THE SHARP LOWER BOUND OF THE FIRST EIGENVALUE OF THE SUB-LAPLACIAN ON A QUATERNIONIC CONTACT MANIFOLD IN DIMENSION 7

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ABSTRACT. A version of Lichnerowicz' theorem giving a lower bound of the eigenvalues of the sub-Laplacian under a lower bound on the $Sp(1)Sp(1)$ component of the qc-Ricci curvature on a compact seven dimensional quaternionic contact manifold is established. It is shown that in the case of a seven dimensional compact 3-Sasakian manifold the lower bound is reached if and only if the quaternionic contact manifold is a round 3-Sasakian sphere.

CONTENTS

1. INTRODUCTION

The purpose of this paper is to give a version of the results of [18] in the case of a seven dimensional quaternionic contact (abbr. qc) manifold. Thus, the main result is the establishment of a lower bound for the first eigenvalue on a seven dimensional qc manifold satisfying a Lichnerowicz type condition extending the classical [26] and CR [14, 27, 12] Lichnerowicz type results to the case of a general compact qc manifold. We also prove an Obata type result [29] characterizing the extremal case in the qc Lichnerowicz type result assuming the extremal case is a (seven dimensional) qc-Einstein manifold of constant qc-scalar curvature.

It should be noted that a seven dimensional qc structure, similarly to a three dimensional strongly pseudoconvex manifold, frequently presents some additional difficulties or requires a different analysis in various geometric and analytic questions, see for example [16] and [15]. Such is the case of this paper. On the other hand, a notable difference are the (conformal) flatness problems, see [5, 11, 24] and [19] respectively, where there is no distinction between the seven and higher dimensional results in the qc case.

Date: October 26, 2012.

2010 *Mathematics Subject Classification.* 53C26, 53C25, 58J60.

Key words and phrases. quaternionic contact structures, qc conformal flatness, qc conformal curvature, Einstein metrics.

This work has been partially funded by.

Another motivation for the current paper comes from a number of Sasakian case results [8, 7, 6], [2] and [10]. These CR results came after Greenleaf [14] proved the CR Lichnerowicz type result on a $2n+1$ dimensional CR manifold for $n \geq 3$. Subsequently, [27] adapted Greenleaf's proof to cover the case $n = 2$, while the case $n = 1$ was treated in [27] assuming further a condition on the covariant derivative of the pseudohermitian torsion. In [12] the author also considered the $n = 1$ case replacing the additional assumption with the *CR-invariant condition* that the CR-Paneitz operator is non-negative. As well known, for $n > 1$ the CR-Paneitz operator is always non-negative, while in the case $n = 1$ the vanishing of the pseudohermitian torsion implies that the CR-Paneitz operator is non-negative, see [12] and[9].

In the current paper we introduce a certain $P-$function in the setting of a qc structure which allows us to obtain in dimension seven the qc Lichnerowicz - Obata type results, except for the latter we require that the extremal qc structure is 3-Sasakian. This condition means that M is a qc-Einstein structure of constant positive qc-scalar curvature, see [15], and also [20] and [21] for the negative scalar curvature case. Note that in higher dimensions the qc-Einstein condition implies the constancy of the qc-scalar curvature [15]. In view of [22], [28] and [23], where the Obata type result was proven for a general CR manifold, our qc Obata-type result is not completely satisfactory, but the non qc-Einstein case seems to present extra difficulties. In addition, the introduced $P-$form is interesting on its own. Here P should remind not only of the Paneitz operator, but also of the $P-$function used in the theory of elliptic partial differential equations, see [30] for references and some results.

Theorem 1.1. *Let $(M, g, \mathbb{Q})$ be a compact quaternionic contact manifold of dimension seven. Suppose there is a positive constant k_0 such that the normalized scalar curvature S and the torsion T^0 satisfy the Lichnerowicz type inequality*

$$6Sg(X,X) + 10T^0(X,X) \geq k_0 g(X,X). \tag{1.1}$$

If, in addition, the $P-$function of any eigenfunction of the sub-Laplacian is non-negative, then for any eigenvalue λ of the sub-Laplacian $\triangle$ we have the inequality

$$\lambda \geq \frac{1}{3}k_0.$$

The P-function of a smooth function f, cf. Definition 3.1, is defined with the help of the Biquard connection, the qc-scalar curvature and the $[-1]$ component of the torsion tensor, see Section 2.1, (2.11) and (2.7). We say that the $P-$function of f is non-negative if

$$-\int_M P_f \, Vol_\eta \geq 0.$$

We note that the Lichnerowicz type assumption in any dimension is

$$2(n+2)Sg(X,X) + \frac{4n^2+14n+12}{2n+1}T^0(X,X) + \frac{4(n+2)^2(2n-1)}{(n-1)(2n+1)}U(X,X) \geq k_0 g(X,X). \tag{1.2}$$

When $n > 1$ as shown in [18] it implies that for any eigenvalue λ of the sub-Laplacian $\triangle$ we have the lower bound

$$\lambda \geq \frac{n}{n+2}k_0.$$

In fact, both the seven and higher dimensional cases follow from the positivity of the P-function.

A stronger condition than the one required in Theorem 1.1 is the assumption that the P-function of any smooth function is non-negative in which case we say that the $C-$operator of M is non-negative. Here C is a fourth-order differential operator on M (independent of f!) defined by

$$Cf = -\nabla^* P_f = (\nabla_{e_a} P_f)(e_a).$$

It turns out that this stronger condition holds on any compact qc manifold of dimension at least eleven, see Theorem 3.3. On the other hand we shall prove in Proposition 3.4 that on a seven dimensional qc-Einstein manifold of constant non-negative scalar curvature the $P-$function of any eigenfunction of the sub-Laplacian is non-negative. This fact implies the following

Corollary 1.2. *Let $(M, g, \mathbb{Q})$ be a compact quaternionic contact manifold of dimension seven which is qc-Einstein and of constant normalized positive scalar curvature $S = 2$. Then for any eigenvalue λ of the sub-Laplacian $\triangle$ we have the inequality*

$$\lambda \geq 4. \tag{1.3}$$

Furthermore, $\lambda = 4$ is an eigenvalue of the sub-Laplacian if and only if M is the unit seven dimensional 3-Sasakian sphere.

In particular, on a seven dimensional 3-Sasakian manifold any eiqnfunction λ of the sub-Laplacian satisfies the inequality (1.3) and the equality is attained only on the seven dimensional 3-Sasakian sphere.

We note that in [17] is given an explicit formula for the eigenfunctions of the above eigenvalue, see also [1].

Convention 1.3.

a) We shall use X, Y, Z, U to denote horizontal vector fields, i.e. $X, Y, Z, U \in H$.
b) $\{e_1, \dots, e_{4n}\}$ denotes a local orthonormal basis of the horizontal space H.
c) The summation convention over repeated vectors from the basis $\{e_1, \dots, e_{4n}\}$ will be used. For example, for a (0,4)-tensor P, the formula $k = P(e_b, e_a, e_a, e_b)$ means

$$k = \sum_{a,b=1}^{4n} P(e_b, e_a, e_a, e_b);$$

d) The triple (i, j, k) denotes any cyclic permutation of $(1, 2, 3)$.
e) s will be any number from the set $\{1, 2, 3\}$, $\quad s \in \{1, 2, 3\}$.

2. Quaternionic contact manifolds

Quaternionic contact manifolds were introduced in [3]. We also refer to [15] and [19] for results and further background.

2.1. **Quaternionic contact structures and the Biquard connection.** A quaternionic contact (qc) manifold $(M, g, \mathbb{Q})$ is a $4n + 3$-dimensional manifold M with a codimension three distribution H equipped with an $Sp(n)Sp(1)$ structure. Explicitly, H is the kernel of a local 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ together with a compatible Riemannian metric g and a rank-three bundle $\mathbb{Q}$ consisting of endomorphisms of H locally generated by three almost complex structures I_1, I_2, I_3 on H satisfying the identities of the imaginary unit quaternions. Thus, we have $I_1 I_2 = -I_2 I_1 = I_3, \quad I_1 I_2 I_3 = -id_{|_H}$ which are hermitian compatible with the metric $g(I_s., I_s.) = g(., .)$ and the following compatibility condition holds

$$2g(I_s X, Y) \ = \ d\eta_s(X, Y), \quad X, Y \in H.$$

On a qc manifold of dimension $(4n + 3) > 7$ with a fixed metric g on the horizontal distribution H there exists a canonical connection defined in [3]. In fact, Biquard showed that there is a unique connection ∇ with torsion T and a unique supplementary subspace V to H in TM, such that:

(i) ∇ preserves the decomposition $H \oplus V$ and the $Sp(n)Sp(1)$ structure on H, i.e. $\nabla g = 0, \nabla\sigma \in \Gamma(\mathbb{Q})$ for a section $\sigma \in \Gamma(\mathbb{Q})$, and its torsion on H is given by $T(X, Y) = -[X, Y]_{|V}$;
(ii) for $\xi \in V$, the endomorphism $T(\xi, .)_{|H}$ of H lies in $(sp(n) \oplus sp(1))^{\perp} \subset gl(4n)$;
(iii) the connection on V is induced by the natural identification φ of V with the subspace $sp(1)$ of the endomorphisms of H, i.e. $\nabla\varphi = 0$.

In ii), the inner product $<, >$ of $End(H)$ is given by $< A, B >= \sum_{i=1}^{4n} g(A(e_i), B(e_i))$, for $A, B \in End(H)$. We shall call the above connection *the Biquard connection*. When the dimension of M is at least eleven [3] also described the supplementary distribution V, which is (locally) generated by the so called Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ determined by

$$\begin{aligned} \eta_s(\xi_k) = \delta_{sk}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \\ (\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}, \end{aligned} \tag{2.1}$$

where $\lrcorner$ denotes the interior multiplication.

If the dimension of M is seven Duchemin shows in [13] that if we assume, in addition, the existence of Reeb vector fields as in (2.1), then the Biquard result holds. *Henceforth, by a qc structure in dimension* 7 *we shall mean a qc structure satisfying* (2.1). This implies the existence of the connection with properties (i), (ii) and (iii) above.

The fundamental 2-forms ω_s of the quaternionic structure Q are defined by

$$2\omega_{s|H} \;=\; d\eta_{s|H}, \qquad \xi\lrcorner\omega_s = 0, \quad \xi\in V, \tag{2.2}$$

and the torsion restricted to H has the form

$$T(X,Y) = -[X,Y]_{|V} = 2\omega_1(X,Y)\xi_1 + 2\omega_2(X,Y)\xi_2 + 2\omega_3(X,Y)\xi_3. \tag{2.3}$$

2.2. **Invariant decompositions.** Any endomorphism Ψ of H can be decomposed with respect to the quaternionic structure $(\mathbb{Q},g)$ uniquely into four $Sp(n)$-invariant parts $\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$, where Ψ^{+++} commutes with all three I_i, Ψ^{+--} commutes with I_1 and anti-commutes with the others two and etc. The two $Sp(n)Sp(1)$-invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \quad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}.$$

They are the projections on the eigenspaces of the Casimir operator

$$\Upsilon = \; I_1\otimes I_1 \; + \; I_2\otimes I_2 \; + \; I_3\otimes I_3, \tag{2.4}$$

corresponding, respectively, to the eigenvalues 3 and -1, see [4]. Note here that each of the three 2-forms ω_s belongs to its [-1]-component, $\omega_s = \omega_{s[-1]}$ and constitute a basis of the lie algebra $sp(1)$. In particular, using that $\left\{\frac{1}{2\sqrt{n}}\omega_s\right\}$ is an orthonormal set in $\Psi_{[-1]}$ we have

$$|(\nabla^2 f)_{[-1]}|^2 \geq \frac{1}{4n}\sum_{s=1}^{3}\left[g\left(\nabla^2 f,\omega_s\right)\right]^2 \tag{2.5}$$

while a projection on $\left\{\frac{1}{2\sqrt{n}}g\right\}$ gives

$$|(\nabla^2 f)_{[3]}|^2 \geq \frac{1}{4n}(\triangle f)^2. \tag{2.6}$$

If $n=1$ then the space of symmetric endomorphisms commuting with all I_s is 1-dimensional, i.e., the [3]-component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_3 = -\frac{tr\Psi}{4}Id_{|H}$. Thus, when $n=1$ we have the following fact.

Lemma 2.1. *The space $\Psi_{[3]}$ is four dimensional and the symmetric tensors in it are proportional to the metric. The space $\Psi_{[-1]}$ is twelve dimensional, in which lies the three dimensional space of the 2-forms ω_i. The latter determines the anti-symmetric part of the $\Psi_{[-1]}$ component.*

2.3. **The torsion tensor.** The torsion endomorphism $T_\xi = T(\xi,\cdot) : H \to H, \quad \xi\in V$ will be decomposed into its symmetric part T^0_ξ and skew-symmetric part $b_\xi, T_\xi = T^0_\xi + b_\xi$. Biquard showed [3] that the torsion T_ξ is completely trace-free, $tr\,T_\xi = tr\,T_\xi\circ I_s = 0$, its symmetric part has the properties $T^0_{\xi_i}I_i = -I_iT^0_{\xi_i} \quad I_2(T^0_{\xi_2})^{+--} = I_1(T^0_{\xi_1})^{-+-}, \quad I_3(T^0_{\xi_3})^{-+-} = I_2(T^0_{\xi_2})^{--+}, \quad I_1(T^0_{\xi_1})^{--+} = I_3(T^0_{\xi_3})^{+--}$, where the upperscript $+++$ means commuting with all three I_i, $+--$ indicates commuting with I_1 and anti-commuting with the other two and etc. The skew-symmetric part can be represented as $b_{\xi_i} = I_iU$, where U is a traceless symmetric (1,1)-tensor on H which commutes with I_1, I_2, I_3. Therefore we have $T_{\xi_i} = T^0_{\xi_i} + I_iU$. If $n=1$ then the tensor U vanishes identically, $U=0$, and the torsion is a symmetric tensor,

$$T_\xi = T^0_\xi.$$

The two $Sp(n)Sp(1)$-invariant trace-free symmetric 2-tensors on H

$$T^0(X,Y) = g((T^0_{\xi_1}I_1 + T^0_{\xi_2}I_2 + T^0_{\xi_3}I_3)X,Y) \;\text{ and }\; U(X,Y) = g(uX,Y) \tag{2.7}$$

were introduced in [15] and enjoy the properties

$$
\begin{aligned}
T^0(X,Y)+T^0(I_1X,I_1Y)+T^0(I_2X,I_2Y)+T^0(I_3X,I_3Y)=0,\\
U(X,Y)=U(I_1X,I_1Y)=U(I_2X,I_2Y)=U(I_3X,I_3Y).
\end{aligned}
\tag{2.8}
$$

From [19, Proposition 2.3] we have

$$
4T^0(\xi_s,I_sX,Y)=T^0(X,Y)-T^0(I_sX,I_sY),
\tag{2.9}
$$

hence, taking into account (2.9) it follows

$$
T(\xi_s,I_sX,Y)=T^0(\xi_s,I_sX,Y)+g(I_suI_sX,Y)=\frac{1}{4}\Big[T^0(X,Y)-T^0(I_sX,I_sY)\Big]-U(X,Y).
\tag{2.10}
$$

Any 3-Sasakian manifold has zero torsion endomorphism, and the converse is true if in addition the qc scalar curvature (see (2.11)) is a positive constant [15].

2.4. **Torsion and curvature.** Let $R=[\nabla,\nabla]-\nabla_{[\ ,\ ]}$ be the curvature tensor of ∇ and the dimension is $4n+3$. We denote the curvature tensor of type (0,4) and the torsion tensor of type (0,3) by the same letter, $R(A,B,C,D):=g(R(A,B)C,D)$, $\quad T(A,B,C):=g(T(A,B),C)$, $A,B,C,D\in\Gamma(TM)$. The Ricci tensor, the normalized scalar curvature and the Ricci 2-forms of the Biquard connection, called *qc-Ricci tensor* Ric, *normalized qc-scalar curvature* S and *qc-Ricci forms* ρ_s,τ_s, respectively, are defined by

$$
\begin{aligned}
&Ric(A,B)=R(e_b,A,B,e_b),\quad 8n(n+2)S=R(e_b,e_a,e_a,e_b),\quad \rho_s(A,B)=\frac{1}{4n}R(A,B,e_a,I_se_a),\\
&\tau_s(A,B)=\frac{1}{4n}R(e_a,I_se_a,A,B,),\quad \zeta_s(A,B)=\frac{1}{4n}R(e_a,A,B,I_se_a).
\end{aligned}
\tag{2.11}
$$

Definition 2.2. *A qc structure is said to be qc Einstein if the horizontal qc-Ricci tensor is a scalar multiple of the metric,*

$$
Ric(X,Y)=2(n+2)Sg(X,Y).
$$

As shown in [15], see also [16, 19, 18] for a different proof, the qc Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection. In this case S is constant and the vertical distribution is integrable provided $n>1$. It is also worth recalling that the horizontal qc-Ricci tensors and the integrability of the vertical distribution can be expressed in terms of the torsion of the Biquard connection [15] (see also [16, 19, 18]). For example, we have

$$
\begin{aligned}
&Ric(X,Y)=(2n+2)T^0(X,Y)+(4n+10)U(X,Y)+2(n+2)Sg(X,Y),\\
&\zeta_s(X,I_sY)=\frac{2n+1}{4n}T^0(X,Y)+\frac{1}{4n}T^0(I_sX,I_sY)+\frac{2n+1}{2n}U(X,Y)+\frac{S}{2}g(X,Y),\\
&T(\xi_i,\xi_j)=-S\xi_k-[\xi_i,\xi_j]_H,\qquad S=-g(T(\xi_1,\xi_2),\xi_3),\\
&g(T(\xi_i,\xi_j),X)=-\rho_k(I_iX,\xi_i)=-\rho_k(I_jX,\xi_j)=-g([\xi_i,\xi_j],X).
\end{aligned}
\tag{2.12}
$$

Note that for $n=1$ the above formulas hold with $U=0$.

We shall also need the general formula for the curvature [19, 21]

$$
\begin{aligned}
R(\xi_i,X,Y,Z)=&-(\nabla_XU)(I_iY,Z)+\omega_j(X,Y)\rho_k(I_iZ,\xi_i)-\omega_k(X,Y)\rho_j(I_iZ,\xi_i)\\
&-\frac{1}{4}\Big[(\nabla_YT^0)(I_iZ,X)+(\nabla_YT^0)(Z,I_iX)\Big]+\frac{1}{4}\Big[(\nabla_ZT^0)(I_iY,X)+(\nabla_ZT^0)(Y,I_iX)\Big]\\
&-\omega_j(X,Z)\rho_k(I_iY,\xi_i)+\omega_k(X,Z)\rho_j(I_iY,\xi_i)-\omega_j(Y,Z)\rho_k(I_iX,\xi_i)+\omega_k(Y,Z)\rho_j(I_iX,\xi_i),
\end{aligned}
\tag{2.13}
$$

where the Ricci two forms are given by, cf. [19, Theorem 3.1] or [21, Theorem4.3.11]

$$
\begin{aligned}
&6(2n+1)\rho_s(\xi_s,X)=(2n+1)X(S)+\frac{1}{2}(\nabla_{e_a}T^0)[(e_a,X)-3(I_se_a,I_sX)]-2(\nabla_{e_a}U)(e_a,X),\\
&6(2n+1)\rho_i(\xi_j,I_kX)=(2n-1)(2n+1)X(S)-\frac{1}{2}(\nabla_{e_a}T^0)[(4n+1)(e_a,X)+3(I_ie_a,I_iX)]\\
&\qquad\qquad -4(n+1)(\nabla_{e_a}U)(e_a,X).
\end{aligned}
\tag{2.14}
$$

2.5. **The Ricci identities.** We shall use repeatedly the following Ricci identities of order two and three, see also [19]. Let ξ_i, $i=1,2,3$ be the Reeb vector fields, $X,Y\in H$ and f a smooth function on the qc manifold M with ∇f its horizontal gradient of f, $g(\nabla f,X)=df(X)$. We have the following Ricci identities

$$
\begin{aligned}
&\nabla^2 f(X,Y)-\nabla^2 f(Y,X)=-2\sum_{s=1}^{3}\omega_s(X,Y)df(\xi_s)\\
&\nabla^2 f(X,\xi_s)-\nabla^2 f(\xi_s,X)=T(\xi_s,X,\nabla f)\\
&\nabla^3 f(X,Y,Z)-\nabla^3 f(Y,X,Z)=-R(X,Y,Z,\nabla f)-2\sum_{s=1}^{3}\omega_s(X,Y)\nabla^2 f(\xi_s,Z)\\
&\nabla^3 f(\xi_i,X,Y)=\nabla^3 f(X,Y,\xi_i)-\nabla^2 f\left(T\left(\xi_i,X\right),Y\right)-\nabla^2 f\left(X,T\left(\xi_i,Y\right)\right)-df\left(\left(\nabla_X T\right)\left(\xi_i,Y\right)\right)\\
&\qquad\qquad -R(\xi_i,X,Y,\nabla f).
\end{aligned}
\tag{2.15}
$$

In particular we have

$$
g(\nabla^2 f,\omega_s)=\nabla^2 f(e_a,I_se_a)=-4ndf(\xi_s). \tag{2.16}
$$

Now, Lemma 2.1 and the Ricci identities show that for any smooth function f on a seven dimensional qc manifold we have

$$
\begin{aligned}
&(\nabla^2 f)_{[3]}(X,Y)=-\frac{\triangle f}{4}g(X,Y),\qquad (\nabla^2 f)_{[3][a]}(X,Y)=\frac{1}{2}\left[(\nabla^2 f)_{[3]}(X,Y)-(\nabla^2 f)_{[3]}(Y,X)\right]=0\\
&(\nabla^2 f)_{[-1][a]}(X,Y)=-df(\xi_i)\omega_i(X,Y).
\end{aligned}
\tag{2.17}
$$

2.6. **The horizontal divergence theorem and qc-normal frames.** Let $(M,g,\mathbb{Q})$ be a qc manifold of dimension $4n+3\geq 7$ For a fixed local 1-form η and a fix $s\in\{1,2,3\}$ the form

$$
Vol_\eta=\eta_1\wedge\eta_2\wedge\eta_3\wedge\omega_s^{2n} \tag{2.18}
$$

is a locally defined volume form. Note that Vol_η is independent on s as well as it is independent on the local one forms η_1,η_2,η_3. Hence it is globally defined volume form denoted with Vol_η.

We define the (horizontal) divergence of a horizontal vector field/one-form $\sigma\in\Lambda^1(H)$ as

$$
\nabla^*\sigma\ =-tr|_H\nabla\sigma=\ -\nabla\sigma(e_a,e_a). \tag{2.19}
$$

The integration by parts formula from [15], see also [31], takes the form

$$
\int_M(\nabla^*\sigma)\ Vol_\eta\ =\ 0. \tag{2.20}
$$

Finally, we recall that an orthonormal frame

$$
\{e_1,e_2=I_1e_1,e_3=I_2e_1,e_4=I_3e_1,\dots,e_{4n}=I_3e_{4n-3},\xi_1,\xi_2,\xi_3\}
$$

is a qc-normal frame (at a point) if the connection 1-forms of the Biquard connection vanish (at that point). Lemma 4.5 in [15] asserts that a qc-normal frame exists at each point of a qc manifold.

3. The $P-$form

Let $(M,g,\mathbb{Q})$ be a compact quaternionic contact manifold of dimension $4n+3$ and f a smooth function on M.

Definition 3.1. *For a fixed f we define a one form $P\equiv P_f\equiv P[f]$ on M, which we call the $P-$form of f, by the following equation*

$$
\begin{aligned}
P_f(X)=&\nabla^3 f(X,e_b,e_b)+\sum_{t=1}^{3}\nabla^3 f(I_tX,e_b,I_te_b)-4nSdf(X)+4nT^0(X,\nabla f)-\frac{8n(n-2)}{n-1}U(X,\nabla f),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } n>1,\\
P_f(X)=&\nabla^3 f(X,e_b,e_b)+\sum_{t=1}^{3}\nabla^3 f(I_tX,e_b,I_te_b)-4Sdf(X)+4T^0(X,\nabla f),\quad \textit{if } n=1.
\end{aligned}
\tag{3.1}
$$

The $P-$function of f is the function $P_f(\nabla f)$. Finally, the $C-$operator is the fourth-order differential operator on M (independent of f!) defined by

$$Cf = -\nabla^* P_f = (\nabla_{e_a} P_f)(e_a). \tag{3.2}$$

We say that the $P-$function of f is non-negative if its integral exists and is non-positive

$$\int_M f\cdot Cf\, Vol_\eta = -\int_M P_f(\nabla f)\, Vol_\eta \geq 0. \tag{3.3}$$

If (3.3) holds for any smooth function of compact support we say that the $C-$operator is non-negative.

One of the key identities which relates the P-function and the Bochner formula is given in the following

Lemma 3.2. *On a qc manifold of dimension $4n+3$ we have*

$$\sum_{s=1}^3 \nabla^2 f(\xi_s, I_s X) = \frac{1}{4n}\sum_{s=1}^3 \nabla^3 f(I_sX, I_se_a, e_a) - \sum_{s=1}^3 T(\xi_s, I_sX, \nabla f). \tag{3.4}$$

In addition, if the manifold is compact, then the next integral formula holds

$$\int_M \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s\nabla f)\, Vol_\eta = \int_M \Big[-\frac{1}{4n}P_n(\nabla f) - \frac{1}{4n}(\triangle f)^2 - S|\nabla f|^2 + \frac{(n+1)}{n-1}U(\nabla f,\nabla f)\Big]\, Vol_\eta. \tag{3.5}$$

Proof. Working in a qc-normal frame, taking into account the $Sp(n)Sp(1)$ invariance of $\nabla^2 f(\xi_s, I_s\nabla f)$, we have

$$\begin{aligned}\sum_{s=1}^3 \nabla^3 f(I_sX, I_se_a, e_a) &= \sum_{s=1}^3 \nabla_{I_sX}[\nabla^2 f(I_se_a, e_a)] = \sum_{s=1}^3 \nabla_{I_sX}[4n df(\xi_s)] = 4n\sum_{s=1}^3 \nabla^2 f(I_sX, \xi_s)\\ &= 4n\sum_{s=1}^3 \nabla^2 f(\xi_s, I_sX) + 4n\sum_{s=1}^3 T(\xi_s, I_sX, \nabla f),\end{aligned}$$

where we used (2.16) for the second equality and the second equality in (2.15) for the fourth one.

For the second part of the lemma, first we express the term $\sum_{s=1}^3 \nabla^3 f(I_sX, I_se_a, e_a)$ in (3.4) by the definition (3.1) of P_f. Then we replace X by ∇f, integrate over M and use the easily checked equality

$$\int_M \nabla^3(\nabla f, e_a, e_a)\, Vol_\eta = -\int_M (\triangle f)^2\, Vol_\eta \tag{3.6}$$

for the term $\nabla^3 f(\nabla f, e_a, e_a)$. Finally we use

$$2\sum_{s=1}^3 T(\xi_s, I_s\nabla f, \nabla f) = 2T^0(\nabla f, \nabla f) - 6U(\nabla f, \nabla f), \tag{3.7}$$

which follows from (2.10) together with (2.8). This proves (3.5). □

3.1. **The non-negativity of the C-operator for $n>1$.** Let B_0 be the trace-free part of the 3-component of the horizontal Hessian.

$$4B_0(X,Y) = \nabla^2 f(X,Y) + \nabla^2 f(I_1X, I_1Y) + \nabla^2 f(I_2X, I_2Y) + \nabla^2 f(I_3X, I_3Y) + \frac{1}{n}\triangle f g(X,Y). \tag{3.8}$$

Theorem 3.3. *On a qc manifold of dimension $4n+3$ we have the formula*

$$4(\nabla_{e_a} B_0)(e_a, X) = \frac{n-1}{n}P_n(X). \tag{3.9}$$

In particular, if the manifold is compact then the $C-$operator is non-negative for any dimension bigger than seven. In this case for any function f the function Cf vanishes exactly when the trace-free part of the 3-component of a function vanishes. In this case the $P-$form of f vanishes as well.

Proof. We have from the Ricci identities that:

$$\nabla^3 f(e_a,e_a,X)=\nabla^3 f(X,e_a,e_a)+Ric(X,\nabla f)+4\sum_{s=1}^3\nabla^2(\xi_s,I_sX)+2\sum_{s=1}^3 T(\xi_s,I_sX,\nabla f) \tag{3.10}$$

Similarly, we obtain

$$\begin{aligned}\nabla^3 f(e_a,I_se_a,I_sX)&=\nabla^3 f(I_sX,e_a,I_se_a)+4n\zeta_s(I_sX,\nabla f)-2\sum_{t=1}^3\omega_t(I_se_a,I_sX)T(\xi_t,e_a,\nabla f)\\&=\nabla^3 f(I_sX,e_a,I_se_a)+4n\zeta_s(I_sX,\nabla f)-2\sum_{s=1}^3 T(\xi_s,I_sX,\nabla f).\end{aligned} \tag{3.11}$$

A substitution of (3.10), (3.11), (3.4) and the first and second equality of (2.12) in (3.8) imply (3.9). The last statement follows by integration by parts using the orthogonality of the components of the horizontal Hessian. □

3.2. **Non-negativity of the $P-$functions of eigenfunctions in dimension seven.** In dimension seven we show the following

Proposition 3.4. *On a seven dimensional compact qc-Einstein manifold of constant positive qc scalar curvature, $S=2$, the $P-$function of any eigenfunction of the sub-Laplacian is non-negative.*

Proof. Suppose f satisfies $\triangle f=-\nabla^2 f(e_a,e_a)=\lambda f$. We calculate

$$\begin{aligned}\int_M|P_f|^2\,Vol_\eta&=\int_M\Big[\nabla^3 f(e_a,e_b,e_b)+\sum_{t=1}^3\nabla^3 f(I_te_a,e_b,I_te_b)-4Sdf(e_a)\Big]^2\,Vol_\eta\\&=-4S\int_M P_1(\nabla f)\,Vol_\eta-\int_M\nabla^2 f(e_b,e_b)Cf\,Vol_\eta\\&\quad-\int_M\sum_{s,t=1}^3\nabla^2 f(e_b,I_se_b)\Big[\nabla^4 f(I_se_a,e_a,e_c,e_c)+\nabla^4 f(I_se_a,I_te_a,e_c,I_te_c)-4S\nabla^2 f(I_se_a,e_a)\Big]\,Vol_\eta\\&=-(\lambda+4S)\int_M P_1(\nabla f)\,Vol_\eta-4S\int_M\sum_{s=1}^3[\nabla^2 f(e_a,I_se_a)]^2\,Vol_\eta\\&\quad-8\int_M\sum_{s=1}^3\nabla^2 f(e_b,I_se_b)[\nabla^2(\xi_s,e_c,e_c)-\nabla^2(e_c,e_c,\xi_s)]\,Vol_\eta-\int_M\sum_{s\neq t}\nabla^2 f(e_b,I_se_b)\nabla^4 f(I_se_a,I_te_a,e_c,I_te_c)\\&=-(\lambda+4S)\int_M P_f(\nabla f)\,Vol_\eta-8^3\int_M\sum_{s=1}^3[df(\xi_s)]^2\,Vol_\eta+8\int_M df(\xi_i)8^2[\nabla^2(\xi_k,\xi_j)-\nabla^2(\xi_j,\xi_k)]\,Vol_\eta,\end{aligned} \tag{3.12}$$

where we used the first Ricci identity in (2.15) several times and the fact that third covariant derivatives commute on a qc-Einstein manifold with constant scalar curvature because of (2.14) and the last of the Ricci identities in (2.15). The assumptions $T^0=U=0, S=2=const$ together with the second equality in (2.14) and the fourth equality in (2.12) imply that the vertical space is integrable, $g([\xi_s,\xi_t],X)=0$. Now, the Ricci identity, the integrability of the vertical space and the third equality in (2.12) yield

$$\nabla^2 f(\xi_k,\xi_j)-\nabla^2 f(\xi_j,\xi_k)=T(\xi_j,\xi_k,df)=-2df(\xi_i).$$

A substitution of this equality in (3.12) gives

$$\int_M|P_f|^2\,Vol_\eta=-(\lambda+8)\int_M P_f(\nabla f)\,Vol_\eta-8^3\int_M\sum_{s=1}^3[df(\xi_s)]^2\,Vol_\eta-2\times8^3\int_M\sum_{s=1}^3[df(\xi_s)]^2\,Vol_\eta \tag{3.13}$$

and the non-negativity of the $P-$function of f follows since $\lambda>0$. □

4. The Lichnerowicz type result for a seven dimensional qc structure

Here we shall prove our main Theorem 1.1. From (2.17) and (2.16) we have

$$|(\nabla^2 f)_{[3]}|^2 = \frac{(\triangle f)^2}{4}, \qquad |(\nabla^2 f)_{[-1][a]}|^2 = 4\sum_{s=1}^{3}(\xi_s f)^2. \tag{4.1}$$

Next, we recall the four identities given by [18, Lemma 3.3], [18, Lemma 3.4], Lemma 3.2, and the Bochner identity [18, (4.1)]. However, in this lowest dimension, Bochner's identity [18, (4.1)] and [18, Lemma 3.3] are identical. Therefore, in dimension seven, using $U = 0$, (4.1) and (3.7) we have the following three identities

$$\begin{aligned}
&\int_M \sum_{s=1}^{3} \nabla^2 f(\xi_s, I_s\nabla f)\, Vol_\eta = -\int_M \Big[|(\nabla^2 f)_{[-1][a]}|^2 + T^0(\nabla f,\nabla f)\Big]\, Vol_\eta,\\
&\int_M \sum_{s=1}^{3} \nabla^2 f(\xi_s, I_s\nabla f)\, Vol_\eta = \int_M \Big[\frac{3}{16}(\triangle f)^2 - \frac{1}{4}|(\nabla^2 f)_{[-1][a]}|^2 - \frac{1}{4}|(\nabla^2 f)_{[-1][s]}|^2\Big]\, Vol_\eta\\
&\qquad\qquad + \int_M \Big[-\frac{3}{2}T^0(\nabla f,\nabla f) - \frac{3}{2}S|\nabla f|^2\Big]\, Vol_\eta,\\
&\int_M \sum_{t=1}^{3} \nabla^2 f(\xi_t, I_t\nabla f)\, Vol_\eta = \int_M \Big[-T^0(\nabla f,\nabla f) - \frac{1}{4}\sum_{t=1}^{3}\nabla^3 f(I_t\nabla f, e_b, I_t e_b)\Big]\, Vol_\eta,
\end{aligned} \tag{4.2}$$

corresponding to [18, Lemma 3.4], [18, Lemma 3.3] and (3.4), respectively.

The a-priori Lichnerowicz type assumption gives the inequality

$$10T^0(\nabla f,\nabla f) + 6S|\nabla f|^2 \ge k_0|\nabla f|^2.$$

From the first two formulas in (4.2) we have

$$0 \;=\; \int_M \Big[\frac{3}{16}(\triangle f)^2 + \frac{3}{4}|(\nabla^2 f)_{[-1][a]}|^2 - \frac{1}{4}|(\nabla^2 f)_{[-1][s]}|^2 - \frac{1}{2}T^0(\nabla f,\nabla f) - \frac{3}{2}S|\nabla f|^2\Big]\, Vol_\eta$$

while the first and the last formula in (4.2) give

$$\int_M |(\nabla^2 f)_{[-1][a]}|^2\, Vol_\eta \;=\; \int_M \frac{1}{4}\sum_{t=1}^{3}\nabla^3 f(I_t\nabla f, e_b, I_t e_b)\, Vol_\eta.$$

A substitution of the last equation in the previous identity brings us to

$$\begin{aligned}
0 = &\int_M \Big[-\frac{3}{16}(\triangle f)^2 - \frac{3}{16}\sum_{t=1}^{3}\nabla^3 f(I_t\nabla f, e_b, I_t e_b) + \frac{1}{4}|(\nabla^2 f)_{[-1][s]}|^2 + \frac{1}{2}T^0(\nabla f,\nabla f) + \frac{3}{2}S|\nabla f|^2\Big]\, Vol_\eta\\
&= \int_M \Big[\Big(\frac{5}{4}T^0(\nabla f,\nabla f) + \frac{3}{4}S|\nabla f|^2 - \frac{3}{8}\lambda|\nabla f|^2\Big) + \frac{1}{4}|(\nabla^2 f)_{[-1][s]}|^2\Big]\, Vol_\eta\\
&+ \int_M \Big[\frac{3}{16}(\triangle f)^2 - \frac{3}{16}\sum_{t=1}^{3}\nabla^3 f(I_t\nabla f, e_b, I_t e_b) + \frac{3}{4}S|\nabla f|^2 - \frac{3}{4}T^0(\nabla f,\nabla f)\Big]\, Vol_\eta\\
&= \int_M \Big[\Big(\frac{5}{4}T^0(\nabla f,\nabla f) + \frac{3}{4}S|\nabla f|^2 - \frac{3}{8}\lambda|\nabla f|^2\Big) + \frac{1}{4}|(\nabla^2 f)_{[-1][s]}|^2 - \frac{3}{16}P_f(\nabla f)\Big]\, Vol_\eta,
\end{aligned} \tag{4.3}$$

where P_f was defined in (3.1). If we assume that the $P-$function of f is non-negative,

$$-\int_M P_f(\nabla f)\, Vol_\eta \ge 0,$$

then we have

$$0 \ge \int_M \Big[\Big(\frac{1}{8}k_0 - \frac{3}{8}\lambda|\nabla f|^2\Big) + \frac{1}{4}|(\nabla^2 f)_{[-1][s]}|^2\Big]\, Vol_\eta. \tag{4.4}$$

Therefore,

$$\lambda \geq k_0/3.$$

This finishes the proof of Theorem 1.1.

Remark 4.1. *Note that in the extremal case in Theorem 1.1 when $\lambda = \frac{1}{3}k_0$ it follows from (4.3) that the corresponding extremal eigenfunction f satisfies the equalities*

$$(\nabla^2 f)_{[-1][s]} = 0, \qquad \int_M P_f(\nabla f)\, Vol_\eta = 0. \tag{4.5}$$

The first equality in (4.5) together with (4.1) imply that the horizontal Hessian of an extremal eigenfunction is given by

$$(\nabla^2 f)(X,Y) = -\frac{k_0}{12} f g(X,Y) - \sum_{s=1}^{3} df(\xi_s)\omega_s(X,Y).$$

5. Proof of Corollary 1.2

In view of Theorem 1.1 and Proposition 3.4 we only have to show that if $\lambda = 4$ is a an eigenvalue then M is the standard unit 3-Sasakian sphere. Since M is a qc-Einstein manifold of constant normalized positive scalar curvature $S = 2$ then by [15, Theorem 1.3] it follows M is locally 3-Sasakian. In other words, locally there exists an $SO(3)$-matrix Ψ with smooth entries, such that, the local qc structure determined by $\Psi \cdot \eta$ is 3-Sasakian. The claim then follows as in the higher dimensional case [18, Theorem 1.2] invoking the (Riemannian) Obata theorem. For this we consider the natural Riemannian metric (also denoted by) g defined using the triple of Reeb vector fields extending g to a metric on M by requiring $span\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and $g(\xi_s, \xi_k) = \delta_{sk}$. The extended metric does not depend on the action of $SO(3)$ on V and the Biquard connection is metric.

On one hand we have the well known fact that a 3-Sasakian manifold of dimension $4n+3$ is Einstein with respect to the extended metric with Riemannian scalar curvature $4n+2$ [25], i.e., the Riemannian Ricci tensor Ric^g is given by

$$Ric^g(A,A) = (4n+2)\, g(A,A). \tag{5.1}$$

By Lichnerowicz' theorem [26] and (5.1) we have

$$\mu \geq 4n+3 = 7, \tag{5.2}$$

where μ is the first eigenvalue of the Riemannian Laplacian of the extended metric.

On the other hand, it follows from the variational characterization of the first eigenvalues and the relation bewteen the Riemannian Laplacian and the sub-Laplacian, see [18, Proposition 5.2] that

$$\mu \leq \lambda + \int_M \sum_{s=1}^{3} (df(\xi_s))^2\, Vol_\eta \tag{5.3}$$

for any smooth function f with $\int_M f^2\, Vol_\eta = 1$.

After a possible rescaling of f and using the divergence formula we have then the following identities

$$\lambda = 4, \quad \triangle f = 4f, \quad \int_M f^2\, Vol_\eta = 1, \quad \int_M |\nabla f|^2\, Vol_\eta = 4 = \frac{1}{4}\int_M (\triangle f)^2\, Vol_\eta. \tag{5.4}$$

For $\lambda = 4$ the first equality in (4.5) combined with the first two equations in (4.2) and (4.1) yield

$$\int_M \sum_{s=1}^{3} (df(\xi_s))^2\, Vol_\eta = 3. \tag{5.5}$$

Now, from (5.5) and (5.3) we have the inequality $\mu \leq 4+3 = 7$ which combined with (5.2) yields the equality $\mu = 4+3 = 7$. Therefore, by Obata's result [29] we conclude that the manifold (M,g) is isometric to the sphere $S^7(1)$ and hence the manifold $(M, g, \mathbb{Q})$ is qc equivalent to the 3-Sasakian sphere of dimension 7. The last statement follows from the fact that any 3-Sasakian manifold satisfies $T^0 = U = 0, S = 2$ [15]. This completes the proof of of Corollary 1.2.

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