SOLUTION OF THE QC YAMABE EQUATION ON A 3-SASAKIAN MANIFOLD AND THE QUATERNIONIC HEISENBERG GROUP

STEFAN IVANOV, IVAN MINCHEV, AND DIMITER VASSILEV

Abstract. A complete solution to the quaternionic contact Yamabe equation on the qc sphere of dimension $4n+3$ as well as on the quaternionic Heisenberg group is given. A uniqueness theorem for the qc Yamabe problem in a compact locally 3-Sasakian manifold is shown.

Contents

1. Introduction

It is well known that the solution of the Yamabe problem on a compact Riemannian manifold is unique in the case of negative or vanishing scalar curvature. The proof of these results, which rely on the maximum principle, extend readily to sub-Riemannian settings such as the CR and quaternionic contact (abbr. qc) Yamabe problems due to the sub-ellipticity of the involved operators. The positive (scalar curvature) case is of continued interest since it presents considerable difficulties due to the possible non-uniqueness. The most important positive case in each of these geometries is given by the corresponding round sphere due to its role in the general existence theorem and also because of its connection with the corresponding L^2 Sobolev type embedding inequality. Through the corresponding Cayley transforms, the sphere cases are equivalent to the problems of finding all solutions to the respective Yamabe equation on the flat models given by Euclidean space or Heisenberg groups. The Riemannian and CR sphere cases were settled in [23] and [21]. It should be noted that the Euclidean case can be handled alternatively by a reduction to a radially symmetric solution [11] and [25]. Furthermore, [23] established a uniqueness result in every conformal class of an Einstein metric. In this paper we solve the qc Yamabe problem on the $4n+3$ dimensional round sphere and quaternionic

Date: June 2, 2015.
1991 *Mathematics Subject Classification.* 58G30, 53C17.
Key words and phrases. Yamabe equation, quaternionic contact structures, Einstein structures, divergence formula.

Heisenberg group and establish a uniqueness result in every qc-conformal class containing a 3-Sasakain metric.

We continue by giving a brief background and the statements of our results. It is well known that the sphere at infinity of a any non-compact symmetric space M of rank one carries a natural Carnot-Carathéodory structure, see [22, 24]. A quaternionic contact (qc) structure, [2, 3], appears naturally as the conformal boundary at infinity of the quaternionic hyperbolic space. Following Biquard, a quaternionic contact structure (*qc structure*) on a real (4n+3)-dimensional manifold M is a codimension three distribution H (*the horizontal distribution*) locally given as the kernel of a $\mathbb{R}^3$-valued one-form $\eta=(\eta_1,\eta_2,\eta_3)$, such that, the three two-forms $d\eta_i|_H$ are the fundamental forms of a quaternionic Hermitian structure on H. The 1-form η is determined up to a conformal factor and the action of $SO(3)$ on $\mathbb{R}^3$, and therefore H is equipped with a conformal class $[g]$ of quaternionic Hermitian metrics. To every metric in the fixed conformal class one can associate a linear connection with torsion preserving the qc structure, see [2], which is called the Biquard connection. For a fixed metric in the conformal class of metrics on the horizontal space one associates the horizontal Ricci-type tensor of the Biquard connection, which is called the qc Ricci tensor. This is a symmetric tensor [2] whose trace-free part is determined by the torsion endomorphism of the Biquard connection [12] while the trace part is determined by the scalar curvature of the qc-Ricci tensor, called the qc-scalar curvature. It was shown in [12] that the torsion endomorphism of the Biquard connection is completely determined by the trace-free part of the horizontal Ricci tensor whose vanishing defines the class of qc-Einstein manifolds. A basic example of a qc manifold is a 3-Sasakian space which can be defined as a $(4n+3)$-dimensional Riemannian manifold whose Riemannian cone is a hyperKähler manifold and the qc structure is induced from that hyperKähler structure. It was shown in [12, 15] that the qc-Einstein manifolds of positive qc-scalar curvature are exactly the locally 3-Sasakian manifolds, up to a multiplication with a constant factor and a $SO(3)$-matrix. In particular, every 3-Sasakian manifold has vanishing torsion endomorphism and is a qc-Einstein manifold.

The quaternionic contact Yamabe problem on a compact qc manifold M is the problem of finding a metric $\bar g\in[g]$ on H for which the qc-scalar curvature is constant. A natural question is to determine the possible uniqueness or non-uniqueness of such qc-Yamabe metrics.

The question reduces to the solvability of the quaternionic contact (qc) Yamabe equation (2.7). Taking the conformal factor in the form $\bar\eta=u^{4/(Q-2)}\eta$, $Q=4n+6$, turns (2.7) into the equation

$$\mathcal{L}u \ \equiv\ 4\frac{Q+2}{Q-2}\,\triangle u-\ u\,Scal\ =\ -\ u^{2^*-1}\ \overline{Scal},$$

where $\triangle$ is the horizontal sub-Laplacian, $\triangle h\ =\ tr^g(\nabla^2 h)$, $Scal$ and $\overline{Scal}$ are the qc-scalar curvatures correspondingly of $(M,\,\eta)$ and $(M,\,\bar\eta)$, and $2^*=\frac{2Q}{Q-2}$, with $Q=4n+6$–the homogeneous dimension.

Another motivation for studying the qc Yamabe equation comes from its connection with the determination of the norm and extremals in the L^2 Folland-Stein [8] Sobolev-type embedding on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$, [10], [27], [26] and completed in [14]. The qc Yamabe equation is essentially the Euler-Lagrange equation of the extremals for the L^2 case of the Folland-Stein inequality [8] on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$.

On a compact quaternionic contact manifold M with a fixed conformal class $[\eta]$ the qc Yamabe equation characterizes the non-negative extremals of the qc Yamabe functional defined by

$$\Upsilon(u)\ =\ \int_M\Big(4\frac{Q+2}{Q-2}\ |\nabla u|^2\ +\ Scal\,u^2\Big)dv_g,\qquad \int_M u^{2^*}\,dv_g\ =\ 1,\ u>0.$$

Here dv_g denotes the Riemannian volume form of the Riemannian metric on M extending in a natural way the horizontal metric associated to η. Considering M equipped with a fixed qc structure, hence, a conformal class $[\eta]$, the Yamabe constant is defined as the infimum

$$\lambda(M) \equiv \lambda(M,[\eta]) = \inf\{\Upsilon(u) : \int_M u^{2^*}\,dv_g = 1,\ u>0\}.$$

The main result of [28] is that the qc Yamabe equation has a solution on a compact qc manifold provided $\lambda(M) < \lambda(S^{4n+3})$, where S^{4n+3} is the standard unit sphere in the quaternionic space $\mathbb{H}^n$.

In this paper we consider the qc Yamabe problem on the unit $(4n+3)$-dimensional sphere in $\mathbb{H}^n$. The standard 3-Sasaki structure on the sphere $\tilde\eta$ has a constant qc-scalar curvature $\widetilde{\mathrm{Scal}} = 16n(n+2)$ and vanishing trace-free part of its qc-Ricci tensor, i.e., it is a qc-Einstein space. The images under conformal quaternionic contact automorphisms are again qc-Einstein structures and, in particular, have constant qc-scalar curvature. In [12] we conjectured that these are the only solutions to the Yamabe problem on the quaternionic sphere and proved it in dimension seven in [13]. One of the main goals of this paper is to prove this conjecture in full generality.

Theorem 1.1. *Let $\tilde\eta = \frac{1}{2h}\eta$ be a qc conformal transformation of the standard qc-structure $\tilde\eta$ on a 3-Sasakian sphere of dimension $4n+3$. If η has constant qc-scalar curvature, then up to a multiplicative constant η is obtained from $\tilde\eta$ by a conformal quaternionic contact automorphism.*

We note that Theorem 1.1 together with the results of [12] allows the determination of *all* solutions of the qc Yamabe problem on the sphere and on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$. In fact, as a consequence of Theorem 1.1, we obtain here that all solutions to the qc Yamabe equation are given by the functions which realize the equality case of the L^2 Folland-Stein inequality found in [14] with the help of the center of mass technique developed for the CR case in [9] and [5].

Recall that the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$ of homogeneous dimension $Q = 4n+6$ is given by $\boldsymbol{G}\,(\mathbb{H}) = \mathbb{H}^n \times \mathrm{Im}\mathbb{H}$, $\quad (q=(t^a,x^a,y^a,z^a)\in\mathbb{H}^n, \omega=(x,y,z)\in\mathrm{Im}\mathbb{H})$ with the group low

$$(q_o,\omega_o)\circ(q,\omega) = (q_o + q, \omega + \omega_o + 2\,\mathrm{Im}\,q_o\,\bar q).$$

The "*standard*" qc contact form in quaternion variables is $\tilde\Theta = (\tilde\Theta_1,\ \tilde\Theta_2,\ \tilde\Theta_3) = \frac12\ (d\omega - q\cdot d\bar q + dq\cdot\bar q)$. The corresponding sub-Laplacian $\triangle_{\tilde\Theta} u = \sum_{a=1}^n \left(T_\alpha^2 u + X_\alpha^2 u + Y_\alpha^2 u + Z_\alpha^2 u\right)$, where T_a, X_a, Y_a, Z_a denote the left-invariant horizontal vector fields on $\boldsymbol{G}\,(\mathbb{H})$. Theorem 1.1 shows, in particular, the following

Corollary 1.2. *If Φ satisfies the qc Yamabe equation on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$,*

$$\frac{4(Q+2)}{Q-2}\triangle_{\tilde\Theta}\Phi = -S_\Theta\,\Phi^{2^*-1},$$

for some constant S_Θ, then up to a left translation the function $\Phi = (2h)^{-(Q-2)/4}$ and h is given by

$$h(q,\omega) = c_0\left[\left(\sigma + |q+q_0|^2\right)^2 + |\omega + \omega_o + 2\ \mathit{Im}\ q_o\,\bar q|^2\right], \tag{1.1}$$

for some fixed $(q_o,\omega_o)\in\boldsymbol{G}\,(\mathbb{H})$ and constants $c_0>0$ and $\sigma>0$. Furthermore, the qc-scalar curvature of Θ is $S_\Theta = 128n(n+2)c_0\sigma$.

This confirms the Conjecture made after [10, Theorem 1.1]. In [10, Theorem 1.6] the above result is proved on all groups of Iwasawa type, but with the assumption of partial-symmetry of the solution. Here with a completely different method from [10] we show that the symmetry assumption is superfluous. The corresponding solutions on the 3-Sasakain sphere are obtained via the Cayley transform, see for example [12, 13, 14], [19, Sections 2.3 & 5.2.1] for an account and history. Finally, it should be observed that the functions (1.1) with $c_0\in\mathbb{R}$ give all conformal factors for which Θ is also qc-Einstein.

We derive Theorem 1.1 from a more general result in which we solve the qc Yamabe problem on a locally 3-Sasakian compact manifolds. By the results of [12] and [15] a qc-Einstein manifold is of constant qc-scalar curvature, hence as far as the qc Yamabe equation is concerned only the uniqueness of solutions needs to be addressed. As mentioned earlier, the interesting case is when the qc-scalar curvature is a positive constant, hence we focus exclusively on the locally 3-Sasakian case.

Theorem 1.3. *Let $(M, \bar\eta)$ be a compact locally 3-Sasakian qc manifold of qc-scalar curvature $16n(n+2)$. If $\eta = 2h\bar\eta$ is qc-conformal to $\bar\eta$ structure which is also of constant qc-scalar curvature, then up to a homothety (M, η) is locally 3-Sasakian manifold. Furthermore, the function h is constant unless $(M, \bar\eta)$ is the unit 3-Sasakian sphere.*

The proof of Theorem 1.3 consists of two steps. The first step is a divergence formula Theorem 4.1 which shows that if $\bar\eta$ is of constant qc-curvature and is qc-conformal to a locally 3-Sasakian manifold, then $\bar\eta$ is also a locally 3-Sasakian manifold. The general idea to search for such a divergence formula goes back to Obata [23] where the corresponding result on a Riemannian manifold was proved for a conformal transformation of an Einstein space. However, our result is motivated by the (sub-Riemannian) CR case where a formula of this type was introduced in the ground-breaking paper of Jerison and Lee [21]. As far as the qc case is concerned in [12, Theorem 1.2] a weaker results was shown, namely Theorem 1.3 holds provided the vertical space of η is integrable. In dimension seven, the $n = 1$ case, this assumption was removed in [13, Theorem 1.2] where the result was established with the help of a suitable divergence formula. The general case $n > 1$ treated here presents new difficulties due to the extra non-zero torsion terms that appear in the higher dimensions, which complicate considerably the search of a suitable divergence formula. In the seven dimensional case the [3]-component of the traceless qc-Ricci tensor vanishes which decreases the number of torsion components.

The proof of the second part of Theorem 1.3 builds on ideas of Obata in the Riemannian case, who used that the gradient of the (suitably taken) conformal factor is a conformal vector field and the characterization of the unit sphere through its first eigenvalue of the Laplacian among all Einstein manifolds. We show a similar, although a more complicated relation between the conformal factor and the existence of an infinitesimal qc automorphism (qc vector field). Our divergence formula found in Theorem 4.1 involves a smooth function f, c.f. (4.7), expressed in terms of the conformal factor and its horizontal gradient. Remarkably, we found that the horizontal gradient of f is precisely the horizontal part of the qc vector field mentioned above and the sub-Laplacian of f is an eigenfuction of the sub-Laplacian with the smallest possible eigenvalue $-4n$ thus showing a geometric nature of f (cf Remark 5.3). Then we use the characterization of the 3-Sasakian sphere by its first eigenvalue of the sub-Laplacian among all locally 3-Sasakian manfolds established in [17, Theorem 1.2] for $(n > 1)$ and in [16, Corollary 1.2] for $n = 1$.

Remark 1.4. *Remarkably, a similar arguments also work in the CR case describing the geometric nature of the mysterious function in the Jerison-Lee's divergence formula in* [21]. *Indeed, the CR-Laplacian of the real part of the function f defined in* [21, Proposition 3.1] *turns out to be an eigenfuction of the CR-Laplacian with the smallest possible eigenvalue $-2n$ thus showing a geometric nature of the real part of f.*

Convention 1.5. *We use the following*

1. *$\{e_1, \dots, e_{4n}\}$ denotes an orthonormal basis of the horizontal space H;*
2. *The capital letters X,Y,Z... denote horizontal vectors, $X, Y, Z... \in H$.*
3. *The summation convention over repeated vectors from the basis $\{e_1, \dots, e_{4n}\}$ will be used. For example, for a (0,4)-tensor P, $k = P(e_b, e_a, e_a, e_b)$ means $k = \sum_{a,b=1}^{4n} P(e_b, e_a, e_a, e_b)$.*
4. *The triple (i, j, k) denotes any cyclic permutation of $(1, 2, 3)$.*

Acknowledgements S.Ivanov is visiting University of Pennsylvania, Philadelphia. S.I. thanks UPenn for providing the support and an excellent research environment during the whole stages of the paper. S.I. and I.M. are partially supported by Contract DFNI I02/4/12.12.2014 and Contract 168/2014 with the Sofia University "St.Kl.Ohridski". I.M. is supported by a SoMoPro II Fellowship which is co-funded by the European Commission[1] from "People" specific programme (Marie Curie Actions) within the EU Seventh Framework Programme on the basis of the grant agreement REA No. 291782. It is further co-financed by the South-Moravian Region. D.V. was partially supported by Simons Foundation grant #279381.

2. Quaternionic contact manifolds

In this section we will briefly review the basic notions of quaternionic contact geometry and recall some results from [2] and [12], see [19] for a more leisurely exposition.

A quaternionic contact (qc) manifold $(M, \eta, g, \mathbb{Q})$ is a $4n+3$-dimensional manifold M with a codimension three distribution H locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$. In addition H has an $Sp(n)Sp(1)$ structure, that is, it is equipped with a Riemannian metric g and a rank-three bundle $\mathbb{Q}$ consisting of endomorphisms of H locally generated by three almost complex structures I_1, I_2, I_3 on H satisfying the identities of the imaginary unit quaternions, $I_1I_2 = -I_2I_1 = I_3, \quad I_1I_2I_3 = -id_{|_H}$ which are hermitian compatible with the metric $g(I_s.,I_s.) = g(.,.)$ and the following contact condition holds

$$2g(I_sX,Y) \;=\; d\eta_s(X,Y).$$

A special phenomena, noted in [2], is that the contact form η determines the quaternionic structure and the metric on the horizontal distribution in a unique way.

The transformations preserving a given quaternionic contact structure η, i.e., $\bar\eta = \mu\Psi\eta$ for a positive smooth function μ and an $SO(3)$ matrix Ψ with smooth functions as entries are called *quaternionic contact conformal (qc-conformal) transformations*. If the function μ is constant $\bar\eta$ is called qc-homothetic to η. The qc conformal curvature tensor W^{qc}, introduced in [18], is the obstruction for a qc structure to be locally qc conformal to the standard 3-Sasakian structure on the $(4n+3)$-dimensional sphere [12, 18].

Definition 2.1. *A diffeomorphism ϕ of a QC manifold $(M,[g],\mathbb{Q})$ is called a* conformal quaternionic contact automorphism (conformal qc-automorphism) *if ϕ preserves the QC structure, i.e.*

$$\phi^*\eta = \mu\Phi\cdot\eta,$$

for some positive smooth function μ and some matrix $\Phi \in SO(3)$ with smooth functions as entries and $\eta = (\eta_1,\eta_2,\eta_3)^t$ is a local 1-form considered as a column vector of three one forms as entries.

On a qc manifold with a fixed metric g on H there exists a canonical connection defined first by O. Biquard in [2] when the dimension $(4n+3) > 7$, and in [7] for the 7-dimensional case. Biquard showed that there is a unique connection ∇ with torsion T and a unique supplementary subspace V to H in TM, such that:

(i) ∇ preserves the decomposition $H \oplus V$ and the $Sp(n)Sp(1)$ structure on H, i.e. $\nabla g = 0, \nabla\sigma \in \Gamma(\mathbb{Q})$ for a section $\sigma \in \Gamma(\mathbb{Q})$, and its torsion on H is given by $T(X,Y) = -[X,Y]_{|V}$;

(ii) for $\xi \in V$, the endomorphism $T(\xi,.)_{|H}$ of H lies in $(sp(n)\oplus sp(1))^\perp \subset gl(4n)$;

(iii) the connection on V is induced by the natural identification φ of V with the subspace $sp(1)$ of the endomorphisms of H, i.e. $\nabla\varphi = 0$.

[1] This article reflects only the author's views and the EU is not liable for any use that may be made of the information contained therein.

This canonical connection is also known as *the Biquard connection*. When the dimension of M is at least eleven [2] also described the supplementary distribution V, which is (locally) generated by the so called Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ determined by

$$\eta_s(\xi_k) = \delta_{sk}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \qquad (\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}, \tag{2.1}$$

where $\lrcorner$ denotes the interior multiplication. If the dimension of M is seven Duchemin shows in [7] that if we assume, in addition, the existence of Reeb vector fields as in (2.1), then the Biquard result holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2.1).

The fundamental 2-forms ω_s of the quaternionic contact structure Q are defined by

$$2\omega_{s|H} \;=\; d\eta_{s|H}, \qquad \xi \lrcorner \omega_s = 0, \quad \xi \in V.$$

Notice that equations (2.1) are invariant under the natural $SO(3)$ action. Using the triple of Reeb vector fields we extend the metric g on H to a metric h on TM by requiring $span\{\xi_1, \xi_2, \xi_3\} = V \perp H$ and $h(\xi_s, \xi_k) = \delta_{sk}$. The Riemannian metric h as well as the Biquard connection do not depend on the action of $SO(3)$ on V, but both change if η is multiplied by a conformal factor [12]. Clearly, the Biquard connection preserves the Riemannian metric on $TM, \nabla h = 0$.

The properties of the Biquard connection are encoded in the torsion endomorphism $T_\xi \in (sp(n) + sp(1))^\perp$. We recall the $Sp(n)Sp(1)$ invariant decomposition. An endomorphism Ψ of H can be decomposed with respect to the quaternionic structure $(\mathbb{Q}, g)$ uniquely into four $Sp(n)$-invariant parts $\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$, where the superscript $+++$ means commuting with all three I_i, $+--$ indicates commuting with I_1 and anti-commuting with the other two and etc. The two $Sp(n)Sp(1)$-invariant components $\Psi_{[3]} = \Psi^{+++}, \quad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$ are determined by

$$\begin{aligned} \Psi = \Psi_{[3]} \quad &\Longleftrightarrow 3\Psi + I_1\Psi I_1 + I_2\Psi I_2 + I_3\Psi I_3 = 0, \\ \Psi = \Psi_{[-1]} \quad &\Longleftrightarrow \Psi - I_1\Psi I_1 - I_2\Psi I_2 - I_3\Psi I_3 = 0. \end{aligned}$$

With a short calculation one sees that the $Sp(n)Sp(1)$-invariant components are the projections on the eigenspaces of the Casimir operator $\Upsilon = \; I_1 \otimes I_1 \; + \; I_2 \otimes I_2 \; + \; I_3 \otimes I_3$ corresponding, respectively, to the eigenvalues 3 and -1, see [6]. If $n = 1$ then the space of symmetric endomorphisms commuting with all I_s is 1-dimensional, i.e. the [3]-component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_{[3]} = -\frac{tr\Psi}{4} Id_{|H}$. Note here that each of the three 2-forms ω_s belongs to its [-1]-component, $\omega_s = \omega_{s[-1]}$ and constitute a basis of the Lie algebra $sp(1)$.

2.1. **The torsion tensor.** Decomposing the endomorphism $T_\xi \in (sp(n) + sp(1))^\perp$ into its symmetric part T^0_ξ and skew-symmetric part $b_\xi, T_\xi = T^0_\xi + b_\xi$, O. Biquard shows in [2] that the torsion T_ξ is completely trace-free, $tr\, T_\xi = tr\, T_\xi \circ I_s = 0$, its symmetric part has the properties $T^0_{\xi_i} I_i = -I_i T^0_{\xi_i} \quad I_2(T^0_{\xi_2})^{+--} = I_1(T^0_{\xi_1})^{-+-}, \quad I_3(T^0_{\xi_3})^{-+-} = I_2(T^0_{\xi_2})^{--+}, \quad I_1(T^0_{\xi_1})^{--+} = I_3(T^0_{\xi_3})^{+--}$. The skew-symmetric part can be represented as $b_{\xi_i} = I_i u$, where u is a traceless symmetric (1,1)-tensor on H which commutes with I_1, I_2, I_3. Therefore we have $T_{\xi_i} = T^0_{\xi_i} + I_i u$. If $n = 1$ then the tensor u vanishes identically, $u = 0$, and the torsion is a symmetric tensor, $T_\xi = T^0_\xi$.
The two $Sp(n)Sp(1)$-invariant trace-free symmetric 2-tensors $T^0(X,Y) = g((T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3)X, Y)$, $U(X,Y) = g(uX,Y)$ on H, introduced in [12], have the properties:

$$\begin{aligned} T^0(X,Y) + T^0(I_1X, I_1Y) + T^0(I_2X, I_2Y) + T^0(I_3X, I_3Y) = 0, \\ U(X,Y) = U(I_1X, I_1Y) = U(I_2X, I_2Y) = U(I_3X, I_3Y). \end{aligned} \tag{2.2}$$

In dimension seven ($n = 1$), the tensor U vanishes identically, $U = 0$.

These tensors determine completely the torsion endomorphism of the Biquard connection due to the following identity [18, Proposition 2.3] $4T^0(\xi_s, I_sX, Y) = T^0(X,Y) - T^0(I_sX, I_sY)$ which implies

$$4T(\xi_s, I_sX, Y) = 4T^0(\xi_s, I_sX, Y) + 4g(I_suI_sX, Y) = T^0(X,Y) - T^0(I_sX, I_sY) - 4U(X,Y).$$

2.2. **The qc-Einstein condition and Bianchi identities.** We explain briefly the consequences of the Bianchi identities and the notion of qc-Einstein manifold introduced in [12] since it plays a crucial role in solving the Yamabe equation in the quaternionic sphere (see [13] for dimension seven). For more details see [12].

Let $R = [\nabla, \nabla] - \nabla_{[\ ,\ ]}$ be the curvature of the Biquard connection ∇. The Ricci tensor and the scalar curvature, called *qc-Ricci tensor* and *qc-scalar curvature*, respectively, are defined by

$$Ric(X,Y) = g(R(e_a, X)Y, e_a), \qquad Scal = Ric(e_a, e_a) = g(R(e_b, e_a)e_a, e_b).$$

According to [2] the Ricci tensor restricted to H is a symmetric tensor. If the trace-free part of the qc-Ricci tensor is zero we call the quaternionic structure *a qc-Einstein manifold* [12]. It is shown in [12] that the qc-Ricci tensor is completely determined by the components of the torsion. Theorem 1.3, Theorem 3.12 and Corollary 3.14 in [12] imply that on a qc manifold $(M^{4n+3}, g, \mathbb{Q})$ the qc-Ricci tensor and the qc-scalar curvature satisfy

$$\begin{aligned} Ric(X,Y) &= (2n+2)T^0(X,Y) + (4n+10)U(X,Y) + \frac{Scal}{4n}g(X,Y) \\ Scal &= -8n(n+2)g(T(\xi_1,\xi_2),\xi_3) \end{aligned}$$

Hence, the qc-Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection and in this case the qc scalar curvature is constant [12, 15]. If $Scal > 0$ the latter holds exactly when the qc-structure is locally 3-Sasakian up to a multiplication by a constant and an $SO(3)$-matrix with smooth entries. We remind that a (4n+3)-dimensional Riemannian manifold (M,g) is called 3-Sasakian if the cone metric $g_N = t^2g + dt^2$ on $N = M \times \mathbb{R}^+$ is a hyperkähler metric, namely, it has holonomy contained in $Sp(n+1)$. The 3-Sasakian manifolds are Einstein with positive Riemannian scalar curvature.

The following vectors will be important for our considerations,

$$A_i = I_i[\xi_j, \xi_k], \qquad A = A_1 + A_2 + A_3. \tag{2.3}$$

We denote with the same letter the corresponding horizontal 1-form and recall the action of I_s on it,

$$A(X) = g(I_1[\xi_2,\xi_3] + I_2[\xi_3,\xi_1] + I_3[\xi_1,\xi_2], X), \quad I_sA(X) = -A(I_sX).$$

The horizontal divergence ∇^*P of a (0,2)-tensor field P on M with respect to Biquard connection is defined to be the (0,1)-tensor field $\nabla^*P(.) = (\nabla_{e_a}P)(e_a, .)$. We have from [12, Theorem 4.8] that on a $(4n+3)$-dimensional QC manifold with constant qc-scalar curvature the next identities hold

$$\nabla^*T^0 = (n+2)A, \qquad \nabla^*U = \frac{1-n}{2}A. \tag{2.4}$$

For any smooth function h on a qc manifold with constant qc scalar curvature the following formulas are valid [13, Lemma 4.1]

$$\begin{aligned} \nabla^*\Big(\sum_{s=1}^3 dh(\xi_s)I_sA_s\Big) &= \sum_{s=1}^3 \nabla dh\,(I_se_a,\xi_s)A_s(e_a); \\ \nabla^*\Big(\sum_{s=1}^3 dh(\xi_s)I_sA\Big) &= \sum_{s=1}^3 \nabla dh\,(I_se_a,\xi_s)A(e_a). \end{aligned} \tag{2.5}$$

2.3. **Qc conformal transformations.** Let h be a positive smooth function on a qc manifold (M,η). Let $\bar\eta=\frac{1}{2h}\eta$ be a conformal deformation of the qc structure η. We will denote the objects related to $\bar\eta$ by over-lining the same object corresponding to η. Thus, $d\bar\eta=-\frac{1}{2h^2}\,dh\wedge\eta\;+\;\frac{1}{2h}d\eta$ and $\bar g=\frac{1}{2h}g$. The new triple $\{\bar\xi_1,\bar\xi_2,\bar\xi_3\}$ is determined by the conditions defining the Reeb vector fields as follows $\bar\xi_s\;=\;2h\,\xi_s\;+\;I_s\nabla h$, where ∇h is the horizontal gradient defined by $g(\nabla h,X)=dh(X)$. The components of the torsion tensor transform according to the following formulas from [12, Section 5]

$$\begin{aligned}\overline{T}^0(X,Y)\;&=\;T^0(X,Y)\;+\;h^{-1}\,[\nabla dh]_{[sym][-1]}(X,Y),\\ \bar U(X,Y)\;&=\;U(X,Y)\;+\;(2h)^{-1}[\,\nabla dh-2h^{-1}dh\otimes dh]_{[3][0]}(X,Y),\end{aligned}\tag{2.6}$$

where the symmetric part is given by

$$[\,\nabla dh]_{[sym]}(X,Y)\;=\;\nabla dh(X,Y)\;+\;\sum_{s=1}^3 dh(\xi_s)\,\omega_s(X,Y)$$

and $_{[3][0]}$ indicates the trace free part of the [3]-component of the corresponding tensor.

In addition, the qc-scalar curvature changes according to the formula [2]

$$\overline{\text{Scal}}\;=\;2h\,(\text{Scal})\;-\;8(n+2)^2\,h^{-1}|\nabla h|^2\;+\;8(n+2)\,\triangle h.\tag{2.7}$$

3. Qc conformal transformations on qc Einstein manifolds

Throughout this section h is a positive smooth function on a qc manifold $(M,g,\mathbb{Q})$ with constant qc-scalar curvature $Scal=16n(n+2)$ and $\bar\eta\;=\;\frac{1}{2h}\,\eta$ is a qc Einstein structure which is a conformal deformation of the qc structure η. We recall some formulas from [13] which we need here.

First we write the expressions of the 1-forms A_s,A in terms of h (see [13, Lemma])

$$\begin{aligned}A_i(X)\;=\;&-\frac12 h^{-2}dh(X)\;-\;\frac12 h^{-3}|\nabla h|^2dh(X)-\;\frac12 h^{-1}\Big(\;\nabla dh(I_jX,\xi_j)\;+\;\;\nabla dh(I_kX,\xi_k)\Big)\\ &+\;\frac12 h^{-2}\Big(dh(\xi_j)\,dh(I_jX)\;+\;dh(\xi_k)\,dh(I_kX)\Big)+\;\frac14 h^{-2}\Big(\;\nabla dh(I_jX,I_j\nabla h)\;+\;\;\nabla dh(I_kX,I_k\nabla h)\Big).\end{aligned}\tag{3.1}$$

Thus, we have also

$$\begin{aligned}A(X)\;=\;&-\frac32 h^{-2}dh(X)\;-\;\frac32 h^{-3}|\nabla h|^2dh(X)\\ &-\;h^{-1}\sum_{s=1}^3\;\nabla dh(I_sX,\xi_s)\;+\;h^{-2}\sum_{s=1}^3 dh(\xi_s)\,dh(I_sX)\;+\;\frac12 h^{-2}\sum_{s=1}^3\;\nabla dh(I_sX,I_s\nabla h)\end{aligned}\tag{3.2}$$

Second we consider the following one-forms

$$D_s(X)=-\frac{1}{2h}\Big[T^0(X,\nabla h)+T^0(I_sX,I_s\nabla h)\Big]\tag{3.3}$$

For simplicity, using the musical isomorphism, we will denote with D_1, D_2, D_3 the corresponding (horizontal) vector fields, for example $g(D_1,X)=D_1(X)$. Using (2.2), we set

$$D\;=\;D_1\;+\;D_2\;+\;D_3\;=\;-h^{-1}\,T^0(X,\nabla h).\tag{3.4}$$

Setting $\bar T^0=0$ in (2.6), we obtain from equations (3.3) the expressions (cf. [13] or [18])

$$\begin{aligned}D_i(X)\;=\;h^{-2}\,dh(\xi_i)\,dh(I_iX)+\;\frac14 h^{-2}\,[\;\nabla dh\,(X,\nabla h)+\;\nabla dh\,(I_iX,I_i\nabla h)\\ -\;\nabla dh\,(I_jX,I_j\nabla h)\;-\;\nabla dh\,(I_kX,I_k\nabla h)\big].\end{aligned}\tag{3.5}$$

The equalities (3.4) together with (3.5) yield [13, Lemma 4.2]

$$D(X) = \frac{1}{4}h^{-2}\Big(3\,\nabla dh(X,\nabla h) - \sum_{s=1}^{3}\nabla dh(I_sX, I_s\nabla h)\Big) + h^{-2}\sum_{s=1}^{3} dh(\xi_s)\,dh(I_sX). \tag{3.6}$$

Third, we consider the following one-forms (and corresponding vectors)

$$F_s(X) = -h^{-1}\,T^0(X, I_s\nabla h).$$

From the definition of F_i and (3.3) we find

$$F_i(X) = -h^{-1}T^0(X, I_i\nabla h) = -D_i(I_iX) + D_j(I_iX) + D_k(I_iX). \tag{3.7}$$

We recall the next divergence formulas established in [13, Lemma 4.2, Lemma 4.3] with the help of the contracted second Bianchi identity (2.4).

$$\nabla^* D = |T^0|^2 - h^{-1}g(dh, D) - h^{-1}(n+2)\,g(dh, A). \tag{3.8}$$

$$\begin{aligned}\nabla^*\Big(\sum_{s=1}^{3} dh(\xi_s)F_s\Big) &= \sum_{s=1}^{3}\Big[\nabla dh\,(I_se_a,\xi_s)F_s(I_se_a)\Big]\\ &\quad + h^{-1}\sum_{s=1}^{3}\Big[dh(\xi_s)dh(I_se_a)D(e_a) + (n+2)\,dh(\xi_s)dh(I_se_a)\,A(e_a)\Big].\end{aligned} \tag{3.9}$$

4. The divergence formula

Following is our main technical result. As mentioned in the introduction, we were motivated to seek a divergence formula of this type based on the Riemannian, CR and seven dimensional qc cases of the considered problem. The main difficulty was to find a suitable vector field with non-negative divergence containing the norm of the torsion. The fulfilment of this task was facilitated by the results of [12]. In particular, similarly to the CR case, but unlike the Riemannian case, we were not able to achieve a proof based purely on the Bianchi identities, see [12, Theorem 4.8].

Using $\overline{Scal} = Scal = 16n(n+2)$ in the Yamabe equation (2.7) we have

$$\triangle h = 2n - 4nh + h^{-1}(n+2)|\nabla h|^2. \tag{4.1}$$

The equation (2.6) in the case $\bar T^0 = \bar U = 0$ and (4.1) motivate the definition of the following symmetric (0,2) tensors

$$\mathbf{D}(X,Y) = -T^0(X,Y) = \frac{h^{-1}}{4}\Big[3\nabla^2h(X,Y) - \sum_{s=1}^{3}\nabla^2h(I_sX,I_sY) + 4\sum_{s=1}^{3}dh(\xi_s)\omega_s(X,Y)\Big] \tag{4.2}$$

$$\begin{aligned}\mathbf{E}(X,Y) = -2U(X,Y) &= \frac{h^{-1}}{4}\Big[\nabla^2h(X,Y) + \sum_{s=1}^{3}\nabla^2h(I_sX,I_sY)\Big]\\ &\quad - \frac{2h^{-2}}{4}\Big[dh(X)dh(Y) + \sum_{s=1}^{3}dh(I_sX)dh(I_sY)\Big] - \frac{h^{-1}}{4}\Big(2 - 4h + h^{-1}|\nabla h|^2\Big)g(X,Y).\end{aligned} \tag{4.3}$$

The one form D defined in (3.4) and expressed in terms of h in (3.6) satisfies $D(X) = h^{-1}\mathbf{D}(X,\nabla h)$.

Consider the 1-form $E(X)=h^{-1}\mathbf{E}(X,\nabla h)$. We obtain from (4.2) and (4.3) the expression

$$E(X)=\frac{h^{-2}}{4}\Big[\nabla^2h(X,\nabla h)+\sum_{s=1}^3\nabla^2h(I_sX,I_s\nabla h)+\Big(-2+4h-3h^{-1}|\nabla h|^2\Big)dh(X)\Big]. \tag{4.4}$$

We also define the (0,3)-tensors $\mathbb{D}$ and $\mathbb{E}$ by

$$\mathbb{D}(X,Y,Z)=-\frac{h^{-1}}{8}\Big[dh(X)T^0(Y,Z)+dh(Y)T^0(X,Z) \\ +\sum_{s=1}^3dh(I_sX)T^0(I_sY,Z)+\sum_{s=1}^3dh(I_sY)T^0(I_sX,Z)\Big] \tag{4.5}$$

$$\mathbb{E}(X,Y,Z)=\frac{h^{-1}}{8}\Big\{dh(X)\mathbf{E}(Y,Z)+dh(Y)\mathbf{E}(X,Z) \\ +\sum_{s=1}^3dh(I_sX)\mathbf{E}(I_sY,Z)+\sum_{s=1}^3dh(I_sY)\mathbf{E}(I_sX,Z)\Big\}. \tag{4.6}$$

After this preparations we are ready to state the main result.

Theorem 4.1. *Suppose (M^{4n+3},η) is a quaternionic contact structure conformal to a 3-Sasakian structure $(M^{4n+3},\bar\eta)$, $\tilde\eta = \frac{1}{2h}\eta$. If $Scal_\eta=Scal_{\tilde\eta}=16n(n+2)$, then with f given by*

$$f = \frac12 + h + \frac14 h^{-1}|\nabla h|^2, \tag{4.7}$$

the following identity holds

$$\nabla^*\Big(f(D+E)+\sum_{s=1}^3dh(\xi_s)I_sE+\sum_{s=1}^3dh(\xi_s)\,F_s+4\sum_{s=1}^3dh(\xi_s)I_sA_s-\frac{10}{3}\sum_{s=1}^3dh(\xi_s)\,I_sA\Big) \\ = \Big(\frac12+h\Big)\Big(|T^0|^2+|\boldsymbol{E}|^2\Big)+2h|\mathbb{D}+\mathbb{E}|^2+h\,\langle QV,\,V\rangle. \tag{4.8}$$

where Q is equal to

$$Q:=\begin{bmatrix} \frac52 & -\frac12 & -\frac12 & -\frac12 & -2 & -2 & -2\\ -\frac12 & \frac52 & -\frac12 & -\frac12 & \frac{10}{3} & -\frac23 & -\frac23\\ -\frac12 & -\frac12 & \frac52 & -\frac12 & -\frac23 & \frac{10}{3} & -\frac23\\ -\frac12 & -\frac12 & -\frac12 & \frac52 & -\frac23 & -\frac23 & \frac{10}{3}\\ -2 & \frac{10}{3} & -\frac23 & -\frac23 & \frac{22}{3} & -\frac23 & -\frac23\\ -2 & -\frac23 & \frac{10}{3} & -\frac23 & -\frac23 & \frac{22}{3} & -\frac23\\ -2 & -\frac23 & -\frac23 & \frac{10}{3} & -\frac23 & -\frac23 & \frac{22}{3} \end{bmatrix}$$

Here, Q is a positive definite matrix with eigenvalues 1, $\frac92\pm\frac{\sqrt{73}}{2}$ and $\frac{11}{2}\pm\frac{\sqrt{89}}{2}$ and $V=(E,D_1,D_2,D_3,A_1,A_2,A_3)$ with E, D_s, A_s defined, correspondingly, in (4.4) (3.3) and (2.3).

Proof. For the sake of making some formulas more compact, in the proof we will use sometimes the notation $XY = g(X,Y)$ for the product of two horizontal vector fields X and Y and the similar abbreviation for horizontal 1-forms.

We begin by recalling (3.6), (4.4) and (3.2), which imply

$$A(X) = \frac{3E(X) - D(X)}{2} - h^{-1}\sum_{s=1}^{3}\nabla^2 h(I_sX,\xi_s) \\ + \frac{3}{2}h^{-2}\sum_{s=1}^{3} dh(\xi_s)dh(I_sX) - \frac{3}{2}h^{-2}\Big(\frac{1}{2} + h + \frac{1}{4}h^{-1}|\nabla h|^2\Big)dh(X). \tag{4.9}$$

Using the function f defined in (4.7), we write (4.9) in the form

$$2\sum_{s=1}^{3}\nabla^2 h(I_sX,\xi_s) = h(3E(X) - D(X) - 2A(X)) + 3h^{-1}\sum_{s=1}^{3} dh(\xi_s)dh(I_sX) - 3h^{-1}f dh(X). \tag{4.10}$$

The sum of (3.6) and (4.4) yields

$$(E+D)(X) = h^{-2}\nabla^2 h(X,\nabla h) + h^{-2}\sum_{s=1}^{3} dh(\xi_s)dh(I_sX) + \frac{h^{-2}}{4}\Big(-2+4h-3h^{-1}|\nabla h|^2\Big)dh(X)\Big]. \tag{4.11}$$

Using (4.7) and (4.11), we obtain

$$2\nabla_X f = h(E+D)(X) - h^{-1}\sum_{s=1}^{3} dh(\xi_s)dh(I_sX) + h^{-1}f dh(X). \tag{4.12}$$

We calculate the divergences of E using (2.4) as follows

$$\begin{aligned}\nabla^*E &= 2h^{-2}dh(e_a)U(e_a,\nabla h) - 2h^{-1}(\nabla_{e_a}U)(e_a,\nabla h) - 2h^{-1}U(e_a,e_b)\nabla^2 h(e_a,e_b) \\ &= -h^{-1}(1-n)A(\nabla h) + U(e_a,e_b)(-2h^{-1})\Big[\nabla^2 h(e_a,e_b) - 2dh(e_a)dh(e_b)\Big] + h^{-1}E(\nabla h) \\ &= |\mathbf{E}|^2 + h^{-1}dh(e_a)E(e_a) - h^{-1}(1-n)dh(e_a)A(e_a).\end{aligned} \tag{4.13}$$

Similarly, we have

$$\begin{aligned}-\nabla^*I_sE &= 2h^{-2}dh(e_a)U(I_se_a,\nabla h) + 2h^{-1}(\nabla_{e_a}U)(e_a,I_s\nabla h) - 2h^{-1}U(I_se_a,e_b)\nabla^2 h(e_a,e_b) \\ &= h^{-1}(1-n)A(I_s\nabla h) + U(I_se_a,e_b)(-2h^{-1})\Big[\nabla^2 h(e_a,e_b) - 2dh(e_a)dh(e_b)\Big] + h^{-1}E(I_s\nabla h) \\ &= U(I_se_a,e_b)U(e_a,e_b) - h^{-1}(1-n)dh(I_se_a)A(e_a) = -h^{-1}(1-n)dh(I_se_a)A(e_a),\end{aligned} \tag{4.14}$$

since $U(I_se_a,e_b)U(e_a,e_b) = E(I_s\nabla h) = 0$ due to (2.2).

Now we are prepared to calculate the divergence of the first four terms. Using (3.8), (3.9), (4.13), (4.12), (4.14) and (4.10), we have

$$
\begin{aligned}
(4.15)\quad & \nabla_{e_a}\Big[f(D+E)(e_a)-\sum_{s=1}^3 dh(\xi_s)E(I_se_a)+\sum_{s=1}^3 dh(\xi_s)F_s(e_a)\Big]\\
&= \Big(\frac{h}{2}(E+D)(e_a)-\frac{h^{-1}}{2}\sum_{s=1}^3 dh(\xi_s)dh(I_se_a)+\frac{h^{-1}}{2}f dh(e_a)\Big)(D+E)(e_a)\\
&+f\Big[-h^{-1}D(\nabla h)-h^{-1}(n+2)A(\nabla h)+|T^0|^2+|E|^2+h^{-1}dh(e_a)E(e_a)-h^{-1}(1-n)dh(e_a)A(e_a)\Big]\\
&+h^{-1}(1-n)\sum_{s=1}^3 dh(\xi_s)dh(I_se_a)A(e_a)+\sum_{s=1}^3\nabla^2h(I_se_a,\xi_s)E(e_a)\\
&+\sum_{s=1}^3\nabla^2h\,(I_se_a,\xi_s)F_s(I_se_a)+\ h^{-1}\sum_{s=1}^3 dh(\xi_s)dh(I_se_a)D(e_a)\ +(n+2)\sum_{s=1}^3 dh(\xi_s)dh(I_se_a)\,A(e_a)\\
&= f(|T^0|^2+|\mathbf{E}|^2)+\frac{h}{2}|D+E|^2+\frac{h}{2}(3E-D-2A)(e_a)E(e_a)\\
&+h^{-1}\Big[\sum_{s=1}^3 dh(\xi_s)dh(I_se_a)-f dh(e_a)\Big]\Big(\frac{1}{2}D(e_a)+3A(e_a)\Big)+\sum_{s=1}^3\nabla^2h\,(I_se_a,\xi_s)F_s(I_se_a).
\end{aligned}
$$

Applying (2.5) and (4.10) we obtain

$$
\begin{aligned}
(4.16)\quad & \nabla_{e_a}\Big[f(D+E)(e_a)-\sum_{s=1}^3 dh(\xi_s)E(I_se_a)+\sum_{s=1}^3 dh(\xi_s)F_s(e_a)-2\sum_{s=1}^3 dh(\xi_s)I_sA(e_a)\Big]\\
&= f(|T^0|^2+|\mathbf{E}|^2)+\frac{h}{2}|D+E|^2+\frac{h}{2}(3E-D-2A)E-h(3E-D-2A)A\\
&+\frac{h^{-1}}{2}\Big[\sum_{s=1}^3 dh(\xi_s)dh(I_se_a)-f dh(e_a)\Big]D(e_a)+\sum_{s=1}^3\nabla^2h\,(I_se_a,\xi_s)F_s(I_se_a)
\end{aligned}
$$

According to (3.7), the last term in (4.16) reads

$$
\begin{aligned}
(4.17)\quad & \sum_{s=1}^3\nabla^2h\,(I_se_a,\xi_s)F_s(I_se_a)=D_1(e_a)\Big[\nabla^2h(I_1e_a,\xi_1)-\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_3e_a,\xi_3)\Big]\\
&+D_2(e_a)\Big[-\nabla^2h(I_1e_a,\xi_1)+\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_3e_a,\xi_3)\Big]\\
&+D_3(e_a)\Big[-\nabla^2h(I_1e_a,\xi_1)-\nabla^2h(I_2e_a,\xi_2)+\nabla^2h(I_3e_a,\xi_3)\Big].
\end{aligned}
$$

Using (4.17) we rewrite the last line in (4.16) as follows

$$\Big[\frac{h^{-1}}{2}\sum_{s=1}^3 dh(\xi_s)dh(I_se_a)-\frac{h^{-1}}{2}f dh(e_a)\Big]D(e_a)+\sum_{s=1}^3\nabla^2h\,(I_se_a,\xi_s)F_s(I_se_a) \tag{4.18}$$
$$=D_1(e_a)\Big[\nabla^2h(I_1e_a,\xi_1)-\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_3e_a,\xi_3)+\frac{h^{-1}}{2}\sum_{s=1}^3 dh(\xi_s)dh(I_se_a)-\frac{h^{-1}}{2}f dh(e_a)\Big]$$
$$+D_2(e_a)\Big[-\nabla^2h(I_1e_a,\xi_1)+\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_3e_a,\xi_3)+\frac{h^{-1}}{2}\sum_{s=1}^3 dh(\xi_s)dh(I_se_a)-\frac{h^{-1}}{2}f dh(e_a)\Big]$$
$$+D_3(e_a)\Big[-\nabla^2h(I_1e_a,\xi_1)-\nabla^2h(I_2e_a,\xi_2)+\nabla^2h(I_3e_a,\xi_3)+\frac{h^{-1}}{2}\sum_{s=1}^3 dh(\xi_s)dh(I_se_a)-\frac{h^{-1}}{2}f dh(e_a)\Big].$$

The equalities (4.4), (3.5) and (3.1) imply

$$\nabla^2h(I_2X,\xi_2)+\nabla^2h(I_3X,\xi_3) \tag{4.19}$$
$$=h(E-D_1-2A_1)(X)+h^{-1}\sum_{s=1}^3 dh(\xi_s)dh(I_sX)-h^{-1}f dh(X),$$

Subtracting two times (4.19) from (4.10) we obtain

$$\nabla^2h(I_1e_a,\xi_1)-\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_3e_a,\xi_3)+\frac{h^{-1}}{2}\sum_{s=1}^3 dh(\xi_s)dh(I_se_a)-\frac{h^{-1}}{2}f dh(e_a) \tag{4.20}$$
$$=\frac{h}{2}\Big[-E-D+4D_1-2A+8A_1\Big](e_a)$$

The left-hand side of the above identity is the second line in (4.18). The other two lines are evaluated similarly and the formulas are obtained from the above by a cyclic rotation of $\{1,2,3\}$. A substitution of the resulting new form of (4.18) in (4.16) give

$$\nabla_{e_a}\Big[f(D+E)(e_a)-\sum_{s=1}^3 dh(\xi_s)E(I_se_a)+\sum_{s=1}^3 dh(\xi_s)F_s(e_a)-2\sum_{s=1}^3 dh(\xi_s)I_sA(e_a)\Big] \tag{4.21}$$
$$=f\Big(|T^0|^2+|\mathbf{E}|^2\Big)+\frac{4h}{2}\Big[E^2+A^2+D_1^2+D_2^2+D_3^2-2AE+2A_1D_1+2A_2D_2+2A_3D_3\Big].$$

In view of (2.5) for any non-zero constant c we calculate the following divergences as follows

$$\nabla_{e_a}\Big(c\sum_{s=1}^3 dh(\xi_s)I_sA_s(e_a)-\frac{c}{3}\sum_{s=1}^3 dh(\xi_s)I_sA(e_a)\Big) \tag{4.22}$$
$$=\frac{c}{3}\Big[2\nabla^2h(I_1e_a,\xi_1)-\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_3e_a,\xi_3)\Big]A_1(e_a)$$
$$+\frac{c}{3}\Big[2\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_1e_a,\xi_1)-\nabla^2h(I_3e_a,\xi_3)\Big]A_2(e_a)$$
$$+\frac{c}{3}\Big[2\nabla^2h(I_3e_a,\xi_3)-\nabla^2h(I_2e_a,\xi_2)-\nabla^2h(I_1e_a,\xi_1)\Big]A_3(e_a)$$

subtracting (4.19) from twice (4.10) yields

$$2\nabla^2 h(I_1e_a,\xi_1)-\nabla^2 h(I_2e_a,\xi_2)-\nabla^2 h(I_3e_a,\xi_3) = h\Big[2D_1-D_2-D_3+4A_1-2A_2-2A_3\Big](e_a) \tag{4.23}$$

Now, taking into account (4.23), (4.22) and (4.21) we obtain

$$\begin{aligned}
&\nabla^*\Big[f(D+E)(X)-\sum_{s=1}^3 dh(\xi_s)E(I_sX)+\sum_{s=1}^3 dh(\xi_s)F_s(X)-2\sum_{s=1}^3 dh(\xi_s)I_sA(X)\Big]\\
&\qquad+\nabla^*\Big[c\quad\sum_{s=1}^3 dh(\xi_s)I_sA_s(X)-\frac{c}{3}\sum_{s=1}^3 dh(\xi_s)I_sA(X)\Big]\\
&=f\Big(|T^0|^2+|\mathbf{E}|^2\Big)+\frac{4h}{2}\Big[E^2+A^2+D_1^2+D_2^2+D_3^2-2AE+2A_1D_1+2A_2D_2+2A_3D_3\Big]\\
&+h\frac{c}{3}\Big[(2D_1-D_2-D_3+4A_1-2A_2-2A_3)A_1\Big]+h\frac{c}{3}\Big[(2D_2-D_1-D_3+4A_2-2A_1-2A_3)A_2\Big]\\
&\qquad+h\frac{c}{3}\Big[(2D_3-D_2-D_1+4A_3-2A_2-2A_1)A_3\Big]
\end{aligned} \tag{4.24}$$

In the next Lemma, as in the proof of Theorem 4.1, we shall use again the notation $XY=g(X,Y)$ for the product of two horizontal vector fields X and Y and the similar abbreviation for horizontal 1-forms.

Lemma 4.2. *For the (0,3)-tensors $\mathbb{D}$ and $\mathbb{E}$ defined by (4.5) and (4.6) we have*

$$\begin{aligned}
|\mathbb{D}|^2 &= \frac{1}{8}h^{-2}|\nabla h|^2|T^0|^2-\frac{1}{4}\sum_{s=1}^3|D_s|^2+\frac{1}{2}(D_1D_2+D_1D_3+D_2D_3),\\
|\mathbb{E}|^2 &= \frac{1}{8}h^{-2}|\nabla h|^2|\mathbf{E}|^2-\frac{1}{4}|E|^2, \qquad \mathbb{D}\mathbb{E} = \frac{1}{4}\sum_{s=1}^3 ED_s.
\end{aligned} \tag{4.25}$$

Consequently,

$$\begin{aligned}
\frac{1}{4}h^{-2}|\nabla h|^2(|T^0|^2+|\mathbf{E}|^2) &= 2|\mathbb{D}+\mathbb{E}|^2-\sum_{s=1}^3 ED_s\\
&\quad+\frac{1}{2}|E|^2+\frac{1}{2}\sum_{s=1}^3|D_s|^2-(D_1D_2+D_1D_3+D_2D_3)
\end{aligned} \tag{4.26}$$

Proof. We shall repeatedly apply (2.2), the defining equations (4.5), (4.6), (2.3) and (3.4). We have

$$\begin{aligned}
|\mathbb{D}|^2 &= \frac{h^{-2}}{8}|\nabla h|^2|T^0|^2+\frac{h^{-2}}{8^2}\Big(2T^0(\nabla h,e_c)T^0(\nabla h,e_c)\\
&\quad-4\sum_{s=1}^3 T^0(I_s\nabla h,e_c)T^0(I_s\nabla h,e_c)+2\sum_{s,t=1}^3 T^0(I_sI_t\nabla h,e_c)T^0(I_tI_s\nabla h,e_c)\Big)\\
&= \frac{h^{-2}}{8}|\nabla h|^2|T^0|^2+\frac{1}{4}\Big(-\sum_{s=1}^3 D_s^2+2(D_1D_2+D_1D_3+D_2D_3)\Big)
\end{aligned} \tag{4.27}$$

which is the first line of (4.25). For example, the third term in (4.27) is calculated as follows

$$\sum_{s,t=1}^{3} T^0(I_sI_t\nabla h,e_c)T^0(I_tI_s\nabla h,e_c)=\sum_{s=1}^{3}\Big[T^0(\nabla h,e_c)T^0(\nabla h,e_c)-2T^0(I_s\nabla h,e_c)T^0(I_s\nabla h,e_c)\Big]$$
$$=6|D|^2-12\sum_{s=1}^{3}D_s^2+8(D_1D_2+D_1D_3+D_2D_3)=-6\sum_{s=1}^{3}D_s^2+20(D_1D_2+D_1D_3+D_2D_3).$$

Similarly, we obtain the second line of (4.25). The equality (4.26) follows from (4.25) which completes the proof of Lemma 4.2. □

Finally, the proof of Theorem 4.1 follows by letting $c=4$ in (4.24)and using (4.26) and (2.3). □

5. Proof of Theorem[1.3] and Theorem[1.1]

We begin with the proof of Theorem 1.3. The first step of the proof relies on Theorem 4.1. By a homothety we can suppose that both qc-scalar curvatures are equal to $16n(n+2)$. Integrating the divergence formula of Theorem 4.1 and then using the divergence theorem established in [12, Proposition 8.1] shows that the integral of the left-hand side is zero. Thus, the right-hand side vanishes as well, which shows that the quaternionic contact structure $\bar\eta$ has vanishing torsion, i.e., it is also qc-Einstein according to [12, Proposition 4.2]. This proves the first part of Theorem 1.3.

To prove the second part, we develop a sub-Riemannian extension of the result of [23], see also [4] and the review [20, Theorem 2.6], on the relation between the Yamabe equation and the Lichnerowicz-Obata first eigenvalue estimate. We begin by recalling some results from [12, Section 7.2]. A vector field Q on a qc manifold (M,η) is a *qc vector field* if its flow preserves the horizontal distribution $H=\ker\eta$. Since the conformal class of the qc structure on span$\{\eta_1,\eta_2,\eta_3\}$ is uniquely determined by H (cf. [2]), we have that

$$\mathcal{L}_Q\,\eta=(\nu I+O)\cdot\eta,$$

where ν is a smooth function and $O\in so(3)$ is a matrix valued function with smooth entries. Since the exterior derivative d commutes with the Lie derivative $\mathcal{L}_Q$, any qc vector field Q satisfies

$$\mathcal{L}_Q\,g=\nu g,\qquad \mathcal{L}_Q\,I=O\cdot I,\qquad I=(I_1,I_2,I_3)^t,$$

which is equivalent to saying that the flow of Q preserves the conformal class $[g]$ of the horizontal metric and the quaternionic structure $\mathbb{Q}$ on H. The function ν can be easily expressed in terms of the divergence (with respect to g) of the horizontal part Q_H of the vector field Q. Indeed, from [12, Lemma 7.12] we have

$$g(\nabla_XQ_H,Y)\ +\ g(\nabla_YQ_H,X)\ +\ 2\eta_s(Q)g(T^0_{\xi_s}X,Y)=\nu\,g(X,Y),$$

hence

$$\nu=\frac{1}{2n}\nabla^*Q_H.$$

This gives a geometric interpretation for the quantity (∇^*Q_H), namely, the flow of a qc vector field Q preserves a fixed metric $g\in[g]$ if and only if $\nabla^*Q_H=0$.

As an infinitesimal version of the qc Yamabe equation we obtain the following general fact concerning the divergence of a QC vector field.

Lemma 5.1. *Let (M,η) be a qc manifold. For any qc vector field Q on M we have*

$$\Delta(\nabla^*Q_H)\ =\ -\ \frac{n}{2(n+2)}Q(Scal)\ -\ \frac{Scal}{4(n+2)}\nabla^*Q_H,$$

where Scal, ∇^*, Δ and the projection Q_H correspond to the contact form η.

Proof. Suppose Q is a qc vector field and let ϕ_t be the corresponding (local) 1-parameter group of diffeomorphisms generated by its flow. Then

$$\phi_t^*(\eta) \;=\; \frac{1}{2h_t}\,\eta \quad \text{and} \quad \phi_t^*(g) \;=\; \frac{1}{2h_t}\,g$$

for some positive function h_t, depending smoothly on the parameter t. The qc scalar curvature Scal_t of the pull back contact form $\phi_t^*(\eta)$ is given by $\mathrm{Scal}_t = \mathrm{Scal}\circ\phi_t$. Then, formula (2.7) yields

$$\mathrm{Scal}\circ\phi_t \;=\; 2h_t\,(\mathrm{Scal}) \;-\; 8(n+2)^2\,h_t^{-1}|\nabla h_t|^2 \;+\; 8(n+2)\,\triangle h_t. \tag{5.1}$$

Clearly, we have $h_0 = \frac{1}{2}$, and from

$$\frac{1}{2n}(\nabla^* Q_H)\,g \;=\; \mathcal{L}_Q\,g \;=\; \frac{d}{dt}|_{t=0}\left(\frac{1}{2h_t}g\right) \;=\; -\,\frac{h_0'}{2h_0}\,g \;=\; -\,2h_0'\,g$$

we obtain that $h_0' = -\frac{1}{4n}\nabla^* Q_H$, where h_0' denotes the derivative of h_t at $t=0$. A differentiation at $t=0$ in (5.1) gives the lemma. $\square$

Lemma 5.2. *Let (M,η) and $(M,\bar\eta)$ be qc-Einsten manifolds with equal qc-scalar curvatures $16n(n+2)$. If η and $\bar\eta$ are qc conformal to each other, $\overline{\eta} = \frac{1}{2h}\eta$ for some smooth positive function h, then*

$$Q = \frac{1}{2}\nabla f + \sum_{s=1}^{3} dh(\xi_s)\xi_s \tag{5.2}$$

is a qc vector field on M, where the function f is defined in (4.12).

Proof. The assumption of the lemma implies that $E = D = D_s = A_s = 0$. Using (4.19), (4.20) and (4.12) we obtain $\nabla^2 h(I_sX,\xi_s) = -df(X)$ and thus $\nabla^2 h(X,\xi_s) = df(I_sX)$. It follows that

$$\sum_{s=1}^{3}\nabla_X(dh(\xi_s)\xi_s) = \sum_{s=1}^{3} df(I_sX)\xi_s.$$

To show that the flow of the vector field Q, defined by (5.2), preserves the horizontal distribution H, for any $X\in H$, we have

$$\mathcal{L}_Q(X) = \frac{1}{2}\,[\nabla f, X] + \sum_{s=1}^{3}[dh(\xi_s)\xi_s, X] = \frac{1}{2}\nabla_{\nabla f}X \;-\; \frac{1}{2}\nabla_X(\nabla f) \;-\; \sum_{s=1}^{3}\omega_s(\nabla f, X)\xi_s$$
$$+\sum_{s=1}^{3}[dh(\xi_s)\nabla_{\xi_s}X - \nabla_X(dh(\xi_s)\xi_s) - dh(\xi_s)T_{\xi_s}(X)] = \frac{1}{2}\nabla_{\nabla f}X - \frac{1}{2}\nabla_X(\nabla f) + \sum_{s=1}^{3} dh(\xi_s)\nabla_{\xi_s}X \;\in H.$$

$\square$

At this point we are ready to complete the proof of Theorem 1.3. Consider the qc vector field Q defined in Lemma 5.2. By Lemma 5.1, the function $\phi = \frac{1}{2}\triangle f$ is either an eigenfunction of the sub-Laplacian with eigenvalue $-4n$, $\triangle\phi = -4n\phi$, or it vanishes identically. In the first case, using the quaternionic contact version of the Lichnerowicz-Obata eigenfunction sphere theorem [16, Theorem 1.2] and [17, Corollary 1.2] (see also [1]), we conclude that (M,η) is the 3-Sasakain sphere. In the other case, we have that $\Delta f = 0$, hence the function $f = \frac{1}{2} + h + \frac{1}{4}h^{-1}|\nabla h|^2 = \mathit{const}$ since M is compact. It follows that $h = 1/2$ by considering the points where h achieves its minimum and maximum and taking into account the qc Yamabe equation (4.1). The proof of Theorem 1.3 is complete.

Remark 5.3. *Lemma 5.2 provides also a certain geometric insight for the mysterious function* f *in* (4.7). *In fact, up to an additive constant,* f *is the unique function on* M *for which* $Q_H = \frac{1}{2}\nabla f$ *is the horizontal part of a qc vector field* Q *with vertical part* $Q_V = dh(\xi_s)\xi_s$, $Q = Q_H + Q_V$. *This assertion is an easy consequence of the computation given in the proof of Lemma 5.2. Moreover, it implies that on the 3-Sasakain sphere* $\phi = \triangle f$ *is an eigenfuction of the sub-Laplacian realizing the smallest possible eigenvalue* $-4n$ *on a compact locally 3-Sasakian manifold.*

Theorem 1.1 is a direct corollary from Theorem 1.3. Alternatively, as in the proof of Theorem 1.3, we can use in the first step Theorem 4.1 which shows that the "new" structure is also qc-Einstein. The second step of the proof of Theorem 1.1 follows then also by taking into account [12, Theorem 1.2] where all locally 3-Sasakian structures of positive constant qc-scalar curvature which are qc-conformal to the standard 3-Sasakian structure on the sphere were classified (we note that this classification extends easily to the case when no sign condition of the "new" qc-structure is assumed, see [20]).

References

[1] Baudoin, F., & Kim, B., *Sobolev, Poincaré, and isoperimetric inequalities for subelliptic diffusion operators satisfying a generalized curvature dimension inequality*. Rev. Mat. Iberoam. 30 (2014), no. 1, 109–131. 16

[2] Biquard, O., *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **265** (2000). 2, 5, 6, 7, 8, 15

[3] Biquard, O., *Quaternionic contact structures*, Quaternionic structures in mathematics and physics (Rome, 1999), 23–30 (electronic), Univ. Studi Roma "La Sapienza", Roma, 1999. 2

[4] Bourguignon, J.-P., $ Ezin, J.-P., *Scalar curvature functions in a conformal class of metrics and conformal transformations*. Trans. Amer. Math. Soc. 301 (1987), no. 2, 723–736. 15

[5] Branson, T. P., Fontana, L., & Morpurgo, C., *Moser-Trudinger and Beckner-Onofri's inequalities on the CR sphere*, Ann. of Math. (2) 177 (2013), no. 1, 1–52. 3

[6] Capria, M. & Salamon, S., *Yang-Mills fields on quaternionic spaces* Nonlinearity **1** (1988), no. 4, 517–530. 6

[7] Duchemin, D., *Quaternionic contact structures in dimension 7*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 4, 851–885. 5, 6

[8] Folland, G. B. & Stein, E. M., *Estimates for the* $\bar\partial_b$ *Complex and Analysis on the Heisenberg Group*, Comm. Pure Appl. Math., **27** (1974), 429–522. 2

[9] Frank, R. L. & Lieb, E. H., *Sharp constants in several inequalities on the Heisenberg group*, Ann. of Math. (2) **176** (2012), no. 1, 349381. 3

[10] Garofalo, N. & Vassilev, D., *Symmetry properties of positive entire solutions of Yamabe type equations on groups of Heisenberg type*, Duke Math J, **106** (2001), no. 3, 411–449. 2, 3

[11] Gidas, B., Ni, W. M. & Nirenberg, L., *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., **68** (1979), 209–243. 1

[12] Ivanov, St., Minchev, I., & Vassilev, D., *Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem*, Memoirs Amer. Math. Soc. **vol. 231**, number 1086; http://dx.doi.org/10.1090/memo/1086 2, 3, 4, 5, 6, 7, 8, 9, 15, 17

[13] Ivanov, St., Minchev, I., & Vassilev, D., *Extremals for the Sobolev inequality on the seven dimensional quaternionic Heisenberg group and the quaternionic contact Yamabe problem*, J. Eur. Math. Soc. (JEMS) 12 (2010), no. 4, 1041–1067. 3, 4, 7, 8, 9

[14] Ivanov, St., Minchev, I., & Vassilev, D., *The optimal constant in the* L^2 *Folland-Stein inequality on the quaternionic Heisenberg group*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **Vol. XI** (2012), 635-652; 2, 3

[15] Ivanov, S., Minchev, I., & Vassilev, *Quaternionic contact Einstein structures*, arXiv:1306.0474, to appear in Math. Res. Lett. 2, 3, 7

[16] Ivanov, S., Petkov, A., & Vassilev, D., *The sharp lower bound of the first eigenvalue of the sub-Laplacian on a quaternionic contact manifold in dimension 7*, Nonlinear Anal-Theory 93 (2013) 51–61. 4, 16

[17] ———, *The sharp lower bound of the first eigenvalue of the sub-Laplacian on a quaternionic contact manifold*, J. Geom. Anal. 24 (2014), no. 2, 756–778.4, 16

[18] Ivanov, St., & Vassilev, D., *Conformal quaternionic contact curvature and the local sphere theorem*, J. Math. Pures Appl. **93** (2010), 277–307. 5, 7, 8

[19] ———, *Extremals for the Sobolev Inequality and the Quaternionic Contact Yamabe Problem*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011. xviii+219 pp. 3, 5

[20] ———, *The Lichnerowicz and Obata first eigenvalue theorems and the Obata uniqueness result in the Yamabe problem on CR and quaternionic contact manifolds*, preprint 2014, arxiv15 15, 17

[21] Jerison, D., & Lee, J., *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc., **1** (1988), no. 1, 1–13. 1, 4

[22] Mostow, G. D., *Strong rigidity of locally symmetric spaces*, Annals of Mathematics Studies, No. 78. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973. v+195 pp. 2

[23] Obata, M., *The conjecture of conformal transformations in Riemannian manifolds*, J. Diff. Geom., **6** (1971), 247–258. 1, 4, 15

[24] Pansu, P., *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. (2) 129 (1989), no. 1, 1–60. 2

[25] Talenti, G., *Best constant in the Sobolev inequality*, Ann. Mat. Pura Appl., **110** (1976), 353–372. 1

[26] Vassilev, D, *Yamabe type equations on Carnot groups*, Ph. D. thesis Purdue University, 2000. 2

[27] Vassilev, D., *Regularity near the characteristic boundary for sub-laplacian operators*, Pacific J Math, **227** (2006), no. 2, 361–397. 2

[28] Wang, W., *The Yamabe problem on quaternionic contact manifolds*, Ann. Mat. Pura Appl., **186** (2007), no. 2, 359–380. 3

(Stefan Ivanov) University of Sofia, Faculty of Mathematics and Informatics, blvd. James Bourchier 5, 1164, Sofia, Bulgaria, & Institute of Mathematics and Informatics, Bulgarian Academy of Sciences

and Department of Mathematics, University of Pennsylvania, DRL 209 South 33rd Street Philadelphia, PA 19104-6395

E-mail address: ivanovsp@fmi.uni-sofia.bg

(Ivan Minchev) University of Sofia, Faculty of Mathematics and Informatics, blvd. James Bourchier 5, 1164 Sofia, Bulgaria; Department of Mathematics and Statistics, Masaryk University, Kotlarska 2, 61137 Brno, Czech Republic

E-mail address: minchev@fmi.uni-sofia.bg

(Dimiter Vassilev) Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87131-0001

E-mail address: vassilev@unm.edu