NON-UMBILICAL QUATERNIONIC CONTACT HYPERSURFACES IN HYPER-KÄHLER MANIFOLDS

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ABSTRACT. We show that any compact quaternionic contact (abbr. qc) hypersurfaces in a hyper-Kähler manifold which is not totally umbilical has an induced qc structure, locally qc homothetic to the standard 3-Sasakian sphere. We also show that any nowhere umbilical qc hypersurface in a hyper-Kähler manifold is endowed with an involutive 7-dimensional distribution whose integral leaves are locally qc-conformal to the standard 3-Sasakian sphere.

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1. INTRODUCTION

Any real hypersurface in a complex manifold carries a natural CR structure which in the case of a strictly positive Levi form endows the surface with a natural pseudo-Hermitian structure. The goal of this paper is to consider a hyper-Kähler manifold and describe the real hypersurfaces which carry a natural quaternionic contact (qc) structure. The concept of a qc structure was originally introduced by O. Biquard [2] as a model for the conformal boundary at infinity of the quaternionic hyperbolic space. According to a result in [2, 4], every real analytic qc structure is the conformal infinity of a unique (asymptotically hyperbolic) quaternionic-Kähler metric defined in a neighborhood of the qc structure. Similar to the CR case the question of embedded quaternionic contact hypersurfaces is a natural one, but in contrast to the CR case it imposes a rather strong conditions on the hypersurface. The situation has the flavor of the Kähler versus the hyper-Kähler case. As well known any complex submanifold of a Kähler manifold is a Kähler manifold and a Kähler metric is locally given by a Kähler potential. In contrast, a hyper-complex manifold of a hyper-Kähler manifold must be totally geodesic and (in general) there is no hyper-Kähler potential (the structure is rigid). This suggests that we can expect that there are few quaternionic contact hypersurfaces in a hyper-Kähler manifold. Indeed, we showed in [9] that given a connected qc-hypersurface M in the flat quaternion space $\mathbb{H}^{n+1}$, then, up to a quternionic affine transformation of $\mathbb{H}^{n+1}$, M is contained in one of the following three hyperquadrics (the 3-Sasakain sphere, the hyperboloid and the quaternionic Heisenberg group):

$$(1.1)\quad (i)\ |q_1|^2+\dots+|q_n|^2+|p|^2=1,\quad (ii)\ |q_1|^2+\dots+|q_n|^2-|p|^2=-1,\quad (iii)\ |q_1|^2+\dots+|q_n|^2+\mathbb{R}e(p)=0,$$

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where $(q_1, q_2, \dots q_n, p)$ denote the standard quaternionic coordinates of $\mathbb{H}^{n+1}$. We recall that the above three examples are locally qc-conformal. Furthermore, it was shown [9] in the general hyper-Kähler case that the Riemannian curvature of the ambient space has to be degenerate along the normal to the qc-hypersurface vector field.

The notion of qc-hypersurface was first defined by Duchemin [5] in the general setting of quaternionic manifold. A manifold K is called *quaternionic* if K is endowed with a 3-dimensional sub-bundle $\mathbb{Q}^K \subset End(TK)$ locally generated by a pointwise quaternionic structure J_1, J_2, J_3 together with a torsion free connection that preserves $\mathbb{Q}^K$.

An embedding $\iota : M \to K$ of a qc manifold M with a horizontal space H equipped with a quaternion structure $\mathbb{Q}^H$, see Section 2.1 for precise definition, into a quaternionic manifold $(K, \mathbb{Q}^K)$ is called a *qc embedding* if the differential ι_* intertwines $\mathbb{Q}^K$ and $\mathbb{Q}^H$, i.e., if

$$\mathbb{Q}^H = \iota_*^{-1}\, \mathbb{Q}^K\, \iota_*$$

is satisfied at each point of M, where $\mathbb{Q}^H$ denotes the point-wise quaternionic structure of the horizontal distribution $H \subset TM$. In particular, the image $\iota_*(H)$ coincides with the maximal $\mathbb{Q}^K-$invariant subspace of $\iota_*(TM) \subset TK$. A real hypersurface $M \subset K$ in a quaternionic manifold K is called a *qc hypersurface* if there exists a qc structure on M for which the inclusion map is a qc embedding. Notice that, if such a qc structure exists, then it is unique, since the qc distribution H is the maximal $\mathbb{Q}^K$ invariant subspace of TM.

Duchemin [5]showed that a real analytic qc manifold can be realized as a qc-hypersurface in an appropriate quaternionic manifold.

In this paper we consider qc-hypersurfaces in a hyper-Kähler manifold. Our main result in the case of a compact embedded qc-hypersurface is the following.

Theorem 1.1. *Let M be a compact qc-hypersurface of a hyper-Kähler manifold. If M is not a totally umbilical hypersurface, then the qc-conformal class of the embedded qc structure contains a qc-Einstein structure of positive qc-scalar curvature which is locally qc-equivalent to the 3-Sasakian sphere.*

We note that the existence of a conformal factor leading to a qc-Einstein structure, called calibrated qc-structure, was established earlier by the authors, see [9, Theorem 1.2]. Thus, the main new result here is the qc-conformal flatness of the calibrated qc-Einstein structure. In the connected simply-connected case the above Theorem implies that the qc-conformal class of the embedded qc structure contains a qc structure qc-equivalent to the round 3-Sasakian sphere, see also Theorem 4.1. It is well known that any totally umbilical hypersurface of a hyper-Kähler manifold is a qc-hypersurface whose qc structure is generated by its induced 3-Sasakian metric. Furthermore, a 3-Sasakian space can be embedded as a totally umbilical qc-hypersurface in a hyper-Kähler manifold, namely in its metric cone. The hyperquadric

$$|q_1|^2 + \cdots + |q_n|^2 + 2|p|^2 = 1$$

in $\mathbb{H}^{n+1}$ is an example of a compact qc-hypersurface which is not totally umbilical with respect to the standard flat hyper-Kähler metric of $\mathbb{H}^{n+1}$.

The case of a local qc-embedding is considered in Section 5 where we prove results which in the seven dimensional case give the following theorem.

Theorem 1.2. *A seven dimensional everywhere non-umbilical qc-hypersurface M embedded in a hyper-Kähler manifold is qc-conformal to a qc-Einstein structure which is locally qc-equivalent to the 3-Sasakian sphere, the quaternionic Heisenberg group or the hyperboloid.*

The proofs of the main results rely on the known and some new properties of the "calibratng" qc-conformal factor. More precisely, as shown in [9], given a qc-hypersurface M in a hyper-Kähler manifold K there is a positive function f on M called "calibrating" function so that the qc structure on M obtained from the embedded one with f as a qc-conformal factor is qc-Einstein, see [9, Lemma 3.7]. Furthermore, if II is the second fundamental form of M, then the (0,2) tensor $f\,II$ extends to a covariant constant along M, see [9, Theorem 3.1]. The new key points for the results of the current paper are certain identities for the second and third order (horizontal) covariant derivative of the calibrating function f. Using the bracket generating condition and the relation between the Biquard and Levi-Civita connections these identities lead to a third order differential system on M well studied in the Riemannian case by several authors, see [17, 6, 18, 16]. In the compact case, this system is known to have the remarkable property that it admits a non-constant solution only on Riemannian manifolds which are locally isometric to the round sphere.

Convention 1.3. *Throughout the paper, unless explicitly stated otherwise, we will use the following notation.*

a) All manifolds are assumed to be C^∞ and connected.
b) The triple (i, j, k) denotes any positive permutation of $(1, 2, 3)$.
c) s, t are any numbers from the set $\{1, 2, 3\}$, $s, t \in \{1, 2, 3\}$.
d) For a given decomposition $TM = V \oplus H$ we denote by $[.]_V$ and $[.]_H$ the corresponding projections to V and H.
e) A, B, C, etc. will denote sections of the tangent bundle of M, $A, B, C \in TM$.
f) X, Y, Z, U will denote horizontal vector fields, $X, Y, Z, U \in H$.

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2. Preliminaries

2.1. **Quaternionic contact manifolds.** Here, we recall briefly the relevant facts and notation needed for this paper and refer to [2], [7] and [13] for a more detailed exposition. A quaternionic contact (qc) manifold is a $(4n+3)$-dimensional manifold M with a codimension three distribution H equipped with an $Sp(n)Sp(1)$ structure locally defined by an $\mathbb{R}^3$-valued 1-form $\eta = (\eta_1, \eta_2, \eta_3)$. Thus, $H = \cap_{s=1}^3 Ker\, \eta_s$ carries a positive definite symmetric tensor g, called the horizontal metric, and a compatible rank-three bundle $\mathbb{Q}^H$ consisting of endomorphisms of H locally generated by three orthogonal almost complex structures I_s, satisfying the unit quaternion relations: (i) $I_1I_2 = -I_2I_1 = I_3$, $\quad I_1I_2I_3 = -id_{|_H}$; (ii) $g(I_s., I_s.) = g(., .)$; and (iii) the compatibility conditions $2g(I_sX, Y) = d\eta_s(X, Y)$, $X, Y \in H$ hold true. Unlike the CR case, in the qc case the horizontal space determines uniquely the qc-conformal class, cf. [9]. For this reason very often we will identify the qc structure with the $\mathbb{R}^3$-valued 1-form η while supressing the remaining data. We also note that by virtue of its definition a quaternionic contact manifold is orientable.

Two qc structures η and $\bar\eta$ on a manifold M are called *qc-conformal* to each other if $\bar\eta = \mu\Psi\eta$ for a positive smooth function μ and an $SO(3)$ matrix Ψ with smooth functions as entries. A diffeomorphism F between two qc manifolds M and $\bar M$ is called *quaternionic contact conformal (qc-conformal) transformation* if $F * \bar\eta = \mu\Psi\eta$. The qc-conformal curvature tensor W^{qc}, introduced in [11], is the obstruction for a qc structure to be locally qc-conformally to the standard 3-Sasakian structure on the $(4n+3)$-dimensional sphere [11, 13]. As already noted in the introduction the 3-Sasakain sphere, the hyperboloid and the quaternionic Heisenberg group are all locally qc-conformal to each other.

As shown in [2], there is a "canonical" connection associated to every qc manifold of dimension at least eleven. In the seven dimensional case the existence of such a connection requires the qc structure to be integrable [4]. The integrability condition is equivalent to the existence of Reeb vector fields [4], which (locally) generate the supplementary to H distribution V. The Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ are determined by [2]

$$\eta_s(\xi_t) = \delta_{st}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \quad (\xi_s \lrcorner d\eta_t)_{|H} = -(\xi_t \lrcorner d\eta_s)_{|H}, \tag{2.1}$$

where $\lrcorner$ denotes the interior multiplication. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2.1) and refer to the "canonical" connection as *the Biquard connection.* The Biquard connection is the unique linear connection preserving the decomposition $TM = H \oplus V$ and the

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$Sp(n)Sp(1)$ structure on H with torsion T determined by $T(X,Y) = -[X,Y]_{|V}$ while the endomorphisms $T(\xi_s,.) : H \to H$ belong to the orthogonal complement $(sp(n) + sp(1))^{\perp} \subset GL(4n,R)$.

The covariant derivatives with respect to the Biquard connection of the endomorphisms I_s and the Reeb vector fields are given by

$$\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j, \qquad \nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j. \tag{2.2}$$

The $\mathfrak{sp}(1)$-connection 1-forms $\alpha_1, \alpha_2, \alpha_3$, defined by the above equations satisfy [2]

$$\alpha_i(X) = d\eta_k(\xi_j, X) = -d\eta_j(\xi_k, X), \qquad X \in H.$$

Let $R = [\nabla,\nabla] - \nabla_{[.,.]}$ be the curvature tensor of ∇ and $R(A,B,\mathbb{C},D) = g(R_{A,B}\mathbb{C},D)$ be the corresponding curvature tensor of type (0,4). The qc Ricci tensor Ric, the qc-Ricci forms ρ_s and the normalized qc scalar curvature S are defined by

$$Ric(A,B) = \sum_{a=1}^{4n} R(e_a,A,B,e_a), \quad 4n\rho_s(A,B) = \sum_{a=1}^{4n} R(A,B,e_a,I_se_a), \quad 8n(n+2)S = Scal = \sum_{a=1}^{4n} Ric(e_a,e_a),$$

where $e_1, \dots, \mathbf{e}_{4n}$ of H is an g-orthonormal frame on H.

We say that (M,η) is a qc-Einstein manifold if the restriction of the qc-Ricci tensor to the horizontal space H is trace-free, i.e.,

$$Ric(X,Y) = \frac{Scal}{4n} g(X,Y) = 2(n+2)Sg(X,Y), \quad X,Y \in H.$$

The qc-Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection, $T(\xi_s,X) = 0$ [7]. It is also known [7, 8] that the qc-scalar curvature of a qc Einstein manifold is constant and the vertical distribution is integrable.

By [8, Theorem 5.1], see also [12] and [13, Theorem 4.4.4] for alternative proofs in the case $Scal \neq 0$, a qc-Einstein structure is characterised by either of the following equivalent conditions

i) locally, the given qc structure is defined by 1-form (η_1,η_2,η_3) such that for some constant S, we have

$$d\eta_i = 2\omega_i + S\eta_j \wedge \eta_k; \tag{2.3}$$

ii) locally, the given qc structure is defined by a 1-form (η_1,η_2,η_3) such that the corresponding connection 1-forms vanish on H and (cf. the proof of Lemma 4.18 of [7])

$$\alpha_s = -S\eta_s. \tag{2.4}$$

2.1.1. *The correspondin (pseudo) Riemannian geometry.* Let M be a qc-Einstein manifold. Note that, by applying an appropriate qc homothetic transformation, we can aways reduce a general qc-Einstein structure to one whose normalized qc-scalar curvature S equals $0, 2$ or -2. Consider the one-parameter family of (pseudo) Riemannian metrics h^μ, $\mu \neq 0$ on M by letting

$$h^\mu = g|_H + \mu(\eta_1^2 + \eta_2^2 + \eta_3^2). \tag{2.5}$$

Let ∇^μ be the Levi-Civita connection of h^μ. Note that h^μ is a positive-definite metric when $\mu > 0$ and has signature $(4n,3)$ when $\mu < 0$. The difference $L = \nabla^\mu - \nabla$ between the Levi-Cevita connection ∇^μ and the Biquard connection ∇ is given by [7, 8]

$$L(A,B) \equiv \nabla^\mu_A B - \nabla_A B = \frac{S}{2}[A]_V \times [B]_V + \sum_{s=1}^{3} \Big\{ -\omega_s(A,B)\xi_s + \mu\eta_s(A)I_sB + \mu\eta_s(B)I_sA \Big\}, \tag{2.6}$$

where $._V \times ._V$ is the standard vector cross product on the 3-dimensional vertical space V.

2.2. **Quaternionic contact hypersurfaces.** In this section we summarize some results from [9] which are the starting point of the subject of the current paper. For ease of reading we follow [9] closely.

2.2.1. *qc-hypersurfaces.* Let K be a hyper-Kähler manifold with hyper-complex structure (J_1, J_2, J_3), quaternionic bundle $\mathcal{Q}^K$, and hyper-Kähler metric G. In particular, the Levi-Civita connection D is a torsion free connection on K preserving $\mathcal{Q}^K$.

For a real hypersurface $M \subset K$ the maximal $\mathcal{Q}^K$-invariant subspace TM is denoted by H and refereed to as the horizontal distribtution. If $\iota : M \to K$ is the natural inclusion map, then M is a qc-hypersurface if it is a qc manifold with respect to the induced quaternionic structure $\iota_*^{-1}(Q^K)\iota_*$ on H. In order to simplify the notation we shell identify the corresponding points and tensor fields on M with their images through ι in K. An equivalent characterization of a qc-hypersurface M is that the restriction of the second fundamental form of M to the horizontal space H is a definite symmetric form, which is invariant with respect to the induced quaternion structure, see [5, Proposition 2.1]. After choosing the unit normal vector N to M appropriately, we will assume that the second fundamental form of M,

$$II(A,B) = -G(D_A N, B), \quad A, B \in TM,$$

is negative definite on the horizontal space H. The defining tensors of the embedded qc structure on M are given by

$$\hat\eta_s(A) = G(J_s N, A), \quad \hat\xi_s = J_s N + \hat r_s, \quad \hat\omega_s(X,Y) = -II(I_s X, Y), \quad \hat g(X,Y) = -\hat\omega_s(I_s X, Y), \tag{2.7}$$

where $I_s = J_s|_H$ and $\hat\xi_s$, are the Reeb vector fields corresponding to $\hat\eta_s$, see [9, Section 2.2].

2.2.2. *The calibrating function.* Let $\hat\omega_s$ be the fundamental 2-forms corresponding to $\hat\eta_s$, given by $2\hat\omega_s(X,Y) = d\hat\eta_s(X,Y)$, $X,Y \in H$ and $\hat\xi_t \lrcorner \hat\omega_s = 0$, $s,t = 1,2,3$. Following [9, Section 3.1], consider the complex 2-forms on M,

$$\hat\gamma_i = \hat\omega_j + \sqrt{-1}\,\hat\omega_k, \qquad \Gamma_i(A,B) \;=\; G(J_j A, B) + \sqrt{-1}\, G(J_k A, B).$$

Using a type decomposition argument it was shown in [9, Section 3.1] that

$$\Gamma_s^n \equiv \mu_s \hat\gamma_s^n \mod \{\hat\eta_1, \hat\eta_2, \hat\eta_3\}, \tag{2.8}$$

for $s = 1,2,3$ and some complex valued functions μ_s and, in fact, $\mu_1 = \mu_2 = \mu_3 = \mu$ for a positive (real valued) function μ on M. The *calibrating* function of M was defined by

$$f = \mu^{\frac{1}{n+2}}.$$

2.2.3. *The calibrated qc structure.* The qc structure

$$(\eta_1, \eta_2, \eta_3) \overset{def}{=} f(\hat\eta_1, \hat\eta_2, \hat\eta_3)$$

is called *calibrated*. As shown in [9], it satisfies the structure equations (2.3). In particular, it is a qc-Einstein structure. Moreover, by [9, Lemma 3.9] the horizontal metric g of the calibrated qc structure is related to the second fundamental form of the qc-embedding by the formula

$$g(A'', B'') = -f\, II(A,B) - \frac{S}{2}\sum_{s=1}^{3} \eta_s(A)\eta_s(B), \quad A, B \in TM, \tag{2.9}$$

where for $A \in TM$ we let $A'' = A - \sum_{s=1}^3 \eta_s(A)\xi_s$ be the horizontal part of A. The corresponding Reeb vector fields ξ_s are given by

$$\xi_s = J_s\Big(f^{-1}N + r\Big), \tag{2.10}$$

where $r \in H$ is determined by $II(r, X) = f^{-2} df(X)$, $X \in H$. In fact, we have [9, Lemma 3.8]

$$r = -f^{-1}\nabla f, \tag{2.11}$$

$$df(\xi_s) = 0, \qquad s = 1,2,3, \tag{2.12}$$

where $\nabla f \in H$ denotes the horizontal gradient of f, $df(X) = g(\nabla f, X)$.

The *calibrated transversal* to M vector field is defined by

$$\xi = f^{-1}N + r. \tag{2.13}$$

From (2.10) and (2.13) we have

$$\xi_s = J_s\xi. \tag{2.14}$$

With the obvious identifications, the bundle $TK|_M \to M$ decomposes as a direct sum,

$$TK|_M = H \oplus V \oplus \mathbb{R}\xi, \tag{2.15}$$

where V is the span of the Reeb vector fields ξ_s of the calibrated qc structure on M. For $v \in T_pK$ we define

$$v' = v - \lambda(v)\xi(p) \in T_pM = H_p \oplus V_p, \qquad v'' = \pi v = v' - \sum_{s=1}^{3}\eta_s(v')\xi_s \in H_p, \tag{2.16}$$

where $\lambda = fG(N,.)$ so that v' is the projection of v on $T_pM = H_p \oplus V_p$ parallel to the calibrated transversal field ξ and $\pi : TK|_M \to H$ is the projection on the horizontal space using the decomposition (2.15). Thus, for $v \in TK|_M$ we have

$$\lambda(J_sv) = \eta_s(v') \tag{2.17}$$

and the decomposition

$$v = \pi v + \sum_{s=1}^{3}\eta_s(v')\xi_s + \lambda(v)\xi \ \in H \oplus V \oplus \mathbb{R}\xi. \tag{2.18}$$

Following [9, (3.23)] consider the symmetric bilinear form $\mathfrak{W} \in T^*K|_M \otimes T^*K|_M$,

$$\mathfrak{W}(v,w) \overset{def}{=} -fII(v',w') + \frac{S}{2}\lambda(v)\lambda(w) = g(\pi v, \pi w) + \frac{S}{2}\sum_{s=1}^{3}\eta_s(v')\eta_s(w') + \frac{S}{2}\lambda(v)\lambda(w). \tag{2.19}$$

Clearly, $\mathfrak{W}(J_s.,.J_s.) = \mathfrak{W}(.,.)$, $s = 1,2,3$, and $\mathfrak{W}$ as the unique J_s-invariant extension of the symmetric bilinear form $-fII$ on TM to a symmetric bilinear form on $TK|_M$. A very important property of the calibrated qc structure is that $\mathfrak{W}$ is constant along M with respect to the Levi-Civita connection D of the hyper-Kähler metric G, see [9, Theorem 3.1]), i.e., we have

$$D_A\mathfrak{W} = 0, \quad A \in TM. \tag{2.20}$$

Finally, we record an important relation between the calibrating function and the parallel bilinear form, see [9, (2.16)]

$$\mathfrak{W}(N,A) = -fII(N',A) = f^2II(r,A) = -fg(r,A'') = df(A'') = df(A). \tag{2.21}$$

3. The system of differential equations for the calibrating function

We begin with a lemma relating the Levi-Civita connection D of the hyper-Kähler metric G to the Biquard connection ∇ of the calibrated qc structure on M.

Lemma 3.1. *For any $A \in TM$ and $X \in H$ we have:*
i) $D_AX = \nabla_AX + \sum_{t=1}^{3}\Big((S/2)\eta_t(A)I_tX - \omega_t(\pi A, X)\xi_t\Big) - g(\pi A, X)\xi.$
ii) $D_A\xi = (S/2)A$ *and* $D_A\xi_s = (S/2)J_sA.$

Proof. First we shall prove the formula in part *i)* for a horizontal vector field A,

$$D_XY = \nabla_XY - \omega_s(X,Y)\xi_s - g(X,Y)\xi. \tag{3.1}$$

We start with the computation of the horizontal part of D_XY,

$$\nabla_XY = \pi(D_XY), \qquad X,Y \in H, \tag{3.2}$$

recalling that π is the projection on the horizontal space, see (2.18). From (2.19) and (2.20) we have

$$\begin{aligned}0 = (D_X\mathfrak{W})(Y,Z) = X\Big(\mathfrak{W}(Y,Z)\Big) - \mathfrak{W}(D_XY,Z) - \mathfrak{W}(Y,D_XZ)&\\ = X\Big(g(Y,Z)\Big) - g\Big(\pi(D_XY),Z\Big) - g\Big(Y,\pi(D_XZ)\Big).&\end{aligned}$$

Letting $F(X,Y) \overset{def}{=} \nabla_X Y - \pi(D_X Y)$, we compute

$$0 = (\nabla_X g)(Y,Z) = X\Big(g(Y,Z)\Big) - g\Big(\pi(D_X Y) + F(X,Y), Z\Big) - g\Big(Y, \pi(D_X Z) + F(X,Z)\Big)$$
$$= -g\Big(F(X,Y),Z\Big) - g\Big(F(X,Z),Y\Big),$$

while on the other hand

$$0 = g\Big(\pi(T(X,Y)),Z\Big) = g\Big(\nabla_X Y - \nabla_Y X - \pi([X,Y]),Z\Big) = g\Big(\nabla_X Y - \nabla_Y X - \pi(D_X Y - D_Y X),Z\Big)$$
$$= g\Big(F(X,Y),Z\Big) - g\Big(F(Y,X),Z\Big).$$

Thus, the tensor $g\Big(F(X,Y),Z\Big)$ is both symmetric in X,Y and skew-symmetric in Y,Z which implies that it vanishes.

The remaining part of $D_X Y$ in the decomposition based on (2.18) can be computed easily as follows,

$$\lambda(D_X Y) = -fG(D_X N, Y) = fII(X,Y) = -g(X,Y)$$

and

$$\eta_s((D_X Y)') = -\lambda(J_s D_X Y) = -\lambda(D_X(J_s Y)) = g(X, J_s Y) = -\omega_s(X,Y).$$

From the above the formula in part *i)* in the case when A is a horizontal vector field follows.

Next we prove the formula

$$D_X N = \nabla_X \nabla f + \frac{Sf}{2} X - df(J_s X)\xi_s. \tag{3.3}$$

In order to determine the horizontal part of $D_X N$ we recall (2.20) and then compute the (horizontal) Hessian of f as follows

$$\nabla^2 f(X,Y) = X(df(Y)) - df(\nabla_X Y) = X(\mathfrak{W}(N,Y)) - df(\nabla_X Y)$$
$$= \mathfrak{W}(D_X N, Y) + \mathfrak{W}(N, D_X Y) - df(\nabla_X Y) = \mathfrak{W}(D_X N, Y) + \mathfrak{W}(N, D_X Y - \nabla_X Y)$$

using (2.21) in the last equality. From (2.19) and (3.1) it follows

$$\nabla^2 f(X,Y) = g(\pi(D_X N), Y) - \frac{1}{2} fSg(X,Y)$$

noting that $\mathfrak{W}(N,\xi) = \frac{1}{2} fS$. The vertical part of $D_X N$ is computed with the the help of (2.9) and (2.13)

$$\eta_s(D_X N) = -fG\Big(N, J_s(D_X N),\Big) = fG\Big(D_X N, J_s N\Big) = -fII(X, J_s N) = -df(J_s X).$$

The proof of formula (3.3) is complete.

An immediate consequence of (2.13), (2.11), (3.1) and (3.3) is the following formula

$$D_X \xi \overset{(2.10)}{=} \frac{1}{2} SX. \tag{3.4}$$

At this point we can complete the proof of part *i)*. Since the calibrated qc structure is qc-Einstein and the 1-forms η_s satisfy the structure equations (2.3), we have $\nabla_{\xi_s} X = [\xi_s, X]$. Therefore,

$$\nabla_{\xi_s} X = [\xi, X] = D_{\xi_s} X - D_X \xi_s = D_{\xi_s} X - J_s(D_X \xi) \overset{(3.4)}{=} D_{\xi_s} X - \frac{S}{2} J_s X. \tag{3.5}$$

Finally, we compute

$$D_A X = D_{\pi A} X + \eta_s(A) D_{\xi_s} X \overset{(3.1),(3.5)}{=} \nabla_{\pi A} X - \omega_s(\pi A, X)\xi_s - g(\pi A, X)\xi + \eta_s(A)\Big(\nabla_{\xi_s} X + \frac{S}{2} J_s X\Big)$$
$$= \nabla_A X - \omega_s(\pi A, X)\xi_s - g(\pi A, X)\xi + \frac{S}{2}\eta_s(A) J_s X.$$

Turning to the proof of *ii)*, we have from (2.9) and (2.17) the formula

$$G(D_{\xi_s} N, A) = -II(\xi_s, A) = \frac{1}{2} fS\eta_s(A) = \frac{S}{2} G(J_s N, A),$$

hence

$$D_{\xi_s} N = \frac{1}{2} S J_s N = \frac{1}{2} S \xi_s - \frac{1}{2} S J_s \nabla r. \tag{3.6}$$

From (2.12) and $T(\xi_s, X) = 0$ it follows $\nabla_{\xi_s} \nabla f = 0$, hence (2.13), (2.11), (3.6) and (2.10) give

$$D_{\xi_s} \xi = \frac{S}{2} \xi_s,$$

which together with (3.4) completes the proof of part *ii)* after recalling (2.14). □

Corollary 3.2. *M is a totally umbilical qc-hypersurface of a hyper-Kähler manifold iff the calibrating function is locally constant.*

Proof. In view of (3.6) and (2.11) it follows the horizontal gradient of f vanishes $\nabla f = 0$, hence f is locally constant taking into account that the horizontal space is bracket generating. □

As customary, let $\mathfrak{W} : M \to End(TK)|_M$ also denote the (1,1) tensor corresponding to the symmetric bilinear form $\mathfrak{W}$, i.e., $G(\mathfrak{W}u, v) = \mathfrak{W}(u, v)$ for all $u, v \in TK|_M$. Then $\mathfrak{W} J_s = J_s \mathfrak{W}$ and, since both G and $\mathfrak{W}$ are D-parallel along M, we also have

$$(D_A \mathfrak{W})(u) = 0, \qquad A \in TM, \ u \in TK|_M. \tag{3.7}$$

An almost immediate corollary of the proof of Lemma 3.1 is the following formula for $\mathfrak{W}$ in terms of the calibrating function.

Lemma 3.3. *For $X \in H$ we have:*
i) $\mathfrak{W}X = f \nabla_X \nabla f + (Sf^2/2) X + df(X) \nabla f - f \sum_{s=1}^3 df(I_s X) \xi_s + f df(X) \xi$;
ii) $\mathfrak{W}\xi = (Sf/2) \nabla f + (Sf^2/2) \xi$;
iii) $\mathfrak{W}\xi_s = (Sf/2) I_s \nabla f + (Sf^2/2) \xi_s$, $s = 1, 2, 3$.

Proof. By definition (2.19), recall also (2.16), we have

$$\begin{aligned} \mathfrak{W}(X, u) = -f II(X, u') = f G(D_X N, u') = f G(D_X N, u) - f G(D_X N, \xi) \lambda(u) \\ = f G(D_X N, u') = f G(D_X N, u) - f^2 G(D_X N, \xi) G(N, u). \end{aligned}$$

Now, the formula of part *i)* follows by a direct substitution using (2.13), (2.11) and (3.3). Finally, part iii) follows from $J_s \xi = \xi_s$, see after equation (2.13).

Part*ii)* is proved similarly with the help of (3.6) instead of (3.3). □

After the preceding technical lemmas we turn to the key result which gives a system of partial differential equations for the calibrating function. With the help of (2.6) is then expressed in terms of Levi-Civita connection in the subsequent lemma.

Lemma 3.4. *The function $\phi \overset{def}{=} \frac{1}{2} f^2$ satisfies the following equations*

$$d\phi(\xi_s) = 0; \tag{3.8}$$

$$\nabla^2 \phi(X, Y) = \nabla^2 \phi(I_s X, I_s Y); \tag{3.9}$$

$$\begin{aligned} \nabla^3 \phi(X, Y, Z) + S d\phi(X) g(Y, Z) + \frac{S}{2} d\phi(Y) g(Z, X) + \frac{S}{2} d\phi(Z) g(X, Y) \\ = \frac{S}{2} \sum_{s=1}^3 \Big[d\phi(I_s Y) \omega_s(X, Z) + d\phi(I_s Z) \omega_s(X, Y) \Big]. \end{aligned} \tag{3.10}$$

Proof. Since $d\phi = f df$ and $\nabla^2 \phi = f \nabla^2 f + df \otimes df$, (2.21) gives (3.8). Recalling the decomposition (2.18), see also (2.16), by Lemma 3.3 we have

$$g\Big((\mathfrak{W}X)'', Y \Big) = \nabla^2 \phi(X, Y) + S \phi g(X, Y). \tag{3.11}$$

From $\mathfrak{W} J_s = J_s \mathfrak{W}$ and $g(I_s X, I_s Y) = g(X, Y)$ for $s = 1, 2, 3$, (3.9) follows from

$$0 = g\Big((\mathfrak{W}X)'', Y \Big) - g\Big((\mathfrak{W} I_s X)'', I_s Y \Big) = \nabla^2 \phi(X, Y) - \nabla^2 \phi(I_s X, I_s Y).$$

We turn to the proof of (3.10). A differentiation of (3.11) gives

$$\nabla^3\phi(X,Y,Z)+Sd\phi(X)g(Y,Z)=g\left(\nabla_X\left(\mathfrak{W}Y\right)'',Z\right)-g\left(\pi\mathfrak{W}\nabla_XY,Z\right). \tag{3.12}$$

Taking into account (3.2), (2.18) and (3.7) we can rewrite the first term in the right-hand side of the above identity as follows

$$g\left(\nabla_X\left(\mathfrak{W}Y\right)'',Z\right)=g\left(\pi D_X\left(\mathfrak{W}Y\right)'',Z\right)=g\left(\pi WD_XY,Z\right)-\eta_s\left((\mathfrak{W}Y)'\right)g\left(\pi D_X\xi_s,Z\right)-\lambda(\mathfrak{W}Y)g\left(\pi D_X\xi,Z\right).$$

Now we use Lemma 3.3 to compute

$$\eta_s\left((\mathfrak{W}Y)'\right)=-f df(I_sY)\quad\text{and}\quad\lambda(\mathfrak{W}Y)=f df(Y).$$

A substitution of the last two equations in (3.12) gives

$$\begin{aligned}\nabla^3\phi(X,Y,Z)+Sd\phi(X)g(Y,Z)&=g\left(\pi\mathfrak{W}\left(D_XY-\nabla_XY\right),Z\right)+\frac{Sf}{2}\omega_s(X,Z)df(I_sY)-\frac{Sf}{2}g(X,Z)df(Y)\\&=\frac{S}{2}\Big(\omega_s(X,Y)d\phi(I_sZ)-g(X,Y)d\phi(Z)+\omega_s(X,Z)d\phi(I_sY)-g(X,Z)d\phi(Y)\Big)\end{aligned}$$

using Lemma 3.1 and Lemma 3.3 in the last equality. The proof of Lemma 3.4 is complete. □

We continue with our main technical result, which allows the partial reduction to a Riemannian geometry problem.

Proposition 3.5. *Let (M,η,Q) be a (4n+3)-dimensional qc-Einstein space with constant qc-scalar curvature $S\neq 0$ and ϕ be a smooth function which satisfies identities (3.8) , (3.9) and (3.10). With respect to the Levi-Civita connection ∇^S of the (pseudo) Riemannian metric given by (2.5) for $\mu=\frac{S}{2}$, the function ϕ satisfies the following identity*

$$(\nabla^S)^3\phi(A,B,C)+Sd\phi(A)h^S(B,C)+\frac{S}{2}d\phi(B)h^S(C,A)+\frac{S}{2}d\phi(C)h^S(A,B)=0,\quad A,B,C\in\Gamma(TM). \tag{3.13}$$

Proof. From (3.8), the properties of the Biquard connection, the Ricci identities, the vanishing of the torsion of the Biquard connection and the integrability of the vertical space we have the following equalities

$$0=\nabla^2\phi(X,\xi_s)=\nabla^2\phi(\xi_s,X)=\nabla^2\phi(\xi_s,\xi_t), \tag{3.14}$$

$$\nabla^2\phi(X,Y)-\nabla^2\phi(Y,X)=2\sum_{s=1}^3\omega_s(X,Y)d\phi(\xi_s)=0. \tag{3.15}$$

Next, using the equality (2.6) together with the Ricci identities for the Levi-Civita connection, (3.14) gives the identities

$$(\nabla^S)^2\phi(Y,X)=(\nabla^S)^2\phi(X,Y)=\nabla^2\phi(X,Y)-d\phi(L(X,Y))=\nabla^2\phi(X,Y). \tag{3.16}$$

$$(\nabla^S)^2\phi(X,\xi_s)=(\nabla^S)^2\phi(\xi_s,X)=\nabla^2\phi(\xi_s,X)-d\phi(L(\xi_s,X)=-Sd\phi(I_sX); \tag{3.17}$$

$$(\nabla^S)^2\phi(\xi_s,\xi_t)=(\nabla^S)^2\phi(\xi_t,\xi_s)=\nabla^2\phi(\xi_s,\xi_t)-d\phi(L(\xi_s,\xi_t)=0. \tag{3.18}$$

Now we turn to the computation of the third derivative. Using (3.16) and (3.17) we obtain the identities

$$\begin{aligned}(\nabla^S)^3\phi(X,Y,Z)&=\nabla^3\phi(X,Y,Z)-(\nabla^S)^2\phi(L(X,Y),Z)-(\nabla^S)^2\phi(Y,L(X,Z))\\&=\nabla^3\phi(X,Y,Z)+\sum_{s=1}^3\Big[\omega_s(X,Y)(\nabla^S)^2\phi(\xi_s,Z)+\omega_s(X,Z)(\nabla^S)^2\phi(Y,\xi_s)\Big]\\&=\nabla^3\phi(X,Y,Z)-\frac{S}{2}\sum_{s=1}^3\Big[\omega_s(X,Y)d\phi(I_sZ)+\omega_s(X,Z)d\phi(I_sY)\Big]\\&=Sdf(X)g(Y,Z)+\frac{S}{2}df(Y)g(Z,X)+\frac{S}{2}df(Z)g(X,Y),\end{aligned} \tag{3.19}$$

where we used (3.10) in the last equality. Proceeding in the same fashion, we obtain

$$(3.20)\quad (\nabla^S)^3\phi(\xi_s,Y,Z) = \nabla^3\phi(\xi_s,Y,Z) - (\nabla^S)^2\phi(L(\xi_s,Y),Z) - \frac{S}{2}(\nabla^S)^2\phi(Y,L(\xi_s,Z)) \\ = 0 - \frac{S}{2}(\nabla^S)^2\phi(I_sY,Z) - (\nabla^S)^2\phi(Y,I_sZ) = -\frac{S}{2}\nabla^2\phi(I_sY,Z) - \frac{S}{2}\nabla^2\phi(Y,I_sZ) = 0,$$

where we used (3.9) in the last equality. A similar computation shows

$$(3.21)\quad (\nabla^S)^3\phi(Y,Z,\xi_s) = (\nabla^S)^3\phi(Y,\xi_s,Z) \\ = \nabla(\nabla^S)^2(Y,\xi,Z) - (\nabla^S)^2\phi(L(Y,\xi_s),Z) - S(\nabla^S)^2\phi(\xi,L(Y,Z)) \\ = -\frac{S}{2}\nabla^2\phi(Y,I_sZ) - \frac{S}{2}(\nabla^S)^2\phi(I_sY,Z) = -\frac{S}{2}\nabla^2\phi(I_sY,Z) - \frac{S}{2}\nabla^2\phi(Y,I_sZ) = 0,$$

where we used (3.9) in the last equality, and also

$$(3.22)\quad (\nabla^S)^3\phi(Y,\xi_s,\xi_s) = \nabla^3\phi(Y,\xi_s,\xi_s) - 2(\nabla^S)^2\phi(L(Y,\xi_s)\xi_s) = -2\nabla^2\phi(I_sY,\xi_s) = -Sd\phi(Y);$$

$$(3.23)\quad (\nabla^S)^3\phi(Y,\xi_s,\xi_t) = \nabla^3\phi(Y,\xi_s,\xi_t) - (\nabla^S)^2\phi(L(Y,\xi_s),\xi_t) - (\nabla^S)^2\phi(\xi_s,L(Y,\xi_t)) = 0.$$

Finally, we calculate

$$(3.24)\quad (\nabla^S)^3\phi(\xi_s,\xi_s,Y) = \nabla^3\phi(\xi_s,\xi_s,Y) - (\nabla^S)^2\phi((L(\xi_s,\xi_s),Y) - (\nabla^S)^2\phi(\xi_s,L(\xi_s,Y)) = -\frac{S}{2}d\phi(Y);$$

$$(3.25)\quad (\nabla^S)^3\phi(\xi_s,\xi_t,Y) = \nabla^3\phi(\xi_s,\xi_t,Y) - (\nabla^S)^2\phi(L(\xi_s,\xi_t),Y) - (\nabla^S)^2\phi(\xi_s,L(\xi_t,Y))$$

$$(3.26)\quad = -\frac{S}{2}d\phi(I_sI_tY) - \frac{S}{2}d\phi(I_tI_sY) = 0.$$

Equations (3.16)-(3.25) show the validity of (3.13) for all $A,B,C \in \Gamma(TM)$. This completes the proof of the Proposition. □

4. Compact qc-hypersurfaces

4.1. **Proof of Theorem 1.1.**

Proof. We begin by showing that if a function ϕ satisfies (3.10), then $h \overset{def}{=} \triangle\phi$ is necessarily an eigenfunction for the sub-Laplacian $\triangle h = tr^g(\nabla^2 h)$. Indeed, see [18, (2.7)] for the analogous calculation in the Riemannian case, taking a trace in (3.10) we obtain that $X(\triangle\phi) = -4(n+1)Sd\phi(X)$ which yields $\nabla^2\triangle\phi(X,Y) = -4(n+1)S\,\nabla^2\phi(X,Y)$ and $\triangle h = -4(n+1)Sh$. Since M is compact it follows $S \geq 0$.

If the qc-scalar curvature vanishes, $S = 0$, then it follows $\phi = const$, which contradicts our assumption that M is non-umbilic, see Corollary 3.2. Thus, we have $S > 0$. In fact, after a qc-homothety, we can assume that $S = 2$. Let $h \overset{def}{=} h^S$ be the corresponding Riemannian metric on M. Now, in view of (3.13), by Gallot-Obata-Tanno's theorem [18, 6, 17] it follows that the Riemannian manifold (M,h) is isometric to the round sphere of radius 1. Therefore, the curvature tensor R^h of the Levi-Civita connection ∇^h of h is given by

$$(4.1)\quad R^h(A,B,C,D) = h(B,C)h(A,B) - h(B,D)h(A,C).$$

The relation between the curvatures of the Levi-Civita connection and the Biquard connection for qc-Einstein spaces with $S = 2$ (i.e., 3-Sasakian spaces) [7, Corollary 4.13] or [13, Theorem 4.4.3] together with (4.1) yields

$$(4.2)\quad R(X,Y,Z,W) = R^h(X,Y,Z,W) \\ + \sum_{s=1}^{3}\Big[\omega_s(Y,Z)\omega_s(X,W) - \omega_s(X,Z)\omega_s(Y,W) - 2\omega_s(X,Y)\omega_s(Z,W)\Big] \\ = h(Y,Z)h(X,W) - h(Y,W)h(X,Z) + \sum_{s=1}^{3}\Big[\omega_s(Y,Z)\omega_s(X,W) - \omega_s(X,Z)\omega_s(Y,W) - 2\omega_s(X,Y)\omega_s(Z,W)\Big].$$

According to [11, Proposition 4.2], the qc conformal curvature tensor W^{qc} can by expressed in terms of the curvature R of the Biquard connection, in general, on a qc-Einstein spaces with qc scalar curvature S by the formula

$$W^{qc}(X,Y,Z,W) = R(X,Y,Z,W) + \frac{S}{2}\Big\{ - g(X,W)g(Y,Z) + g(X,Z)g(Y,W) + \sum_{s=1}^{3}\Big[- \omega_s(X,W)\omega_s(Y,Z) + \omega_s(X,Z)\omega_s(Y,W) + 2\omega_s(X,Y)\omega_s(Z,W)\Big]\Big\}. \tag{4.3}$$

Then, since in our case $g(X,Y) = h(X,Y)$ and $S = 2$, (4.2) implies that $W^{qc} = 0$ and therefore, (M,η) is qc-conformally flat (cf. [11, Theorem 1.3]). Now, the result follows by Theorem 6.1.

Let us remark that the final step of the proof is similar to an argumentation that had been already used before in the proof of [10, Theorem 1.3].

□

In the case of a positive qc-scalar curvature of the calibrated qc structure we can substitute the compactness with completeness assumption of the Riemannian metric noting that the Gallot-Obata-Tanno's theorem holds for a complete Riemannian manifold. In particular, the manifold is compact. In addition, the local qc-conformal maps considered in the proof of Theorem 1.1 define a global qc-conformality to the round sphere, see Theorem 6.1. Therefore, we have

Theorem 4.1. *Let M be a simply connected qc hypersurface of a hyper-Kähler manifold which is not totally umbilical. Suppose that the calibrated qc structure (η_1,η_2,η_3) on M has a positive qc-scalar curvature and that it is complete with respect to the natural Riemannian metric $h = g + \eta_1^2 + \eta_2^2 + \eta_3^2$. Then the calibrated qc structure on M is qc-homothetic to the standard 3-Sasakian sphere.*

5. Locally embedded qc-hypersurfaces

In the non-compact case we show

Theorem 5.1. *Let M be a qc-hypersurface in a hyper-Kähler manifold such that all points of M are non-umbilic. Then there exists a 7 dimensional involutive distribution $\mathcal{D}$ on M such that the induced qc structure on each integral leaf of $\mathcal{D}$ is locally qc-conformal to the standard 7-dimensional 3-Sasakian sphere.*

Proof. We achieve Theorem 5.1 with a series of lemmas. We begin with the following

Lemma 5.2. *Let M be a qc Einstein space with local qc 1-forms η_1,η_2,η_3 satisfying the structure equations (2.3) and let ξ_1,ξ_2,ξ_3 be the corresponding Reeb vector fields. If there exists a function ϕ with a nowhere vanishing horizontal gradient $\nabla\phi$ on M, satisfying (3.10)-(3.8), then the 7-dimensional distribution $D = span\{\xi_1,\xi_2,\xi_3,\nabla\phi, I_1\nabla\phi, I_2\nabla\phi, I_3\nabla\phi\}$ is integrable.*

Proof. Since η_1,η_2,η_3 satisfy (2.3), the vertical distribution spand$\{\xi_1,\xi_2,\xi_3\}$ is integrable and we have $\nabla_X\xi_s = 0$. Moreover,

$$\begin{aligned}[\nabla\phi, I_i\nabla\phi] &= \nabla_{\nabla\phi}(I_i\nabla\phi) - \nabla_{I_i\nabla\phi}(\nabla\phi) - T(\nabla\phi, I_i\nabla\phi)\\ &= -I_i\nabla_{\nabla\phi}(\nabla\phi) - \nabla_{I_i\nabla\phi}(\nabla\phi) - 2\sum_{t=1}^{3}\omega_t(\nabla\phi, I_i\nabla\phi)\xi_t \overset{(3.9)}{=} -2g(\nabla\phi,\nabla\phi)\xi_i.\end{aligned} \tag{5.1}$$

We have also that $T(\xi_s, X) = 0$, which leads to

$$[\xi_s,\nabla\phi] = \nabla_{\xi_s}\nabla\phi - \nabla_{\nabla\phi}\xi_s - T(\xi_s,\nabla\phi) = \nabla^2\phi(\xi_s,e_a)e_a - \nabla_{\nabla\phi}\xi_s \overset{(3.8)}{=} \nabla_{\nabla\phi}\xi_s \subset D.$$

Similarly, $[\xi_s, I_t\nabla\phi] \subset D$ and thus the integrability of the distribution D is proved. □

We need the following

Lemma 5.3. *The qc-conformal curvature of a qc-Einstein space has the property*

$$W^{qc}(X,Y,Z,U) = W^{qc}(Z,U,X,Y) = W^{qc}(X,Y,I_sZ,I_sU) = W^{qc}(I_sX,I_sY,Z,U).$$

Proof. The first equality in the lemma is already known, see e.g. [8]. The second equality follows after a small calculation using formula (4.3) combined with

$$\rho_s = -S\omega_s, \quad R(X,Y,Z,W) = R(Z,W,X,Y) \tag{5.2}$$

(cf. [8, (3.28)] and [11, Theorem 3.1]). □

We proceed with

Lemma 5.4. *Let M be a 7-dimensional qc Einstein space with local qc 1-forms η_1, η_2, η_3, satisfying the structure equations (2.3), corresponding Reeb vector fields ξ_1, ξ_2, ξ_3 and Biquard connection ∇. If there exists a function ϕ on M satisfying at each point: (i) $\nabla\phi \neq 0$, (ii) $d\phi(\xi_1) = d\phi(\xi_2) = d\phi(\xi_3) = 0$ and (iii) $\nabla^2\phi(X,Y) = hg(X,Y)$, for a smooth function h on M, then M is locally qc-conformally flat.*

Proof. Since we assume that the qc 1-forms η_s satisfy (2.3), we have $\nabla_X \xi_s = 0$ and thus

$$\nabla^2\phi(\xi_s, X) = \nabla^2\phi(X, \xi_s) = X\Big(d\phi(\xi_s)\Big) = 0. \tag{5.3}$$

By differentiating *(iii)* we get

$$\nabla^3\phi(X,Y,Z) = dh(X)g(Y,Z). \tag{5.4}$$

The Ricci identity for the Biquard connection ∇ implies that

$$\nabla^2_{X,Y}\nabla\phi - \nabla^2_{Y,X}\nabla\phi = R(X,Y)\nabla\phi - \nabla_{T(X,Y)}\nabla\phi = R(X,Y)\nabla\phi - 2\omega_s(X,Y)\nabla_{\xi_s}\nabla\phi \overset{(5.3)}{=} R(X,Y)\nabla\phi,$$

which by means of (5.4) gives

$$R(X,Y,Z,\nabla\phi) = -\nabla^3\phi(X,Y,Z) + \nabla^3\phi(Y,X,Z) = -dh(X)g(Y,Z) + dh(Y)g(X,Z). \tag{5.5}$$

We take a trace in (5.5) to obtain

$$Ric(X,\nabla\phi) = -3dh(X). \tag{5.6}$$

On the other hand, since M is qc Einstein, $Ric(X,Y) = 6Sg(X,Y)$, hence $Ric(X,\nabla\phi) = 6Sd\phi(X)$. Therefore,

$$2Sd\phi(X) + dh(X) = 0. \tag{5.7}$$

The qc-conformal curvature tensor is given by (4.3), which, by (5.5) and (5.7), implies that

$$\begin{aligned} W^{qc}(X,Y,Z,\nabla\phi) &= R(X,Y,Z,\nabla\phi) + 2S\Big(-d\phi(X)g(Y,Z) + d\phi(Y)g(X,Z)\Big) \\ &\quad - \Big(2Sd\phi(X) + dh(X)\Big)g(Y,Z) + \Big(2Sd\phi(Y) + dh(Y)\Big)g(X,Z) = 0. \end{aligned} \tag{5.8}$$

Since the dimension of M is seven and since by assumption $\nabla\phi \neq 0$ on M, the vector fields $\nabla\phi, I_1\nabla\phi, I_2\nabla\phi, I_3\nabla\phi$ form an orthogonal frame of the 4-dimensional horizontal distribution H. Then, by (5.8) and Lemma 5.3, we have $W^{qc}(X,Y,Z,I_s\nabla\phi) = -W^{qc}(X,Y,I_sZ,\nabla\phi) = 0$ which implies that $W^{qc}(X,Y,Z,W) = 0$, i.e. M is locally qc conformally flat.

□

The next lemma together with [11, Theorem 3.1] completes the proof of Theorem 5.1.

Lemma 5.5. *Let M be a qc Einstein space, ϕ be the non-constant function satisfying (3.10)-(3.8) and $D = span\{\xi_1, \xi_2, \xi_3, \nabla\phi, I_1\nabla\phi, I_2\nabla\phi, I_3\nabla\phi\}$ be the integrable distribution from Lemma 5.2. Then each integral manifold $\iota : N \to M$ of D caries an induced qc structure, defined locally by the 1-forms $\iota^*(\eta_1), \iota^*(\eta_2), \iota^*(\eta_3)$, which is qc conformally flat and qc-Einstein with qc-scalar curvature with the same sign as the qc-scalar curvature of M.*

Proof. Let $\mathfrak{j} : N \to M$ be any integral manifold of D. Then the pull-back 1-forms $\mathfrak{j}^*(\eta_1), \mathfrak{j}^*(\eta_2), \mathfrak{j}^*(\eta_3)$ on N define a qc structure on N with Reeb vector fields $\tilde{\xi}_s = \mathfrak{j}_*^{-1}(\xi_s)$. The horizontal distribution on N is then just $\tilde{H} = \mathfrak{j}_*^{-1}(H)$ and the corresponding quaternionic structure on it is given by the endomorphisms $\tilde{I}_s = \mathfrak{j}_*^{-1} I_s \mathfrak{j}_*$. Moreover, the pull-backs of the structure equations (2.3) remain satisfied on N and thus the induced qc structure on N is again qc Einstein with the same qc scalar curvature as M. Let us denote the corresponding Biquard connection on N by $\tilde{\nabla}$ and consider the function $\tilde{\phi} = \mathfrak{j}^*\phi$. Then, clearly, $\tilde{\nabla}\,(\tilde{\phi}) = \mathfrak{j}_*^{-1}\nabla\phi$ and thus, for any $s = 1, 2, 3$,

$$[\tilde{\nabla}\tilde{\phi}, \tilde{I}_s\tilde{\nabla}\tilde{\phi}] = \mathfrak{j}_*^{-1}[\nabla\phi, I_s\nabla\phi] \overset{(5.1)}{=} \mathfrak{j}_*^{-1}\Big(-2g(\nabla\phi, \nabla\phi)\xi_s \Big) = -2\tilde{g}(\tilde{\nabla}\tilde{\phi}, \tilde{\nabla}\tilde{\phi})\tilde{\xi}_s.$$

Therefore,

$$-2\tilde{g}(\tilde{\nabla}\tilde{\phi}, \tilde{\nabla}\tilde{\phi})\tilde{\xi}_s = [\tilde{\nabla}\tilde{\phi}, \tilde{I}_s\tilde{\nabla}\tilde{\phi}] = \tilde{I}_s(\tilde{\nabla}_{\tilde{\nabla}\tilde{\phi}}\tilde{\nabla}\tilde{\phi}) - \tilde{\nabla}_{\tilde{I}_s\tilde{\nabla}\tilde{\phi}}\tilde{\nabla}\tilde{\phi} - 2\sum_{t=1}^{3}\tilde{\omega}_t(\tilde{\nabla}\tilde{\phi}, \tilde{I}_s\tilde{\nabla}\tilde{\phi})\tilde{\xi}_t,$$

i.e. we have

$$\tilde{\nabla}^2\tilde{\phi}(\tilde{\nabla}\tilde{\phi}, \tilde{I}_s X) = -\tilde{\nabla}^2\tilde{\phi}(\tilde{I}_s\tilde{\nabla}\tilde{\phi}, X), \qquad X \in \tilde{H}.$$

Since the four vector fields $\tilde{\nabla}\tilde{\phi}, \tilde{I}_1\tilde{\nabla}\tilde{\phi}, \tilde{I}_2\tilde{\nabla}\tilde{\phi}, \tilde{I}_3\tilde{\nabla}\tilde{\phi}$ define a frame for the distribution $\tilde{H}$ we obtain that

$$\tilde{\nabla}^2\phi(\tilde{I}_s X, \tilde{I}_s Y) = \tilde{\nabla}^2\phi(X, Y)$$

for any $X, Y \in \tilde{H}$ and $s = 1, 2, 3$. This implies that $\tilde{\nabla}^2\tilde{\phi}(X, Y) = h\tilde{g}(X, Y)$ and thus the function $\tilde{\phi}$ satisfies the assertions of Lemma 5.4. Therefore, the integral manifold N is locally qc-conformally flat. □

□

We finish the section with the prof of Theorem 1.2.

Proof. The proof is similar to that of Theorem 1.1 noting that, here, the qc-conformal flatness follows from Lemma 5.4. However, the (constant) qc-scalar curvature is not necessarily positive. The proof is complete taking into account Theorem 6.1. □

6. Appendix.

In the course of the paper we used several times the fact that a qc-Einstein qc-conformally flat manifold is locally qc-homothetic to one of the standard model qc-spaces (1.1). As indicated below, this fact has been essentially proved before, but due to its independent interest we formulate it explicitly. Furthermore, we include an argument for global equivalence.

Theorem 6.1. *A qc-conformally flat qc-Einstein manifold M is locally qc-homothetic to one of the following three model spaces: the 3-Sasakian sphere S^{4n+3}, the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$ or the hyperboloid S^{4n}_3 depending on the sign of the qc-scalar curvature, respectively. If in addition M is connected, simply connected with complete Biquard connection then we have a global qc-homothety with the model spaces* (1.1).

Proof. By a qc-homothety, depending on the sign of the qc-scalar curvature, we can reduce the claim to one of the cases $S = 2$, $S = 0$ or $S = -2$. We recall that the model spaces (1.1) are qc-Einstein qc-conformally flat manifolds with positive qc-scalar curvature $S = 2$ in the case *i)* of the 3-Sasakian sphere [7, 10], flat in the case of the quaternionic Heisenberg group *iii)* [7], and negative qc-scalar curvature $S = -2$, [9], for the hyperboloid *ii)*.

One proof of the local equivalence goes as follows. Due to the local qc-conformality with the quaternionic Heisenberg group, with the help of [14, Theorem 6.2], see [7, Theorem 1.2] for the positive qc-scalar curvature case, we can determine the exact form of the conformal factor relating the invariant qc structure on the Heisenberg group to the image by a qc-conformal transformation of the given qc-Einstein structure. The proof of the local equivalence statement in Theorem 1.1 follows, for more details see [7, Theorem 1.2] in the case of positive qc-scalar curvature, the paragraph after [14, Lemma 8.6] in the zero qc-scalar curvature case, while the negative qc-scalar curvature case follows analogously. The global result in the case of a compact manifold is achieved by a monodromy argument and Liouville's theorem [14, Theorem 8.5], [3]. Below is an argument using that in our case Biquard's connection is an affine connection with parallel torsion and parallel curvature, hence we can invoke the results in [15, Chapter VI].

For a qc-Einstein manifold we have from [7, 8] $T^0 = U = 0$, the qc-scalar curvature is constant, $S = const$ and the vertical space is integrable. As a consequence, on a qc-Einstein manifold we have [7, 11, 13, 8]

$$T(X,Y) = 2\sum_{s=1}^{3}\omega_s(X,Y)\xi_s; \quad T(\xi_i,\xi_j) = -S\xi_k, \tag{6.1}$$

$$R(\xi_s,X,Y,Z) = R(\xi_s,\xi_t,X,Y) = 0, \quad R(A,B)\xi = -2S\sum_{s=1}^{3}\omega_s(A,B)\xi_s\times\xi. \tag{6.2}$$

Using (2.2), we obtain from (6.1) that the torsion of the Biquard connection is parallel, $\nabla T = 0$. Similarly, (6.2) implies that $\nabla R(\xi_s,A,B,C) = \nabla R(A,B,C,\xi_s) = 0$.

For the horizontal part of R we apply the second condition of the qc-conformal flatness, $W^{qc} = 0$. A substitution of (5.2) into (4.3) gives

$$\begin{aligned} R(X,Y,Z,W) = \frac{S}{2}\Big[g(Y,Z)g(X,W) - g(Y,W)g(X,Z)\Big] \\ + \frac{S}{2}\sum_{s=1}^{3}\Big[\omega_s(Y,Z)\omega_s(X,W) - \omega_s(X,Z)\omega_s(Y,W) - 2\omega_s(X,Y)\omega_s(Z,W)\Big]. \end{aligned} \tag{6.3}$$

Hence, by (2.2), it follows that the horizontal curvature of the Biquard connection is parallel as well, i.e., we have $\nabla T = \nabla R = 0$.

Let F be a linear isomorphism between the tangent spaces $T_p(M)$ and $T'_p(M')$ of a point p in M and a point p' in the model space (1.1) of same qc-scalar curvature, such that, F maps an orthonormal basis $\{e_a, I_1e_a, I_2e_a, I_3e_a\}_{a=1}^n$ of the horizontal space at p to the an orthonormal basis $\{e'_a, I'_1e'_a, I'_2e'_a, I'_3e_a\}_{a=1}^n$ of the horizontal space at p' and also sends the corresponding Reeb vector fields at p to those at p'. Thus, F preserves the horizontal and vertical spaces $F(H_p) = H'_q, \quad F(V_p) = V'_q$, and the $Sp(n)Sp(1)$-structure, i.e., it maps the tensors $g_p, (I_s)|_p, (\xi_s)|_p$ at the point $p \in M$ into the tensors $g'_q, (I'_s)|_q, (\xi'_s)|_q$. Taking into account $S = S'$, (6.1) together with (6.2), and (6.3) show that F maps the torsion T_p and the curvature R_p at p into the torsion T'_q and the curvature R'_q at $q \in M'$, respectively.

Now, we can apply the affine equivalence theorem [15, Theorem 7.4] to obtain an affine local isomorphism between M and the coresponding model space. Since the qc structure $(H \oplus V, \mathbb{Q}, g)$ is parallel the affine local isomorphism is a qc-homothety.

Finally, if in addition M is connected, simply connected with a complete Biquard connections then [15, Theorem 7.8] gives us a global qc-homothety to the corresponding model case. We note that the Biquard connection in each of the model cases is complete since the 3-Sasakian spere is compact, the Biquard connection on the qc Heisenberg group is an invariant connection of a homogeneous space, while the hyperboloid is $Sp(n,1)Sp(1)/Sp(n)Sp(1)$, see e.g. [1, Theorem 5.1], with the invariant Biquard connection determined by (2.6). $\square$

References

[1] Alekseevsky, D., Kamishima, Y., *Pseudo-conformal quaternionic CR structure on $4n+3$-dimensional manifolds*, Ann. Mat. Pura Appl. (4) 187 (2008), no. 3, 487-529. 14

[2] Biquard, O., *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **265** (2000). 1, 3, 4

[3] Čap, A., & Slovák, J., Parabolic geometries. I. Background and general theory. Mathematical Surveys and Monographs, 154. American Mathematical Society, Providence, RI, 2009. x+628 pp. 13

[4] Duchemin, D., *Quaternionic contact structures in dimension 7*, Ann. Inst. Fourier, Grenoble **56**, 4 (2006) 851–885. 1, 3

[5] Duchemin, D., *Quaternionic contact hypersurfaces*, math.DG/0604147. 2, 5

[6] Gallot, S., *Equations differentielles de la sphere*, Ann. scient. Ec. Norm. Sup. 4^e serie, **12** (1979), 235-267. 3, 10

[7] Ivanov, S., Minchev, I., & Vassilev, *Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem*, Memoirs Amer. Math. Soc. vol. 231, number 1086; http://dx.doi.org/10.1090/memo/1086 3, 4, 10, 13, 14

[8] Ivanov, S., Minchev, I., & Vassilev, *Quaternionic contact Einstein structures*, to appear in Math. Res. Lett., arXiv:1306.0474. 4, 12, 14

[9] Ivanov, S., Minchev, I., Vassilev, D., *Quaternionic contact hypersurfaces in hyper-Kähler manifolds*, to appear in Ann. Mat. Pura Appl., arXiv:1406.4256. 1, 2, 3, 4, 5, 6, 13

[10] Ivanov, S., Petkov, A., Vassilev, D., *The Obata sphere theorems on quaternionic contact manifold of dimension bigger than 7*, to appear in J. Spectral Theory, arXive:1303.0409. 11, 13

[11] Ivanov, S., Vassilev, D., *Conformal quaternionic contact curvature and the local sphere theorem*, J. Math. Pure Appl, **93** (2010), 277–307. 3, 11, 12, 14

[12] Ivanov, S., & Vassilev, D., *Quaternionic contact manifolds with a closed fundamental 4-form*, Bull. London Math. Soc. **42** (2010), 1021-1030. 4

[13] Ivanov, S., & Vassilev, D., *Extremals for the Sobolev Inequality and the Quaternionic Contact Yamabe Problem*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011. xviii+219 pp. 3, 4, 10, 14

[14] Ivanov, S., & Vassilev, D., *The Lichnerowicz and Obata first eigenvalue theorems and the Obata uniqueness result in the Yamabe problem on CR and quaternionic contact manifolds*, Nonlinear Anal. 126 (2015), 262–323. 13

[15] Kobayashi, S., Nomizu,K., Foundations of Differential geometry volume I, Interscience Publishers, John Willey and Sons, New-York, London, 1963 14

[16] Matveev, V., Mounoud, P., *Gallot-Tanno theorem on closed incomplete pseudo-Riemannian manifolds and applications*, Ann. Glob. Analysis Geom. **38** no.3 (2010), 259-271. 3

[17] Obata, M., *Riemannian manifolds admitting admitting a solutions of a certain system of differential equations*, Proc. US-Japan Seminar in Differential Geometry (Kyoto, 1965), pp. 101-114. 3, 10

[18] Tanno, S., *Some differential equations on Riemannian manifolds*, J. Math. Soc. Japan **30**no. 3, 509-531. 3, 10

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