Prof. D. Vassilev Fall 2015

HOMEWORK PROBLEMS, MATH 431-535

The odd numbered homework are due the Monday following week at the beginning of class. Please check again the homework problems after class as advanced postings of homework could change.

1. Homework 1

(final version)

Problem 1.1. *Describe the relation (does one contain the other?) between the closed ball $C(x,r)$ and the closure $\overline{B(x,r)}$ of the open ball $B(x,r)$. In particular, determine if $C(x,r) = \overline{B(x,r)}$.*

Problem 1.2. *Let U, V, W and U_α, $\alpha \in A$-some index set, be subsets of some set X. Prove the following identities:*
a) $(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$;
b) $(U \cap V) \setminus W = (U \setminus W) \cap (V \setminus W)$;
c) Does $U \setminus (V \setminus W) = (U \setminus V) \setminus W$*?*
d) $X \setminus \cap_{\alpha \in A} U = \cup_{\alpha \in A}(X \setminus U_\alpha)$;

Problem 1.3. *Show that two metrics on a set X are equivalent if and only if they define the same convergent sequences. Recall: equivalent metrics are metrics which define the same open sets.*

Problem 1.4. *Show that every metric on a set X is equivalent to a bounded metric, i.e., if d is a metric on X, then there is a metric ρ on X equivalent to d and such that $\rho(x,y) \leq 1$, $x, y \in X$.*

Problem 1.5. *Find all the limit points of the following subsets of the real line $\mathbb{R}$(i.e. we consider them with the distance function "inherited" from $\mathbb{R}$):*
(a) $\left\{\frac{1}{n}\, sin n \mid n \in \mathbb{N}\right\}$;
(b) $\left\{\frac{1}{m} + \frac{1}{n} \mid m, n \in \mathbb{N}\right\}$.

Problem 1.6. *Let $\mathcal{B}$ be the set of binary sequences $\{x = (x_1, x_2, \dots, x_n, dots) \mid n \in \mathbb{N},\ x_n = 0 \text{ or } 1\}$. For two points $x,\ y \in \mathcal{B}$ define*

$$d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}.$$

a) Show that d is a metric on $\mathcal{B}$.
b) Let $\mathcal{F}$ be the subset of $\mathcal{B}$ containing all binary sequences which have only finitely many non-zero elements. Show that $\mathcal{F}$ is dense in $\mathcal{B}$.

Problem 1.7. *Finish the proof of Baire's theorem, i.e., show that the sequence of balls we constructed $\overline{B(y_n, r_n)} \subset U_n \cap B(y_{n-1}, r_{n-1})$, $0 < r_n < 1/n$, $n \in \mathbb{N}$, $y_0 = x$, $r_0 = \epsilon$ allows us to conclude $y_n \to y \in B(x,r)$, where $y_1 = x$, $r_1 = r$ and $B(x,r)$ is an arbitrary chosen ball.*

Problem 1.8. *Suppose (X,d) is a complete metric space. Show that a subspace E of X is closed iff E is a complete metric space.*

Problem 1.9. *(**1.2**/ 5 from the book) Prove that any countable union of sets of the first category in X is again of the first category in X. Here X is a complete metric space.*

Problem 1.10. *Show that if f is an infinitely many times differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that for for every $x \in \mathbb{R}$ there some derivative $f^{(n_x)}$ vanishing at x, $f^{(n_x)}(x) = 0$, then f is a polynomial function on some interval.*
Remark: Actually f is a polynomial on $\mathbb{R}$, but this requires another use of Baire's theorem.

Date: October 13, 2017.

2. Homework 2

(final version)

Problem 2.1. *Let f be real valued function on $\mathbb{R}$. Show that there exist $M > 0$ and a nonempty open subset U of $\mathbb{R}$ such that for any $s \in U$, there is a sequence $\{s_n\}$ satisfying $s_n \to s$ and $|f(s_n)| \le M$ for all $n \in \mathbb{N}$.*

Problem 2.2. *Consider M_2-the metric space of all 2×2 matrices equipped with the Euclidean metric, i.e.,*

$$d(A,B) = \left(\sum_{i=1}^{4}(a_i - b_i)^2\right)^{1/2}, \quad A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

a) Determine if $SL(2)$- the set (subgroup) of all matrices in M_2 of determinant equal to one is a compact.
b) Determine if $O(2)$- the set (subgroup) of all orthogonal matrices in M_2 is a compact.

Problem 2.3. *Let $\{U_\alpha \mid \alpha \in A\}$ be a finite open cover of a compact metric space (X, d).*
a) Show that there exists $\epsilon > 0$ such that for every $x \in X$ the ball $B(x, \epsilon)$ is contained in some U_α. Such an ϵ is called a Lebesgue number of the cover.
b) Show that if all of the U_α's are proper subsets of X, then there is a largest Lebesgue number for this cover.

Problem 2.4. *Let E be the subspace of $\mathbb{R}^2$ obtained from the circle centered at $(0, 1/2)$ of radius $1/2$ by deleting the point $(0, 1)$. Define a function h from $\mathbb{R}$ to E so that $h(s)$ is the point at which the line segment from $(s, 0)$ to $(0, 1)$ meets E.*
(a) Show that h is a homeomorphism from $\mathbb{R}$ to E.
(b) Show that $\rho(s,t) = ||h(s) - h(t)||$ $s, t \in \mathbb{R}$ defines a metric on $\mathbb{R}$ which is equivalent to the usual metric on $\mathbb{R}$. Here, $||h(s) - h(t)||$ is the Euclidean distance.
(c) Show that $(\mathbb{R}, \rho)$ is totally bounded, but not complete.
(Can you identify the completion of $(\mathbb{R}, \rho)$?)

Problem 2.5. *Prove that a metric space is compact iff every continuous real-valued function on X is bounded.*

Problem 2.6. *Show that the space of continuous real-valued functions on the unit interval $[0, 1]$ is not complete in the norm $||f||_1 = \int_0^1 |f(x)dx$.*

Problem 2.7. *Let (X, d) be a metric space and S a set. Show that $\mathcal{B}(S : X)$ is a complete metric space iff X is complete. Here, $\mathcal{B}(S : X)$ is the space of bounded functions from S to X with the metric defined by $\rho(f, g) = \sup_{s \in S} d(f(s), g(s))$.*

Problem 2.8. *Let (X, d) be a metric space. Show that the metric $d : X \times X \to \mathbb{R}$ is a continuous function on $X \times X$. Here we regard $X \times X$ with the product metric $\rho(\xi, \xi') = \max\{d(x, y), d(x', y')\}$ where $\xi = (x, y)$, $\xi' = (x', y') \in X \times X$.*

Problem 2.9. *Let (X, d) be a metric space and A a subset of X.*
a) Show that the function $d_A(x) = \inf\{d(x, y) \mid y \in A\}$ is a continuous function.
b) Show that of F is a closed subset of X then $d_F(x) = 0$ iff $x \in F$.
c) Give an example where $d_A(x) = 0$ but $x \notin A$.

Problem 2.10. *Let X and Y be metric spaces such that X is complete. Show that if $\{f_\alpha\}_{\alpha \in A}$ is a family of continuous functions from X to Y such that for each $x \in X$ the set $\{f_\alpha(x) : \alpha \in A\}$ is a bounded subset of Y, then there exists a nonempty open subset U of X such that $\{f_\alpha\}_{\alpha \in A}$ is uniformly bounded on U, i.e., the set $\{f_\alpha(x) : \alpha \in A, x \in U\}$ is a bounded subset of Y.*

3. Homework 3

(final version)

Problem 3.1. *Let (K, d_K) and (Y, d_Y) be metric spaces with K a compact. Show that if a subset $\mathcal{F} \subset \mathcal{C}(K, Y)$ is totally bounded, then the image of K under the family $\mathcal{F}$ is totally bounded subset of Y, i.e., $\mathcal{F}(K) = \{f(x) \mid x \in K,\ f \in \mathcal{F}\}$ is totally bounded subspace of Y. Hint: the image of a compact is compact; for $\epsilon > 0$ consider the finitely many functions f_j in $\mathcal{F}$ whose epsilon neighborhoods cover $\mathcal{F}$ and then the sets $f_j(K)$.*

Problem 3.2. *Let (K, d_K) and (Y, d_Y) be metric spaces with K a compact. Show that a subset $\mathcal{F} \subset \mathcal{C}(K, Y)$ is totally bounded if $\mathcal{F}$ is equicontinuous and $\mathcal{F}(K) = \{f(x) \mid x \in K,\ f \in \mathcal{F}\}$ is totally bounded subspace of Y. In other words assuming the properties claimed in* both *Problems 3.1 and 3.3 is equivalent to the hypothesis in these problems.*

Problem 3.3. *Let (K, d_K) and (Y, d_Y) be metric spaces with K a compact. Show that if a subset $\mathcal{F} \subset \mathcal{C}(K, Y)$ is totally bounded, then $\mathcal{F}$ is equicontinuous. Hint: recall a "uniform" property of a continuous function on a compact; for $\epsilon > 0$ consider the finitely many functions f_j in $\mathcal{F}$ whose epsilon neighborhoods cover $\mathcal{F}$.*

Problem 3.4. *Suppose that F is a function from a non empty compact metric space (K, d) to itself such that $d(F(x), F(x')) < d(x, x')$. Show that for any $\epsilon > 0$, there is a constant $c = c(\epsilon) < 1$ such that $d(F(x), F(x')) \leq cd(x, x')$ for all $x, y \in K$ satisfying $d(x, x') \geq \epsilon$.*

Problem 3.5. *Show that if $F : X \to X$ is mapping on the compact metric space (X, d) such that*

$$d\left(F\left(x\right), F\left(y\right)\right) < d(x, y), \quad x \neq y,$$

then F has a unique fixed point. You can assume the claim of the last problem.

Problem 3.6. *Let E^n be a linear space of dimension n. Show that all norms on E^n are equivalent, i.e., if $||\,.\,||_1$ and $||\,.\,||_2$ are two norms on E^n then there are constants m and M such that*

$$m||x||_1 \leq ||x||_2 \leq M||x||_1$$

for all $x \in E^n$. You can do the proof for $\mathbb{R}^n$ if you are more comfortable working in the Euclidean space. Hint: Use that the closed unit balls (in a finite dimensional normed vector space) are compact and the norms are continuous functions, hence you get boundedness there. Then use dilations.

Problem 3.7. *Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ a normed linear space. Show that if T_n is a sequence of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$, such that for all $x \in \mathcal{X}$ the sequence $\{T_n x\}$ converges as $n \to \infty$, then $Tx = \lim_{n\to\infty} T_n x$ defines a bounded linear operator T from $\mathcal{X}$ to $\mathcal{Y}$. Furthermore, $||T|| \leq \liminf ||T_n||$. Hint: First show that T is linear and then that T is bounded using the uniform boundedness principle. Use that if for two sequences of numbers we have $a_n \leq b_n$ then $\liminf a_n \leq \liminf b_n$.*

Problem 3.8. *Show that a linear operator T between two normed linear spaces $(X, ||.||_X)$ and $(Y, ||.||_Y$ is bounded iff T is continuous at 0.*

Problem 3.9. *Let (X, d) be a metrics space. Show that $\rho(x, y) = min\{1, d(x, y)\}$ is a metric equivalent to d.*

Problem 3.10. *Let X be as in Problem 3.9. Is X complete with respect to d the same as completeness with respect to ρ?*

4. Homework 4

(final version)

Problem 4.1. *Let $X \neq \emptyset$ be a set and $p \in X$. Show that the family of all subsets of X containing p together with the empty set defines a topology on X, called the particular point topology. Determine int(A) and $\overline{A}$ for subsets of X containing or not containing p in the particular point topology at p.*

Problem 4.2. *(a) Show that the family of all subsets of $\mathbb{R}$ of the type (a, ∞) together with the empty set and $\mathbb{R}$ is a topology on $\mathbb{R}$, called the half-line topology.*
(b) Determine the closure of the $(0, 1)$ in the half-line topology.

Problem 4.3. *Let S be a subset of a topological space X in which singletons (sets of one point) are closed sets. A point $x \in X$ is a* limit point *of S if every neighborhood of x contains a point of S other than x itself. A point $x \in X$ is an* isolated point *of S if there is a neighborhood U of x such that $U \cap S = \{x\}$.*
(a) Show that the set of limit points of S is closed.
(b) Show that $\bar{S}$ is the disjoint union of the set of limit points of S and the isolated points of S

Problem 4.4. *Let S be a subset of a topological space X. Show that*
(a) $\overline{X \setminus S} = X \setminus int(S)$; *(b) $int(X \setminus S) = X \setminus \overline{S}$.*

Problem 4.5. *Show that the intersection $\mathfrak{F}$ of a family of topologies $\{\mathfrak{F}_\alpha\}$, $\alpha \in A$, for a set X is a topology for X, i.e., $\mathfrak{F} = \{U \subset X : U \in \mathfrak{F}_\alpha \text{ for all } \alpha \in A\}$ defines a topology on X. This is the largest topology contained in each of the given topologies, i.e., if $U \in \mathfrak{F}$ then $U \in \mathfrak{F}_\alpha$ for all $\alpha \in A$, and $\mathfrak{F}$ is the largest topology with this porperty.*

Problem 4.6. *Let $\{\mathfrak{F}_\alpha\}$, $\alpha \in A$, be a family of topologies on a set X. Show that there is a unique (smallest) topology $\mathfrak{F}$ containing each of the given topologies, i.e., if $U \in \mathfrak{F}_\alpha$ for some α then $U \in \mathfrak{F}$, and $\mathfrak{F}$ is the smallest topology with this property.*

Problem 4.7. *Show that if we consider $\mathbb{R}$ with the co-finite topology, then every sequence $\{x_n\}$ of different real numbers is convergent and to every real number. In particular, limits are not unique.*

Problem 4.8. *Show that a function f between two topological spaces X and Y is a continuous function iff the pre-image of every closed set is a closed set.*

Problem 4.9. *Let $\mathfrak{T}$ be the smallest topology on $\mathbb{R}$ which contains all sets of the type $[a, b)$, $a, b \in \mathbb{R}$, i.e., $\mathfrak{T}$ is the intersection of all topologies on $\mathbb{R}$ in which all intervals $[a, b)$ are open (for example the discrete topology is such). Determine if the set $[0, 1)$ is closed in this topology, called sometimes the lower limit topology.*

Problem 4.10. *Prove that a set S in a topological space X is open iff every relatively open subset of S is open in X. Is this statement true if "open" is replaced by "closed"?*

5. Homework 5

(final version)

Problem 5.1. *Let A and B be two subspaces of the topological space X. Prove that the closure of $A \cap B$ in the relative topology of B is a subset of $\overline{A} \cap B$. Give an example where this inclusion is proper. i.e., show that* $\overline{A \cap B}^B \subseteq \bar{A} \cap B$ *and there are examples when* $\overline{A \cap B}^B \neq \overline{A} \cap B$.

Problem 5.2. *Describe the Zariski topology on $\mathbb{R}$ - when is a set closed/ open? This is a topology you know!*

Problem 5.3. *Let X be a topological space and A and B two subsets of X, $A \subset B \subset X$. Show that the subspace topology (=relative topology) of A as a subset of X coincides with the subspace topology of A as a subset of B.*

Problem 5.4. *Determine if the following statement is True or False? The relative topology on $\mathbb{Q}$ induced from the standard topology on $\mathbb{R}$ coincides with the discrete topology on $\mathbb{Q}$.*

Problem 5.5. *(a) Prove that if the topological space X does not have the discrete topology, then there is a function from X to a topological space Y which is not continuous.*
(b) Prove that if the topological space Y does not have the indiscrete topology, then there is a function from a topological space X to Y which is not continuous.

Problem 5.6. *Let X be a set and $\mathcal{S}$ a family of subsets of X.*
(a) Show that there exists unique topology $\mathcal{T}$, such that $\mathcal{S} \subset \mathcal{T}$ (i.e. all of the sets of $\mathcal{S}$ are open in $\mathcal{T}$) and $\mathcal{T}$ is smaller (coarser) than any other topology containing $\mathcal{S}$. This topology is called the topology generated by $\mathcal{S}$.
(b) Let $\mathcal{B}$ be the family of subsets of X consisting of X, $\emptyset$, and all finite intersections of sets in $\mathcal{S}$. Show that $\mathcal{B}$ is a base of a topology of X.

Problem 5.7. *Prove that the family of all arithmetic progressions in $\mathbb{Z}$ is a basis for a topology on $\mathbb{Z}$. In other words, show that $\mathcal{B} = \{\{..., m-2n, m-n, m, m+n, m+2n, ...\} : m, n \in \mathbb{Z}\}$ is a base of topology on $\mathbb{Z}$. Here $\{..., m-2n, m-n, m, m+n, m+2n, ...\}$ is a subset of $\mathbb{Z}$, so $\mathcal{B}$ is a family of subsets of $\mathbb{Z}$.*

Problem 5.8. *(a) Can you compare the* vertical interval topology *on $\mathbb{R}^2$ with the standard topology on $\mathbb{R}^2$? The vertical interval topology is the topology with a basis consisting of the sets of the form $\{a\} \times (b, c)$, where a, b, $c \in \mathbb{R}$.*
(b) Determine the closure and interior of the $x-$axis and $y-$axis in the vertical interval topology on $\mathbb{R}^2$.

Problem 5.9. *(a) Show that the family of all sets of the type $[a, b)$, $a, b \in \mathbb{R}$ is a base of the lower limit topology (also called half-open interval topology) considered in Problem 4.9.*
(b) Show that a function f on the real line is continuous from the real line with the lower limit topology to the real line with the standard topology iff f is continuous from the right, that is, for all $x \in \mathbb{R}$ we have $f(x) = \lim_{\epsilon \to 0^+} f(x + \epsilon)$.

Problem 5.10. *Let X be a topological space with the cofinite topology.*
(a) Show that X is separable. *(b) When is X second-countable?*

6. Homework 6

(final version)

Problem 6.1. *Let $\iota : X \to Y$ be an embedding. Show that if Y has any of the following properties then the same is true for X. a) T_1. b) T_2. c) Regular.*

Problem 6.2. *Show that every regular second countable space is normal.* ***Hint****: Very similarly to the metric case, but using the regularity, you can construct open covers of the given disjoint closed sets such that the closure of each of the sets in the covers does not intersect the other closed set. Use the second countability property to extract countable covers $\{U_n\}_{n\in\mathbb{N}}$ and $\{V_n\}_{n\in\mathbb{N}}$. The sets $U_n' = U_n \setminus \cup_{j=1}^n \overline{V_j}$ and similarly defined V_n' are helpful.*

Problem 6.3. *Let $p : X \to Y$ be a quotient map.*
a) Show that Y is a T_1 space iff $\{f^{-1}(y) : y \in Y\}$ is a closed subset (notice that the preimage of a point by f is the "equivalent class" of points that are identified by the map f).
b) Give an example when the T_1 property is not preserved under a quotient map.

Problem 6.4. *Let $p : X \to Y$ be a quotient map (hence surjective by definition).*
a) Show that p defines an equivalence relation $\sim$ on X by defining $x \sim x'$ iff $p(x) = p(x')$.
b) Let $X/\sim$ be the quotient set defined by $\sim$ (or p), i.e., it is the set of equivalent classes $\{f^{-1}(y) : y \in Y\}$ and $\pi : X \to X/\sim$ the quotient map assigning to each point the set of all points equivalent to it. Show that Y and $X/\sim$ are homeomorphic when equipped with the quotient topologies.

Problem 6.5. *Let F be a closed set in a normal topological space X. Show that every continuous function $f \in \mathcal{C}(F : \mathbb{R})$ can be extended to a real valued continuous function defined on the whole space X. Note: Notice that here we do not assume that f is bounded on F.*
Hint*: First compose f with a homeomorphism between the real line and $(-1, 1)$, and then extend this composition to a map $\bar{f}$ from X to $[-1, 1]$. Explain how to handle the places where this composition takes the values ± 1 by using Urysohn's lemma and the disjoint sets $\bar{f}^{-1}(\{-1, 1\})$ and F.*

Problem 6.6. *Give an example when the T_2 property is not preserved but the T_1 property is preserved under a quotient map. For this, you can consider $X = [0, 1] \times [0, 1]$ and the equivalence relation $(x, y) \sim (x', y')$ iff $y = y' > 0$. Remark: we will give a characterization later.*

Problem 6.7. *Let K be a compact Hausdorff topological space and $\{U_\alpha\}_{\alpha\in A}$ an open cover of K. Show that there is a partition of unity subordinate (to the cover $\{U_\alpha\}_{\alpha\in A}$). In other words, show that there exist a finite number of continuous real-valued functions $h_1, h_2, \dots, h_m$ such that*

(1) $0 \le h_j \le 1$, $j = 1, \dots, m$.
(2) $\sum_{j=1}^m h_j = 1$.
(3) For each $j = 1, \dots, m$, there is an open set from the given cover containing the support of h_j, i.e., the set $\operatorname{supp} h_j = \overline{\{x \in K : h_j(x) > 0\}}$.

Problem 6.8. *Let K be a compact. Prove the Arzela-Ascoli theorem, that a subset $F \subset \mathcal{C}(K : \mathbb{R})$ is a compact if and only if F is (i) closed, (ii) bounded, and (iii) equicontinuous. Here, we consider $\mathcal{C}(K : \mathbb{R})$ with the $\sup$ metric, $d(f, g) = sup_{x\in K}|f(x) - g(x)|$, which defines the uniform convergence on K. Note: By the last problem we can can replace (ii) with point-wise bounded.*

Problem 6.9. *Let K be a compact and $F \subset \mathcal{C}(K : \mathbb{R})$. Show that if F is (i) equicontinuous, i.e., for any $\epsilon > 0$ and $x_0 \in K$ there is an open neighborhood U of x_0 such that $|f(x) - f(x_0)| < \epsilon$ for all $f \in F, x \in U$; and (ii) point-wise bounded, i.e., for every $x \in K$ we have $\sup_{f\in F} |f(x)| < \infty$, then F is (uniformly) bounded: $\sup_{f\in F,\, x\in K} |f(x)| < \infty$.*

Problem 6.10. *Suppose $Y \subset X$ and $\sim$ is an equivalence relation on X (hence also on Y). Show that if Y is open and the quotient map $\pi : X \to X/\sim$ is an open map (images of open sets are open sets), then $Y/\sim$ is homeomorphic to $\pi(Y)$, where $\pi(Y)$is equipped with the relative topology of $Y/\sim$.*

7. Homework 7

(final version)

Problem 7.1. *Prove that a Hausdorff topological space X is locally compact iff for every $x \in X$ and open neighborhood U of x, there is an open neighborhood V of x such that $x \in V \subset \overline{V} \subseteq U$ and $\overline{V}$ is compact. In particular, open subsets of a locally compact and Hausdorff spaces are also locally compact.*

Problem 7.2. *Prove that local compactness is a topological property.*

Problem 7.3. *Prove that a locally compact Hausdorff space is regular.*

Problem 7.4. *A point x of a topological space X is called a cut point if $X \setminus \{x\}$ is disconnected.*
(a) Show that the property of having a cut point is a topological property.
(b) Can two homeomorphic spaces have different number (finite) of cut points? Explain.

Problem 7.5. *A topological space X is called locally connected if, for each point $x \in X$ and open neighborhood U of x, there is a connected open neighborhood V of x contained in U. Show that a connected component of a locally connected topological space is open. (Note: recall, each connected component is closed.)*

Problem 7.6. *Show by counterexample that a connected component of a topological space is not necessarily open. (It is closed as we showed in class)*

Problem 7.7. *Let X be a locally compact Hausdorff space that is not compact. Let Y be the one point compactification of X.*
(a) If X is connected is Y connected?
(b) If Y is connected is X connected?

Problem 7.8. *Find the connected components of the real line equipped with the lower limit topology, cf. Problem 4.9.*

Problem 7.9. *Show that if X is path connected and $f \in C(X : Y)$ then $f(X) \subset Y$ is path connected.*

Problem 7.10. *Show that an open set of $\mathbb{R}^n$ is connected iff it is path-connected.*

8. Homework 8

(final version)

Problem 8.1. *Show that an open set of $\mathbb{R}^n$ is connected iff it is path-connected.*

Problem 8.2. *Give an example showing that the path components of a topological space are not necessarily closed, nor are they necessarily open.*

Problem 8.3. *Prove that if X is locally connected and locally path-connected, then every path component of X is open and closed.*

Problem 8.4. *Let $f_\alpha \in \mathcal{C}(X_\alpha : Y_\alpha)$, $\alpha \in A$. Show that $f = \prod f_\alpha \in \mathcal{C}(X_\alpha : Y_\alpha)$, where for $x \in \prod_{\alpha\in A} X_\alpha$ we let $f(x)(\alpha) = f_\alpha(x(\alpha))$.*

Problem 8.5. *Prove that the product of regular spaces is a regular space.*

Problem 8.6. *Prove that a product of two normal space is not necessarily a normal space. For this, show that the product of two real lines each considered with the half-open topology (see homework problems 4.9 and 5.9) is a topological space $(\mathbb{R}^2, \mathcal{P})$ which is not a normal space. For this you can do the following steps.*
(a) Prove that the bisect of the 2nd and 4th quadrants is a closed subset of $(\mathbb{R}^2, \mathcal{P})$.
(b) Take a sequence S in $\mathbb{R}^2$ which is dense in $\mathbb{R}^2$ when considered with the standard topology. Show that the sets $E = \{(x, -x) : x \in S\}$ and $F = \{(x, -x) : x \in \mathbb{R} \setminus S\}$ are disjoint closed sets of $(\mathbb{R}^2, \mathcal{P})$.
(c) Using the Baire category theorem applied to $\mathbb{R}$, show that if V is open set in $(\mathbb{R}^2, \mathcal{P})$ such that $F \subset V$, then there exist $\epsilon > 0$, $b \in S$, an a sequence $\{x_n\}$ in $\mathbb{R}\setminus S$ such that (i) $|x_n - b| \to 0$ and (ii) $[x_n, x_n+\epsilon)\times[x_n, x_n+\epsilon) \subset V$ for all n.
(d) Prove that there are no disjoint open neighborhoods of E and F in $(\mathbb{R}^2, \mathcal{P})$.

Problem 8.7. *Suppose $X = X_1 \times \cdots \times X_n$, where each X_j is a non-empty topological space. Show the following assertions.*
(a) If X is Hausdorff then each X_j is Hausdorff. (b) If X is regular then each X_j is regular. (c) If X is normal then each X_j is normal. (d) If X is compact then each X_j is compact.

Problem 8.8. *Suppose $X = X_1 \times \cdots \times X_n$, where each X_j is a non-empty topological space. Show the following assertions.*
(a) If X is connected then each X_j is connected. (b) If X is path-connected then each X_j is path-connected.

Problem 8.9. *Prove that the connected components of a finite number of topological spaces are the sets of the type $E_1 \times \cdots \times E_n$, where each E_j is a connected component of X_j, $j = 1, \ldots, n$. Note: A similar result holds for path connected components.*

Problem 8.10. *Prove that $\prod X_\alpha$ need not be compact in the box-topology when each X_α is compact. Hint: Use the discrete topology on a two point space for each α.*

9. Homework 9

(final version)

Problem 9.1. *Let X be a compact Hausdorff space. For $A \subset X$ let X/A be the quotient space defined by declaring $x \sim x'$ iff $x, x' \in A$ (thus X/A is obtained from X by "collapsing" A to a point). Show that if A is closed in X then X/A is a Hausdorff space.*

Problem 9.2. *Let $f : X \to Y$ be a surjective map. A subset A of X is called saturated by f if $A = f^{-1}(B)$ for some $B \subset Y$. Show that f is a quotient map iff f is continuous and maps saturated open (closed) sets in X to open (closed) sets in Y.*

Problem 9.3. *Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean (every point has an open neighbourhood homeomorphic to an open set of a Euclidean space) and second countable, but not Hausdorf. [This space is called the line with two origins.]*

Problem 9.4. *Show that the map $f : R^2 \backslash \{O\} \to (0, +\infty)$, $f(x, y) = (x^2+y^2)^{1/2}$ is a quotient map. a)Determine if f is an open or closed map. b) Can you think of a group such that f is the quotient map $f : R^2 \setminus \{O\} \to R^2 \setminus \{O\}/G$?*

Problem 9.5. *For $n \geq 1$, define $P^n = S^n/\sim$, where $\sim$ is the equivalence relation defined on S^n by $x \sim y$ iff $x = y$ or $x = -y$ (i.e. identify antipodal points). Show similarly to the complex case done in class that*
(a) P^n is a compact Hausdorff space; Hint: here, you can really "see" the separating neighbourhoods!
(b) the projection $\pi : S^n \to P^n$ is a local homeomorphism, that is, each point $x \in S^n$ has an open neighborhood that is mapped homeomorphically by π onto an open neighborhood of $\pi(x)$;
(c) P^1 is homeomorphic to the circle S^1.

Problem 9.6. *Regard the unit sphere S^{2n+1} as the set of all point $z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ such that $|z_0|^2 + \cdots + |z_n|^2 = 1$.*
a) Show that for the obvious projection $\pi : S^{2n+1} \to \mathbb{C}P^n$ we have $\pi^{-1}([z])$ is homeomorphic to S^1 for all $x \in \mathbb{C}P^n$.
b) Show that each $[z] \in \mathbb{C}P^n$ has an open neighborhood U such that $\pi^{-1}(U)$ is homeomorphic to the product space $U \times S^1$.

Problem 9.7. *a) Show that the map $p : \mathbb{C}_\times \times \mathbb{C} \to \mathbb{C}$ defined by $p(z_0, z_1) = z_1/z_0$ is open. Hint: enough to show that images of open sets from a base are mapped to open sets.*
b) Show that the map $p : \mathbb{C}_\times \times \mathbb{C}^n \to \mathbb{C}^n$ defined by $p(z_0, z_1, \ldots, z_n) = (z_1/z_0, \ldots, z_n/z_0)$ is open.

Problem 9.8. *Let G_n be the group of n-th roots of unity. Regard the unit sphere S^{2n+1} as the set of all point $z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ such that $|z_0|^2 + \cdots + |z_n|^2 = 1$. Show that the action of G_n on S^{2n+1} defined by $(z_0, \ldots, z_n) \mapsto (\zeta z_0, \ldots, \zeta z_n)$ for $\zeta \in G_n$ is an even action., i.e., every point $z \in S^{2n+1}$ has a neighborhood U such that $U \cap \zeta U = \emptyset$ for every $\zeta \in G_n$.*

Problem 9.9. *Consider $\mathbb{Z}$ acting on $\mathbb{R}$ by $n(x) = n + x$, $n \in \mathbb{Z}$. Determine if this is an even action. Show that $\mathbb{R}/\mathbb{Z} \cong S^1$-the unit circle.*

Problem 9.10. *Show that if G is a group which acts evenly on a topological space X then the quotient map $\pi : X \to X/G$ is a covering map, i.e., every point $y \in X/G$ has an open neighborhood $V \subset X/G$ such that: (i) $\pi^{-1}(V)$ is the disjoint union of open sets U_α, $\pi^{-1}(V) = \cup U_\alpha$; and (ii) each of the maps $p : U_\alpha \to V$ is a homeomorphism. Here, as usual, X/G is the space of orbits.*

10. Homework 10

(final version)

Problem 10.1. *Show that if a finite group G acts on a Hausdorff topological space X without fixed points, i.e., if $g \neq e$ (identity in G) then $g(x) \neq x$ for all $x \in X$, then the action is even provided G is finite.*

Problem 10.2. *Let $G =< \tau >$ be the group generated by the translation of the complex plane $\mathbb{R}^2 = \mathbb{C}$, $\tau(z) = i - \bar{z}$. Show that this action is even.*

Problem 10.3. *Let $G =< \tau, >$ be the group generated by the isometries of the complex plane $\mathbb{R}^2 = \mathbb{C}$, $\tau(z) = \bar{z} - 1$. Show that this action is even. The quotient is the (infinite) Mobius band.*

Problem 10.4. *Show that the polar coordinate map $p : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^2_*$ defined by $p(r, \phi) = (r\cos\phi, r\sin\phi)$ is a covering map. Note: $\mathbb{R}_+$ is the set of all positive real numbers and $\mathbb{R}^2_* = \mathbb{R}^2 \setminus \{(0,0)\}$.*

Problem 10.5. *Show that the polar coordinate map in the last problem is the quotient of the right half-plane by a suitable $\mathbb{Z}$-action (which you will have to find). Is this an even action?*

Problem 10.6. *Recall that the conjugate of the quaternion q is $\bar{q} = t - ix - jy - kz$.*
a) Show that conjugation is an involutive anti-automorphism of $\mathbb{H}$, i.e., it is $\mathbb{R}$-linear, $\bar{\bar{q}} = q$, and $\overline{q \cdot q'} = \bar{q}' \cdot \bar{q}$.
b) Show that $q^{-1} = \frac{\bar{q}}{|q|^2}$ for a non-zero quaternion q, where as in class $|q| = \sqrt{q\bar{q}}$.

Problem 10.7. *a) Write the matrix version of the "left" and "right" unit quaternion actions $L_u(q) = uq$ and $R_u(q) = q\bar{u}$ where $u \in Sp(1)$ and $q \in \mathbb{H}$, i.e., find 4 by 4 matrices A and B representing the linear transformation L_u and R_u of $\mathbb{R}^4$.*
b) Do the same for the restrictions of L_u and R_u on $Im(\mathbb{H}) \equiv \mathbb{R}^3$ - these are all quaternions with vanishing real part.

Problem 10.8. *Show that $Sp(1)$ is isomorphic to the special unitary group $SU(2)$ using the following identification*

$$q = t + ix + jy + kz \in Sp(1) \qquad \to \qquad t\sigma_0 - xi\sigma_1 - yi\sigma_2 - zi\sigma_3 \in SU(2),$$

where $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In other words we let $1 \longleftrightarrow \sigma_0, i \longleftrightarrow \sigma_i = -i\sigma_1, j \longleftrightarrow \sigma_j = -i\sigma_2$ and $k \longleftrightarrow \sigma_k = -i\sigma_3$ and then extend naturally.

Problem 10.9. *Show that if $p : \tilde{X} \to X$ is a covering and X is connected, then all fibers have the same cardinality, i.e., there is a bijection between $p^{-1}(x)$ and $p^{-1}(y)$ for any $x, y \in X$. Hint: think of how the fiber $p^{-1}(x)$ intersects the sets covering evenly a neighbourhood U of x.*

Problem 10.10. *Show that $Sp(1) \times Sp(1)$ is a double cover of $SO(4)$, i.e., there is a covering map which is a group homomorphism with the "right" kernel, which you will determine. NOTE: You can use that any rotation of $\mathbb{R}^4 = \mathbb{H}$ is of the form $q \mapsto u_1 q u_2$ for some $u_1, u_2 \in Sp(1)$.*

11. Homework 11

(final version)

Problem 11.1. *Let X and Y be Hausdorff locally compact spaces. Show that if $\phi : X \to Y$ is a surjective local homeomorphism which is a proper map (i.e. the preimage of every compact is compact) then ϕ is a covering map. You can follow the following steps: a) start by showing that $\phi^{-1}(y)$ is a finite set for every $y \in Y$; b) for each of the finitely many points x_j in the preimage $f^{-1}(y)$ construct an open set U_j so that the sets U_j are disjoint and p is a homeomorphism between each U_j and its image $V_j = f(U_j)$; c) let W be an open neighbourhood of y such that $\overline{W}$ is compact and $\overline{W} \subset \cap V_j$; remove from $p^{-1}(W)$ the points that are not in any of the sets U_j (do it first for the case when there is only one or two x_j's) in order to get an even cover of some neighbourhood of y.*

Problem 11.2. *Show that if ω is a unit vector in $\mathbb{R}^3$ which we identify with a purely imaginary quaternion also denoted by ω (as we did in class), then for any $\phi \in \mathbb{R}$, the quaternion $u = cos\phi + sin\phi \cdot \omega$ is a unit quaternion, $u \in Sp(1)$. Show that the associated map of $\mathbb{R}^3$, defined in class as $T_u(q) = uq\bar{u}$ for $q \in Im\mathbb{H}$ is a rotation in $\mathbb{R}^3$ about the axis defined by ω through the angle 2ϕ.*

Problem 11.3. *Suppose the group G acts evenly on the topological space X and H is a subgroup of G.*
a) Show that H also acts evenly on X.
b) Show that the natural map between orbits (you should define it) $j : X/H \to X/G$ is a covering map. If the index of H is n this is a $n-$sheeted covering.

Problem 11.4. *Consider the additive group $G = \mathbb{Z} \times Z$ acting on $\mathbb{R}^2$ by $(m, n)(x, y) = (m + x, n + y,\ m, n \in \mathbb{Z}$. Show that the quotient map is a covering map and $\mathbb{R}^2/G \cong S^1 \times S^1$, i.e., the quotient space is homeomorphic to a 2-torus. Hint: see Problem 9.9; here the map $p : (x, y) \mapsto (e^{2\pi ix}, e^{2\pi iy})$ will be useful.*

Problem 11.5. *Consider the action of $G =< \zeta >$ on the unit circle S^1 defined with the help of a real number t acting as a rotation by angle t on the unit circle, thus $\zeta(e^{i\phi}) = e^{it+i\phi}$.*
a) Show that if $\frac{t}{2\pi}$ is a rational number then the orbit of every point is periodic, S^1/G is homeomorphic to S^1 and $S^1 \to S^1/G$ is a covering map. Note: if you take $\zeta = i2\pi n$, $n = pq$ for two primes p and q you have Z_p is a subgroup of Z_{pq}, so you have an example of the set up of Problem 11.3 - an n-cover of the circle together with two more p and q-fold subcovers.
b) Show that if $\frac{t}{2\pi}$ is an irrational number then the orbit of every point is a dense subset of S^1. Is the quotient map $S^1 \to S^1/G$ a covering map?

Problem 11.6. *For a fixed $a \notin \mathbb{Q}$ consider the action of the additive group $\mathbb{R}$ on the 2-torus $S^1 \times S^1$ defined by $t(e^{2\pi ix}, e^{2\pi iy}) = (e^{2\pi i(x+t)}, e^{2\pi i(y+at)})$. Show that the orbit of any point is a dense subset of the 2-torus. Is the action even? Is the quotient space Hausdorff?*

Problem 11.7. *Let the group G act on the topological space X in a way so that for every x, $x' \in X$ which are not in the same orbit there are open neighbourhoods U and U' containing x and x' respectively and such that $gU \cap hU' = \emptyset$ for all $g, h \in G$. Show that the space of orbits X/G is a Hausdorff space.*

Problem 11.8. *Show that the 3-dimensional real projective space P^3, see Problem 9.5, is homeomorphic to $SO(3)$.*

Problem 11.9. *Let $\mathbb{Z}_p =< \zeta = e^{2\pi i/p} >$ be the group of p-th roots of unity. Let $m, n \in \mathbb{Z}$ be two integers each of them relatively prime to p. Show that $\mathbb{Z}_p$ acts evenly on S^3 (can be generalized to any odd-dimensional sphere) via $S^3 = \{z = (z_1, z_2) \in \mathbb{C}^n \mid |z_1|^2 + |z_2|^2 = 1\}$ using the formula*

$$\zeta \cdot (z_1, z_2) = (\zeta^m z_1, \zeta^n z_2)$$

for $\zeta \in S^1$. Is the quotient space Hausdorff? The orbit space S^3/Z is the so called lense space $L(p; m, n)$. Hint: you should recall Problem 10.1.

Problem 11.10. *Consider the action of $Sp(1)$ on the $4n + 3$ dimensional sphere $S^{4n+3} = \{Q = (q_0, \dots, q_n) \in \mathbb{H}^n \mid |q_0|^2 + \cdots + |q_n|^2 = 1\}$ defined by the formula*

$$u \cdot (q_1, \dots, q_n) = (uq_0, \dots, uq_n).$$

Show that this is an even action (the quotient space is the "left" quaternion projective space $\mathbb{H}P^n$). Is $\mathbb{H}P^n$ a Hausdorff space?

12. Homework 12

(final version)

Problem 12.1. *Suppose that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for any three paths α, β and γ in X for which the product is defined. Show that each path component of X consists of a single point.*

Problem 12.2. *Let X be path-connected and let $b \in X$. Show that every path in X is homotopic with endpoints fixed to a path passing through b.*

Problem 12.3. *Let D be an open subset in $\mathbb{R}^n$, let α be a path in D from x to y, and set*

$$d = \inf\{|\alpha(s) - w| : \ w \in \partial D, 0 \le s \le 1\}.$$

Show that if β is any path in D from x to y such that $|\alpha(s) - \beta(s)| < d$, $0 \le s \le 1$, then β is homotopic to α with endpoints fixed.

Problem 12.4. *Show that every path in an open set of $\mathbb{R}^n$ is homotopic with endpoints fixed to a polygonal path. A polygonal path is a path of finitely many segments, i.e., α is polygonal if there is a subdivision $0 = s_0 < s_1 \cdots < s_m = 1$ of $[0,1]$ such that α maps each $[s_j, s_{j+1}]$ onto the segment connecting $\alpha(s_j)$ and $\alpha(s_{j+1})$. Hint: Justify carefully why there is a finite subdivision of $[0,1]$ such that on each of the intervals of the subdivision the path is contained inside a ball of radius d centered at the beginning for a suitable $d > 0$. Uniform continuity might be of some help.*

Problem 12.5. *Let (X,d) be a compact metric space and let $a, b \in X$. Let $\mathcal{P}$ be the set of all paths in X from a to b with the metric*

$$\rho(\alpha, \beta) = \sup\{d(\alpha(s), \beta(s)) : 0 \le s \le 1\}.$$

Show that two paths α, $\beta \in \mathcal{P}$ are homotopic with endpoints fixed if and only if α and β lie in the same path component of $\mathcal{P}$.

Problem 12.6. *a) Prove that if $n \ge 2$ then the n-dimensional sphere S^n is simply connected, in particular, the fundamental group is trivial. Hint: Reduce to the case of a loop that misses a point on the sphere by showing that every loop is homotopic to a loop formed by finitely many arcs lying on great circles. For a loop that misses a point you can use a stereographic projection from that point to get a loop on $\mathbb{R}^n$.*
b) Prove that if $n \ge 3$ then $\mathbb{R}^n \setminus \{0\}$ is simply connected.

Problem 12.7. *Let X be the comb space, that is, the compact subset of $\mathbb{R}^2$ consisting of the horizontal interval $\{(x,0) : 0 \le x \le 1\}$ and the closed vertical intervals of unit length with lower endpoints at $(0,0)$ and at $(0, 1/n)$, $n \in \mathbb{N}$.*
a) Show that X is contractible to $(0,0)$ with $(0,0)$ held fixed.
b) Show that X is not contractible to $(0,1)$ with $(0,1)$ held fixed.

Problem 12.8. *Show that a star-shaped set of $\mathbb{R}^n$ is contractible to a point. A subset of $\mathbb{R}^n$ is star shaped if there is a point a in the set such that whenever b is another point in the set the segment joining a to b lies in the set.*

Problem 12.9. *Let X be a path-connected topological space and b, $c \in X$. Let B^2 be the closed unit disk in $\mathbb{R}^2$ with boundary circle S^1. Show that the following are equivalent*
a) X is simply connected.
b) Any two paths from b to c are homotopic with endpoints fixed.
c) Every continuous map $f : S^1 \to X$ extends to a continuous map $F : B^2 \to X$.

Problem 12.10. *A subspace A of a topological space X is a retract of X if there is a map $f : X \to A$ such that $f(y) = y$ for all $y \in A$ (f is called a retraction of X onto A). Show that the unit sphere S^n in $\mathbb{R}^{n+1}$ is a retract of $\mathbb{R}^{n+1} \setminus \{0\}$.*

13. Homework 13

(final version)

Problem 13.1. *Use the Seiferet - van Kampen's theorem to obtain another proof that* $\pi_1(S^n) = \{e\}$ *for* $n \geq 2$.

Problem 13.2. *Let* $f, g \in C(X : Y)$ *be two maps that are homotopic relative to some point* $x_0 \in X$, $f \sim g, \quad rel\ \{x_0\}$. *Show that* $f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $y_0 = f(x_0) = g(x_0)$.

Problem 13.3. *Let* f *be retraction of* X *onto* A, $x_o \in A$, *and* $j : A \to X$ *the inclusion map. Prove the following*

a) $j_* : \pi_1(A, x_o) \to \pi_1(X, x_o)$ *is one-to-one.*

b) $f_* : \pi_1(X, x_o) \to \pi_1(A, x_o)$ *is onto.*

c) If X *is simply connected, then* A *is simply connected. Thus, a retract of a simply connected spaces is a simply connected space.*

Problem 13.4. *Let* $f : S^1 \to S^1$ *be the map* $f(x) = -x$. *Show that* f *is homotopic to the identity.*

Problem 13.5. *Show that the fundamental group of a product of two spaces is the product of the fundamental groups of the spaces,* $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

Problem 13.6. *Find* $\pi_1(T^n)$ *of the n-torus* $T^n = S^1 \times S^1 \times \cdots \times S^1$.

Problem 13.7. *Show that if the spaces* X *and* Y *are homotopically equivalent and* $f : (X, x_0) \to (Y, y_0)$ *is a homotopy equivalence, then the fundamental groups* $\pi_1(X, x_0)$ *and* $\pi_1(Y, y_0)$ *are isomorphic via* f_*.

Problem 13.8. *Show that homotopy equivalence between maps on the space* $\mathcal{C}(X : Y)$, *denoted by* $f \sim g$ *for* $f, g \in \mathcal{C}(X : Y)$, *is an equivalence relation.*

Problem 13.9. *Show that if* $j : (X, x_0) \to (Y, y_0)$ *embeds* X *in* Y *as a deformation retract, then* $j_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ *is an isomorphism. Hint: use Problem 13.2.*

Problem 13.10. *Show that* $A, B \in O(n+1)$ *considered as maps of* S^n *to itself are homotopic iff they have the same determinant (+1 or -1). In particular, any orthogonal matrix with determinant +1 can be connected by a path to the identity matrix.*

14. Homework 14

(final version)
(DUE WEDNESDAY AFTER THANKSGIVING)

Problem 14.1. *Let $p : (\tilde{X}, \tilde{a}) \to (X, a)$ be a covering map and γ a loop at a. Show that the (unique) lift $\tilde{\gamma}$ of γ starting at $\tilde{a}$ is a loop at $\tilde{a}$ if and only if $[\gamma] \in p_*\left(\pi_1(\tilde{X}, \tilde{a})\right)$. Hint: an immediate corollary of the monodromy theorem.*

Problem 14.2. *Let $p : (\tilde{X}, \tilde{e}) \to (X, e)$ be a covering map and $\tilde{X}$ be a simply connected space. Suppose that $\tilde{X}$ and X are groups with identities $\tilde{e}$ and e, respectively, and that p is a homomorphism. Suppose finally that for each fixed $\tilde{x} \in \tilde{X}$ the group multiplication $\tilde{y} \mapsto \tilde{x} * \tilde{y}$ is continuous. Prove that the fiber $p^{-1}(e) = Ker\, p$ is a subgroup of $\tilde{X}$ and $\pi_1(X, e) \cong p^{-1}(e)$.*

Problem 14.3. *Let $p : \tilde{X} \to X$ be a covering map and $a \in X$.*
a) Show that $\pi_1(X, a)$ acts on $p^{-1}(a)$ on the right *by the monodromy action defined as follows: for $[\alpha] \in \pi_1(X, a)$ and $\tilde{a} \in p^{-1}(a)$*

$$[\alpha](\tilde{a}) \overset{def}{=} \tilde{\alpha}(1),$$

where $\tilde{\alpha}$ is the (unique) lift of α starting at $\tilde{a}$.
b) Show that the monodromy action is transitive on the fiber $p^{-1}(a)$ when $\tilde{X}$ is path-connected.

Problem 14.4. *Let (Y, b) be a pointed topological space such that Y is locally path-connected and simply connected, let $p : (\tilde{X}, \tilde{a}) \to (X, a)$ be a covering map. Show that every continuous map $f : (Y, b) \to (X, a)$ can be uniquely lifted to a continuous map $\tilde{f} : (Y, b) \to (\tilde{X}, \tilde{a})$.*

Problem 14.5. *Let $p : (\tilde{X}, \tilde{a}) \to (X, a)$ be a covering map. Show that the isotropy sub-group of $\tilde{a}$ under the monodromy action of $\pi_1(X, a)$ on $p^{-1}(a)$ is $p_*\left(\pi_1(\tilde{X}, \tilde{a})\right)$ (a sub-group of $\pi_1(X, a)$).*

Problem 14.6. *Show that a simply connected covering space of a locally path-connected space is unique up to a homeomorphism. Hint: With the notation of Problem 14.4, suppose in addition that $\tilde{X}$ is locally path-connected and simply connected. Show that if f is a covering map, then $\tilde{f}$ is a homeomorphism of Y and $\tilde{X}$.*

Problem 14.7. *Compute the fundamental group of the infinite Möbius band $M = \mathbb{R}^2/G$, where G was defined in Problem 10.3.*

Problem 14.8. *Using Seiferet - van Kampen's theorem find $\pi_1(X)$ for X- the boundary of an n-leaf clover.*

Problem 14.9. *Find $\pi_1(T^2 \setminus a)$ for some $a \in T^2$, T^2 is the 2-dimensional torus.*

Problem 14.10. *Compute $\pi_1(SO(4))$. Hint: use Problem 10.10.*