THE OBATA SPHERE THEOREMS ON A QUATERNIONIC CONTACT MANIFOLD OF DIMENSION BIGGER THAN SEVEN

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Abstract. On a compact quaternionic contact (qc) manifold of dimension bigger than seven and satisfying a Lichnerowicz type lower bound estimate we show that if the first positive eigenvalue of the sub-Laplacian takes the smallest possible value then, up to a homothety of the qc structure, the manifold is qc equivalent to the standard 3-Sasakian sphere. The same conclusion is shown to hold on a non-compact qc manifold which is complete with respect to the associated Riemannian metric assuming the existence of a function with traceless horizontal Hessian.

Contents

1. Introduction

Motivated by the classical Lichnerowicz [55] and Obata [62] theorems, earlier papers of the authors [36, 37] established a Lichnerowicz type lower bound estimate for the first eigenvalue of the sub-Laplacian on a compact quaternionic contact (qc) manifold. The case of equality in the lower bound estimate (Obata-type theorem) was settled in the special case of a 3-Sasakian compact manifold where it was shown that the lower bound for the first eigenvalue of the sub-Laplacian is achieved if and only if the 3-Sasakian manifold is isometric to the standard 3-Sasakian sphere. Quaternionic contact (qc) structures were introduced by O.

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Biquard [6] and are modeled on the conformal boundary at infinity of the quaternionic hyperbolic space. Thus, manifolds equipped with a qc structure are examples of sub-Riemannian geometries. The (locally) 3-Sasakian manifolds were characterized in [32, 40] by the vanishing of the torsion tensor of the Biquard connection. The qc geometry was a crucial geometric tool in finding the extremals and the best constant in the L^2 Folland-Stein Sobolev-type embedding, [23, 24], completely described on the quaternionic Heisenberg groups, [34, 35].

In this paper we prove the full qc version of Obata's results for a general qc manifold of dimension bigger than seven. We find that the equality case of Lichnerowicz' type inequality on a compact qc manifold of dimension at least eleven can be achieved only on the 3-Sasakian spheres. More general, we show that on a complete with respect to the associated Riemannian metric qc manifold a certain (horizontal) Hessian equation, cf. (1.6), allows a non-trivial solution if and only if the manifold is qc homothetic to the standard 3-Sasakian sphere.

The qc seven dimensional case was considered in [37], however, the general qc Obata results in dimension seven remain open.

Turning to some details, let us recall the mentioned classical results. Using the classical Bochner-Weitzenböck formula Lichnerowicz [55] showed that on a compact Riemannian manifold (M, h) of dimension n for which the Ricci curvature satisfies $Ric(X, Y) \geq (n-1)h(X, Y)$ the first positive eigenvalue λ_1 of the (positive) Laplace operator satisfies the inequality $\lambda_1 \geq n$. Subsequently, Obata [62] proved that equality is achieved if and only if the Riemannian manifold is isometric to the round unit sphere. Obata observed that the trace-free part of the Riemannian Hessian of an eigenfunction f with eigenvalue $\lambda = n$ vanishes, i.e., it satisfies the system

$$(\nabla^h)^2 f = -fh \tag{1.1}$$

after which he defined an isometry using analysis based on the geodesics and Hessian comparison of the distance function from a point. In fact, Obata showed that on a complete Riemannian manifold (M, h) equation (1.1) allows a non-constant solution if and only if the manifold is isometric to the round unit sphere. In this case, the eigenfunctions corresponding to the first eigenvalue are the solutions of (1.1). Later, Gallot [26] generalized these results to statements involving the higher eigenvalues and corresponding eigenfunctions of the Laplace operator.

The interest in relations between the spectrum of the Laplacian and geometric quantities justified the interest in Lichnerowicz-Obata type theorems in other geometric settings such as Riemannian foliations (and the eigenvalues of the basic Laplacian) [51, 50], [47] and [63], to CR geometry (and the eigenvalues of the sub-Laplacian) [29], [4], [16, 14, 15], [17], [19], [52], and to general sub-Riemannian geometries, see [5] and [31]. In the CR case, Greenleaf [29] gave a version of Lichnerowicz' result showing that if a compact strongly pseudo-convex CR manifold M of dimension $2n+1$, $n \geq 3$ satisfies a Lichnerowicz type inequality

$$Ric(X, X) + 4A(X, JX) \geq (n+1)g(X, X)$$

for all horizontal vectors X, where Ric and A are, correspondingly, the Ricci curvature and the Webster torsion of the Tanaka-Webster connection (in the notation from [44, 41]), then the first positive eigenvalue λ_1 of the sub-Laplacian satisfies the inequality $\lambda_1 \geq n$. The standard (Sasakian) CR structure on the sphere achieves equality in this inequality. Following [29] the above cited results on a compact CR manifold focused on adding a corresponding inequality for $n = 1, 2$ or characterizing the equality case mainly in the vanishing Webster-torsion case (the Sasakian case). The general case on a compact CR manifold satisfying the Lichnerowicz type condition was proved in [53, 54] using the results and the method of [42]. This was achieved by introducing a new integration by parts step proving the vanishing of the Webster torsion assuming the first eigenvalue is equal to n (for the three dimensional case see [43]). On the other hand, a generalization of the Obata result in the complete non-compact case was achieved in [42], where the standard Sasakian structure on the unit sphere was characterized through the existence of a non-trivial solution of a (horizontal) Hessian equation on a complete with respect to the associated Riemannian metric CR manifold with a divergence free Webster torsion. To the best of our knowledge the case of a general torsion remains still open.

The main purpose of this paper is to prove the qc version of both results of Obata under no extra assumptions on the Biquard' torsion when the dimension of the qc manifold is at least eleven, cf. Theorem 1.2 and Theorem 1.3. In particular, completeness rather than compactness is required in the second result, cf. Theorem 1.3, in contrast to the currently known CR case as mentioned in the previous paragraph.

The quaternionic contact version of the Lichnerowicz' result was found in [36] in dimensions grater than seven and in [37] in the seven dimensional case. The following result of [36] gives a lower bound on the positive eigenvalues of the sub-Laplacian on a qc manifold.

Theorem 1.1 ([36])**.** *Let $(M, \eta, g, \mathbb{Q})$ be a compact quaternionic contact manifold of dimension $4n + 3 > 7$. Suppose that there is a positive constant k_0 such that the qc-Ricci tensor and torsion of the Biquard connection satisfy the inequality*

$$Ric(X,X) + \frac{2(4n+5)}{2n+1}T^0(X,X) + \frac{6(2n^2+5n-1)}{(n-1)(2n+1)}U(X,X) \geq k_0 g(X,X), \tag{1.2}$$

where Ric, T^0, U are, correspondingly, the Ricci curvature and the components of the torsion of the Biquard connection and X is a horizontal vector.

Then, any eigenvalue λ of the sub-Laplacian $\triangle$ satisfies the inequality

$$\lambda \geq \frac{n}{n+2}k_0.$$

The equality case of Theorem 1.1 is achieved on the 3-Sasakian sphere. It was shown in [35], see also [2], that the eigenspace of the first non-zero eigenvalue of the sub-Laplacian on the unit 3-Sasakian sphere in Euclidean space is given by the restrictions to the sphere of all linear functions.

The main results of this paper are the following two theorems.

Theorem 1.2. *Let $(M, \eta, g, \mathbb{Q})$ be a compact quaternionic contact manifold of dimension $4n + 3 > 7$ whose qc-Ricci tensor and torsion of the Biquard connection satisfy the inequality (1.2). Then, the first positive eigenvalue λ of the sub-Laplacian $\triangle$ satisfies the equality*

$$\lambda = \frac{n}{n+2}k_0 \tag{1.3}$$

if and only if the qc manifold $(M, g, \mathbb{Q})$ is qc-homothetic to the unit (4n+3)-dimensional 3-Sasakian sphere.

According to [36, Remark 4.1], under the conditions of Theorem 1.1, an eigenfunction f corresponding to the first non-zero eigenvalue as in (1.3), $\triangle f = \frac{n}{n+2}k_0 f$, satisfies a linear PDE system, namely, the horizontal Hessian of f is given by (see Corollary 4.2 in the Appendix)

$$\nabla df(X,Y) = -\frac{1}{4(n+2)}k_0 f g(X,Y) - \sum_{s=1}^{3} df(\xi_s)\omega_s(X,Y), \tag{1.4}$$

where ξ_1, ξ_2, ξ_3 and $\omega_1, \omega_2, \omega_3$ are the vertical Reeb vector fieds and the fundamental 2-forms, respectively.

This brings us to our second main result, in which no compactness of M is assumed a-priori,

Theorem 1.3. *Let $(M, \eta, g, \mathbb{Q})$ be a quaternionic contact manifold of dimension $4n+3 > 7$ which is complete with respect to the associated Riemannian metric*

$$h = g + (\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2. \tag{1.5}$$

Suppose there exists a non-constant smooth function f whose horizontal Hessian satisfies

$$\nabla df(X,Y) = -fg(X,Y) - \sum_{s=1}^{3} df(\xi_s)\omega_s(X,Y). \tag{1.6}$$

Then the qc manifold $(M, \eta, g, \mathbb{Q})$ is qc homothetic to the unit (4n+3)-dimensional 3-Sasakian sphere.

Clearly Theorem 1.3 implies Theorem 1.2 since any Riemannian metric on a compact manifold is complete and a qc-homothety allows us to reduce to the case $k_0 = 4(n + 2)$, which turns (1.4) in (1.6).

We prove Theorem 1.3 by showing first that M is isometric to the unit sphere S^{4n+3} and then that M is qc-equivalent to the standard 3-Sasakian structure on S^{4n+3}. To this effect we show that the torsion of the Biquard connection vanishes and in this case the Riemannian Hessian satisfies (1.1) after which we invoke the classical Obata theorem showing that M is isometric to the unit sphere. In order to prove the qc-equivalence part we show that the qc-conformal curvature vanishes, which gives the local qc-conformal equivalence with the 3-Sasakian sphere due to [39, Theorem 1.3]. The existence of a global qc-conformal map between M and the 3-Sasakian sphere follows, for example, from a qc Liouville-type result on the extension of a local (qc-conformal) automorphism to a global one, see [10, Proposition 1.5.2] for a general statement in the setting of Cartan geometries.

In the Appendix, for completeness, we recall the notion of the P-function introduced in [37] and give a different proof of Theorem 1.1 based on the positivity of the P-function in the case $n > 1$ established in [37, Theorem 3.3]. As a corollary of the proof, we show the validity of (1.4) for any eigenfunction of the sub-Laplacian with eigenvalue given by (1.3).

Convention 1.4.

a) We shall use X, Y, Z, U *to denote horizontal vector fields, i.e.* $X, Y, Z, U \in H$.

b) $\{e_1, \dots, e_{4n}\}$ *denotes a local orthonormal basis of the horizontal space* H.

c) The summation convention over repeated vectors from the basis $\{e_1, \dots, e_{4n}\}$ *will be used. For example, for a (0,4)-tensor* P*, the formula* $k = P(e_b, e_a, e_a, e_b)$ *means* $k = \sum_{a,b=1}^{4n} P(e_b, e_a, e_a, e_b)$.

d) The triple (i, j, k) *denotes any cyclic permutation of* $(1, 2, 3)$.

e) The sum $\sum_{(ijk)}$ *means the cyclic sum. For example,*

$$\sum_{(ijk)} df(I_iX)\omega_j(Y,Z) = df(I_1X)\omega_2(Y,Z) + df(I_2X)\omega_3(Y,Z) + df(I_3X)\omega_1(Y,Z).$$

e) s *will be any number from the set* $\{1, 2, 3\}$, $\quad s \in \{1, 2, 3\}$.

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2. Quaternionic contact manifolds

In this section we will briefly review the basic notions of quaternionic contact geometry and recall some results from [6], [32] and [39] which we will use in this paper.

It is well known that the sphere at infinity of a non-compact symmetric space M of rank one carries a natural Carnot-Carathéodory structure, see [58, 60]. In the real hyperbolic case one obtains the conformal class of the round metric on the sphere. In the remaining cases, each of the complex, quaternion and octonionic hyperbolic metrics on the unit ball induces a Carnot-Carathéodory structure on the unit sphere. This defines a conformal structure on a sub-bundle of the tangent bundle of co-dimension $\dim_{\mathbb{R}} \mathbb{K} - 1$, where $\mathbb{K} = \mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. In the complex case the obtained geometry is the well studied standard CR structure on the unit sphere in complex space. Quaternionic contact (qc) structure were introduced by O. Biquard, see [6], and are modeled on the conformal boundary at infinity of the quaternionic hyperbolic space. Biquard showed that the infinite dimensional family [49] of complete quaternionic-Kähler deformations of the quaternion hyperbolic metric have conformal infinities which provide an infinite dimensional family of examples of qc structures. Conversely, according to [6] every real analytic qc structure on a manifold M of dimension at least eleven is the conformal infinity of a unique quaternionic-Kähler metric defined in a neighborhood of M. Furthermore, [6] considered CR and qc structures as boundaries of infinity of Einstein metrics rather than only as boundaries at infinity of Kähler-Einstein and quaternionic-Kähler metrics, respectively. In fact, in [6] it was shown that in each of the three cases (complex, quaternionic, octoninoic) any small perturbation of the standard Carnot-Carathéodory structure on the boundary is the conformal infinity of an essentially unique Einstein metric on the unit ball, which is asymptotically symmetric. In the Riemannian case the corresponding question was posed in [22] and the perturbation result was proven in [28].

Another natural extension of an interesting Riemannian problem is the quaternionic contact Yamabe problem, a particular case of which [27, 65, 32, 34] amounts to finding the best constant in the L^2 Folland-Stein Sobolev-type embedding and the functions for which the equality is achieved, [23] and [24], with a complete solution on the quaternionic Heisenberg groups given in [34, 35].

2.1. **Quaternionic contact structures and the Biquard connection.** A quaternionic contact (qc) manifold $(M, \eta, g, \mathbb{Q})$ is a $4n+3$-dimensional manifold M with a codimension three distribution H locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$. In addition H has an $Sp(n)Sp(1)$ structure, that is, it is equipped with a Riemannian metric g and a rank-three bundle $\mathbb{Q}$ consisting of endomorphisms of H locally generated by three almost complex structures I_1, I_2, I_3 on H satisfying the identities of the imaginary unit quaternions, $I_1I_2 = -I_2I_1 = I_3, \quad I_1I_2I_3 = -id_{|_H}$ which are hermitian compatible with the metric $g(I_s., I_s.) = g(.,.)$ and the following compatibility condition holds $\quad 2g(I_sX, Y) = d\eta_s(X, Y), \quad X, Y \in H.$

The transformations preserving a given quaternionic contact structure η, i.e., $\bar\eta = \mu\Psi\eta$ for a positive smooth function μ and an $SO(3)$ matrix Ψ with smooth functions as entries are called *quaternionic contact conformal (qc-conformal) transformations*. If the function μ is constant $\bar\eta$ is called qc-homothetic to η. The qc conformal curvature tensor W^{qc}, introduced in [39], is the obstruction for a qc structure to be locally qc conformal to the standard 3-Sasakian structure on the $(4n+3)$-dimensional sphere [32, 39].

A special phenomena, noted in [6], is that the contact form η determines the quaternionic structure and the metric on the horizontal distribution in a unique way.

On a qc manifold with a fixed metric g on H there exists a canonical connection defined first by O. Biquard in [6] when the dimension $(4n+3) > 7$, and in [21] for the 7-dimensional case. Biquard showed that there is a unique connection ∇ with torsion T and a unique supplementary subspace V to H in TM, such that:

(i) ∇ preserves the decomposition $H \oplus V$ and the $Sp(n)Sp(1)$ structure on H, i.e. $\nabla g = 0, \nabla\sigma \in \Gamma(\mathbb{Q})$ for a section $\sigma \in \Gamma(\mathbb{Q})$, and its torsion on H is given by $T(X, Y) = -[X, Y]_{|V}$;
(ii) for $\xi \in V$, the endomorphism $T(\xi, .)_{|H}$ of H lies in $(sp(n) \oplus sp(1))^\perp \subset gl(4n)$;
(iii) the connection on V is induced by the natural identification φ of V with the subspace $sp(1)$ of the endomorphisms of H, i.e. $\nabla\varphi = 0$.

This canonical connection is also known as *the Biquard connection*. When the dimension of M is at least eleven [6] also described the supplementary distribution V, which is (locally) generated by the so called Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ determined by

$$\eta_s(\xi_k) = \delta_{sk}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \qquad (\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}, \tag{2.1}$$

where $\lrcorner$ denotes the interior multiplication. If the dimension of M is seven Duchemin shows in [21] that if we assume, in addition, the existence of Reeb vector fields as in (2.1), then the Biquard result holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2.1).

Notice that equations (2.1) are invariant under the natural $SO(3)$ action. Using the triple of Reeb vector fields we extend the metric g on H to a metric h on TM by requiring $span\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and $h(\xi_s, \xi_k) = \delta_{sk}$. The Riemannian metric h as well as the Biquard connection do not depend on the action of $SO(3)$ on V, but both change if η is multiplied by a conformal factor [32]. Clearly, the Biquard connection preserves the Riemannian metric on $TM, \nabla h = 0$. Since the Biquard connection is metric it is connected with the Levi-Civita connection ∇^h of the metric h by the general formula

$$h(\nabla_A B, C) = h(\nabla^h_A B, C) + \frac{1}{2}\Big[h(T(A, B), C) - h(T(B, C), A) + h(T(C, A), B)\Big], \quad A, B, C \in \Gamma(TM). \tag{2.2}$$

The covariant derivative of the qc structure with respect to the Biquard connection and the covariant derivative of the distribution V are given by

$$\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j, \quad \nabla\xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j. \tag{2.3}$$

The vanishing of the $sp(1)$-connection 1-forms on H implies the vanishing of the torsion endomorphism of the Biquard connection (see [32]).

The fundamental 2-forms ω_s of the quaternionic structure $\mathbb{Q}$ are defined by

$$2\omega_{s|H} = d\eta_{s|H}, \qquad \xi\lrcorner\omega_s = 0, \quad \xi\in V. \tag{2.4}$$

Due to (2.4), the torsion restricted to H has the form

$$T(X,Y) = -[X,Y]_{|V} = 2\omega_1(X,Y)\xi_1 + 2\omega_2(X,Y)\xi_2 + 2\omega_3(X,Y)\xi_3. \tag{2.5}$$

2.2. **Invariant decompositions.** An endomorphism Ψ of H can be decomposed with respect to the quaternionic structure $(\mathbb{Q},g)$ uniquely into four $Sp(n)$-invariant parts $\Psi=\Psi^{+++}+\Psi^{+--}+\Psi^{-+-}+\Psi^{--+}$, where Ψ^{+++} commutes with all three I_i, Ψ^{+--} commutes with I_1 and anti-commutes with the others two and etc. The two $Sp(n)Sp(1)$-invariant components $\Psi_{[3]}=\Psi^{+++}$, $\quad\Psi_{[-1]}=\Psi^{+--}+\Psi^{-+-}+\Psi^{--+}$ are determined by

$$\begin{aligned}\Psi=\Psi_{[3]} \quad &\Longleftrightarrow 3\Psi+I_1\Psi I_1+I_2\Psi I_2+I_3\Psi I_3=0,\\ \Psi=\Psi_{[-1]} \quad &\Longleftrightarrow \Psi-I_1\Psi I_1-I_2\Psi I_2-I_3\Psi I_3=0.\end{aligned}$$

With a short calculation one sees that the $Sp(n)Sp(1)$-invariant components are the projections on the eigenspaces of the Casimir operator $\Upsilon = I_1\otimes I_1 + I_2\otimes I_2 + I_3\otimes I_3$ corresponding, respectively, to the eigenvalues 3 and -1, see [11]. If $n=1$ then the space of symmetric endomorphisms commuting with all I_s is 1-dimensional, i.e. the [3]-component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_{[3]}=-\frac{tr\Psi}{4}Id_{|H}$. Note here that each of the three 2-forms ω_s belongs to its [-1]-component, $\omega_s=\omega_{s[-1]}$ and constitute a basis of the Lie algebra $sp(1)$.

2.3. **The torsion tensor.** The properties of the Biquard connection are encoded in the properties of the torsion endomorphism $T_\xi = T(\xi,\cdot): H\rightarrow H$, $\quad \xi\in V$. Decomposing the endomorphism $T_\xi\in(sp(n)+sp(1))^\perp$ into its symmetric part T^0_ξ and skew-symmetric part $b_\xi, T_\xi=T^0_\xi+b_\xi$, O. Biquard shows in [6] that the torsion T_ξ is completely trace-free, $tr\,T_\xi = tr\,T_\xi\circ I_s=0$, its symmetric part has the properties $T^0_{\xi_i}I_i = -I_iT^0_{\xi_i}$ $\quad I_2(T^0_{\xi_2})^{+--}=I_1(T^0_{\xi_1})^{-+-}$, $\quad I_3(T^0_{\xi_3})^{-+-}=I_2(T^0_{\xi_2})^{--+}$, $\quad I_1(T^0_{\xi_1})^{--+}=I_3(T^0_{\xi_3})^{+--}$, where the superscript $+++$ means commuting with all three I_i, $+--$ indicates commuting with I_1 and anti-commuting with the other two and etc. The skew-symmetric part can be represented as $b_{\xi_i}=I_iu$, where u is a traceless symmetric (1,1)-tensor on H which commutes with I_1,I_2,I_3. Therefore we have $T_{\xi_i}=T^0_{\xi_i}+I_iu$. If $n=1$ then the tensor u vanishes identically, $u=0$, and the torsion is a symmetric tensor, $T_\xi=T^0_\xi$.

Any 3-Sasakian manifold has zero torsion endomorphism, $T_\xi=0$, and the converse is true if in addition the qc scalar curvature (see (2.6)) is a positive constant [32] (the case of negative qc-scalar curvature can be treated very similarly, see [40, 41]). We remind that a $(4n+3)$-dimensional Riemannian manifold (M,g) is called 3-Sasakian if the cone metric $g_c=t^2h+dt^2$ on $C=M\times\mathbb{R}^+$ is a hyper Kähler metric, namely, it has holonomy contained in $Sp(n+1)$ [9]. A 3-Sasakian manifold of dimension $(4n+3)$ is Einstein with positive Riemannian scalar curvature $(4n+2)(4n+3)$ [48] and if complete it is a compact manifold with a finite fundamental group (see [8] for a nice overview of 3-Sasakian spaces).

2.4. **Torsion and curvature.** Let $R=[\nabla,\nabla]-\nabla_{[\ ,\ ]}$ be the curvature tensor of ∇ and the dimension is $4n+3$. We denote the curvature tensor of type (0,4) and the torsion tensor of type (0,3) by the same letter, $R(A,B,C,D):=h(R(A,B)C,D)$, $\quad T(A,B,C):=h(T(A,B),C)$, $A,B,C,D\in\Gamma(TM)$. The *qc-Ricci* tensor Ric, the normalized *qc-scalar* curvature S, the *qc-Ricci* 2-forms ρ_s, and the *qc-Ricci* type-tensors ζ_s are given by

$$\begin{aligned}&Ric(A,B)=R(e_b,A,B,e_b), \quad 8n(n+2)S=R(e_b,e_a,e_a,e_b),\\ &\rho_s(A,B)=\frac{1}{4n}R(A,B,e_a,I_se_a), \quad \zeta_s(A,B)=\frac{1}{4n}R(e_a,A,B,I_se_a).\end{aligned} \tag{2.6}$$

The $sp(1)$-part of R is determined by the Ricci 2-forms and the connection 1-forms by

$$R(A,B,\xi_i,\xi_j)=2\rho_k(A,B)=(d\alpha_k+\alpha_i\wedge\alpha_j)(A,B), \qquad A,B\in\Gamma(TM). \tag{2.7}$$

The two $Sp(n)Sp(1)$-invariant trace-free symmetric 2-tensors $T^0(X,Y) = g((T^0_{\xi_1}I_1 + T^0_{\xi_2}I_2 + T^0_{\xi_3}I_3)X,Y)$, $U(X,Y) = g(uX,Y)$ on H, introduced in [32], have the properties:

$$\begin{aligned} T^0(X,Y) + T^0(I_1X,I_1Y) + T^0(I_2X,I_2Y) + T^0(I_3X,I_3Y) = 0,\\ U(X,Y) = U(I_1X,I_1Y) = U(I_2X,I_2Y) = U(I_3X,I_3Y). \end{aligned} \tag{2.8}$$

In dimension seven ($n = 1$), the tensor U vanishes identically, $U = 0$.

We shall need the following identity taken from [39, Proposition 2.3] $4T^0(\xi_s, I_sX,Y) = T^0(X,Y) - T^0(I_sX,I_sY)$ which implies the formula

$$T(\xi_s, I_sX,Y) = T^0(\xi_s, I_sX,Y) + g(I_suI_sX,Y) = \frac{1}{4}\Big[T^0(X,Y) - T^0(I_sX,I_sY)\Big] - U(X,Y). \tag{2.9}$$

We recall that a qc structure is said to be *qc-Einstein* if the horizontal qc-Ricci tensor is a scalar multiple of the metric, $Ric(X,Y) = 2(n+2)Sg(X,Y)$. The horizontal Ricci-type tensor can be expressed in terms of the torsion of the Biquard connection [32] (see also [34, 39]). We collect below the necessary facts from [32, Theorem 1.3, Theorem 3.12, Corollary 3.14, Proposition 4.3 and Proposition 4.4] with slight modification presented in [39]

$$\begin{aligned} &Ric(X,Y) = (2n+2)T^0(X,Y) + (4n+10)U(X,Y) + 2(n+2)Sg(X,Y),\\ &\rho_s(X,I_sY) = -\frac{1}{2}\Big[T^0(X,Y) + T^0(I_sX,I_sY)\Big] - 2U(X,Y) - Sg(X,Y),\\ &\zeta_s(X,I_sY) = \frac{2n+1}{4n}T^0(X,Y) + \frac{1}{4n}T^0(I_sX,I_sY) + \frac{2n+1}{2n}U(X,Y) + \frac{S}{2}g(X,Y),\\ &T(\xi_i,\xi_j) = -S\xi_k - [\xi_i,\xi_j]_{|H}, \qquad S = -h(T(\xi_1,\xi_2),\xi_3),\\ &g(T(\xi_i,\xi_j),X) = -\rho_k(I_iX,\xi_i) = -\rho_k(I_jX,\xi_j) = -h([\xi_i,\xi_j],X). \end{aligned} \tag{2.10}$$

For $n = 1$ the above formulas hold with $U = 0$. Hence, the qc-Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection. In this case the normalized qc scalar curvature S is constant and the vertical distribution V is integrable [32] for $n > 1$ and [33] for $n = 1$. If $S > 0$ then the qc manifold is locally 3-Sasakian [32], (see [40] for the negative qc scalar curvature).

We shall also need the general formula for the curvature [39, 41]

$$\begin{aligned} R(\xi_i,X,Y,Z) = &-(\nabla_XU)(I_iY,Z) + \omega_j(X,Y)\rho_k(I_iZ,\xi_i) - \omega_k(X,Y)\rho_j(I_iZ,\xi_i)\\ &-\frac{1}{4}\Big[(\nabla_YT^0)(I_iZ,X) + (\nabla_YT^0)(Z,I_iX)\Big] + \frac{1}{4}\Big[(\nabla_ZT^0)(I_iY,X) + (\nabla_ZT^0)(Y,I_iX)\Big]\\ &-\omega_j(X,Z)\rho_k(I_iY,\xi_i) + \omega_k(X,Z)\rho_j(I_iY,\xi_i) - \omega_j(Y,Z)\rho_k(I_iX,\xi_i) + \omega_k(Y,Z)\rho_j(I_iX,\xi_i), \end{aligned} \tag{2.11}$$

where the Ricci two forms are given by, cf. [39, Theorem 3.1] or [41, Theorem4.3.11]

$$\begin{aligned} &6(2n+1)\rho_s(\xi_s,X) = (2n+1)X(S) + \frac{1}{2}(\nabla_{e_a}T^0)[(e_a,X) - 3(I_se_a,I_sX)] - 2(\nabla_{e_a}U)(e_a,X),\\ &6(2n+1)\rho_i(\xi_j,I_kX) = (2n-1)(2n+1)X(S) - \frac{1}{2}(\nabla_{e_a}T^0)[(4n+1)(e_a,X) + 3(I_ie_a,I_iX)]\\ &\qquad\qquad -4(n+1)(\nabla_{e_a}U)(e_a,X). \end{aligned} \tag{2.12}$$

2.5. **The Ricci identities, the divergence theorem.** We shall use repeatedly the following Ricci identities of order two and three, see also [39] and [36]. Let ξ_s be the Reeb vector fields, f a smooth function on the qc manifold M and ∇f its horizontal gradient, $g(\nabla f, X) = df(X)$. We have:

$$
\begin{aligned}
&\nabla^2 f(X,Y) - \nabla^2 f(Y,X) = -2\sum_{s=1}^{3}\omega_s(X,Y)df(\xi_s),\\
&\nabla^2 f(X,\xi_s) - \nabla^2 f(\xi_s,X) = T(\xi_s,X,\nabla f),\\
&\nabla^3 f(X,Y,Z) - \nabla^3 f(Y,X,Z) = -R(X,Y,Z,\nabla f) - 2\sum_{s=1}^{3}\omega_s(X,Y)\nabla^2 f(\xi_s,Z),\\
&\nabla^3 f(X,Y,\xi_i) - \nabla^3 f(Y,X,\xi_i) = -2df(\xi_j)\rho_k(X,Y) + 2df(\xi_k)\rho_j(X,Y) - 2\sum_{s=1}^{3}\omega_s(X,Y)\nabla^2 f(\xi_s,\xi_i),\\
&\nabla^3 f(\xi_s,X,Y) - \nabla^3 f(X,\xi_s,Y) = -R(\xi_s,X,Y,\nabla f) - \nabla^2 f(T(\xi_s,X),Y),\\
&\nabla^3 f(\xi_s,X,Y) - \nabla^3 f(X,Y,\xi_s) = -\nabla^2 f\left(T\left(\xi_s,X\right),Y\right) - \nabla^2 f\left(X,T\left(\xi_s,Y\right)\right) - df\left(\left(\nabla_X T\right)\left(\xi_s,Y\right)\right)\\
&\hspace{9cm} - R(\xi_s,X,Y,\nabla f).
\end{aligned}
\tag{2.13}
$$

The sub-Laplacian $\triangle f$ and the norm of the horizontal gradient ∇f of a smooth function f on M are defined respectively by

$$\triangle f \ = -\ tr^g_H(\nabla^2 f) \ = \nabla^* df = -\ \nabla^2 f(e_a,e_a), \qquad |\nabla f|^2 \ = \ df(e_a)\, df(e_a).$$

The function f is an eigenfunction with eigenvalue λ of the sub-Laplacian if, for some constant λ we have

$$\triangle f = \lambda f. \tag{2.14}$$

From the Ricci identities we have the following formulas for the traces through the almost complex structures of the Hessian

$$g(\nabla^2 f, \omega_s) = \nabla^2 f(e_a, I_s e_a) = -4n df(\xi_s). \tag{2.15}$$

For a fixed local 1-form η and a fix $s \in \{1,2,3\}$ the form $Vol_\eta = \eta_1\wedge\eta_2\wedge\eta_3\wedge\omega_s^{2n}$ is a locally defined volume form. Note that Vol_η is independent of s and the local one forms η_1,η_2,η_3 and therefore it is a globally defined volume form denoted with Vol_η. The (horizontal) divergence of a horizontal vector field/one-form $\sigma\in\Lambda^1(H)$ defined by $\nabla^*\sigma \ = -tr|_H\nabla\sigma = \ -\nabla\sigma(e_a,e_a)$ supplies the "integration by parts" over compact M formula [32], see also [65],

$$\int_M (\nabla^*\sigma)\ Vol_\eta \ = \ 0. \tag{2.16}$$

3. Proof of the main Theorems

The proof of Theorem 1.3 is lengthy and requires a number of steps which we present in the following subsections. Throughout this section we will work with the assumptions of Theorem 1.3. In particular, f is a non-constant smooth function whose horizontal Hessian satisfies (1.6). Our first step is to show the vanishing of the torsion tensor, $T^0=0$ and $U=0$. We start by expressing the remaining parts of the Hessian (w.r.t. the Biquard connection) in terms of the torsion tensors and show that f satisfies an elliptic equation on M. A simple argument shows that $T^0(I_s\nabla f,\nabla f) = U(I_s\nabla f,\nabla f)=0$, $s=1,2,3$. Furthermore, using the $[-1]$-component of the curvature tensor we show that $T^0(I_s\nabla f, I_t\nabla f)=0$, $s,\, t\in\{1,2,3\}$, $s\neq t$. In addition, we determine the torsion tensors T^0 and U in terms of the horizontal gradient of f and the tensor $U(\nabla f,\nabla f)$. The analysis proceeds by finding formulas of the same type for the covariant derivatives of T^0 and U. Thus, the crux of the matter in showing that the torsion vanishes is the proof that $U(\nabla f,\nabla f)=0$. This fact will be achieved with the help of the Ricci identities, the contracted Bianchi second identity and thus far established results. In the next step of the proof of Theorem 1.3 we compute the Riemannian Hessian of f, with respect to the Levi-Civita connection of the metric (1.5) which allow us to invoke Obata's result thus proving that M equipped with the Riemannian metric (1.5) is homothetic to the unit sphere in quaternion space. The final step is to show that M is qc-homothetic to the $(4n+3)$-dimensional 3-Sasakian unit sphere. Here, we employ a standard monodromy argument showing that a compact simply connected locally qc-conformally flat manifold is globally qc-conformal to the 3-Sasakian unit sphere. For this we invoke the Liouville theorem

[10], showing that every qc-conformal transformation between open subsets of the 3-Sasakian unit sphere is the restriction of a global qc-conformal transformation, i.e., an element of the group $PSp(n+1,1)$.

3.1. **Some basic identities.** We start our analysis by finding a formula for the third covariant derivative of a function which satisfies (1.6).

Lemma 3.1. *With the assumptions of Theorem 1.3 we have the following formula for the third covariant derivative of the function f,*

$$\nabla^3 f(A,X,Y) = -df(A)g(X,Y) - \sum_{s=1}^{3} \omega_s(X,Y)\nabla^2 f(A,\xi_s), \quad A \in \Gamma(TM). \tag{3.1}$$

Proof. The claimed formula is obtained by differentiating the Hessian equation (1.6). Indeed, the covariant derivative along $A \in \Gamma(TM)$ of (1.6) gives

$$\nabla^3 f(A,X,Y) = -df(A)g(X,Y) - \sum_{s=1}^{3}\left[\nabla^2 f(A,\xi_s)\omega_s(X,Y) + df(\nabla_A\xi_s)\omega_s(X,Y) + df(\xi_s)\left(\nabla_A\omega_s\right)(X,Y)\right],$$

which together with (2.3) gives the identity, cf. also Convention 1.4 e),

$$\begin{aligned}
\nabla^3 f(A,X,Y) &= -df(A)g(X,Y) - \sum_{(ijk)}\left[\nabla^2 f(A,\xi_i)\omega_i(X,Y) + df(\nabla_A\xi_i)\omega_i(X,Y) + df(\xi_i)\left(\nabla_A\omega_i\right)(X,Y)\right]\\
&= -df(A)g(X,Y) - \sum_{t=1}^{3}\left[\nabla^2 f(A,\xi_t)\omega_t(X,Y)\right]\\
&\quad - \sum_{(ijk)}\left[-\alpha_j(A)df(\xi_k) + \alpha_k(A)df(\xi_j)\right]\omega_i(X,Y) - \sum_{(ijk)}\left[-\alpha_j(A)\omega_k(X,Y) + \alpha_k(A)\omega_j(X,Y)\right]df(\xi_i)\\
&= -df(A)g(X,Y) - \sum_{t=1}^{3}\left[\nabla^2 f(A,\xi_t)\omega_t(X,Y)\right],
\end{aligned}$$

which completes the proof. ∎

After this technical Lemma, our first goal is to find a formula for the curvature tensor $R(Z,X,Y,\nabla f)$, for f satisfying (1.6), using Lemma 3.1 with $A = Z$, the Ricci identities (2.13), and the properties of the torsion. In fact, after some standard calculations it follows

$$\begin{aligned}
R(Z,X,Y,\nabla f) &= \Big[df(Z)g(X,Y) - df(X)g(Z,Y)\Big]\\
&\quad + \sum_{s=1}^{3}\Big[\nabla df(\xi_s,Z)\omega_s(X,Y) - \nabla df(\xi_s,X)\omega_s(Z,Y) - 2\nabla df(\xi_s,Y)\omega_s(Z,X)\Big]\\
&\quad + \sum_{s=1}^{3}\Big[T(\xi_s,Z,\nabla f)\omega_s(X,Y) - T(\xi_s,X,\nabla f)\omega_s(Z,Y)\Big].
\end{aligned} \tag{3.2}$$

By taking traces in (3.2) we can derive formulas for the various contracted tensors (2.6). We shall use the following,

$$\begin{aligned}
&Ric(Z,\nabla f) = (4n-1)df(Z) - \sum_{s=1}^{3} T(\xi_s, I_sZ, \nabla f) - 3\sum_{s=1}^{3}\nabla df(\xi_s, I_sZ),\\
&4n\zeta_i(I_iZ,\nabla f) = -df(Z) + (4n-1)T(\xi_i, I_iZ,\nabla f) + T(\xi_j, I_jZ, \nabla f) + T(\xi_k, I_kZ, \nabla f)\\
&\qquad + (4n+1)\nabla df(\xi_i, I_iZ) - \nabla df(\xi_j, I_jZ) - \nabla df(\xi_k, I_kZ).
\end{aligned} \tag{3.3}$$

The above formulas imply some other basic identities to which we turn next. Note that with the help of (2.10) we can rewrite the Lichnerowicz type assumption (1.2) in the form

$$L(X,X) \stackrel{def}{=} 2(n+2)Sg(X,X)+\alpha'_n T^0(X,X)+\beta'_n U(X,X)\geq k_0 g(X,X), \quad X\in H, \tag{3.4}$$
$$\alpha'_n=\frac{2(2n+3)(n+2)}{2n+1}, \qquad \beta'_n=\frac{4(2n-1)(n+2)^2}{(2n+1)(n-1)},$$

which allows to write the first claim of the following Lemma in the form $L(Z,\nabla f)=0$ for all $Z\in H$ whenever f satisfies (1.6) taking $k_0=4(n+2)$.

Lemma 3.2. *With the assumptions of Theorem 1.3, the next identity holds true*

$$(S-2)df(Z)+\frac{2n+3}{2n+1}T^0(Z,\nabla f)+\frac{2(2n-1)(n+2)}{(2n+1)(n-1)}U(Z,\nabla f)=0. \tag{3.5}$$

Furthermore, we have

$$T^0(I_s\nabla f,\nabla f)=0, \qquad U(I_s\nabla f,\nabla f)=0. \tag{3.6}$$

Proof. The first equations in (3.3) and (2.10) together with (2.9) imply

$$3\sum_{s=1}^3\nabla df(\xi_s,I_sZ)=\Big[4n-1-(2n+4)S\Big]df(Z)-(2n+3)T^0(Z,\nabla f)-(4n+7)U(Z,\nabla f). \tag{3.7}$$

The sum over $1,2,3$ of the second equality in (3.3) together with the third equality of (2.10) and (2.9) gives

$$(4n-1)\sum_{s=1}^3\nabla df(\xi_s,I_sZ)=(3-6nS)df(Z)-(2n+3)T^0(Z,\nabla f)-3U(Z,\nabla f). \tag{3.8}$$

Subtracting (3.7) from (3.8) we obtain

$$4(n-1)\sum_{s=1}^3\nabla df(\xi_s,I_sZ)=4(1-n)(1+S)df(Z)+4(n+1)U(Z,\nabla f),$$

which for $n>1$ yields

$$\sum_{s=1}^3\nabla df(\xi_s,I_sZ)=-(1+S)df(Z)+\frac{n+1}{n-1}U(Z,\nabla f). \tag{3.9}$$

The sum of (3.7) and (3.8) gives

$$(2n+1)\sum_{s=1}^3\nabla df(\xi_s,I_sZ)=(2n+1)(1-2S)df(Z)-(2n+3)T^0(Z,\nabla f)-(2n+5)U(Z,\nabla f). \tag{3.10}$$

Equalities (3.9) and (3.10) imply (3.5). Letting $Z=I_s\nabla f$ in the latter it follows $T^0(I_s\nabla f,\nabla f)=0$ since $U(I_s\nabla f,\nabla f)=0$. □

3.2. **Formulas for the derivatives of f.** By assumption, the second order horizontal derivatives of f satisfy the Hessian equation (1.6). We derive next formulas for the second order derivatives involving a horizontal and a vertical directions.

Lemma 3.3. *With the assumptions of Theorem 1.3 we have*

$$\nabla df(\xi_i,I_iZ)=-df(Z)+\frac{2n+3}{4(2n+1)}\Big[T^0(Z,\nabla f)-T^0(I_iZ,I_i\nabla f)\Big]+\frac{2n^2+3n-1}{(2n+1)(n-1)}U(Z,\nabla f) \tag{3.11}$$

and

$$\nabla df(Z,\xi_i)=df(I_iZ)-\frac{n+1}{2n+1}\Big[T^0(I_iZ,\nabla f)+T^0(Z,I_i\nabla f)\Big]-\frac{4n}{(2n+1)(n-1)}U(I_iZ,\nabla f). \tag{3.12}$$

Proof. The second equality of (3.3) can be written in the form

$$
\begin{aligned}
(3.13)\quad 4n\zeta_i(I_iZ,\nabla f) &= -df(Z) + (4n-2)T(\xi_i,I_iZ,\nabla f) + \sum_{s=1}^{3} T(\xi_s,I_sZ,\nabla f) \\
&\quad + (4n+2)\nabla df(\xi_i,I_iZ) - \sum_{s=1}^{3}\nabla df(\xi_s,I_sZ) \\
&= -df(Z) + (4n-2)\Big[\frac{1}{4}\Big(T^0(Z,\nabla f) - T^0(I_iZ,I_i\nabla f)\Big) - U(Z,\nabla f)\Big] + T^0(Z,\nabla f) - 3U(Z,\nabla f) \\
&\quad + (1+S)df(Z) - \frac{n+1}{n-1}U(Z,\nabla f) + (4n+2)\nabla df(\xi_i,I_iZ),
\end{aligned}
$$

where we used (2.9) and (3.9). Now, equalities (3.13), (3.5) and the third equality in (2.10) imply

$$
\begin{aligned}
(3.14)\quad \nabla df(\xi_i,I_iZ) &= -\frac{S}{2}df(Z) - \frac{2n+3}{4(2n+1)}\Big[T^0(Z,\nabla f) + T^0(I_iZ,I_i\nabla f)\Big] + \frac{1}{(2n+1)(n-1)}U(Z,\nabla f) \\
&= -df(Z) + \frac{2n+3}{4(2n+1)}\Big[T^0(Z,\nabla f) - T^0(I_iZ,I_i\nabla f)\Big] + \frac{2n^2+3n-1}{(2n+1)(n-1)}U(Z,\nabla f).
\end{aligned}
$$

Finally, the Ricci identity, (2.9) and (3.11) yield

$$
\begin{aligned}
(3.15)\quad \nabla^2 f(Z,\xi_i) &= \nabla df(\xi_i,Z) + T(\xi_i,Z,\nabla f) \\
&= \frac{S}{2}df(I_iZ) + \frac{1}{2(2n+1)}T^0(I_iZ,\nabla f) - \frac{n+1}{2n+1}T^0(Z,I_i\nabla f) + \frac{2n^2-n-2}{(2n+1)(n-1)}U(I_iZ,\nabla f) \\
&= df(I_iZ) - \frac{n+1}{2n+1}\Big[T^0(I_iZ,\nabla f) + T^0(Z,I_i\nabla f)\Big] - \frac{4n}{(2n+1)(n-1)}U(I_iZ,\nabla f),
\end{aligned}
$$

which completes the proof. ∎

Next, we compute the second vertical derivatives of f. We start with a basic useful identity involving only vertical derivatives.

Lemma 3.4. *With the assumptions of Theorem 1.3 the following identity holds*

$$
\begin{aligned}
(3.16)\quad \nabla^2 f(\xi_i,\xi_i) &= -f - \frac{n+1}{4n(2n+1)}\Big[(\nabla_{e_a}T^0)(e_a,\nabla f) - (\nabla_{e_a}T^0)(I_ie_a,I_i\nabla f)\Big] \\
&\quad - \frac{1}{(2n+1)(n-1)}(\nabla_{e_a}U)(e_a,\nabla f).
\end{aligned}
$$

Proof. Differentiating (3.12), using (1.6) and (2.3) we obtain

$$
\begin{aligned}
(3.17)\quad & \nabla^3 f(X,Y,\xi_i) - \alpha_j(X)\nabla^2 f(Y,\xi_k) + \alpha_k(X)\nabla^2 f(Y,\xi_j) \\
&= -\frac{n+1}{2n+1}\Big[(\nabla_X T^0)(I_iY,\nabla f) + (\nabla_X T^0)(Y,I_i\nabla f)\Big] - \frac{4n}{(2n+1)(n-1)}(\nabla_X U)(I_iZ,\nabla f) \\
&\quad + f\Big\{\omega_i(X,Y) + \frac{n+1}{2n+1}\Big[T^0(X,I_iY) + T^0(I_iX,Y)\Big] + \frac{4n}{(2n+1)(n-1)}U(X,I_iY)\Big\} \\
&\quad + df(\xi_i)\Big\{-g(X,Y) + \frac{n+1}{2n+1}\Big[T^0(I_iX,I_iY) - T^0(X,Y)\Big] + \frac{4n}{(2n+1)(n-1)}U(X,Y)\Big\} \\
&\quad + df(\xi_j)\Big\{\omega_k(X,Y) + \frac{n+1}{2n+1}\Big[T^0(I_jX,I_iY) + T^0(I_kX,Y)\Big] + \frac{4n}{(2n+1)(n-1)}U(X,I_kY)\Big\} \\
&\quad + df(\xi_k)\Big\{-\omega_j(X,Y) + \frac{n+1}{2n+1}\Big[T^0(I_kX,I_iY) - T^0(I_jX,Y)\Big] - \frac{4n}{(2n+1)(n-1)}U(X,I_jY)\Big\} \\
&\quad - \alpha_j(X)\Big[df(I_kY) - \frac{n+1}{2n+1}T^0(I_kY,\nabla f) - \frac{n+1}{2n+1}T^0(Y,I_k\nabla f) - \frac{4n}{(2n+1)(n-1)}U(I_kY\nabla f)\Big] \\
&\quad + \alpha_k(X)\Big[df(I_jY) - \frac{n+1}{2n+1}T^0(I_jY,\nabla f) - \frac{n+1}{2n+1}T^0(Y,I_j\nabla f) - \frac{4n}{(2n+1)(n-1)}U(I_jY,\nabla f)\Big].
\end{aligned}
$$

Applying again (3.12) to the second and the third terms in the first line we see that the terms involving the connection 1-forms cancel and (3.17) takes the following form

$$
\begin{aligned}
(3.18)\quad & \nabla^3 f(X,Y,\xi_i)\\
&= -\frac{n+1}{2n+1}\Big[(\nabla_X T^0)(I_iY,\nabla f)+(\nabla_X T^0)(Y,I_i\nabla f)\Big]-\frac{4n}{(2n+1)(n-1)}(\nabla_X U)(I_iZ,\nabla f)\\
&\quad + f\Big\{\omega_i(X,Y)+\frac{n+1}{2n+1}\Big[T^0(X,I_iY)+T^0(I_iX,Y)\Big]+\frac{4n}{(2n+1)(n-1)}U(X,I_iY)\Big\}\\
&\quad + df(\xi_i)\Big\{-g(X,Y)+\frac{n+1}{2n+1}\Big[T^0(I_iX,I_iY)-T^0(X,Y)\Big]+\frac{4n}{(2n+1)(n-1)}U(X,Y)\Big\}\\
&\quad + df(\xi_j)\Big\{\omega_k(X,Y)+\frac{n+1}{2n+1}\Big[T^0(I_jX,I_iY)+T^0(I_kX,Y)\Big]+\frac{4n}{(2n+1)(n-1)}U(X,I_kY)\Big\}\\
&\quad + df(\xi_k)\Big\{-\omega_j(X,Y)+\frac{n+1}{2n+1}\Big[T^0(I_kX,I_iY)-T^0(I_jX,Y)\Big]-\frac{4n}{(2n+1)(n-1)}U(X,I_jY)\Big\}.
\end{aligned}
$$

On the other hand, the skew-symmetric part of (3.18) and the Ricci identity listed in the fourth line of (2.13) yield

$$
\begin{aligned}
(3.19)\quad & \nabla^3 f(X,Y,\xi_i)-\nabla^3 f(Y,X,\xi_i)\\
&= -\frac{n+1}{2n+1}\Big[(\nabla_X T^0)(I_iY,\nabla f)+(\nabla_X T^0)(Y,I_i\nabla f)-(\nabla_Y T^0)(I_iX,\nabla f)-(\nabla_Y T^0)(X,I_i\nabla f)\Big]\\
&\quad -\frac{4n}{(2n+1)(n-1)}\Big[(\nabla_X U)(I_iY,\nabla f)-(\nabla_Y U)(I_iX,\nabla f)\Big]+2f\Big[\omega_i(X,Y)+\frac{4n}{(2n+1)(n-1)}U(X,I_iY)\Big]\\
&\quad +2df(\xi_j)\Big\{\omega_k(X,Y)+\frac{n+1}{2n+1}\Big[T^0(I_kX,Y)-T^0(X,I_kY)\Big]+\frac{4n}{(2n+1)(n-1)}U(X,I_kY)\Big\}\\
&\quad +2df(\xi_k)\Big\{-\omega_j(X,Y)+\frac{n+1}{2n+1}\Big[T^0(X,I_jY)-T^0(I_jX,Y)\Big]-\frac{4n}{(2n+1)(n-1)}U(X,I_jY)\Big\}\\
&= -2df(\xi_j)\rho_k(X,Y)+2df(\xi_k)\rho_j(X,Y)-2\sum_{s=1}^{3}\omega_s(X,Y)\nabla^2 f(\xi_s,\xi_i).
\end{aligned}
$$

The trace $X=e_a, Y=I_ie_a$ of (3.19) and the second equality of (2.10) give (3.16), which completes the proof. ☐

Remark 3.5. *The detailed proof of* (3.18) *shows a particular consequence of* (2.3) *which is that a covariant derivative of identities that are not $Sp(1)$ invariant can lead to formulas which do not involve the connection one-forms. In the rest of the paper we shall usually skip many straightforward calculations some of which rely on a similar use of* (2.3).

3.3. **The elliptic eigenvalue problem.** In this sub-section we will show that (1.6) implies that f satisfies an elliptic PDE. Let $\triangle^h$ be the Riemannian Laplacian of the metric (1.5).

Lemma 3.6. *On a qc manifold of dimension bigger than seven any smooth function satisfying* (1.6) *obeys the following identity*

$$
\triangle^h f=(4n+3)f+\frac{n+1}{n(2n+1)}(\nabla_{e_a}T^0)(e_a,\nabla f)+\frac{3}{(2n+1)(n-1)}(\nabla_{e_a}U)(e_a,\nabla f). \tag{3.20}
$$

Proof. It is shown in [36, Lemma 5.1] that the Riemannian Laplacian $\triangle^h$ and the sub-Laplacian $\triangle$ of a smooth function f are connected by

$$
\triangle^h f=\triangle f-\sum_{s=1}^{3}\nabla^2 f(\xi_s,\xi_s). \tag{3.21}
$$

Equation (3.21) is a consequence of the formula (2.2), $\triangle^h f=-\sum_{a=1}^{4n}\nabla^h df(e_a,e_a)-\sum_{s=1}^{3}\nabla^h df(\xi_s,\xi_s)$, and the identities $T(e_a,A,e_a)=T(\xi_s,A,\xi_s)=0,\ A\in\Gamma(TM)$ which follow from the properties of the torsion tensor T of ∇ listed in (2.10). Lemma 3.4 and (2.8) imply

$$\sum_{s=1}^{3}\nabla^2 f(\xi_s,\xi_s) = -3f - \frac{n+1}{n(2n+1)}(\nabla_{e_a}T^0)(e_a,\nabla f) - \frac{3}{(2n+1)(n-1)}(\nabla_{e_a}U)(e_a,\nabla f). \tag{3.22}$$

A substitution of (3.22) in (3.21), taking into account that f satisfies (1.6) hence $\triangle f = 4nf$, we obtain (3.20) which proves the lemma. ∎

A consequence of Lemma 3.6 and Aronszajn's unique continuation result [1], is that $|\nabla f|$ cannot vanish on any open set. We note this important fact in the next remark.

Remark 3.7. *If M and f are as in Theorem 1.3 then $|\nabla f| \neq 0$ in a dense set since $f \neq const$.*

3.4. **Formulas for the torsion tensors.** In this sub-section we derive formulas for the components T^0 and U of the torsion tensor.

Lemma 3.8. *With the assumption of Theorem 1.3 the following identities hold true for any $X,Y,Z \in H$*

$$T^0(I_s\nabla f, I_t\nabla f) = 0, \quad s\neq t,\ s,\ t\in\{1,2,3\}, \tag{3.23}$$

$$T^0(\nabla f,\nabla f) = -\frac{6n}{n-1}U(\nabla f,\nabla f), \qquad T^0(I_s\nabla f, I_s\nabla f) = \frac{2n}{n-1}U(\nabla f,\nabla f), \qquad s\in\{1,2,3\}, \tag{3.24}$$

$$|\nabla f|^2 T^0(Z,\nabla f) = -\frac{6n}{n-1}U(\nabla f,\nabla f)df(Z), \qquad |\nabla f|^2 U(Z,\nabla f) = U(\nabla f,\nabla f)df(Z), \tag{3.25}$$

$$|\nabla f|^4 T^0(X,Y) = -\frac{2n}{n-1}U(\nabla f,\nabla f)\Big[3df(X)df(Y) - \sum_{s=1}^{3} df(I_sX)df(I_sY)\Big], \tag{3.26}$$

$$|\nabla f|^4 U(Z,X) = -\frac{1}{n-1}U(\nabla f,\nabla f)\Big[|\nabla f|^2 g(Z,X) - n\Big(df(Z)df(X) + \sum_{s=1}^{3} df(I_sZ)df(I_sX)\Big)\Big]. \tag{3.27}$$

Proof. To determine the torsion tensors T^0 and U we are going to apply the following identity [39, 41] for the $[-1]$ component of the curvature

$$\begin{aligned}
&3R(Z,X,Y,\nabla f) - R(I_1Z,I_1X,Y,\nabla f) - R(I_2Z,I_2X,Y,\nabla f) - R(I_3Z,I_3X,Y,\nabla f)\\
&\quad = 2\Big[g(X,Y)T^0(Z,\nabla f) + g(Z,\nabla f)T^0(Y,X) - g(Y,Z)T^0(X,\nabla f) - g(\nabla f,X)T^0(Y,Z)\Big]\\
&- 2\sum_{s=1}^{3}\Big[\omega_s(X,Y)T^0(Z,I_s\nabla f) + \omega_s(Z,\nabla f)T^0(X,I_sY) - \omega_s(Z,Y)T^0(X,I_s\nabla f) - \omega_s(X,\nabla f)T^0(Z,I_sY)\Big]\\
&\quad + \sum_{s=1}^{3}\Big[2\omega_s(Z,X)\Big(T^0(Y,I_s\nabla f) - T^0(I_sY,\nabla f)\Big) - 8\omega_s(Y,\nabla f)U(I_sZ,X) - 4S\omega_s(Z,X)\omega_s(Y,\nabla f)\Big].
\end{aligned} \tag{3.28}$$

With the help of the Ricci identity, cf. the second equality of (2.13), we write the curvature tensor given by (3.2) in the form

$$\begin{aligned}
R(Z,X,Y,\nabla f) &= \Big[df(Z)g(X,Y) - df(X)g(Z,Y)\Big]\\
&\quad + \sum_{s=1}^{3}\Big[\nabla df(Z,\xi_s)\omega_s(X,Y) - \nabla df(X,\xi_s)\omega_s(Z,Y) - 2\nabla df(\xi_s,Y)\omega_s(Z,X)\Big].
\end{aligned} \tag{3.29}$$

A calculation shows

$$\sum_{t=1}^{3} R(I_tZ,I_tX,Y,\nabla f)=\sum_{s=1}^{3}\Big[df(I_sZ)\omega_s(X,Y)-df(I_sX)\omega_s(Z,Y)\Big] \tag{3.30}$$
$$+\sum_{s,t=1}^{3}\Big[\nabla df(I_tZ,\xi_s)\omega_s(I_tX,Y)-\nabla df(I_tX,\xi_s)\omega_s(I_tZ,Y)-2\nabla df(\xi_s,Y)\omega_s(I_tZ,I_tX)\Big]$$
$$=\sum_{s=1}^{3}\Big[df(I_sZ)\omega_s(X,Y)-df(I_sX)\omega_s(Z,Y)+2\nabla df(\xi_s,Y)\omega_s(Z,X)\Big]$$
$$-g(X,Y)\sum_{s=1}^{3}\nabla df(I_sZ,\xi_s)+g(Z,Y)\sum_{s=1}^{3}\nabla df(I_sX,\xi_s)-\sum_{(ijk)}\omega_i(X,Y)\Big[\nabla df(I_jZ,\xi_k)-\nabla df(I_kZ,\xi_j)\Big]$$
$$+\sum_{(ijk)}\omega_i(Z,Y)\Big[\nabla df(I_jX,\xi_k)-\nabla df(I_kX,\xi_j)\Big],$$

where $\sum_{(ijk)}$ denotes the cyclic sum. Now, (3.29) and (3.30) together with (3.11) and (3.12) yield

$$3R(Z,X,Y,\nabla f)-R(I_1Z,I_1X,Y,\nabla f)-R(I_2Z,I_2X,Y,\nabla f)-R(I_3Z,I_3X,Y,\nabla f) \tag{3.31}$$
$$=g(X,Y)\Big[3df(Z)+\sum_{s=1}^{3}\nabla^2f(I_sZ,\xi_s)\Big]-g(Z,Y)\Big[3df(X)+\sum_{s=1}^{3}\nabla^2f(I_sX,\xi_s)\Big]-8\sum_{s=1}^{3}\omega_s(Z,X)\nabla df(\xi_s,Y)$$
$$+\sum_{(ijk)}\omega_i(X,Y)\Big[3\nabla^2f(Z,\xi_i)-df(I_iZ)+\nabla^2f(I_jZ,\xi_k)-\nabla^2f(I_kZ,\xi_j)\Big]$$
$$-\sum_{(ijk)}\omega_i(Z,Y)\Big[3\nabla^2f(X,\xi_i)-df(I_iX)+\nabla^2f(I_jX,\xi_k)-\nabla^2f(I_kX,\xi_j)\Big]$$
$$=g(X,Y)\Big[\frac{4n+4}{2n+1}T^0(Z,\nabla f)+\frac{12n}{(2n+1)(n-1)}U(Z,\nabla f)\Big]$$
$$-g(Z,Y)\Big[\frac{4n+4}{2n+1}T^0(X,\nabla f)+\frac{12n}{(2n+1)(n-1)}U(X,\nabla f)\Big]$$
$$-\sum_{s=1}^{3}\omega_s(Z,X)\Big\{4Sdf(I_sY)+\frac{4n+6}{2n+1}\Big[T^0(I_sY,\nabla f)-T^0(Y,I_s\nabla f)\Big]-\frac{8}{(2n+1)(n-1)}U(I_sY,\nabla f)\Big\}$$
$$-\sum_{s=1}^{3}\omega_s(X,Y)\Big\{\frac{4n+4}{2n+1}T^0(Z,I_s\nabla f)+\frac{4n}{(2n+1)(n-1)}U(I_sZ,\nabla f)\Big\}$$
$$+\sum_{s=1}^{3}\omega_s(Z,Y)\Big\{\frac{4n+4}{2n+1}T^0(X,I_s\nabla f)+\frac{4n}{(2n+1)(n-1)}U(I_sX,\nabla f)\Big\}.$$

Subtracting (3.28) from (3.31) and applying (3.11), (3.12) and the properties of the torsion we come to

$$0=g(X,Y)\Big[T^0(Z,\nabla f)+\frac{6n}{n-1}U(Z,\nabla f)\Big]-g(Z,Y)\Big[T^0(X,\nabla f)+\frac{6n}{n-1}U(X,\nabla f)\Big] \tag{3.32}$$
$$-\sum_{s=1}^{3}\omega_s(X,Y)\Big[T^0(Z,I_s\nabla f)+\frac{2n}{n-1}U(I_sZ,\nabla f)\Big]+\sum_{s=1}^{3}\omega_s(Z,Y)\Big[T^0(X,I_s\nabla f)+\frac{2n}{n-1}U(I_sX,\nabla f)\Big]$$
$$-\sum_{s=1}^{3}\omega_s(Z,X)\Big[2T^0(I_sY,\nabla f)-2T^0(Y,I_s\nabla f)-\frac{4}{n-1}U(I_sY,\nabla f)\Big]$$
$$-(2n+1)\sum_{s=1}^{3}\Big[df(I_sX)T^0(Z,I_sY)-df(I_sZ)T^0(X,I_sY)-4df(I_sY)U(I_sZ,X)\Big]$$
$$+(2n+1)df(X)T^0(Z,Y)-(2n+1)df(Z)T^0(X,Y).$$

Setting $Z=\nabla f$ into (3.32), after some calculations, we obtain

$$
\begin{aligned}
(2n+1)|\nabla f|^2T^0(X,Y) &= (2n+1)df(X)T^0(\nabla f,Y)\\
&+g(X,Y)\Big[T^0(\nabla f,\nabla f)+\frac{6n}{n-1}U(\nabla f,\nabla f)\Big]-df(Y)\Big[T^0(X,\nabla f)+\frac{6n}{n-1}U(X,\nabla f)\Big]\\
&-\sum_{s=1}^{3}df(I_sY)\Big[T^0(X,I_s\nabla f)+\frac{8n^2-2n-4}{n-1}U(I_sX,\nabla f)\Big]\\
&-\sum_{s=1}^{3}df(I_sX)\Big[2T^0(Y,I_s\nabla f)+(2n-1)T^0(I_sY,\nabla f)+\frac{4}{n-1}U(I_sY,\nabla f)\Big].
\end{aligned}\tag{3.33}
$$

Letting $Y=\nabla f$ in (3.33), then using (3.34) and (3.6) shows

$$
|\nabla f|^2T^0(X,\nabla f)=T^0(\nabla f,\nabla f)df(X)+\frac{3n}{(n+1)(n-1)}\Big[U(\nabla f,\nabla f)df(X)-|\nabla f|^2U(X,\nabla f)\Big].\tag{3.34}
$$

On the other hand, letting $X=I_1\nabla f$ in (3.33), using (3.6) and (3.34) gives

$$
\begin{aligned}
0 &= -df(I_1Y)\Big[T^0(\nabla f,\nabla f)+T^0(I_1\nabla f,I_1\nabla f)-\frac{8n^2-8n-4}{n-1}U(\nabla f,\nabla f)\Big]\\
&-df(I_2Y)T^0(I_1\nabla f,I_2\nabla f)-df(I_3Y)T^0(I_1\nabla f,I_3\nabla f)\\
&-(2n-1)|\nabla f|^2\Big[T^0(Y,I_1\nabla f)-T^0(I_1Y,\nabla f)\Big]+\frac{4}{n-1}|\nabla f|^2U(I_1Y,\nabla f).
\end{aligned}\tag{3.35}
$$

From (3.35) with $Y=I_2\nabla f$ and (3.8) the identity (3.23) follows since $|\nabla f|^2\neq 0$.

Setting $Y=I_1\nabla f$ into (3.35) implies

$$
T^0(\nabla f,\nabla f)+T^0(I_1\nabla f,I_1\nabla f)=-\frac{4n}{n-1}U(\nabla f,\nabla f).\tag{3.36}
$$

The latter equality together with the (2.8) yield (3.24).

The equalities (3.35), (3.23) and (3.24) imply

$$
\begin{aligned}
(2n-1)|\nabla f|^2T^0(Y,I_s\nabla f) &= (2n-1)|\nabla f|^2T^0(I_sY,\nabla f)+\frac{4}{n-1}|\nabla f|^2U(I_sY,\nabla f)\\
&-df(I_sY)\Big[T^0(\nabla f,\nabla f)+T^0(I_s\nabla f,I_s\nabla f)-\frac{8n^2-8n-4}{n-1}U(\nabla f,\nabla f)\Big]\\
&=(2n-1)|\nabla f|^2T^0(I_sY,\nabla f)+\frac{4}{n-1}|\nabla f|^2U(I_sY,\nabla f)+4(2n+1)df(I_sY)U(\nabla f,\nabla f).
\end{aligned}\tag{3.37}
$$

Let $Y=I_1\nabla f$ in (3.32) in order to see

$$
\begin{aligned}
2(2n+1)|\nabla f|^2U(I_1Z,X) &= \omega_1(Z,X)\Big[T^0(\nabla f,\nabla f)+T^0(I_1\nabla f,I_1\nabla f)-\frac{2}{n-1}U(\nabla f,\nabla f)\Big]\\
&+ndf(X)\Big[T^0(Z,I_1\nabla f)-\frac{1}{n-1}U(I_1Z,\nabla f)\Big]-ndf(Z)\Big[T^0(X,I_1\nabla f)-\frac{1}{n-1}U(I_1X,\nabla f)\Big]\\
&+ndf(I_1X)\Big[T^0(Z,\nabla f)-\frac{3}{n-1}U(Z,\nabla f)\Big]-ndf(I_1Z)\Big[T^0(X,\nabla f)-\frac{3}{n-1}U(X,\nabla f)\Big]\\
&+ndf(I_2X)\Big[T^0(Z,I_3\nabla f)-\frac{1}{n-1}U(I_3Z,\nabla f)\Big]-ndf(I_2Z)\Big[T^0(X,I_3\nabla f)-\frac{1}{n-1}U(I_3X,\nabla f)\Big]\\
&-ndf(I_3X)\Big[T^0(Z,I_2\nabla f)-\frac{1}{n-1}U(I_2Z,\nabla f)\Big]+ndf(I_3Z)\Big[T^0(X,I_2\nabla f)-\frac{1}{n-1}U(I_2X,\nabla f)\Big].
\end{aligned}\tag{3.38}
$$

Letting $X=\nabla f$ in (3.38) and applying (3.24) we obtain

$$
(4n^2-n-2)U(Z,\nabla f)|\nabla f|^2=(6n^2-n-2)U(\nabla f,\nabla f)df(Z)-n(n-1)T^0(I_1Z,I_1\nabla f)|\nabla f|^2.\tag{3.39}
$$

Therefore,

$$
|\nabla f|^2\Big[n(n-1)T^0(Z,\nabla f)-3(4n^2-n-2)U(Z,\nabla f)\Big]=-3(6n^2-n-2)U(\nabla f,\nabla f)df(Z).\tag{3.40}
$$

On the other hand, taking into account (3.24), equality (3.34) yields

$$|\nabla f|^2\Big[(n^2-1)T^0(Z,\nabla f)+3nU(Z,\nabla f)\Big]=-3n(2n+1)U(\nabla f,\nabla f)df(Z). \tag{3.41}$$

Solving the system of equations (3.40) and (3.41), we obtain (3.25).

A substitution of (3.25) and (3.43) in (3.33) gives (3.26). Now, a substitution of (3.25) and (3.43) in (3.38) shows (3.27). □

We finish this section with a few useful facts. As a direct corollary from (3.5), (3.25) and (3.43) it follows

$$|\nabla f|^2(S-2)=\frac{4(n+1)}{n-1}U(\nabla f,\nabla f). \tag{3.42}$$

In addition, from (3.25) and (3.37) it follows

$$|\nabla f|^2T^0(Z,I_s\nabla f)=-\frac{2n}{n-1}U(\nabla f,\nabla f)df(I_sZ),\quad |\nabla f|^2T^0(I_sY,I_s\nabla f)=\frac{2n}{n-1}U(\nabla f,\nabla f)df(Z). \tag{3.43}$$

The equalities (3.25) and (3.43) yield

$$T^0(I_sZ,\nabla f)=3T^0(Z,I_s\nabla f),\quad T^0(Z,\nabla f)=-3T^0(I_sZ,I_s\nabla f),\quad T^0(Z,\nabla f)=-\frac{6n}{n-1}U(Z,\nabla f), \tag{3.44}$$

3.5. **Formulas for the covariant derivatives of the torsion tensors.** Here we shall prove formulas for the covariant derivative of the torsion tensor.

Lemma 3.9. *If M and f are as in Theorem 1.3, then we have the following identities for the covariant derivatives of the torsion tensor at the points where $|\nabla f|\neq 0$,*

$$\begin{aligned}
|\nabla f|^2(\nabla_ZT^0)(X,Y)&=\frac{4n+2}{n+2}fdf(Z)T^0(X,Y)\\
&-\frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}f\Big[-3df(Y)g(X,Z)-3df(X)g(Y,Z)+\sum_{s=1}^3\Big(df(I_sY)\omega_s(X,Z)+df(I_sX)\omega_s(Y,Z)\Big)\Big]\\
&-\frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\sum_{(ijk)}df(\xi_i)\Big[3df(Y)\omega_i(X,Z)+df(I_iY)g(X,Z)-df(I_jY)\omega_k(X,Z)+df(I_kY)\omega_j(X,Z)\Big]\\
&-\frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\sum_{(ijk)}df(\xi_i)\Big[3df(X)\omega_i(Y,Z)+df(I_iX)g(Y,Z)-df(I_jX)\omega_k(Y,Z)+df(I_kX)\omega_j(Y,Z)\Big]
\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}
|\nabla f|^2(\nabla_ZU)(X,Y)&-2fdf(Z)U(X,Y)-2\sum_{s=1}^3df(\xi_s)df(I_sZ)U(X,Y)=\frac{2n-2}{n+2}fdf(Z)U(X,Y)\\
&-2\sum_{s=1}^3df(\xi_s)df(I_sZ)U(X,Y)-\frac{1}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big[-2fdf(Z)-2\sum_{s=1}^3df(\xi_s)df(I_sZ)\Big]g(X,Y)\\
&-\frac{n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}f\Big[df(Y)g(X,Z)+df(X)g(Y,Z)+\sum_{s=1}^3\Big(df(I_sY)\omega_s(X,Z)+df(I_sX)\omega_s(Y,Z)\Big)\Big]\\
&+\frac{n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\sum_{(ijk)}df(\xi_i)\Big[df(Y)\omega_i(X,Z)-df(I_iY)g(X,Z)+df(I_jY)\omega_k(X,Z)-df(I_kY)\omega_j(X,Z)\Big]\\
&+\frac{n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\sum_{(ijk)}df(\xi_i)\Big[df(X)\omega_i(Y,Z)-df(I_iX)g(Y,Z)+df(I_jX)\omega_k(Y,Z)-df(I_kX)\omega_j(Y,Z)\Big],
\end{aligned} \tag{3.46}$$

where $\sum_{(ijk)}$ means the cyclic sum, cf. Convention 1.4.

Proof. The contracted Bianchi identity reads [32, 41]

$$(\nabla_{e_a}T^0)(e_a,X)+\frac{2n+4}{n-1}(\nabla_{e_a}U)(e_a,X)-(2n+1)dS(X)=0. \tag{3.47}$$

After taking the trace in the covariant derivatives of (3.25) and (3.42) we obtain

$$\begin{aligned}(\nabla_{e_a}T^0)(e_a,\nabla f)&=-\frac{6n}{n-1}\nabla f\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big)+\frac{24n^2}{n-1}f\frac{U(\nabla f,\nabla f)}{|\nabla f|^2},\\ (\nabla_{e_a}U)(e_a,\nabla f)&=\nabla f\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big)-4nf\frac{U(\nabla f,\nabla f)}{|\nabla f|^2},\\ \nabla f(S)&=\frac{4n+4}{n-1}\nabla f\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big).\end{aligned} \tag{3.48}$$

The system (3.48) and (3.47) imply

$$\nabla f\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big)=2\frac{n-1}{n+2}f\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}. \tag{3.49}$$

Similarly, using in addition (3.44), we have

$$\begin{aligned}(\nabla_{e_a}T^0)(e_a,I_s\nabla f)&=\frac{2n}{n-1}I_s\nabla f\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big)+\frac{8n^2}{n-1}df(\xi_s)\frac{U(\nabla f,\nabla f)}{|\nabla f|^2},\\ (\nabla_{e_a}U)(e_a,I_s\nabla f)&=I_s\nabla f\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big)+4ndf(\xi_s)\frac{U(\nabla f,\nabla f)}{|\nabla f|^2},\\ I_s\nabla f(S)&=\frac{4n+4}{n-1}I_s\nabla f\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big).\end{aligned} \tag{3.50}$$

Since the differentiation of (3.43) involves covariant derivatives of the almost complex structures the derivation of (3.50) requires some care we do it explicitly again, cf. Remark 3.5. We start with the proof of the first formula in (3.50). Differentiating the first equation in (3.43), taking into account (2.3), we have

$$\begin{aligned}&\left(\nabla_XT^0\right)(Z,I_i\nabla f)-\alpha_j(X)T^0(Z,I_k\nabla f)+\alpha_k(X)T^0(Z,I_j\nabla f)+T^0(Z,I_i\nabla_X(\nabla f))\\ =\;&-\frac{2n}{n-1}\left[X\left(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\right)df(I_iZ)+\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}g\left(\nabla_X\left(\nabla f\right),I_iZ\right)\right]\\ &-\frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\left[-\alpha_j(X)df\left(I_kZ\right)+\alpha_k(X)df\left(I_jZ\right)\right].\end{aligned}$$

The formula for the Hessian (1.6) gives

$$\begin{aligned}&\left(\nabla_XT^0\right)(Z,I_i\nabla f)-\alpha_j(X)T^0(Z,I_k\nabla f)+\alpha_k(X)T^0(Z,I_j\nabla f)-fT^0(I_iX,Z)-\sum_{s=1}^3df\left(\xi_s\right)T^0(I_iI_sX,Z)\\ =\;&-\frac{2n}{n-1}\left[X\left(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\right)df(I_iZ)-\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\left(fg\left(X,I_iZ\right)+\sum_{s=1}^3df\left(\xi_s\right)g\left(I_sX,I_iZ\right)\right)\right]\\ &-\frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\left[-\alpha_j(X)df\left(I_kZ\right)+\alpha_k(X)df\left(I_jZ\right)\right].\end{aligned}$$

Taking the trace in the above identity and then applying the first equation in (3.43) to the obtained equality we see that the terms involving the connection 1-forms cancel, which gives the first identity in (3.50).

The second line in (3.50) follows similarly.

The system (3.50) and (3.47) yields

$$I_s\nabla f\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big)=2df(\xi_s)\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}. \tag{3.51}$$

We calculate the divergence of T^0 differentiating (3.26), taking the trace in the obtained equality and applying (3.49), (3.51). After a short computation we obtain

$$
\begin{aligned}
(3.52)\quad & |\nabla f|^2(\nabla_{e_a}T^0)(e_a,Y) - 2fT^0(Y,\nabla f) + 2\sum_{s=1}^3 df(\xi_s)T^0(Y,I_s\nabla f) = \\
& -\frac{2n}{n-1}\left[3\nabla f\left(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\right)df(Y) + \sum_{s=1}^3 I_s\nabla f\left(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\right)df(I_sY)\right] \\
& -\frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\left[3\nabla^2 f(e_a,e_a)df(Y) - \sum_{s=1}^3 \nabla^2 f(e_a,I_se_a)df(I_sY)\right] \\
& -\frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\left[3\nabla^2 f(\nabla f,Y) + \sum_{s=1}^3 \nabla^2 f(I_s\nabla f,I_sY)\right].
\end{aligned}
$$

Applying (1.6), (3.25), (3.43), (3.49) and (3.51) to (3.52), we get

$$
\begin{aligned}
(3.53)\quad & |\nabla f|^2(\nabla_{e_a}T^0)(e_a,Y) = \\
& -\frac{12n}{n-1}f\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}df(Y) + \frac{4n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\sum_{s=1}^3 df(\xi_s)df(I_sY) - \frac{12n}{n+2}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}fdf(Y) \\
& -\frac{4n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\sum_{s=1}^3 df(\xi_s)df(I_sY) + \frac{24n^2}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}fdf(Y) - \frac{8n^2}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\sum_{s=1}^3 df(\xi_s)df(I_sY) \\
& +\frac{6n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}fdf(Y) - \frac{6n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\sum_{s=1}^3 df(\xi_s)df(I_sY) \\
& -\frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\sum_{i=1}^3\left[-fdf(Y) + df(\xi_i)df(I_iY) - df(\xi_j)df(I_jY) - df(\xi_k)df(I_kY)\right] \\
& = \frac{12n\,(n+1)\,(2n+1)}{(n+2)\,(n-1)}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}fdf(Y) - 4n\frac{2n+1}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\sum_{s=1}^3 df(\xi_s)df(I_sY).
\end{aligned}
$$

Applying (3.25) and (3.43) to (3.53), we derive

$$
(3.54)\qquad |\nabla f|^2(\nabla_{e_a}T^0)(e_a,Y) = -\frac{(4n+2)(n+1)}{n+2}fT^0(Y,\nabla f) + (4n+2)\sum_{s=1}^3 df(\xi_s)T^0(Y,I_s\nabla f).
$$

Now, we calculate the divergence of U differentiating (3.27), taking the trace in the obtained equality and applying (1.6), (3.49) and (3.51). We have

$$
\begin{aligned}
(3.55)\quad & |\nabla f|^2(\nabla_{e_a}U)(e_a,Y)-2fU(Y,\nabla f)+2\sum_{s=1}^{3}df(\xi_s)U(Y,I_s\nabla f)=\\
&-\frac{1}{n-1}Y\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big)|\nabla f|^2+\frac{n}{n-1}\Big[\nabla f\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big)df(Y)-\sum_{s=1}^{3}I_s\nabla f\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big)df(I_sY)\Big]\\
&-\frac{1}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big[2\nabla^2 f(Y,\nabla f)-n\nabla^2 f(e_a,e_a)df(Y)-n\sum_{s=1}^{3}\nabla^2 f(e_a,I_se_a)df(I_sY)\Big]\\
&+\frac{n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big[\nabla^2 f(\nabla f,Y)-\sum_{s=1}^{3}\nabla^2 f(I_s\nabla f,I_sY)\Big]\\
&=-\frac{1}{n-1}Y\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big)|\nabla f|^2+\frac{2n}{n-1}\Big[\frac{n-1}{n+2}fU(Y,\nabla f)-\sum_{s=1}^{3}df(\xi_s)U(I_sY,\nabla f)\Big]\\
&+\frac{2n}{n-1}fU(Y,\nabla f)+\frac{2n}{n-1}\sum_{s=1}^{3}df(\xi_s)U(I_sY,\nabla f)-\frac{4n^2-2}{n-1}\Big[fU(Y,\nabla f)+\sum_{s=1}^{3}df(\xi_s)U(I_sY,\nabla f)\Big].
\end{aligned}
$$

Thus, from (3.55) we obtain

$$
\begin{aligned}
(3.56)\quad |\nabla f|^2(\nabla_{e_a}U)(e_a,Y)=&-\frac{1}{n-1}Y\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big)|\nabla f|^2\\
&-\frac{4n^2+6n}{n+2}fU(Y,\nabla f)-\frac{4n^2-2n}{n-1}\sum_{s=1}^{3}df(\xi_s)U(I_sY,\nabla f).
\end{aligned}
$$

A substitution of (3.54), (3.56) and

$$
|\nabla f|^2Y(S)=\frac{4n+4}{n-1}Y\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big)|\nabla f|^2
$$

in (3.47) implies

$$
(3.57)\qquad Y\Big(\frac{U(\nabla f,\nabla f)}{|\nabla f|^2}\Big)|\nabla f|^2=\frac{2n-2}{n+2}fU(Y,\nabla f)-2\sum_{s=1}^{3}df(\xi_s)U(I_sY,\nabla f).
$$

The equalities (3.56) and (3.57) yield

$$
(3.58)\qquad |\nabla f|^2(\nabla_{e_a}U)(e_a,Y)=-\frac{2(n+1)(2n+1)}{n+2}fU(Y,\nabla f)-2(2n+1)\sum_{s=1}^{3}df(\xi_s)U(I_sY,\nabla f).
$$

We calculate from (3.26) using (1.6), (3.25) and (3.57) that

$$
\begin{aligned}
|\nabla f|^2(\nabla_Z T^0)(X,Y) - 2f\,df(Z)T^0(X,Y) - 2\sum_{s=1}^{3} df(\xi_s)df(I_sZ)T^0(X,Y)&\\
= \frac{2n-2}{n+2} f\,df(Z)T^0(X,Y) - 2\sum_{s=1}^{3} df(\xi_s)df(I_sZ)T^0(X,Y)&\\
- \frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2} f\Big[-3df(Y)g(X,Z) + \sum_{s=1}^{3} df(I_sY)\omega_s(X,Z)\Big]&\\
- \frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2} f\Big[-3df(X)g(Y,Z) + \sum_{s=1}^{3} df(I_sX)\omega_s(Y,Z)\Big]&\\
- \frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2} df(\xi_1)\Big[3df(Y)\omega_1(X,Z) + df(I_1Y)g(X,Z) - df(I_2Y)\omega_3(X,Z) + df(I_3Y)\omega_2(X,Z)\Big]&\\
- \frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2} df(\xi_1)\Big[3df(X)\omega_1(Y,Z) + df(I_1X)g(Y,Z) - df(I_2X)\omega_3(Y,Z) + df(I_3X)\omega_2(Y,Z)\Big]&\\
- \frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2} df(\xi_2)\Big[3df(Y)\omega_2(X,Z) + df(I_2Y)g(X,Z) + df(I_1Y)\omega_3(X,Z) - df(I_3Y)\omega_1(X,Z)\Big]&\\
- \frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2} df(\xi_2)\Big[3df(X)\omega_2(Y,Z) + df(I_2X)g(Y,Z) + df(I_1X)\omega_3(Y,Z) - df(I_3X)\omega_1(Y,Z)\Big]&\\
- \frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2} df(\xi_3)\Big[3df(Y)\omega_3(X,Z) + df(I_3Y)g(X,Z) - df(I_1Y)\omega_2(X,Z) + df(I_2Y)\omega_1(X,Z)\Big]&\\
- \frac{2n}{n-1}\frac{U(\nabla f,\nabla f)}{|\nabla f|^2} df(\xi_3)\Big[3df(X)\omega_3(Y,Z) + df(I_3X)g(Y,Z) - df(I_1X)\omega_2(Y,Z) + df(I_2X)\omega_1(Y,Z)\Big]&.
\end{aligned}
$$

The last equality yields (3.45). Finally, equation (3.46) follows from (3.27) using (1.6), (3.25) and (3.57). $\square$

In the next, key step of the proof, where we show that the torsion tensor vanishes, we shall use the following particular cases of Lemma 3.9. For $Z=\nabla f$, (3.45) gives

$$
\begin{aligned}
(\nabla_{\nabla f} T^0)(X,Y) &= \frac{2n-2}{n+2} fT^0(X,Y) + df(\xi_1)\Big[T^0(I_1X,Y) + T^0(X,I_1Y)\Big]\\
&\quad + df(\xi_2)\Big[T^0(I_2X,Y) + T^0(X,I_2Y)\Big] + df(\xi_3)\Big[T^0(I_3X,Y) + T^0(X,I_3Y)\Big].
\end{aligned} \tag{3.59}
$$

Similarly, letting $Z = I_i\nabla f$ in (3.45) we obtain

$$
\begin{aligned}
(\nabla_{I_i\nabla f} T^0)(X,Y) &= 2df(\xi_i)T^0(X,Y) + f\Big[T^0(I_iX,Y) + T^0(X,I_iY)\Big]\\
&\quad - df(\xi_j)\Big[T^0(I_kX,Y) + T^0(X,I_kY)\Big] + df(\xi_k)\Big[T^0(I_jX,Y) + T^0(X,I_jY)\Big].
\end{aligned} \tag{3.60}
$$

The substitution of $Y=\nabla f$ in (3.54) taking into account (3.24) and Lemma 3.2 gives

$$
|\nabla f|^2(\nabla_{e_a}T^0)(e_a,\nabla f) = \frac{12n(n+1)(2n+1)}{(n+2)(n-1)} fU(\nabla f,\nabla f), \tag{3.61}
$$

while the substitution $Z=e_a, X=I_ie_a, Y=I_i\nabla f$ in (3.45) and (3.43) gives

$$
|\nabla f|^2(\nabla_{e_a}T^0)(I_ie_a, I_i\nabla f) = -\frac{4n\,(n+1)\,(2n+1)}{(n+2)\,(n-1)} fU(\nabla f,\nabla f). \tag{3.62}
$$

Finally, letting $Z=\nabla f, I_s\nabla f$ in (3.46) shows the next equalities

$$
(\nabla_{\nabla f}U)(X,Y) = \frac{2n-2}{n+2} fU(X,Y), \qquad (\nabla_{I_s\nabla f}U)(X,Y) = 2df(\xi_s)U(X,Y). \tag{3.63}
$$

3.6. **Vanishing of the torsion.** In this section we show the vanishing of the torsion, $T^0 = U = 0$, by calculating in two ways the mixed third covariant derivatives of a function satisfying (1.6).

Lemma 3.10. *If M satisfies the assumptions of Theorem 1.3, then the torsion tensor vanishes, $T^0 = 0$, $U = 0$, i.e., M is a qc-Einstein manifold.*

The proof occupies the remaining part of this sub-section.

3.6.1. *The Ricci identities.* We are going to use the sixth line in (2.13). A substitution of the contracted Bianchi identity (3.47) in the second formula of (2.12) gives

$$
\begin{aligned}
(3.64)\quad (2n+1)\rho_i(\xi_j, I_kX) &= -(2n+1)\rho_i(\xi_k, I_jX)\\
&= -\frac{1}{4}[(\nabla_{e_a}T^0)(e_a,X) + (\nabla_{e_a}T^0)(I_ie_a, I_iX)] + \frac{n}{n-1}(\nabla_{e_a}U)(e_a,X).
\end{aligned}
$$

Let $Z = \nabla f$ in (2.11), and then substitute the obtained equality in the sixth formula of (2.13), after which use (3.64) in order to see

$$
\begin{aligned}
(3.65)\quad & \nabla^3 f(\xi_i, X, Y) - \nabla^3 f(X, Y, \xi_i)\\
&= -\nabla^2 f(T(\xi_i, X), Y) - \nabla^2 f(X, T(\xi_i, Y)) - df((\nabla_X T)(\xi_i, Y)) + (\nabla_X U)(I_iY, \nabla f)\\
&+ \frac{1}{4}\left[(\nabla_Y T^0)(I_i\nabla f, X) + (\nabla_Y T^0)(\nabla f, I_iX)\right] - \frac{1}{4}\left[(\nabla_{\nabla f}T^0)(I_iY, X) + (\nabla_{\nabla f}T^0)(Y, I_iX)\right]\\
&+ \frac{1}{2n+1}\left[-\frac{1}{4}(\nabla_{e_a}T^0)\left[(e_a, I_k\nabla f) - (I_ke_a, \nabla f)\right] + \frac{n}{n-1}(\nabla_{e_a}U)(e_a, I_k\nabla f)\right]\omega_j(X,Y)\\
&- \frac{1}{2n+1}\left[-\frac{1}{4}(\nabla_{e_a}T^0)\left[(e_a, I_j\nabla f) - (I_je_a, \nabla f)\right] + \frac{n}{n-1}(\nabla_{e_a}U)(e_a, I_j\nabla f)\right]\omega_k(X,Y)\\
&+ \frac{1}{2n+1}\left[\frac{1}{4}(\nabla_{e_a}T^0)\left[(e_a, I_kY) - (I_ke_a, Y)\right] - \frac{n}{n-1}(\nabla_{e_a}U)(e_a, I_kY)\right]df(I_jX)\\
&- \frac{1}{2n+1}\left[\frac{1}{4}(\nabla_{e_a}T^0)\left[(e_a, I_jY) - (I_je_a, Y)\right] - \frac{n}{n-1}(\nabla_{e_a}U)(e_a, I_jY)\right]df(I_kX)\\
&+ \frac{1}{2n+1}\left[\frac{1}{4}(\nabla_{e_a}T^0)\left[(e_a, I_kX) - (I_ke_a, X)\right] - \frac{n}{n-1}(\nabla_{e_a}U)(e_a, I_kX)\right]df(I_jY)\\
&- \frac{1}{2n+1}\left[\frac{1}{4}(\nabla_{e_a}T^0)\left[(e_a, I_jX) - (I_je_a, X)\right] - \frac{n}{n-1}(\nabla_{e_a}U)(e_a, I_jX)\right]df(I_kY).
\end{aligned}
$$

Note that from (2.9) we have

$$
(3.66)\qquad T(\xi_i, X, Y) = -\frac{1}{4}\left[T^0(I_iX, Y) + T^0(X, I_iY)\right] - U(X, I_iY).
$$

Differentiating the above formula we find, applying (2.3),

$$
(3.67)\qquad df\left((\nabla_X T)(\xi_i, Y)\right) = -\frac{1}{4}\left[(\nabla_X T^0)(I_iY, \nabla f) + (\nabla_X T^0)(Y, I_i\nabla f)\right] + (\nabla_X U)(I_iY, \nabla f).
$$

Using (1.6), (3.66), (3.67) and the properties of torsion tensor listed in (2.8), we obtain from (3.65)

$$
\begin{aligned}
(3.68)\quad &\nabla^3 f(\xi_i,X,Y)-\nabla^3 f(X,Y,\xi_i)\\
&=\frac14\left[(\nabla_X T^0)(I_iY,\nabla f)+(\nabla_X T^0)(Y,I_i\nabla f)\right]+\frac14\left[(\nabla_Y T^0)(I_iX,\nabla f)+(\nabla_Y T^0)(X,I_i\nabla f)\right]\\
&\quad-\frac14\left[(\nabla_{\nabla f}T^0)(X,I_iY)+(\nabla_{\nabla f}T^0)(I_iX,Y)\right]-\frac12\left[T^0(I_iX,Y)+T^0(X,I_iY)\right]f\\
&\quad+\left[\frac12\left[T^0(X,I_kY)-T^0(I_kX,Y)\right]+2U(X,I_kY)\right]df(\xi_j)\\
&\quad+\left[\frac12\left[T^0(I_jX,Y)-T^0(X,I_jY)\right]+2U(I_jX,Y)\right]df(\xi_k)\\
&\quad+\frac{1}{2n+1}\left[-\frac14(\nabla_{e_a}T^0)\left[(e_a,I_k\nabla f)-(I_ke_a,\nabla f)\right]+\frac{n}{n-1}(\nabla_{e_a}U)(e_a,I_k\nabla f)\right]\omega_j(X,Y)\\
&\quad-\frac{1}{2n+1}\left[-\frac14(\nabla_{e_a}T^0)\left[(e_a,I_j\nabla f)-(I_je_a,\nabla f)\right]+\frac{n}{n-1}(\nabla_{e_a}U)(e_a,I_j\nabla f)\right]\omega_k(X,Y)\\
&\quad+\frac{1}{2n+1}\left[\frac14(\nabla_{e_a}T^0)\left[(e_a,I_kY)-(I_ke_a,Y)\right]-\frac{n}{n-1}(\nabla_{e_a}U)(e_a,I_kY)\right]df(I_jX)\\
&\quad-\frac{1}{2n+1}\left[\frac14(\nabla_{e_a}T^0)\left[(e_a,I_jY)-(I_je_a,Y)\right]-\frac{n}{n-1}(\nabla_{e_a}U)(e_a,I_jY)\right]df(I_kX)\\
&\quad+\frac{1}{2n+1}\left[\frac14(\nabla_{e_a}T^0)\left[(e_a,I_kX)-(I_ke_a,X)\right]-\frac{n}{n-1}(\nabla_{e_a}U)(e_a,I_kX)\right]df(I_jY)\\
&\quad-\frac{1}{2n+1}\left[\frac14(\nabla_{e_a}T^0)\left[(e_a,I_jX)-(I_je_a,X)\right]-\frac{n}{n-1}(\nabla_{e_a}U)(e_a,I_jX)\right]df(I_kY).
\end{aligned}
$$

For $X=I_i\nabla f$, $Y=\nabla f$ equation (3.68) together with (3.59), (3.60), (3.6) and (3.23) imply

$$
(3.69)\qquad \nabla^3 f(\xi_i,I_i\nabla f,\nabla f)-\nabla^3 f(I_i\nabla f,\nabla f,\xi_i)=0.
$$

On the other hand, a subtraction of (3.18) from (3.1) with $A=\xi_i$ gives

$$
\begin{aligned}
(3.70)\quad &\nabla^3 f(\xi_i,X,Y)-\nabla^3 f(X,Y,\xi_i)=\frac{n+1}{2n+1}\Big[(\nabla_X T^0)(I_iY,\nabla f)+(\nabla_X T^0)(Y,I_i\nabla f)\Big]\\
&+\frac{4n}{(2n+1)(n-1)}(\nabla_X U)(I_iY,\nabla f)-f\frac{n+1}{2n+1}\Big[T^0(X,I_iY)+T^0(I_iX,Y)\Big]+\frac{4n}{(2n+1)(n-1)}fU(X,I_iY)\\
&\quad-df(\xi_i)\frac{n+1}{2n+1}\Big[T^0(I_iX,I_iY)-T^0(X,Y)\Big]+\frac{4n}{(2n+1)(n-1)}df(\xi_i)U(X,Y)\\
&\quad-df(\xi_j)\frac{n+1}{2n+1}\Big[T^0(I_jX,I_iY)+T^0(I_kX,Y)\Big]+\frac{4n}{(2n+1)(n-1)}df(\xi_j)U(X,I_kY)\\
&\quad-df(\xi_k)\frac{n+1}{2n+1}\Big[T^0(I_kX,I_iY)-T^0(I_jX,Y)\Big]-\frac{4n}{(2n+1)(n-1)}df(\xi_k)U(X,I_jY)\\
&\qquad\qquad-\sum_{s=1}^{3}\left[\nabla^2 f(\xi_i,\xi_s)+f\right]\omega_s(X,Y).
\end{aligned}
$$

Letting $X=I_i\nabla f, Y=\nabla f$ in (3.70) and then applying (3.6) and (3.23) we obtain

$$
\begin{aligned}
(3.71)\quad &\nabla^3 f(\xi_i,I_i\nabla f,\nabla f)-\nabla^3 f(I_i\nabla f,\nabla f,\xi_i)=\frac{2(n+1)}{2n+1}(\nabla_{I_i\nabla f}T^0)(I_i\nabla f,\nabla f)\\
&+\frac{4n}{(2n+1)(n-1)}(\nabla_{I_i\nabla f}U)(I_i\nabla f,\nabla f)-\frac{n+1}{2n+1}f\Big[T^0(I_i\nabla f,I_i\nabla f)-T^0(\nabla f,\nabla f)\Big]\\
&\qquad+\frac{4n}{(2n+1)(n-1)}fU(\nabla f,\nabla f)+\left[\nabla^2 f(\xi_i,\xi_i)+f\right]|\nabla f|^2.
\end{aligned}
$$

Using (3.60), (3.63) as well as (3.24) in (3.71) we conclude

$$\nabla^3 f(\xi_i, I_i\nabla f, \nabla f) - \nabla^3 f(I_i\nabla f, \nabla f, \xi_i) = \frac{4n}{n-1} fU(\nabla f, \nabla f) + \left[\nabla^2 f(\xi_i, \xi_i) + f\right] |\nabla f|^2. \tag{3.72}$$

The formula for the last term is given in (3.16) to whose right-hand side we apply (3.61), (3.62) and (3.58) in order to obtain

$$\nabla^2 f(\xi_i, \xi_i) + f = -\frac{2\,(n+1)\,(2n+1)}{(n+2)\,(n-1)} f \frac{U(\nabla f, \nabla f)}{|\nabla f|^2}. \tag{3.73}$$

Now (3.73) applied to (3.72) allows us to conclude

$$\nabla^3 f(\xi_i, I_i\nabla f, \nabla f) - \nabla^3 f(I_i\nabla f, \nabla f, \xi_i) = \frac{2}{n+2} fU(\nabla f, \nabla f). \tag{3.74}$$

Comparing (3.69) and (3.74) we obtain $fU(\nabla f, \nabla f) = 0$, which implies $U(\nabla f, \nabla f) = 0$ taking into account Remark 3.7. Hence, $T^0 = U = 0$ due to (3.26) and (3.27). This completes the proof of Lemma 3.10.

3.7. **The Riemannian Hessian.** Here we show that if $T^0 = U = 0$ the equality (1.6) implies that the Riemannian Hessian satisfies (1.1) and therefore the manifold is the standard sphere due to the Obata's theorem.

Lemma 3.11. *Let $(M, \eta, g, \mathbb{Q})$ be a qc-Einstein manifold, $T^0 = U = 0$, of dimension $4n+3 > 7$. Let h be the associated Riemannian metric (1.5). If f is a smooth function whose horizontal Hessian satisfies (1.6), then the Riemannian Hessian of f with respect to the metric h satisfies (1.1).*

Proof. Taking into account (2.2) we have the following formula relating the Hessian with respect to the Levi-Civita and the Biquard connections

$$(\nabla^h)^2 f(A,B) = \nabla^2 f(A,B) + \frac{1}{2}\Big[h(T(A,B),df) - h(T(B,df),A) + h(T(df,A),B)\Big], A, B \in \Gamma(TM). \tag{3.75}$$

From (3.75), (2.5) and (1.6) it follows that

$$(\nabla^h)^2 f(X,Y) = -fh(X,Y). \tag{3.76}$$

Let us recall that a qc-Einstein manifold, $T^0 = U = 0$, has integrable vertical space [32] thus the fourth line in (2.10) shows

$$h(T(\xi_s, \xi_t), X) = 0. \tag{3.77}$$

Now, using (3.12) with $T^0 = U = 0$, we calculate from (3.75)

$$\begin{aligned}(\nabla^h)^2 f(X, \xi_i) = df(I_i X) + \frac{1}{2} h(T(X,\xi_i), \nabla f) - \frac{1}{2} h(T(\xi_i, \nabla f), X) - \frac{1}{2} h(T(\xi_i, \sum_{s=1}^{3} df(\xi_s)\xi_s, X) \\ + \frac{1}{2} h(T(\nabla f, X), \xi_i) + \frac{1}{2} h(T(\sum_{s=1}^{3} df(\xi_s)\xi_s, X), \xi_i) = df(I_i X) + \omega_i(\nabla f, X) = 0,\end{aligned} \tag{3.78}$$

taking into account (3.77) and the properties of the torsion (2.9) and (2.5). A similar argument shows the identity

$$(\nabla^h)^2 f(\xi_i, \xi_i) = \nabla^2 f(\xi_i, \xi_i) = -f, \tag{3.79}$$

where we have used (3.16) taken with $T^0 = U = 0$.

Finally, we have to show $(\nabla^h)^2 f(\xi_i, \xi_j) = 0$. The trace with respect to $X = e_a, Y = I_j e_a$ in (3.19) together with the second equality in (2.10) and the condition $T^0 = U = 0$ yields

$$\nabla^2 f(\xi_j, \xi_i) = (1 - S)df(\xi_k). \tag{3.80}$$

Now, the equality (3.75) together with the fourth equality in (2.10), (3.77) and (3.80) imply

$$(\nabla^h)^2 f(\xi_i, \xi_j) = (1-S)df(\xi_k) + \frac{1}{2} S df(\xi_k) = (1 - \frac{1}{2}S) df(\xi_k) = 0, \tag{3.81}$$

since (3.5) shows $S = 2$ in the case $T^0 = U = 0$. ∎

At this point, applying the Obata theorem we conclude that our manifold is isometric to the unit sphere. In order to show that it is qc-equivalent to the sphere we shall use a Liouville-type result in the quaternionic contact case which we present next.

3.8. **Proof of Theorem 1.3.** From Lemma 3.10, Lemma 3.11 and the classical Obata theorem it follows that (M,h) is isometric to the unit sphere in Euclidean space, i.e., there is a diffeomorphism $i: M\rightarrow S^{4n+3}$ such that $h=i^*dx^2$, where dx^2 denotes the round metric on S^{4n+3} which we take to be of constant Riemannian scalar curvature $Scal^h=(4n+3)(4n+2)$. Thus, the curvature tensor R^h of the Levi-Civita connection ∇^h of h is given by

$$R^h(A,B,C,D)=h(B,C)h(A,D)-h(B,D)h(A,C). \tag{3.82}$$

The relation between the curvature tensors of the Levi-Civita and the Biquard connection [32, Corollary 4.13] or [41, Theorem 4.4.3] together with (3.82) yield

$$\begin{aligned} R(X,Y,Z,V)&=g(Y,Z)g(X,V)-g(Y,V)g(X,Z)\\ &\quad+\sum_{s=1}^3\Big[\omega_s(Y,Z)\omega_s(X,V)-\omega_s(X,Z)\omega_s(Y,V)-2\omega_s(X,Y)\omega_s(Z,V)\Big]. \end{aligned} \tag{3.83}$$

In the case $T^0=U=0, S=2$, the formula for the qc-conformal curvature tensor given in [39, Proposition 4.2] reads

$$\begin{aligned} W^{qc}(X,Y,Z,V)&=\frac14\Big[R(X,Y,Z,V)+\sum_{s=1}^3R(I_sX,I_sY,Z,V)\Big]+g(X,Z)g(Y,V)-g(Y,Z)g(X,V)\\ &\quad+\sum_{s=1}^3\Big[\omega_s(X,Z)\omega_s(Y,V)-\omega_s(Y,Z)\omega_s(X,V)\Big]. \end{aligned} \tag{3.84}$$

With a small calculation we see from (3.84), taking into account (3.83), that the qc-conformal curvature tensor vanishes, $W^{qc}=0$ and $(M,g,\eta,\mathbb{Q})$ is locally qc-conformal to the sphere due to [39, Theorem 1.3].

At this point we invoke the Liouville type result showing that a local qc conformal map on the qc 3-Sasakian sphere is the restriction of a global one. A general version of the Liouville theorem in the setting of Cartan geometries was given in [10, Proposition 1.5.2 & Section 4.3.3] and another general result on Carnot groups in [20]; for results in particular geometric settings, see in the Riemannian case [56], [57], [30], [59], [7], [46], [45], [25], in the CR case [64], [3], [61], [12, 13], [18], an alternative proof in the qc setting [38]. Hence, taking into account that M is the round sphere, it follows $(M,g,\eta,\mathbb{Q})$ is qc-conformal to S^{4n+3}, i.e., we have $\eta=\kappa\Psi F^*\tilde\eta$ for some diffeomorphism $F: M\rightarrow S^{4n+3}$, positive smooth function κ and a matrix $\Psi\in SO(3)$ with smooth functions as entries, where $\eta=(\eta_1,\eta_2,\eta_3)^t$ is a local 1-form considered as an element of $\mathbb{R}^3$. Comparing the metrics we obtain $\kappa=1$ which shows that M is qc-homothetic to the 3-Sasakian unit sphere in the $(n+1)$-dimensional quaternion space. This completes the proof of Theorem 1.3.

The proof of Theorem 1.2 follows as already noted after the statement of Theorem 1.3.

4. Appendix

4.1. **The $P-$form.** Let $(M,g,\mathbb{Q})$ be a compact quaternionic contact manifold of dimension $4n+3$ and f a smooth function on M. We recall the notion of a P-function introduced in [37]

Definition 4.1. *[37]*

a) For a fixed smooth function f we define a one form $P\equiv P_f\equiv P[f]$ on M, which we call the $P-$form of f, by the following equation

$$P_f(X)\ =\ \nabla^3f(X,e_b,e_b)+\sum_{t=1}^3\nabla^3f(I_tX,e_b,I_te_b)-4nSdf(X)+4nT^0(X,\nabla f)-\frac{8n(n-2)}{n-1}U(X,\nabla f).$$

b) The $P-$function of f is the function $P_f(\nabla f)$.

c) The $C-$operator is the fourth-order differential operator on M (independent of f!) defined by

$$Cf = -\nabla^* P_f = (\nabla_{e_a} P_f)(e_a).$$

d) We say that the $P-$function of f is non-negative if its integral exists and is non-positive

$$\int_M f \cdot Cf \, Vol_\eta = -\int_M P_f(\nabla f) \, Vol_\eta \geq 0. \tag{4.1}$$

If (4.1) holds for any smooth function of compact support we say that the $C-$operator is non-negative.

The $Sp(n)Sp(1)$-invariant decomposition of the horizontal Hessian $\nabla^2 f$ are given by

$$\begin{aligned} (\nabla^2 f)_{[3]}(X,Y) &= \frac{1}{4}\Big[\nabla^2 f(X,Y) + \sum_{s=1}^3 \nabla^2 f(I_sX, I_sY)\Big] \\ (\nabla^2 f)_{[-1]}(X,Y) &= \frac{1}{4}\Big[3\nabla^2 f(X,Y) - \sum_{s=1}^3 \nabla^2 f(I_sX, I_sY)\Big]. \end{aligned} \tag{4.2}$$

Let $(\nabla^2 f)_{[3][0]}$ be the trace-free part of the 3-component of the horizontal Hessian,

$$(\nabla^2 f)_{[3][0]}(X,Y) = (\nabla^2 f)_{[3]}(X,Y) + \frac{1}{4n}\triangle f g(X,Y). \tag{4.3}$$

The next local formula, established in [37],

$$(\nabla_{e_a}(\nabla^2 f)_{[3][0]})(e_a, X) = \frac{n-1}{4n} P_f(X). \tag{4.4}$$

implies the non-negativity of the $C-$operator on a compact qc manifold of dimension at least eleven [37, Theorem 3.3]. Indeed, using (4.4) we have

$$\frac{n-1}{4n}\int_M f.Cf \, Vol_\eta = -\frac{n-1}{4n}\int_M P_f(\nabla f)\, Vol_\eta = \int_M |(\nabla^2 f)_{[3][0]}|^2 \, Vol_\eta, \tag{4.5}$$

after using an integration by parts and the orthogonality of the components of the horizontal Hessian.

4.2. **A new proof of Theorem 1.1.** Here we use the non-negativity of the P-function $P(\nabla f)$ of a smooth function f established in [37, Theorem 3.3] to give a new proof of Theorem 1.1.

Proof. Let f be an eigenfunction of the sub-Laplacian with eigenvalue λ, i.e., we assume that (2.14) holds. An integration by parts gives

$$\int_M (\triangle f)^2 \; Vol_\eta = \lambda \int_M f\triangle f \; Vol_\eta = \lambda\int_M |\nabla f|^2 \; Vol_\eta. \tag{4.6}$$

We recall the qc-Bochner identity [36, (4.1)]. Applying the first equality in (2.10), (2.9) and the properties of the torsion, (2.8), we can write the qc-Bochner formula [36, (4.1)] in the form

$$\begin{aligned} \frac{1}{2}\triangle|\nabla f|^2 = |\nabla^2 f|^2 - g\left(\nabla(\triangle f), \nabla f\right) + 2(n+2)S|\nabla f|^2 + 2(n+2)T^0(\nabla f, \nabla f) \\ +2(2n+2)U(\nabla f, \nabla f) + 4\sum_{s=1}^3 \nabla^2 f(\xi_s, I_s\nabla f). \end{aligned} \tag{4.7}$$

One of the key identities which relates the P-function and the qc-Bochner formula (4.7) is given by the following equation [37, Lemma 3.2]

$$\int_M \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s\nabla f)\, Vol_\eta = \int_M \Big[-\frac{1}{4n}P_n(\nabla f) - \frac{1}{4n}(\triangle f)^2 - S|\nabla f|^2 + \frac{(n+1)}{n-1}U(\nabla f, \nabla f)\Big]\, Vol_\eta. \tag{4.8}$$

An integration of (4.7) over the compact M, followed by a substitution of (2.14) and (4.8) in the obtained integral equality, and then a use of the divergence formula (2.16) give

$$0=\int_M\Big[|\nabla^2 f|^2-\lambda|\nabla f|^2+2nS|\nabla f|^2+2(n+2)T^0(\nabla f,\nabla f)+\frac{4n(n+1)}{n-1}U(\nabla f,\nabla f)\\ -\frac{1}{n}P_n(\nabla f)-\frac{1}{n}(\triangle f)^2\Big]\,Vol_\eta. \tag{4.9}$$

The latter formula can be written in the form

$$0=\int_M\Big\{|\nabla^2 f|^2-\lambda|\nabla f|^2-S|\nabla f|^2+T^0(\nabla f,\nabla f)-\frac{2(n-2)}{n-1}U(\nabla f,\nabla f)\\ +\frac{2n+1}{2(n+2)}\Big[2(n+2)S|\nabla f|^2+\frac{4n^2+14n+12}{2n+1}T^0(\nabla f,\nabla f)+\frac{4(n+2)^2(2n-1)}{(n-1)(2n+1)}U(\nabla f,\nabla f)\Big]\\ -\frac{1}{n}P_n(\nabla f)-\frac{1}{n}(\triangle f)^2\Big\}\,Vol_\eta. \tag{4.10}$$

Now we invoke the next integral identity proved in [36, Lemma 3.4]

$$\int_M\sum_{s=1}^3\nabla^2 f(\xi_s,I_s\nabla f)\,Vol_\eta=-\int_M\Big[4n\sum_{s=1}^3(df(\xi_s))^2+\sum_{s=1}^3T(\xi_s,I_s\nabla f,\nabla f)\Big]\,Vol_\eta. \tag{4.11}$$

From (4.11) and (4.8) it follows the equality

$$\int_M\Big[-S|\nabla f|^2+T^0(\nabla f,\nabla f)-\frac{2(n-2)}{n-1}U(\nabla f,\nabla f)\Big]\,Vol_\eta=\int_M\Big\{\frac{1}{4n}P_n(\nabla f)+\frac{1}{4n}(\triangle f)^2\\ -\frac{1}{4n}\sum_{s=1}^3[g(\nabla^2 f,\omega_s)]^2\Big\}\,Vol_\eta. \tag{4.12}$$

A substitution of (4.12) in (4.10) yields

$$0=\int_M\Big\{|\nabla^2 f|^2-\frac{1}{4n}\Big[(\triangle f)^2+\sum_{s=1}^3[g(\nabla^2 f,\omega_s)]^2\Big]-\frac{3}{4n}P_n(\nabla f)\\ +\frac{2n+1}{2}\Big[2S|\nabla f|^2+\frac{4n^2+14n+12}{(2n+1)(n+2)}T^0(\nabla f,\nabla f)+\frac{4(n+2)(2n-1)}{(n-1)(2n+1)}U(\nabla f,\nabla f)-\frac{\lambda}{n}|\nabla f|^2\Big]\Big\}\,Vol_\eta. \tag{4.13}$$

The equality (4.13), the Lichnerowicz type assumption (3.4) and (4.6) imply the inequality

$$0\ge\int_M\Big\{|\nabla^2 f|^2-\frac{1}{4n}\Big[(\triangle f)^2+\sum_{s=1}^3[g(\nabla^2 f,\omega_s)]^2\Big]-\frac{3}{4n}P_n(\nabla f)+\frac{2n+1}{2}\Big(\frac{k_0}{n+2}-\frac{\lambda}{n}\Big)|\nabla f|^2\Big\}\,Vol_\eta. \tag{4.14}$$

Note that in the proof of (4.14) we supposed implicitly that $n>1$. But it works also for $n=1$, when $U=0$ trivially, we have only to remove the torsion tensor U (cf. [37]).

Using that $\left\{\frac{1}{2\sqrt{n}}\omega_s\right\}$ is an orthonormal set in $\Psi_{[-1]}$ we have

$$|(\nabla^2 f)_{[-1]}|^2\ge\frac{1}{4n}\sum_{s=1}^3\left[g\left(\nabla^2 f,\omega_s\right)\right]^2 \tag{4.15}$$

while a projection on $\left\{\frac{1}{2\sqrt{n}}g\right\}$ gives

$$|(\nabla^2 f)_{[3]}|^2\ge\frac{1}{4n}(\triangle f)^2. \tag{4.16}$$

Next, using the $Sp(n)Sp(1)$-invariant orthogonal decomposition (4.2) of horizontal Hessian and the estimates (4.15) and (4.16), we obtain the inequality

$$|\nabla^2 f|^2\ge\frac{1}{4n}\Big[(\triangle f)^2+\sum_{s=1}^3[g(\nabla^2 f,\omega_s)]^2\Big]. \tag{4.17}$$

Finally, using (4.17) and the non-negativity of the P-function for $n>1$, see (4.5), we obtain from (4.14) the desired estimate

$$\lambda \geq \frac{n}{n+2}k_0.$$

□

Corollary 4.2. *If the case of equality in Theorem 1.2 holds, i.e., we have*

$$\lambda = \frac{n}{n+2}k_0, \qquad \triangle f = \frac{n}{n+2}k_0 f,$$

then the horizontal Hessian of the eigenfunction f is given by (1.4).

Proof. The result follows from (4.13), (3.4) and (4.17) which asserts that the equalities in (4.15) and (4.16) must hold, which imply (1.4) . □

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