with the $L^1$ distance $d(f, g) = \int_0^1 |f(x) - g(x)| \, dx$ is not a complete metric space. Note: the $L^1$ distance is determined by the norm $||f||_1 = \int_0^1 |f(x)| dx$.

Due Problem 2.1. Let $(X, d)$ be a metric space and $S$ a set. Show that $\mathcal{B}(S : X)$ is a complete metric space iff $X$ is complete. Here, $\mathcal{B}(S : X)$ is the space of bounded functions from $S$ to $X$ with the metric defined by $\rho(f, g) = \sup_{s \in S} d(f(s), g(s))$.

Due Problem 2.2. Let $X$ and $Y$ be metric spaces such that $X$ is complete. Show that if $\{f_\alpha\}_{\alpha \in A}$ is a family of continuous functions from $X$ to $Y$ such that for each $x \in X$ the set $\{f_\alpha(x) : \alpha \in A\}$ is a bounded subset of $Y$, then there exists a nonempty open subset $U$ of $X$ such that $\{f_\alpha\}_{\alpha \in A}$ is uniformly bounded on $U$, i.e., the set $\{f_\alpha(x) : \alpha \in A, x \in U\}$ is a bounded subset of $Y$.

Problem 2.5. Let $f$ be real valued function on $\mathbb{R}$. Show that there exist $M > 0$ and a nonempty open subset $U$ of $\mathbb{R}$ such that for any $s \in U$, there is a sequence $\{s_n\}$ satisfying $s_n \to s$ and $|f(s_n)| \leq M$ for all $n \in \mathbb{N}$.

Problem 2.6. Prove that a continuous real-valued function on a compact metric space achieves its maximum and minimum values.

Problem 2.7. Prove that a metric space is compact iff every continuous real-valued function on $X$ is bounded.

Problem 2.8. Suppose that $F$ is a function from a non empty compact metric space $(K, d)$ to itself such that $d(F(x), F(x')) < d(x, x')$. Show that for any $\epsilon > 0$, there is a constant $c = c(\epsilon) < 1$ such that $d(F(x), F(x')) \leq cd(x, x')$ for all $x, y \in K$ satisfying $d(x, x') \geq \epsilon$. 
Due Problem 3.1. Let \( \{U_\alpha \mid \alpha \in A \} \) be a finite open cover of a compact metric space \((X,d)\).

a) Show that there exists \( \epsilon > 0 \) such that for every \( x \in X \) the ball \( B(x,\epsilon) \) is contained in some \( U_\alpha \). Such an \( \epsilon \) is called a Lebesgue number of the cover.

b) Show that if all of the \( U_\alpha \)'s are proper subsets of \( X \), then there is a largest Lebesgue number for this cover.

Problem 3.1. Let \((X,d)\) be a metric space. Show that the metric \( d : X \times X \to \mathbb{R} \) is a continuous function on \( X \times X \). Here we regard \( X \times X \) with the product metric \( \rho(\xi,\xi') = \max\{d(x,y),d(x',y')\} \) where \( \xi = (x,y) \), \( \xi' = (x',y') \in X \times X \).

Problem 3.2. Let \( X_0, X_1 \) and \( X_2 \) be metric spaces. Show that if \( f : X_0 \to X_1 \) and \( g : X_1 \to X_2 \) are continuous functions, then their composition \( h = g \circ f : X_0 \to X_2 \) is a continuous function.

Problem 3.3. Let \((X,d)\) be a metric space and \( A \) a subset of \( X \).

a) Show that the function \( d_A(x) = \inf\{d(x,y) \mid y \in A\} \) is a continuous function.

b) Show that if \( F \) is a closed subset of \( X \) then \( d_F(x) = 0 \) iff \( x \in F \).

c) Give an example where \( d_A(x) = 0 \) but \( x \not\in A \).

Problem 3.4. Show that if \( F : X \to X \) is mapping on the compact metric space \((X,d)\) such that

\[ d(F(x),F(y)) < d(x,y), \quad x \neq y, \]

then \( F \) has a unique fixed point.

Due Problem 3.2. Let \( E^n \) be a linear space of dimension \( n \). Show that all norms on \( E^n \) are equivalent, i.e., if \( \|\cdot\|_1 \) and \( \|\cdot\|_2 \) are two norms on \( E^n \) then there are constants \( m \) and \( M \) such that

\[ m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 \]

for all \( x \in E^n \). You can do the proof for \( \mathbb{R}^n \) if you are more comfortable working in the Euclidean space. Hint: Use that the closed unit balls are compact and the norms are continuous functions, hence you get boundedness there. Then use dilations.
4. Homework 4

Problem 4.1. Let \( X \neq \emptyset \) be a set and \( p \in X \). Show that the family of all subsets of \( X \) containing \( p \) together with the empty set defines a topology on \( X \), called the particular point topology. Determine \( \text{int}(A) \) and \( \overline{A} \) for subsets of \( X \) containing or not containing \( p \) in the particular point topology at \( p \).

Problem 4.2. (a) Show that the family of all subsets of \( \mathbb{R} \) of the type \((a, \infty)\) together with the empty set and \( \mathbb{R} \) is a topology on \( \mathbb{R} \), called the half-line topology.

(b) Determine the closure of the \((0,1)\) in the half-line topology.

Problem 4.3. Let \( S \) be a subset of a topological space \( X \). Show that

(a) \( \overline{X \setminus S} = X \setminus \text{int}(S) \);

(b) \( \text{int}(X \setminus S) = X \setminus \overline{S} \).

Due Problem 4.1. Let \( S \) be a subset of a topological space \( X \) in which singletons (sets of one point) are closed sets. A point \( x \in X \) is a limit point of \( S \) if every neighborhood of \( x \) contains a point of \( S \) other than \( x \) itself. A point \( x \in X \) is an isolated point of \( S \) if there is a neighborhood \( U \) of \( x \) such that \( U \cap S = \{x\} \).

(a) Show that the set of limit points of \( S \) is closed.

(b) Show that \( \overline{S} \) is the disjoint union of the set of limit points of \( S \) and the isolated points of \( S \).

Problem 4.4. Show that the intersection \( \mathfrak{F} \) of a family of topologies \( \{\mathfrak{F}_\alpha\}, \alpha \in A \), for a set \( X \) is a topology for \( X \), i.e., \( \mathfrak{F} = \{U \subset X : U \in \mathfrak{F}_\alpha \text{ for all } \alpha \in A\} \) defines a topology on \( X \). This is the largest topology contained in each of the given topologies, i.e., if \( U \in \mathfrak{F} \) then \( U \in \mathfrak{F}_\alpha \) for all \( \alpha \in A \), and \( \mathfrak{F} \) is the largest topology with this property.

Problem 4.5. Let \( \{\mathfrak{F}_\alpha\}, \alpha \in A \), be a family of topologies on a set \( X \). Show that there is a unique (smallest) topology \( \mathfrak{F} \) containing each of the given topologies, i.e., if \( U \in \mathfrak{F}_\alpha \) for some \( \alpha \) then \( U \in \mathfrak{F} \), and \( \mathfrak{F} \) is the smallest topology with this property.

Problem 4.6. Show that if we consider \( \mathbb{R} \) with the co-finite topology, then every sequence \( \{x_n\} \) of different real numbers is convergent and to every real number. In particular, limits are not unique.

Due Problem 4.2. Let \( \mathfrak{T} \) be the smallest topology on \( \mathbb{R} \) which contains all sets of the type \([a,b)\), \( a,b \in \mathbb{R} \), i.e., \( \mathfrak{T} \) is the intersection of all topologies on \( \mathbb{R} \) in which all intervals \([a,b)\) are open (for example the discrete topology is such). Determine if the set \([0,1)\) is closed in this topology, called sometimes the lower limit topology.

Problem 4.7. Prove that a set \( S \) in a topological space \( X \) is open iff every relatively open subset of \( S \) is open in \( X \). Is this statement true if ”open” is replaced by ”closed”?

Problem 4.8. Let \( A \) and \( B \) be two subspaces of the topological space \( X \). Prove that the closure of \( A \cap B \) in the relative topology of \( B \) is a subset of \( \overline{A} \cap B \). Give an example where this inclusion is proper. i.e., show that \( \overline{A \cap B} \subseteq \overline{A} \cap B \) and there are examples when \( \overline{A \cap B} \neq \overline{A} \cap B \).

Problem 4.9. (universal property of the subspace topology) Let \( Y \) be a topological space, \( X \) a subset of \( Y \) and \( i \) the inclusion map \( i(x) = x, x \in X \). Show that the subspace topology is the unique topology such that: (i) \( i \) is continuous (NB: injective trivially) and (ii) for any topological space \( Z \) and any map \( f : Z \to X \) we have that \( f \) is a continuous map iff \( i \circ f \) is a continuous map.

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & X \\
\downarrow i & & \downarrow \alpha \\
Y & \xrightarrow{i \circ f} & Y
\end{array}
\]

Problem 4.10. Let \( X \) be a set and \( \{Y_\alpha : \alpha \in A\} \) a family of topological spaces. Suppose \( \{f_\alpha : \alpha \in A\} \) be a family of maps \( f_\alpha : X \to Y_\alpha \). Show that there is a unique topology on \( X \) which makes all the maps continuous and is the smallest topology with this property.
5. Homework 5

Problem 5.1. Let \( X \) be a topological space and \( A \) and \( B \) two subsets of \( X \), \( A \subset B \subset X \). Show that the subspace topology (=relative topology) of \( A \) as a subset of \( X \) coincides with the subspace topology of \( A \) as a subset of \( B \).

Problem 5.2. Determine if the following statement is True or False? The relative topology on \( \mathbb{Q} \) induced from the standard topology on \( \mathbb{R} \) coincides with the discrete topology on \( \mathbb{Q} \).

Problem 5.3. (a) Prove that if the topological space \( X \) does not have the discrete topology, then there is a function from \( X \) to a topological space \( Y \) which is not continuous.

(b) Prove that if the topological space \( Y \) does not have the indiscrete topology, then there is a function from a topological space \( X \) to \( Y \) which is not continuous.

Problem 5.4. Let \( X \) be a set and \( S \) a family of subsets of \( X \).

(a) Show that there exists unique topology \( T \), such that \( S \subset T \) (i.e. all of the sets of \( S \) are open in \( T \)) and \( T \) is smaller (coarser) than any other topology containing \( S \). This topology is called the topology generated by \( S \).

(b) Let \( B \) be the family of subsets of \( X \) consisting of \( X, \emptyset \), and all finite intersections of sets in \( S \). Show that \( B \) is a base of a topology of \( X \).

Problem 5.5. Prove that the family of all arithmetic progressions in \( \mathbb{Z} \) is a basis for a topology on \( \mathbb{Z} \). In other words, show that \( B = \{\{..., m - 2n, m - n, m, m + n, m + 2n, ...\} : m, n \in \mathbb{Z} \} \) is a subset of \( \mathbb{Z} \), so \( B \) is a family of subsets of \( \mathbb{Z} \).

Due Problem 5.1. (a) Show that the family of all sets of the type \([a, b)\), \(a, b \in \mathbb{R}\) is a base of the lower limit topology (also called half-open interval topology) considered in Problem 4.2.

(b) Show that a function \( f \) on the real line is continuous from the real line with the lower limit topology to the real line with the standard topology iff \( f \) is continuous from the right, that is, for all \( x \in \mathbb{R} \) we have \( f(x) = \lim_{\epsilon \to 0^+} f(x + \epsilon) \).

Due Problem 5.2. Let \( p : X \to Y \) be a quotient map (hence surjective by definition).

(a) Show that \( p \) defines an equivalence relation \( \sim \) on \( X \) by defining \( x \sim x' \) iff \( p(x) = p(x') \).

(b) Let \( X/\sim \) be the quotient set defined by \( \sim \) (or \( p \)), i.e., it is the set of equivalent classes \( \{f^{-1}(y): y \in Y\} \) and \( \pi : X \to X/\sim \) the quotient map assigning to each point the set of all points equivalent to it. Show that \( Y \) and \( X/\sim \) are homeomorphic when equipped with the quotient topologies.

Problem 5.7. Let \( X \) be a topological space with the cofinite topology.

(a) Show that \( X \) is separable.

(b) When is \( X \) second-countable?

Problem 5.8. Let \( i : X \to Y \) be an embedding. Show that if \( Y \) has any of the following properties then the same is true for \( X \). a) \( T_1 \). b) \( T_2 \). c) Regular.
Problem 6.1. Let \( p : X \to Y \) be a quotient map.

a) Show that \( Y \) is a \( T_1 \) space iff each equivalent class \( \{ f^{-1}(y) : y \in Y \} \) is a closed subset.

b) Give an example when the \( T_1 \) property is not preserved under a quotient map.

c) Give an example when the \( T_2 \) property is not preserved but the \( T_1 \) property is preserved under a quotient map. For this, you can consider \( X = [0, 1] \times [0, 1] \) and the equivalence relation \((x, y) \sim (x', y')\) iff \( y = y' > 0\).

Problem 6.2. Show that a sequence in a Hausdorff space cannot converge to more than one point.

Problem 6.3. Let \( F \) be a closed set in a normal topological space \( X \). Show that every continuous function \( f \in C(F : \mathbb{R}) \) can be extended to a real-valued continuous function defined on the whole space \( X \). Note: Notice that here we do not assume that \( f \) is bounded on \( F \).

**Hint:** First compose \( f \) with a homeomorphism between the real line and \((-1, 1)\), and then extend this composition to a map \( \tilde{f} \) from \( X \) to \([-1, 1]\). Explain how to handle the places where this composition takes the values \( \pm 1 \) by using Urysohn's lemma and the sets \( f^{-1}(\{0, 1\}) \) and \( F \).

Due Problem 6.1. Let \( K \) be a compact Hausdorff topological space and \( \{ U_\alpha \}_{\alpha \in A} \) an open cover of \( K \). Show that there is a partition of unity subordinate (to the cover \( \{ U_\alpha \}_{\alpha \in A} \)). In other words, show that there exist a finite number of continuous real-valued functions \( h_1, h_2, \ldots, h_m \) such that

1. \( 0 \leq h_j \leq 1, \ j = 1, \ldots, m \).
2. \( \sum_{j=1}^{m} h_j = 1 \).
3. For each \( j = 1, \ldots, m \), there is an open set from the given cover containing the support of \( h_j \), i.e., the set \( \text{supp} h_j = \{ x \in K : h_j(x) > 0 \} \).

**Hint:** Work with a finite sub-cover \( \{ U_\alpha \}_{j=1}^{m} \). First find a smaller cover \( \{ V_j \}_{j=1}^{m}, \ s.t., \ \overline{V}_j \subset U_j \). Then apply this again to get an even smaller cover \( \{ W_j \}_{j=1}^{m}, \ s.t., \ \overline{W}_j \subset V_j \). Use Urysohn’s lemma for each pair of disjoint sets \( \overline{W}_j \) and \( X \setminus V_j \) to get a function \( f_j \). Finally, use \( f_j/(\sum f_j) \).

Problem 6.4. Show that every regular second countable space is normal. **Hint:** Very similarly to the metric case, but using the regularity, you can construct open covers of the given disjoint closed sets such that the closure of each of the sets in the covers does not intersect the other closed set. Use the second countability property to extract countable covers \( \{ U_n \}_{n \in \mathbb{N}} \) and \( \{ V_n \}_{n \in \mathbb{N}} \). The sets \( U'_n = U_n \setminus \bigcup_{j=1}^{n} V_j \) and similarly defined \( V'_n \) are helpful.

Problem 6.5. Let \( K \) be a compact and \( F \subset C(K : \mathbb{R}) \). Show that if \( F \) is \( (i) \) (uniformly) equicontinuous, i.e., for any \( \epsilon > 0 \) and \( x_0 \in K \) there is an open neighborhood \( U \) of \( x_0 \) such that \( |f(x) - f(x_0)| < \epsilon \) for all \( f \in F, x \in U \); and \( (ii) \) point-wise bounded, i.e., for every \( x \in K \) we have \( \sup_{f \in F} |f(x)| < \infty \), then \( F \) is (uniformly) bounded: \( \sup_{f \in F, x \in K} |f(x)| < \infty \).

Due Problem 6.2. Prove that a Hausdorff topological space \( X \) is locally compact iff for every \( x \in X \) and open neighborhood \( U \) of \( x \), there is an open neighborhood \( V \) of \( x \) such that \( \overline{V} \subseteq U \) and \( \overline{V} \) is compact.

Problem 6.6. Prove that local compactness is a topological property.

Problem 6.7. Prove that a locally compact Hausdorff space is regular.
Problem 7.1. Prove that the product of regular spaces is a regular space.

Due Problem 7.1. Prove that a product of two normal space is not necessarily a normal space. For this, show that the product of two real lines each considered with the half-open topology (see homework problems 4.2 and 5.1) is a topological space \((\mathbb{R}^2, \mathcal{P})\) which is not a normal space. For this you can do the following steps.

(a) Prove that the bisect of the 2nd and 4th quadrants is a closed subset of \((\mathbb{R}^2, \mathcal{P})\).
(b) Take a sequence \(S\) in \(\mathbb{R}^2\) which is dense in \(\mathbb{R}^2\) when considered with the standard topology. Show that the sets \(E = \{(x, -x) : x \in S\}\) and \(F = \{(x, -x) : x \in \mathbb{R} \setminus S\}\) are disjoint closed sets of \((\mathbb{R}^2, \mathcal{P})\).
(c) Using the Baire category theorem applied to \(\mathbb{R}\), show that if \(V\) is open set in \((\mathbb{R}^2, \mathcal{P})\) such that \(F \subset V\), then there exist \(\epsilon > 0\), \(b \in S\), an a sequence \(\{x_n\}\) in \(\mathbb{R} \setminus S\) such that (i) \(|x_n - b| \to 0\) and (ii) \([x_n, x_n + \epsilon] \times [x_n, x_n + \epsilon] \subset V\) for all \(n\).
(d) Prove that there are no disjoint open neighborhoods of \(E\) and \(F\) in \((\mathbb{R}^2, \mathcal{P})\).

Problem 7.2. Suppose \(X = X_1 \times \cdots \times X_n\), where each \(X_j\) is a non-empty topological space. Show the following assertions.

(a) If \(X\) is Hausdorff then each \(X_j\) is Hausdorff. (b) If \(X\) is regular then each \(X_j\) is regular. (c) If \(X\) is normal then each \(X_j\) is normal. (d) If \(X\) is compact then each \(X_j\) is compact.

Problem 7.3. Prove that \(\prod X_\alpha\) need not be compact in the box-topology when each \(X_\alpha\) is compact. Hint: Use the discrete topology on a two point space for each \(\alpha\).

Problem 7.4. Let \(f_\alpha \in C(X_\alpha : Y_\alpha), \alpha \in A\). Show that \(f = \prod f_\alpha \in C(X_\alpha : Y_\alpha)\), where for \(x \in \prod_{\alpha \in A} X_\alpha\) we let \(f(x)(\alpha) = f_\alpha(x(\alpha))\).

Problem 7.5. For \(n \geq 1\), define \(P^n = S^n/\sim\), where \(\sim\) is the equivalence relation defined on \(S^n\) by \(x \sim y\) iff \(x = y\) or \(x = -y\) (i.e. identify antipodal points). Show that

(a) \(P^n\) is a compact Hausdorff space; (b) the projection \(\pi : S^n \to P^n\) is a local homeomorphism, that is, each point \(x \in S^n\) has an open neighborhood that is mapped homeomorphically by \(\pi\) onto an open neighborhood of \(\pi(x)\); (c) \(P^1\) is homeomorphic to the circle \(S^1\).

Due Problem 7.2. Let \(X\) be the set of all points \((x, y) \in \mathbb{R}^2\) such that \(y = \pm 1\), and let \(M\) be the quotient of \(X\) by the equivalence relation generated by \((x, -1) \sim (x, 1)\) for all \(x \neq 0\). Show that \(M\) is locally Euclidean (i.e. every point has an open neighborhood homeomorphic to an open set in a Euclidean space) and second countable, but not Hausdorff. [This space is called the line with two origins.]
8. Homework 8

Due Problem 8.1. A point $x$ of a topological space $X$ is called a cut point if $X \setminus \{x\}$ is disconnected.

(a) Show that the property of having a cut point is a topological property.

(b) Can two homeomorphic spaces have different number (finite) of cut points? Explain.

Due Problem 8.2. A topological space $X$ is called locally connected if, for each point $x \in X$ and open neighborhood $U$ of $x$, there is a connected open neighborhood $V$ of $x$ contained in $U$. Show that a connected component of a locally connected topological space is open. (Note: recall, each connected component is closed.)

Problem 8.1. a) Show that a connected open set in $\mathbb{R}$ is an open interval.

b) Show that a connected set in $\mathbb{R}$ is an interval, i.e., a set of the type $(a, b)$, $[a, b]$, $[a, b)$ or $(a, b]$ for some $-\infty \leq a \leq b \leq +\infty$. 
9. Homework 9

Due Problem 9.1. (a) Prove that the path components of a locally path-connected space coincide with the connected components. In particular, an open set of $\mathbb{R}^n$ is connected iff it is path-connected.

(b) Prove that if $X$ is locally connected and locally path-connected, then every path component of $X$ is open and closed.

(c) Give an example showing that the path components of a topological space are not necessarily closed, nor are they necessarily open.

Problem 9.1. Suppose $X = X_1 \times \cdots \times X_n$, where each $X_j$ is a non-empty topological space. Show the following assertions.

(a) If $X$ is connected then each $X_j$ is connected.  
(b) If $X$ is path-connected then each $X_j$ is path-connected.

Problem 9.2. Prove that the connected components of a finite number of topological spaces are the sets of the type $E_1 \times \cdots \times E_n$, where each $E_j$ is a connected component of $X_j$, $j = 1, \ldots, n$. Note: A similar result holds for path connected components.

Problem 9.3. Regard the unit sphere $S^{2n+1}$ as the set of all point $z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ such that $|z_0|^2 + \cdots + |z_n|^2 = 1$.

(a) Show that for the obvious projection $\pi : S^{2n+1} \to \mathbb{C}P^n$ we have $\pi^{-1}([z])$ is homeomorphic to $S^1$ for all $x \in \mathbb{C}P^n$.

(b) Show that each $x \in \mathbb{C}P^n$ has an open neighborhood $U$ such that $\pi^{-1}(U)$ is homeomorphic to the product space $U \times S^1$.

Problem 9.4. Let $X = [0, 2\pi] \times [0, 1] \subset \mathbb{R}^2$ and $Y = \{(x, y, z) \mid x^2 + y^2 = 1, 0 \leq z \leq 1\} \subset \mathbb{R}^3$. Consider the map

$$f(\phi, t) = (\cos \phi, \sin \phi, t).$$

Show that $X/\sim$ is homeomorphic to $Y$, where $\sim$ is the equivalence relation on $X$ defined by $f$, i.e., $x \sim x'$ iff $f(x) = f(x')$.

Problem 9.5. Let $X$ be a topological space and $p : X \to Y$ a quotient map. Prove the following assertions:

(a) If $X$ is compact, then $Y$ is compact.

(b) If $X$ is connected, then $Y$ is connected.

(c) If $X$ is path-connected, then $Y$ is path-connected.

Due Problem 9.2. Suppose $Y \subset X$ and $\sim$ is an equivalence relation on $X$ (hence also on $Y$). Show that if $Y$ is open and the quotient map $\pi : X \to X/\sim$ is an open map, then $Y/\sim$ is homeomorphic to $\pi(Y)$, where $\pi(Y)$ is equipped with the relative topology of $Y/\sim$. 
10. Homework 10

Problem 10.1. Suppose a group $G$ acts on the topological space $X$. Determine the quotient map $\pi : X \to X/G$ is an open map.

Problem 10.2. Determine if the map $\pi : Z \to Z/|Z|$ of $\mathbb{C}^{n+1}_x$ to $S^{2n+1}$ is open or closed.

Problem 10.3. Show that the map $p : \mathbb{C}_x \times \mathbb{C}^n \to \mathbb{C}^n$ defined by $p(z_0, z_1, \ldots, z_n) = (z_1/z_0, \ldots, z_n/z_0)$ is open.

Due Problem 10.1. Let $G_n$ be the group of $n$-th roots of unity. Regard the unit sphere $S^{2n+1}$ as the set of all point $z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ such that $|z_0|^2 + \cdots + |z_n|^2 = 1$. Show that the action of $G_n$ on $S^{2n+1}$ defined by $(z_0, \ldots, z_n) \mapsto (\zeta z_0, \ldots, \zeta z_n)$ for $\zeta \in G_n$ is an even action, i.e., every point $z \in S^{2n+1}$ has a neighborhood $U$ such that $U \cap \zeta U = \emptyset$ for every $\zeta \in G_n$, $\zeta \neq 1$.

Problem 10.4. For a quaternion $q = t + ix + jy + kz \in \mathbb{H}$ let $\bar{q} = t - ix - jy - kz$ be the conjugate quaternion, $|q| = \sqrt{t^2 + x^2 + y^2 + z^2}$ the norm of $q$, and $q \cdot q$ be the quaternion product of two quaternions $q, q' \in \mathbb{H}$ (the latter is defined with the help of $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i, ki = -ik = j$).

a) Show that conjugation is an involutive anti-automorphism of $\mathbb{H}$, i.e., it is $\mathbb{R}$-linear, $\bar{\bar{q}} = q$, and $q \cdot \bar{q} = \bar{q} \cdot q$ for a non-zero quaternion $q$, where as in class $|q| = \sqrt{q\bar{q}}$.

b) Show that $q^{-1} = \frac{\bar{q}}{|q|^2}$ for a non-zero quaternion $q$, where as in class $|q| = \sqrt{q\bar{q}}$.

Problem 10.5. a) Write the matrix version of the "left" and "right" unit quaternion actions $L_u(q) = uq$ and $R_u(q) = qu$ where $u \in Sp(1) = \{q = t + ix + jy + kz \mid t^2 + x^2 + y^2 + z^2 = 1\}$ and $q \in \mathbb{H}$, i.e., find 4 by 4 matrices $A$ and $B$ representing the linear transformation $L_u$ and $R_u$ of $\mathbb{R}^4$.

b) Do the same for the restrictions of $L_u$ and $R_u$ on $Im(\mathbb{H}) \cong \mathbb{R}^3$.

Problem 10.6. Show that $Sp(1) \times Sp(1)$ is a double cover of $SO(4)$, i.e., there is a covering map which is a group homomorphism with the "right" kernel, which you will determine. NOTE: You can use that any rotation of $\mathbb{R}^4 = \mathbb{H}$ is of the form $q \mapsto u_1 q u_2$ for some $u_1, u_2 \in Sp(1)$.

Problem 10.7. Show that $Sp(1)$ is isomorphic to the special unitary group $SU(2)$ using the following identification

$$q = t + ix + jy + kz \in Sp(1) \quad \mapsto \quad t \sigma_0 - x\sigma_1 - y\sigma_2 - z\sigma_3 \in SU(2),$$

where $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

In other words we let $1 \leftrightarrow \sigma_0, i \leftrightarrow \sigma_1 = -i\sigma_3, j \leftrightarrow \sigma_2 = -i\sigma_2$ and $k \leftrightarrow \sigma_3 = -i\sigma_3$ and then extend naturally to all quaternions.

Problem 10.8. Let $G = \langle \tau \rangle$ be the group generated by the translation of the plane $\mathbb{R}^2 = \mathbb{C}$, $\tau(z) = z - 1$. Show that this action is even. Note: The quotient space $X/G$ is a Möbius band.

Due Problem 10.2. Let $G$ be a group acting on the topological space $X$. Suppose that every $x, x' \in X$ that are not in the same orbit of the $G$–action have open neighborhoods $U$ and $U'$ such that $g(U) \cap U' = \emptyset$ for all $g \in G$. a) Show that $X/G$ is a Hausdorff space. b) Is the quotient map $X \to X/G$ a covering map?
11. Homework 11

Problem 11.1. Show that if $X$ is connected and locally connected, and $\pi : \tilde{X} \to X$ is a covering map, then all fibers of $\pi$ have the same cardinality, i.e., there is a bijection between any two of them.

Due Problem 11.1. Let $\tilde{X}$ be a connected space and $\pi : \tilde{X} \to X$ a covering map. Show that the group of covering transformations $G = \text{Aut}(\tilde{X}/X)$ acts evenly on $\tilde{X}$.

Problem 11.2. Let $\tilde{X}$ be a connected space, $X$ a locally connected space and $\pi : \tilde{X} \to X$ a covering map. Show that if the group of covering transformations $G = \text{Aut}(\tilde{X}/X)$ acts transitively on a fiber (hence on all), then $X = \tilde{X}/G$. Hint: Factor $\pi$ through $\tilde{X}/G$, i.e., consider

\[
\begin{array}{ccc}
\tilde{X} & \overset{\pi}{\longrightarrow} & X \\
\downarrow & & \downarrow \\
\tilde{X}/G
\end{array}
\]

and show that $\pi'$ is a covering map (use that $X$ is locally connected), which with the additional assumption of transitivity becomes a homeomorphism.

Due Problem 11.2. a) Show that the group $\mathbb{Z}^n$ (group w.r.t addition of vectors) acts evenly on $\mathbb{R}^n$ by translations $\phi_{(m_1, m_2, \ldots, m_n)}(x_1, \ldots, x_n) = (x_1 + m_1, \ldots, x_n + m_n)$ for $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$ and $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

b) Show that orbit space $\mathbb{R}^n/\mathbb{Z}^n$ is homeomorphic to the $n$-dimensional torus $T^n$ defined as the product of $n$ circles, $T^n = S^1 \times \cdots \times S^1$.

Problem 11.3. Show that $p : S^{n+1} \to \mathbb{R}P^n$ is a principle $\mathbb{Z}_2$-bundle.

Problem 11.4. Show that $z \mapsto e^z$ is a covering map $\exp : \mathbb{C} \to \mathbb{C}_\times$.

Problem 11.5. a) Show that $z \mapsto z^n$ is a covering map $p : \mathbb{C}_\times \to \mathbb{C}_\times$.

b) Is $z \mapsto z^n$ a covering map $\mathbb{C} \to \mathbb{C}$?
Problem 12.1. Consider \( \mathbb{CP}^n \) with the notation from class. Show that for every fixed \( \lambda \in S^1 \), the map \( \sigma_j : U_j \to W_j \), \( \sigma_j(Z) = \lambda |z_j| Z \), is a section of the (trivial) bundle \( \pi_j : W_j \to U_j \).

Due Problem 12.1. Show that a \( G \)-covering \( p : Y \to Y/G = X \) has a section iff it is a trivial \( G \)-covering, i.e., it is isomorphic to \( X \times G \to X \).

Problem 12.2. Show that a covering space \( \pi : Y \to X \) is the same as a locally trivial fiber bundle with a discrete fibre.

Problem 12.3. Show that the Hopf map \( H : S^3 \subset \mathbb{C}^2 \to S^2 \) is indeed a surjection of \( S^3 \) onto \( S^2 \). As defined in class, \( H(z_0, z_1) = (2z_1 \bar{z}_0, |z_0|^2 - |z_1|^2) \in \mathbb{C} \times \mathbb{R} \).

Problem 12.4. Suppose that \( (\alpha \beta) \gamma = \alpha (\beta \gamma) \) for any three paths \( \alpha, \beta \) and \( \gamma \) in \( X \) for which the product is defined. Show that each path component of \( X \) consists of a single point.

Problem 12.5. Let \( X \) be path-connected and let \( b \in X \). Show that every path in \( X \) is homotopic with endpoints fixed to a path passing through \( b \).

Problem 12.6. Let \( D \) be an open subset in \( \mathbb{R}^n \), let \( \alpha \) be a path in \( D \) from \( x \) to \( y \), and set

\[
d = \inf \{ |\alpha(s) - w| : w \in \partial D, 0 \leq s \leq 1 \}.
\]

Show that if \( \beta \) is any path in \( D \) from \( x \) to \( y \) such that \( |\alpha(s) - \beta(s)| < d, 0 \leq s \leq 1 \), then \( \beta \) is homotopic to \( \alpha \) with endpoints fixed.

Due Problem 12.2. Show that every path in an open set of \( \mathbb{R}^n \) is homotopic with endpoints fixed to a polygonal path. A polygonal path is a path of finitely many segments, i.e., \( \alpha \) is polygonal if there is a subdivision \( 0 = s_0 < s_1 \cdots < s_m = 1 \) of \( [0,1] \) such that \( \alpha \) maps each \( [s_j, s_{j+1}] \) onto the segment connecting \( \alpha(s_j) \) and \( \alpha(s_{j+1}) \). Hint: Justify carefully why there is a finite subdivision of \( [0,1] \) such that on each of the intervals of the subdivision the path is contained inside a ball of radius \( d \) centered at the beginning for a suitable \( d > 0 \). Uniform continuity might be of some help.

Problem 12.7. Let \( (X,d) \) be a compact metric space and let \( a, b \in X \). Let \( \mathcal{P} \) be the set of all paths in \( X \) from \( a \) to \( b \) with the metric

\[
\rho(\alpha, \beta) = \sup \{ d(\alpha(s), \beta(s)) : 0 \leq s \leq 1 \}.
\]

Show that two paths \( \alpha, \beta \in \mathcal{P} \) are homotopic with endpoints fixed if and only if \( \alpha \) and \( \beta \) lie in the same path component of \( \mathcal{P} \).

Problem 12.8. Show that simple connectivity is a topological property.

Problem 12.9. a) Prove that if \( n \geq 2 \) then the \( n \)-dimensional sphere \( S^n \) is simply connected, in particular, the fundamental group is trivial. Hint: Reduce to the case of a loop that misses a point on the sphere by showing that every loop is homotopic to a loop formed by finitely many arcs lying on great circles. For a loop that misses a point you can use a stereographic projection from that point to get a loop on \( \mathbb{R}^n \).

b) Prove that if \( n \geq 3 \) then \( \mathbb{R}^n \setminus \{0\} \) is simply connected.
13. Homework 13

Problem 13.1. Let $X$ be the comb space, that is, the compact subset of $\mathbb{R}^2$ consisting of the horizontal interval $\{(x,0) : 0 \leq x \leq 1\}$ and the closed vertical intervals of unit length with lower endpoints at $(0,0)$ and at $(0,1/n)$, $n \in \mathbb{N}$.

a) Show that $X$ is contractible to $(0,0)$ with $(0,0)$ held fixed.

b) Show that $X$ is not contractible to $(0,1)$ with $(0,1)$ held fixed.

Problem 13.2. Let $X$ be a path-connected topological space and $b, c \in X$. Let $B^2$ be the closed unit disk in $\mathbb{R}^2$ with boundary circle $S^1$. Show that the following are equivalent

a) $X$ is simply connected.

b) Any two paths from $b$ to $c$ are homotopic with endpoints fixed.

c) Every continuous map $f : S^1 \to X$ extends to a continuous map $F : B^2 \to X$.

Problem 13.3. Show that the unit sphere $S^n$ in $\mathbb{R}^{n+1}$ is a retract of $\mathbb{R}^{n+1} \setminus \{0\}$.

Problem 13.4. Compute $\pi_1(\text{SO}(4))$.

Due Problem 13.1. a) Show that the fundamental group of a product of two spaces is the product of the fundamental groups of the spaces, $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

b) Find $\pi_1(T^n)$ of the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$.

Problem 13.5. Let $(\tilde{X}, \tilde{p})$ be a $G$-covering of $X$. a) Show that there is a natural homomorphism between $G$ and $\text{Aut}(\tilde{X}/X)$, which is furthermore injective. b) Show that if $\tilde{X}$ is connected, then $G$ and $\text{Aut}(\tilde{X}/X)$ are isomorphic groups.

Problem 13.6. Let $p : (\tilde{X}, \tilde{e}) \to (X, e)$ be a covering map and $\tilde{X}$ be a simply connected space. Suppose that $\tilde{X}$ and $X$ are groups with identities $\tilde{e}$ and $e$, respectively, and that $p$ is a homomorphism. Suppose finally that for each fixed $\tilde{x} \in \tilde{X}$ the group multiplication $\tilde{y} \mapsto \tilde{x} \ast \tilde{y}$ is continuous. Prove that the fiber $p^{-1}(e) = \ker p$ is a subgroup of $\tilde{X}$ and $\pi_1(X,e) \cong p^{-1}(e)$.

Problem 13.7. Show that if $p : (\tilde{X}, \tilde{a}) \to (X, a)$ is a covering map, then the induced homomorphism $p_* : \pi_1(\tilde{X}, \tilde{a}) \to \pi_1(X, a)$ is an injective map.

Problem 13.8. Let $p : \tilde{X} \to X$ be a covering map and $a \in X$.

a) Show that $\pi_1(X,a)$ acts on $p^{-1}(a)$ on the right by the monodromy action defined as follows: for $[\alpha] \in \pi_1(X,a)$ and $\tilde{a} \in p^{-1}(a)$

$$[\alpha](\tilde{a}) \overset{df}{=} \tilde{a}(1),$$

where $\tilde{a}$ is the (unique) lift of $\alpha$ starting at $\tilde{a}$.

b) Show that the monodromy action is transitive on the fiber $p^{-1}(a)$ when $\tilde{X}$ is path-connected.

Problem 13.9. Let $p : (\tilde{X}, \tilde{a}) \to (X, a)$ be a covering map. Show that the isotropy sub-group of $\tilde{a}$ under the monodromy action of $\pi_1(X,a)$ on $p^{-1}(a)$ is $p_* \left( \pi_1(\tilde{X}, \tilde{a}) \right)$ (a sub-group of $\pi_1(X,a)$).

Due Problem 13.2. Let $p : (\tilde{X}, \tilde{a}) \to (X, a)$ be a covering map and $\gamma$ a loop at $a$. Show that the (unique) lift $\tilde{\gamma}$ of $\gamma$ starting at $\tilde{a}$ is a loop at $\tilde{a}$ if and only if $[\gamma] \in p_* \left( \pi_1(\tilde{X}, \tilde{a}) \right)$. 
14. Homework 14

Problem 14.1. Show that a simply connected covering space of a locally path-connected space is unique up to a homeomorphism.

Due Problem 14.1. Let \( p : (\tilde{X}, \tilde{a}) \rightarrow (X, a) \) be a covering map. Let \( \gamma \) and \( \gamma' \) be two paths in \( X \) from \( a \) to \( a' \). Show that the (unique) lifts \( \tilde{\gamma} \) and \( \tilde{\gamma}' \) of \( \gamma \) and \( \gamma' \) starting at \( \tilde{a} \) have the same end point if and only if \( [\gamma' \wedge \gamma^{-1}] \in p_* \left( \pi_1(\tilde{X}, \tilde{a}) \right) \).

Problem 14.2. Let \( A \subset X \) and \( j : A \hookrightarrow X \) be the inclusion map. Suppose that \( A \) is a strong deformation retract of \( X \), i.e., \( A \) is a retract of \( X \) and there is a homotopy \( r_t \) between \( r_0 = \text{id}_X \) and \( r_1 = j \circ r \), where \( r : X \rightarrow A \) is the retract, with \( r_t(a) = a \) for all \( a \in A \). Show that \( j_* : \pi_1(A, a) \rightarrow \pi_1(X, a) \) is an isomorphism for any \( a \in A \). Note: The claim is true for a deformation retract as well, but is a little harder to prove when the points of \( A \) can move during the homotopy. The next problem is even a more general result since \( j \) is a homotopy equivalence.

Problem 14.3. * Let \( f : (X, x_0) \rightarrow (Y, y_0) \) be a homotopy equivalence, i.e., \( f \) is continuous and there is a continuous map \( g : Y \rightarrow X \) such that \( f \circ g \) is homotopic to \( \text{id}_X \) and \( g \circ f \) is homotopic to \( \text{id}_Y \). Show that \( f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0) \) is an isomorphism.

Problem 14.4. Show that the figure eight (two circles joined at a point) is a deformation retract of the two-torus \( T^2 \) with a point removed (hence the two spaces are homotopy equivalent). Hint: you don’t need to write formulas - try to see it; think of a slinky through a circle (which can be fattened) with the two ends of the slinky getting closer.

Problem 14.5. Find the fundamental group of \( S^1 \times S^2 \).

Problem 14.6. Find a compact six-folded cover of the torus \( T^2 = S^1 \times S^1 \).

Due Problem 14.2. Let \( f : \mathbb{R}P^1 \rightarrow \mathbb{R}P^1 \) be a continuous map and \( F : S^1 \rightarrow S^1 \) be a "cover" of \( f \) which is injective. Show that the degree of \( F \) is odd.

\[
\begin{array}{ccc}
S^1 & \xrightarrow{F} & S^1 \\
\downarrow{p} & & \downarrow{p} \\
\mathbb{R}P^1 & \xrightarrow{f} & \mathbb{R}P^1
\end{array}
\]

Problem 14.7. Compute the degree of the reflection map \( f : S^1 \rightarrow S^1 \), \( f(z) = -z \).

Problem 14.8. Show that a map \( f : S^1 \rightarrow S^1 \) of degree one is homotopic to the identity. Hint: the exponential map \( p : \mathbb{R} \rightarrow S^1 = \mathbb{R}/\mathbb{Z} \) has a lift \( \mathbb{R} \rightarrow \mathbb{R} \).
15. Homework 15

**Problem 15.1.** Using Seiferet - van Kampen theorem find $\pi_1(X)$ for $X$ the boundary of an $n$-leaf clover.

**Problem 15.2.** Find $\pi_1(T^2 \setminus a)$ for some $a \in T^2$, $T^2$ is the 2-dimensional torus.

**Problem 15.3.** Use the Seiferet - van Kampen theorem to obtain another proof that $\pi_1(S^n) = \{e\}$ for $n \geq 2$. Hint: Think of $S^2$. Consider for example the complements of the south and the north poles.