EXTREMALS FOR THE SOBOLEV INEQUALITY ON THE SEVEN DIMENSIONAL QUATERNIONIC HEISENBERG GROUP AND THE QUATERNIONIC CONTACT YAMABE PROBLEM

STEFAN IVANOV, IVAN MINCHEV, AND DIMITER VASSILEV

Abstract. A complete solution to the quaternionic contact Yamabe problem on the seven dimensional sphere is given. Extremals for the Sobolev inequality on the seven dimensional Heisenberg group are explicitly described and the best constant in the L^2 Folland-Stein embedding theorem is determined.

Contents

1. Introduction

It is well known that the sphere at infinity of a any non-compact symmetric space M of rank one carries a natural Carnot-Carathéodory structure, see [M, P]. A quaternionic contact (qc) structure, [Biq1, Biq2], appears naturally as the conformal boundary at infinity of the quaternionic hyperbolic space. In this paper, following Biquard, a quaternionic contact structure (*qc structure*) on a real (4n+3)-dimensional manifold M is a codimension three distribution H locally given as the kernel of a $\mathbb{R}^3$-valued one-form $\eta=(\eta_1,\eta_2,\eta_3)$, such that, the three two-forms $d\eta_i|_H$ are the fundamental forms of a quaternionic structure on H. This means that there exists a Riemannian metric g on H and three local almost complex structures I_i on H satisfying the commutation relations of the imaginary quaternions, $I_1I_2I_3=-1$, such that, $d\eta_i|_H=2g(I_i.,.)$. The 1-form η is determined up to a conformal factor and the action of $SO(3)$ on $\mathbb{R}^3$, and therefore H is equipped with a conformal class $[g]$ of Riemannian metrics and a 2-sphere bundle of almost complex structures, the quaternionic bundle $\mathbb{Q}$. The 2-sphere bundle of one forms determines uniquely the associated metric and a conformal change of the metric is equivalent to a conformal change of the one forms. To every

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metric in the fixed conformal class one can associate a linear connection preserving the qc structure, see [Biq1], which we shall call the Biquard connection.

If the first Pontrijagin class of M vanishes then the 2-sphere bundle of $\mathbb{R}^3$-valued 1-forms is trivial [AK], i.e. there is a globally defined 3-contact form η that anihilates H, we denote the corresponding QC manifold (M, η). In this case the 2-sphere of associated almost complex structures is also globally defined on H.

Examples of QC manifolds are given in [Biq1, Biq2, IMV, D1]. In particular, any totally umbilic hypersurface of a quaternionic Kähler or hyperKähler manifold carries such a structure [IMV]. A basic example is provided by any 3-Sasakian manifold which can be defined as a $(4n+3)$-dimensional Riemannian manifold whose Riemannian cone is a hyperKähler manifold. It was shown in [IMV] that the torsion endomorphism of the Biquard connection is the obstruction for a given qc-structure to be locally 3-Sasakian, up to a multiplication with a constant factor and a $SO(3)$-matrix.

For a fixed metric in the conformal class of metrics on the horizontal space one associates the scalar curvature of the associated Biquard connection, called the qc-scalar curvature. Guided by the real (Riemannian) and complex (CR) cases, the quaternionic contact Yamabe problem is: *given a compact QC manifold (M, η), find a conformal 3-contact form for which the qc-scalar curvature is constant.*

In the present paper we provide a solution of this problem on the seven dimensional sphere equipped with its natural quaternionic contact structure. The spheres are important examples of locally quternionic conformally flat qc structures characterized locally in [IV] with the vanishing of a curvature-type tensor invariant and from the point of view of the Yamabe problem play a role similar to their Riemannian and CR counterparts. The question reduces to the solvability of the Yamabe equation (3.4). Taking the conformal factor in the form $\bar{\eta} = u^{4/(Q-2)}\eta$, $Q = 4n+6$, turns (3.4) into the equation

$$\mathcal{L}u \ \equiv\ 4\frac{Q+2}{Q-2}\triangle u - \ u\,Scal \ = \ -\ u^{2^*-1}\ \overline{Scal},$$

where $\triangle$ is the horizontal sub-Laplacian, $\triangle h \ = \ tr^g(\nabla dh)$, $Scal$ and $\overline{Scal}$ are the qc-scalar curvatures correspondingly of $(M,\,\eta)$ and $(M,\,\bar{\eta})$, and $2^* = \frac{2Q}{Q-2}$, with $Q = 4n+6$–the homogeneous dimension. On a compact quaternionic contact manifold M with a fixed conformal class $[\eta]$ the Yamabe equation characterizes the non-negative extremals of the Yamabe functional defined by

$$\Upsilon(u) \ = \ \int_M \Big(4\frac{Q+2}{Q-2}\ |\nabla u|^2 \ + \ \mathrm{Scal}\,u^2\Big)dv_g, \qquad \int_M u^{2^*}\,dv_g \ = \ 1,\ u>0.$$

Considering M equipped with a fixed qc structure, hence, a conformal class $[\eta]$, the Yamabe constant is defined as the infimum

$$\lambda(M) \ \equiv\ \lambda(M,[\eta]) \ = \ \inf\{\Upsilon(u): \int_M u^{2^*}\,dv_g \ = \ 1,\ u>0\}.$$

Here dv_g denotes the Riemannian volume form of the Riemannian metric on M extending in a natural way the horizontal metric associated to η.

When the Yamabe constant $\lambda(M)$ is less than that of the quaternionic sphere with its standard qc structure the existence of solutions to the quaternionic contact Yamabe problem is shown in [W], see also [JL1]. We consider the Yamabe problem on the standard unit $(4n+3)$-dimensional quaternionic sphere. The standard 3-Sasaki structure on the sphere is a qc-Einstein structure $\tilde{\eta}$ having constant qc-scalar curvature $\widetilde{\mathrm{Scal}} = 16n(n+2)$. Its images under conformal quaternionic contact automorphism have again constant qc-scalar curvature. In [IMV] we conjectured that these are the only solutions to the Yamabe problem on the quaternionic sphere. The purpose of this paper is to prove this conjecture when the dimension is equal to seven, i.e., $n=1$.

Theorem 1.1. *Let $\tilde{\eta} = \frac{1}{2h}\eta$ be a conformal deformation of the standard qc-structure $\tilde{\eta}$ on the quaternionic unit sphere S^7. If η has constant qc-scalar curvature, then up to a multiplicative constant η is obtained from $\tilde{\eta}$ by a conformal quaternionic contact automorphism. In particular,*

$\lambda(S^7) = 48\,(4\pi)^{1/5}$ *and this minimum value is achieved only by* $\tilde\eta$ *and its images under conformal quaternionic contact automorphisms.*

In [IMV] a weaker result was shown, namely the conclusion holds (in all dimensions) provided the vertical space of η is integrable. We recall the definition of conformal quaternionic contact automorphism in Definition 2.1.

Another motivation for studying the Yamabe equation comes from its connection with the determination of the norm and extremals in a relevant Sobolev-type embedding on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$, [GV1] and [Va1] and [Va]. As it is well known, the Yamabe equation is essentially the Euler-Lagrange equation of the extremals for the L^2 case of such embedding results. In the considered setting we have the following Theorem due to Folland and Stein [FSt].

Theorem 1.2 (Folland and Stein)**.** *Let* $\Omega\subset \boldsymbol{G}$ *be an open set in a Carnot group* $\boldsymbol{G}$ *of homogeneous dimension* Q *and Haar measure* dH*. For any* $1<p<Q$ *there exists* $S_p=S_p(\boldsymbol{G})>0$ *such that for* $u\in C_o^\infty(\Omega)$

$$\left(\int_\Omega |u|^{p^*}\,dH(g)\right)^{1/p^*} \le S_p\,\left(\int_\Omega |Xu|^p\,dH(g)\right)^{1/p}, \tag{1.1}$$

where $|Xu|=\sum_{j=1}^m |X_ju|^2$ *with* $X_1,\dots,X_m$ *denoting a basis of the first layer of* $\boldsymbol{G}$ *and* $p^*=\frac{pQ}{Q-p}$.

Let S_p be the best constant in the Folland-Stein inequality, i.e., the smallest constant for which (1.1) holds. The second result of this paper is the following Theorem, which determines the extremals and the best constant in Theorem 1.2 when $p=2$ for the case of the seven dimensional quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$. As a manifold $\boldsymbol{G}\,(\mathbb{H}) = \mathbb{H}\times \operatorname{Im}\mathbb{H}$ with the group law given by

$$(q',\omega') = (q_o,\omega_o)\circ(q,\omega) = (q_o + q, \omega + \omega_o + 2\operatorname{Im} q_o\,\bar q),$$

where $q,\ q_o\in\mathbb{H}$ and $\omega,\omega_o\in\operatorname{Im}\mathbb{H}$. The standard quaternionic contact(qc) structure is defined by the left-invariant quaternionic contact form $\tilde\Theta = (\tilde\Theta_1,\ \tilde\Theta_2,\ \tilde\Theta_3) = \frac12\,(d\omega - q'\cdot d\bar q' + dq'\cdot\bar q')$, where . denotes the quaternion multiplication.

Theorem 1.3. *Let* $\boldsymbol{G}\,(\mathbb{H}) = \mathbb{H}\times Im\mathbb{H}$ *be the seven dimensional quaternionic Heisenberg group. The best constant in the* L^2 *Folland-Stein embedding theorem is*

$$S_2 = \frac{2\sqrt3}{\pi^{3/5}}$$

An extremal is given by the function

$$v = \frac{2^{11}\sqrt3}{\pi^{3/5}}[(1+|q|^2)^2 + |\omega|^2]^{-2},\ (q,\omega)\in\boldsymbol{G}\,(\mathbb{H})$$

Any other non-negative extremal is obtained from v *by translations* (5.10) *and dilations* (5.11)*.*

Our result confirms the Conjecture made after [GV1, Theorem 1.1]. In [GV1, Theorem 1.6] the above Theorem is proved in all dimensions, but with the assumption of partial-symmetry. Here with a completely different method from [GV1] we show that the symmetry assumption is superfluous in the case of the first quaternionic Heisenberg group. On the other hand, in [IMV] we proved Theorem 1.1 in all dimensions, but with the 'extra' assumption of the integrability of the vertical distribution. In the present paper we remove the 'extra' integrability assumption in dimension seven. A key step in present result is the establishment of a suitable divergence formula, Theorem 4.4, see [JL2] for the CR case and [Ob], [LP] for the Riemannian case. With the help of this divergence formula we show that the 'new' structure is also qc-Einstein, thus we reduce the Yamabe problem on the 7-sphere from solving the non-linear Yamabe equation to a geometrical system of differential equations describing the qc-Einstein structures conformal to the standard one. Invoking the (quaternionic) Cayley transform, which is a contact conformal diffeomorphism, [IMV], we turn the question to the corresponding system on the quaternionic Heisenberg group. On the latter all global solutions are explicitly described in [IMV] and this allows us to conclude the proof of our results.

Remark 1.4. *With the left invariant basis of Theorem 1.3 the Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$ is not a group of Heisenberg type. If we consider $\boldsymbol{G}\,(\mathbb{H})$ as a group of Heisenberg type then the best constant in the L^2 Folland-Stein embedding theorem is, cf.* [GV1, Theorem 1.6],

$$S_2 \;=\; \frac{15^{1/10}}{\pi^{2/5}\, 2\,\sqrt{2}}.$$

and an extremal is given by the function

$$F(q,\omega) \;=\; \gamma\; \left[(1+|q|^2)^2 \;+\; 16|\omega|^2)\right]^{-2},\; (q,\omega)\in \boldsymbol{G}\,(\mathbb{H})$$

where

$$\gamma \;=\; 32\,\pi^{-17/50}\, 2^{1/5}\, 15^{2/5}.$$

Organization of the paper. The paper uses some results from [IMV]. In order to make the present paper self-contained, in Section 2 we give a review of the notion of a quaternionic contact structure and collect formulas and results from [IMV] that will be used in the subsequent sections.

Section 3 and 4 are of technical nature. In the former we find some transformations formulas for relevant tensors, while in the latter we prove certain divergence formulas. The key result is Theorem 4.4, with the help of which in the last Section we prove the main Theorems.

Convention 1.5. *We use the following conventions:*

- $\{e_1,\dots,e_{4n}\}$ *denotes an orthonormal basis of the horizontal space* H.
- *The summation convention over repeated vectors from the basis* $\{e_1,\dots,e_{4n}\}$ *will be used. For example, for a (0,4)-tensor* P, *the formula* $k=P(e_b,e_a,e_a,e_b)$ *means*

$$k=\sum_{a,b=1}^{4n} P(e_b,e_a,e_a,e_b).$$

- *The triple* (i,j,k) *denotes any cyclic permutation of* $(1,2,3)$.

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2. Quaternionic contact manifolds

In this section we will briefly review the basic notions of quaternionic contact geometry and recall some results from [Biq1] and [IMV].

For the purposes of this paper, a quaternionic contact (QC) manifold $(M,g,\mathbb{Q})$ is a $4n+3$ dimensional manifold M with a codimension three distribution H equipped with a metric g and an Sp(n)Sp(1) structure, i.e., we have

i) a 2-sphere bundle $\mathbb{Q}$ over M of almost complex structures, such that, we have $\mathbb{Q}=\{aI_1+bI_2+cI_3:\ a^2+b^2+c^2=1\}$, where the almost complex structures $I_s: H\to H,\quad I_s^2=-1,\quad s=1,\,2,\,3$, satisfy the commutation relations of the imaginary quaternions $I_1I_2=-I_2I_1=I_3$;

ii) H is the kernel of a 1-form $\eta=(\eta_1,\eta_2,\eta_3)$ with values in $\mathbb{R}^3$ and the following compatibility condition holds

$$2g(I_sX,Y) \;=\; d\eta_s(X,Y),\quad s=1,2,3,\quad X,Y\in H.$$

Correspondingly, given a quaternionic contact manifold we shall denote with η any associated contact form. The associated contact form is determined up to an $SO(3)$-action, namely if $\Psi \in SO(3)$ with smooth functions as entries then $\Psi\eta$ is again a contact form satisfying the above compatibility condition (rotating also the almost complex structures) . On the other hand, if we consider the conformal class $[g]$, the associated contact forms are determined up to a multiplication with a positive function μ and an $SO(3)$-action, namely if $\Psi \in SO(3)$ then $\mu\Psi\eta$ is a contact form associated with a metric in the conformal class $[g]$.

We shall denote with (M, η) a QC manifold with a fixed globally defined contact form. A special phenomena here, noted in [Biq1], is that the 3-contact form η determines the quaternionic structure and the metric on the horizontal bundle in a unique way.

A QC manifold $(M, \bar{g}, \mathbb{Q})$ is called conformal to $(M, g, \mathbb{Q})$ if $\bar{g} \in [g]$. In that case, if $\bar{\eta}$ is a corresponding associated one-form with complex structures $\bar{I}_s$, $s = 1, 2, 3$, we have $\bar{\eta} = \mu\,\Psi\,\eta$ for some $\Psi \in SO(3)$ with smooth functions as entries and a positive function μ. In particular, starting with a QC manifold (M, η) and defining $\bar{\eta} = \mu\,\eta$ we obtain a QC manifold $(M, \bar{\eta})$ conformal to the original one.

Definition 2.1. *A diffeomorphism ϕ of a QC manifold $(M, [g], \mathbb{Q})$ is called a* conformal quaternionic contact automorphism (conformal qc-automorphism) *if ϕ preserves the QC structure, i.e.*

$$\phi^*\eta = \mu\Psi \cdot \eta,$$

for some positive smooth function μ and some matrix $\Psi \in SO(3)$ with smooth functions as entries and $\eta = (\eta_1, \eta_2, \eta_3)^t$ is a local 1-form considered as a column vector of three one forms as entries.

Any endomorphism Ψ of H can be decomposed with respect to the quaternionic structure $(\mathbb{Q}, g)$ uniquely into $Sp(n)$-invariant parts as follows $\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$, where Ψ^{+++} commutes with all three I_i, Ψ^{+--} commutes with I_1 and anti-commutes with the other two and etc. The two $Sp(n)Sp(1)$-invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \qquad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}. \tag{2.1}$$

Denoting the corresponding (0,2) tensor via g by the same letter one sees that the $Sp(n)Sp(1)$-invariant components are the projections on the eigenspaces of the Casimir operator

$$\dagger \ = \ I_1 \otimes I_1 \ + \ I_2 \otimes I_2 \ + \ I_3 \otimes I_3 \tag{2.2}$$

corresponding, respectively, to the eigenvalues 3 and -1, see [CSal]. If $n = 1$ then the space of symmetric endomorphisms commuting with all $I_i, i = 1, 2, 3$ is 1-dimensional, i.e. the [3]-component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_{[3]} = \frac{tr\,(\Psi)}{4} Id_{|H}$.

On a quaternionic contact manifold there exists a canonical connection defined in [Biq1] when the dimension $(4n + 3) > 7$, and in [D] in the 7-dimensional case.

Theorem 2.2. [Biq1] *Let $(M, g, \mathbb{Q})$ be a quaternionic contact manifold of dimension $4n + 3 > 7$ and a fixed metric g on H in the conformal class $[g]$. Then there exists a unique connection ∇ with torsion T on M^{4n+3} and a unique supplementary subspace V to H in TM, such that:*

- *i) ∇ preserves the decomposition $H \oplus V$ and the metric g;*
- *ii) for $X, Y \in H$, one has $T(X, Y) = -[X, Y]_{|V}$;*
- *iii) ∇ preserves the $Sp(n)Sp(1)$-structure on H, i.e., $\nabla g = 0$ and $\nabla\mathbb{Q} \subset \mathbb{Q}$;*
- *iv) for $\xi \in V$, the endomorphism $T(\xi, .)_{|H}$ of H lies in $(sp(n) \oplus sp(1))^{\perp} \subset gl(4n)$;*
- *v) the connection on V is induced by the natural identification φ of V with the subspace $sp(1)$ of the endomorphisms of H, i.e. $\nabla\varphi = 0$.*

We shall call the above connection *the Biquard connection*. Biquard [Biq1] also described the supplementary subspace V explicitly, namely, locally V is generated by vector fields $\{\xi_1, \xi_2, \xi_3\}$, such that

$$\begin{aligned} \eta_s(\xi_k) = \delta_{sk}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \\ (\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}. \end{aligned} \tag{2.3}$$

The vector fields ξ_1, ξ_2, ξ_3 are called Reeb vector fields or fundamental vector fields.

If the dimension of M is seven, the conditions (2.3) do not always hold. Duchemin shows in [D] that if we assume, in addition, the existence of Reeb vector fields as in (2.3), then Theorem 2.2 holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2.3).

Notice that equations (2.3) are invariant under the natural $SO(3)$ action. Using the triple of Reeb vector fields we extend g to a metric on M by requiring $span\{\xi_1,\xi_2,\xi_3\} = V \perp H$ and $g(\xi_s,\xi_k) = \delta_{sk}$. The extended metric does not depend on the action of $SO(3)$ on V, but it changes in an obvious manner if η is multiplied by a conformal factor. Clearly, the Biquard connection preserves the extended metric on $TM, \nabla g = 0$. We shall also extend the quaternionic structure by setting $I_{s|V} = 0$. The fundamental 2-forms $\omega_i, i = 1,2,3$ of the quaternionic structure Q are defined by

$$2\omega_{i|H} \;=\; d\eta_{i|H}, \qquad \xi\lrcorner\omega_i = 0, \quad \xi \in V. \tag{2.4}$$

Due to (2.4), the torsion restricted to H has the form

$$T(X,Y) = -[X,Y]_{|V} = 2\sum_{s=1}^{3}\omega_s(X,Y)\xi_s, \qquad X,Y \in H. \tag{2.5}$$

The properties of the Biquard connection are encoded in the properties of the torsion endomorphism $T_\xi = T(\xi,.) : H \to H, \quad \xi \in V$. Decomposing the endomorphism $T_\xi \in (sp(n)+sp(1))^\perp$ into its symmetric part T^0_ξ and skew-symmetric part b_ξ, $T_\xi = T^0_\xi + b_\xi$, we summarize the description of the torsion due to O. Biquard in the following Proposition.

Proposition 2.3. [Biq1] *The torsion T_ξ is completely trace-free,*

$$tr\, T_\xi = g\,(T_\xi(e_a), e_a) = 0, \quad tr\, T_\xi \circ I = g\,(T_\xi(e_a), Ie_a) = 0, \quad I \in Q,$$

where $e_1 \dots e_{4n}$ is an orthonormal basis of H. Decomposing the torsion into symmetric and anti-symmetric parts, $T_{\xi_i} = T^0_{\xi_i} + b_{\xi_i}, \quad i = 1,2,3$, we have: the symmetric part of the torsion has the properties

$$T^0_{\xi_i}I_i = -I_iT^0_{\xi_i}$$
$$I_2(T^0_{\xi_2})^{+--} = I_1(T^0_{\xi_1})^{-+-}, \quad I_3(T^0_{\xi_3})^{-+-} = I_2(T^0_{\xi_2})^{--+}, \quad I_1(T^0_{\xi_1})^{--+} = I_3(T^0_{\xi_3})^{+--};$$

the skew-symmetric part can be represented in the following way

$$b_{\xi_i} = I_iu,$$

where u is a traceless symmetric (1,1)-tensor on H which commutes with I_1, I_2, I_3.

If $n = 1$ then the tensor u vanishes identically, $u = 0$ and the torsion is a symmetric tensor, $T_\xi = T^0_\xi$.

The covariant derivative of the quaternionic contact structure with respect to the Biquard connection and the covariant derivative of the distribution V are given by

$$\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j, \qquad \nabla\xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j, \tag{2.6}$$

where the $sp(1)$-connection 1-forms α_s on H are given by [Biq1]

$$\alpha_i(X) = d\eta_k(\xi_j, X) = -d\eta_j(\xi_k, X), \quad X \in H, \quad \xi_i \in V, \tag{2.7}$$

while the $sp(1)$-connection 1-forms α_s on the vertical space V are calculated in [IMV]

$$\alpha_i(\xi_s) \;=\; d\eta_s(\xi_j,\xi_k) - \,\delta_{is}\left(\frac{Scal}{16n(n+2)} \;+\; \frac{1}{2}\,\left(\,d\eta_1(\xi_2,\xi_3) \;+\; d\eta_2(\xi_3,\xi_1) \;+\; d\eta_3(\xi_1,\xi_2)\right)\right), \tag{2.8}$$

where $s \in \{1,2,3\}$. The vanishing of the $sp(1)$-connection 1-forms on H is equivalent to the vanishing of the torsion endomorphism of the Biquard connection, see [IMV].

2.1. **The qc-Einstein condition and Bianchi identities.** We explain briefly the consequences of the Bianchi identities and the notion of qc-Einstein manifold introduced in [IMV] since it plays a crucial role in solving the Yamabe equation in the quaternionic seven dimensional sphere. For more details see [IMV].

Let $R=[\nabla,\nabla]-\nabla_{[\ ,\ ]}$ be the curvature tensor of ∇. The Ricci tensor and the scalar curvature *Scal* of the Biquard connection, called *qc-Ricci tensor* and *qc-scalar curvature*, respectively, are defined by

$$Ric(X,Y)=g(R(e_a,X)Y,e_a),\quad X,Y\in H,\qquad Scal=Ric(e_a,e_a)=g(R(e_b,e_a)e_a,e_b).$$

According to [Biq1] the Ricci tensor restricted to H is a symmetric tensor. If the trace-free part of the qc-Ricci tensor is zero we call the quaternionic structure *a qc-Einstein manifold* [IMV]. It is shown in [IMV] that the qc-Ricci tensor is completely determined by the components of the torsion. First, recall the notion of the $Sp(n)Sp(1)$-invariant trace-free symmetric 2-tensors T^0,U on H introduced in [IMV] by

$$T^0(X,Y)\overset{def}{=}g((T^0_{\xi_1}I_1+T^0_{\xi_2}I_2+T^0_{\xi_3}I_3)X,Y),\quad U(X,Y)\overset{def}{=}g(uX,Y),\quad X,Y\in H.$$

The tensor T^0 belongs to $[-1]$-eigenspace while U is in the [3]-eigenspace of the operator $\dagger$ given by (2.2), i.e., they have the properties:

$$T^0(X,Y)+T^0(I_1X,I_1Y)+T^0(I_2X,I_2Y)+T^0(I_3X,I_3Y)=0, \tag{2.9}$$

$$3U(X,Y)-U(I_1X,I_1Y)-U(I_2X,I_2Y)-U(I_3X,I_3Y)=0. \tag{2.10}$$

Theorem 1.3, Theorem 3.12 and Corollary 3.14 in [IMV] imply:

Theorem 2.4. [IMV] *Let $(M^{4n+3},g,\mathbb{Q})$ be a quaternionic contact $(4n+3)$-dimensional manifold, $n>1$. For any $X,Y\in H$ the qc-Ricci tensor and the qc-scalar curvature satisfy*

$$\begin{aligned} Ric(X,Y) &= (2n+2)T^0(X,Y)+(4n+10)U(X,Y)+\frac{Scal}{4n}g(X,Y)\\ Scal &= -8n(n+2)g(T(\xi_1,\xi_2),\xi_3)\end{aligned}$$

For $n=1$ the above formulas hold with $U=0$.

In particular, the qc-Einstein condition is equivalent to the vanishing of the torsion endomorphism of the Biquard connection. If $Scal\neq 0$ the latter holds exactly when the qc-structure is 3-Sasakian up to a multiplication by a constant and an $SO(3)$-matrix with smooth entries.

For the last part of the above Theorem, we remind that a (4n+3)-dimensional Riemannian manifold (M,g) is called 3-Sasakian if the cone metric $g_N=t^2g+dt^2$ on $N=M\times\mathbb{R}^+$ is a hyperkähler metric, namely, it has holonomy contained in $Sp(n+1)$.

The Ricci 2-forms $\rho_s,\ s=1,2,3$ of a quaternionic contact structure are defined by

$$4n\ \rho_s(B,C)=g(R(B,C)e_a,I_se_a),\ B,C\in\Gamma(TM).$$

For ease of reference, in the following Theorem we summarize the properties of the Ricci 2-forms, the scalar curvature and the torsion evaluated on the vertical space established in Lemma 3.11, Corollary 3.14 Proposition 4.3 and Proposition 4.4 of [IMV] .

Theorem 2.5. [IMV] *The Ricci 2-forms satisfy*

$$\begin{aligned} \rho_1(X,Y) &= 2g((T^0_{\xi_2})^{--+}I_3X,Y)-2g(I_1uX,Y)-\frac{Scal}{8n(n+2)}\omega_1(X,Y),\\ \rho_2(X,Y) &= 2g((T^0_{\xi_3})^{+--}I_1X,Y)-2g(I_2uX,Y)-\frac{Scal}{8n(n+2)}\omega_2(X,Y),\\ \rho_3(X,Y) &= 2g((T^0_{\xi_1})^{-+-}I_2X,Y)-2g(I_3uX,Y)-\frac{Scal}{8n(n+2)}\omega_3(X,Y).\end{aligned} \tag{2.11}$$

$$\rho_i(X,\xi_i) = -\frac{X(Scal)}{32n(n+2)} + \frac{1}{2}(\omega_i([\xi_j,\xi_k],X) - \omega_j([\xi_k,\xi_i],X) - \omega_k([\xi_i,\xi_j],X)),$$

$$\rho_i(X,\xi_j) = \omega_j([\xi_j,\xi_k],X), \quad \rho_i(X,\xi_k) = \omega_k([\xi_j,\xi_k],X), \tag{2.12}$$

$$\rho_i(I_kX,\xi_j) = -\rho_i(I_jX,\xi_k) = g(T(\xi_j,\xi_k),I_iX) = \omega_i([\xi_j,\xi_k],X),$$

$$\rho_i(\xi_i,\xi_j) + \rho_k(\xi_k,\xi_j) = \frac{1}{8n(n+2)}\xi_j(Scal). \tag{2.13}$$

The torsion of the Biquard connection restricted to V satisfies the equality

$$T(\xi_i,\xi_j) = -\frac{Scal}{8n(n+2)}\xi_k - [\xi_i,\xi_j]_H, \tag{2.14}$$

where $[\xi_i,\xi_j]_H$ denotes the projection on H parallel to the vertical space V.

We also recall the definition of the $Sp(n)Sp(1)$-invariant vector field A, which appeared naturally in the Bianchi identities investigated in [IMV]

$$A \ = \ I_1[\xi_2,\xi_3] + I_2[\xi_3,\xi_1] + I_3[\xi_1,\xi_2].$$

We shall denote with the same letter the corresponding horizontal one-form, i.e.,

$$A(X) \ = \ g(I_1[\xi_2,\xi_3] + I_2[\xi_3,\xi_1] + I_3[\xi_1,\xi_2],X).$$

The horizontal divergence ∇^*P of a (0,2)-tensor field P on M with respect to Biquard connection is defined to be the (0,1)-tensor field

$$\nabla^*P(.) = (\nabla_{e_a}P)(e_a,.).$$

Then we conclude from [IMV, Theorem 4.8], that

Theorem 2.6. [IMV] *On a $(4n+3)$-dimensional QC manifold with constant qc-scalar curvature we have the formulas*

$$\nabla^*T^0 = (n+2)A, \qquad \nabla^*U = \frac{1-n}{2}A. \tag{2.15}$$

3. Conformal transformations

Note that a conformal quaternionic contact transformation between two quaternionic contact manifold is a diffeomorphism Φ which satisfies

$$\Phi^*\eta = \mu\ \Psi\cdot\eta,$$

for some positive smooth function μ and some matrix $\Psi\in SO(3)$ with smooth functions as entries and η is an $\mathbb{R}^3$-valued one form, $\eta=(\eta_1,\eta_2,\eta_3)^t$ is a column vector with entries one-forms. The Biquard connection does not change under rotations, i.e., the Biquard connection of $\Psi\cdot\eta$ and η coincide. Hence, studying conformal transformations we may consider only transformations $\Phi^*\eta = \mu\ \eta$.

Let h be a positive smooth function on a QC manifold (M,η). Let $\bar\eta = \frac{1}{2h}\eta$ be a conformal deformation of the QC structure η. We will denote the objects related to $\bar\eta$ by over-lining the same object corresponding to η. Thus, $d\bar\eta = -\frac{1}{2h^2}\,dh\wedge\eta + \frac{1}{2h}d\eta$ and $\bar g = \frac{1}{2h}g$. The new triple $\{\bar\xi_1,\bar\xi_2,\bar\xi_3\}$ is determined by the conditions defining the Reeb vector fields. We have

$$\bar\xi_s \ = \ 2h\,\xi_s \ + \ I_s\nabla h, \quad s=1,2,3, \tag{3.1}$$

where ∇h is the horizontal gradient defined by $g(\nabla h,X) = dh(X), \quad X\in H$.
The components of the torsion tensor transform according to the following formulas from [IMV, Section 5]

$$\overline{T}^0(X,Y) \ = \ T^0(X,Y) \ + \ h^{-1}\,[\nabla dh]_{[sym][-1]}(X,Y), \tag{3.2}$$

$$\bar U(X,Y) \ = \ U(X,Y) \ + \ (2h)^{-1}[\ \nabla dh - 2h^{-1}dh\otimes dh]_{[3][0]}(X,Y), \tag{3.3}$$

where the symmetric part is given by, cf. (3.9),

$$[\,\nabla dh]_{[sym]}(X,Y) \;=\; \nabla dh(X,Y) \;+\; \sum_{s=1}^{3} dh(\xi_s)\,\omega_s(X,Y)$$

and $_{[3][0]}$ indicates the trace free part of the [3]-component of the corresponding tensor. In addition, the qc-scalar curvature changes according to the formula [Biq1]

$$\overline{\mathrm{Scal}} \;=\; 2h\,(\mathrm{Scal}) \;-\; 8(n+2)^2\,h^{-1}|\nabla h|^2 \;+\; 8(n+2)\,\triangle h. \tag{3.4}$$

The following vectors will be important for our considerations,

$$A_i \;=\; I_i[\xi_j,\xi_k], \qquad \text{hence} \quad A \;=\; A_1 \;+\; A_2 \;+\; A_3. \tag{3.5}$$

Lemma 3.1. *Let h be a positive smooth function on a QC manifold $(M,g,\mathbb{Q})$ with constant qc-scalar curvature $Scal = 16n(n+2)$ and $\bar\eta \;=\; \frac{1}{2h}\,\eta$ a conformal deformation of the qc structure η. If $\bar\eta$ is a 3-Sasakian structure, then we have the formulas*

$$\begin{aligned}(3.6)\quad A_1(X) \;&=\; -\frac12 h^{-2}dh(X) \;-\; \frac12 h^{-3}|\nabla h|^2 dh(X)\\ -\;\frac12 h^{-1}\Big(\,\nabla dh(I_2X,\xi_2) \;&+\; \nabla dh(I_3X,\xi_3)\Big) + \frac12 h^{-2}\Big(dh(\xi_2)\,dh(I_2X) \;+\; dh(\xi_3)\,dh(I_3X)\Big)\\ &+\;\frac14 h^{-2}\Big(\,\nabla dh(I_2X,I_2\nabla h) \;+\; \nabla dh(I_3X,I_3\nabla h)\Big).\end{aligned}$$

The expressions for A_2 and A_3 can be obtained from the above formula by a cyclic permutation of $(1,2,3)$. Thus, we have also

$$\begin{aligned}A(X) \;&=\; -\frac32 h^{-2}dh(X) \;-\; \frac32 h^{-3}|\nabla h|^2 dh(X)\\ &-\; h^{-1}\sum_{s=1}^{3}\,\nabla dh(I_sX,\xi_s) \;+\; h^{-2}\sum_{s=1}^{3} dh(\xi_s)\,dh(I_sX) \;+\; \frac12 h^{-2}\sum_{s=1}^{3}\,\nabla dh(I_sX,I_s\nabla h).\end{aligned}$$

Proof. First we calculate the $sp(1)$-connection 1-forms of the Biquard connection ∇. For a 3-Sasaki structure we have $d\bar\eta_i(\bar\xi_j,\bar\xi_k) = 2$, $\bar\xi_i \lrcorner d\bar\eta_i = 0$, the non-zero $sp(1)$-connection 1-forms are $\bar\alpha_i(\bar\xi_i) = -2, i = 1,2,3,$ and the qc-scalar curvature $\overline{Scal} = 16n(n+2)$ (see [Example 4.12,[IMV]]). Then (3.1), (2.7), and (2.8) yield

$$\begin{aligned}2d\eta_i(\xi_j,\xi_k) \;&=\; 2h^{-1} \;+\; h^{-2}\|dh\|^2, &\alpha_i(X) \;&=\; -h^{-1}dh(I_iX),\\ \alpha_i(\xi_j) \;=\; -h^{-1}dh(\xi_k) \;&=\; -\alpha_j(\xi_i), &4\alpha_i(\xi_i) \;&=\; -4 \;-\; 2h^{-1} \;-\; h^{-2}\|dh\|^2.\end{aligned} \tag{3.7}$$

From the 3-Sasakian assumption the commutators are $[\bar\xi_i,\bar\xi_j] \;=\; -2\bar\xi_k$. Thus, for $X\in H$ taking also into account (3.1) we have

$$g([\bar\xi_1,\bar\xi_2],\,I_3X) \;=\; -2\,g(\bar\xi_3,\,I_3X) \;=\; -2\,g(2\,h\,\xi_3 \;+\; I_3\,\nabla h,\,I_3\,X) \;=\; -2\,dh\,(X).$$

Therefore, using again (3.1), we obtain

$$\begin{aligned}(3.8)\quad -2\,dh(X) \;&=\; g([\bar\xi_1,\,\bar\xi_2],\,I_3X) \;=\; g\big([2h\xi_1 \;+\; I_1\,\nabla h, 2h\xi_2 \;+\; I_2\,\nabla h],\,I_3X\big)\\ &=\; -4h^2\,A_3(X) \;+\; 2hg(\,[\xi_1,\,I_2\nabla h],\,I_3\,X) \;+\; 2hg(\,[I_1\nabla h,\xi_2],\,I_3\,X)\\ &\qquad +\; g(\,[I_1\,\nabla h,\,I_2\,\nabla h],I_3\,X).\end{aligned}$$

The last three terms are transformed as follows. The first equals

$$\begin{aligned}g\,(\,[\xi_1,\,I_2\nabla h],I_3\,X) \;&=\; g\,(\,(\nabla_{\xi_1}I_2)\,\nabla h \;+\; I_2\,\nabla_{\xi_1}\nabla h,\,I_3X) \;-\; g\,(\,T(\xi_1,\,I_2\nabla h),\,I_3X)\\ &=\; -\alpha_3(\xi_1)\,dh(I_2X) \;+\; \alpha_1(\xi_1)\,dh(X) \;-\; \nabla dh\,(\xi_1,I_1X) \;-\; g\,(\,T(\xi_1,\,I_2\nabla h),\,I_3X),\end{aligned}$$

where we use (2.6) and the fact that ∇ preserves the splitting $H \oplus V$. The second term is

$$g\,([I_1\nabla h, \xi_2], I_3\, X) \;=\; \alpha_2(\xi_2)\, dh(X) \;+\; \alpha_3(\xi_2)\, dh(I_1 X) \;-\; \nabla dh\,(\xi_2, I_2 X) \\ \;-\; g\,(T(I_1\nabla h,\, \xi_2),\, I_3 X),$$

and finally

$$g\,([I_1\nabla h, I_2\nabla h], I_3\, X) \;=\; -\alpha_3(I_1\nabla h)\, dh(I_2 X) \;+\; \alpha_1(I_1\nabla h)\, dh(X) \;-\; \nabla dh\,(I_1\nabla h, I_1 X) \\ +\; \alpha_2(I_2\nabla h)\, dh(X) \;+\; \alpha_3(I_2\nabla h)\, dh(I_1 X) \;-\; \nabla dh\,(I_2\nabla h, I_2 X).$$

Next we apply (3.7) to the last three equalities, then substitute their sum into (3.8), after which we use the commutation relations

$$\begin{aligned} \nabla dh\,(X,Y) - \nabla dh\,(Y,X) \;&=\; -dh(T(X,Y)) = \;-2\sum_{s=1}^{3}\omega_s(X,Y)\, dh(\xi_s), \\ \nabla dh\,(X,\xi) - \nabla dh\,(\xi, X) \;&=\; -dh(T(X,\xi)), \qquad X,\, Y \in H, \quad \xi \in V. \end{aligned} \tag{3.9}$$

The result is the following identity

$$\begin{aligned} 4h^2 A_3(X) \;=\;& (-4h \;+\; h^{-1}\|\nabla h\|^2)\, dh(X) \\ & - 2h\left[\,\nabla dh\,(I_1 X, \xi_1) \;+\; \nabla dh\,(I_2 X, \xi_2)\right] - \left[\,\nabla dh\,(I_1 X, I_1\nabla h)) \;+\; \nabla dh\,(I_2 X, I_2\nabla h)\right] \\ & +\; 2\left[\,dh(\xi_1)\, dh(I_1 X) \;+\; dh(\xi_2)\, dh(I_2 X) \;+\; 2\, dh(\xi_3)\, dh(I_3 X)\right] \\ & +\; 2h\left[\,T(\xi_1, I_1 X, \nabla h) \;+\; T(\xi_2, I_2 X, \nabla h) \;-\; T(\xi_1, I_2 X, I_3\nabla h) \;+\; T(\xi_2, I_1 X, I_3\nabla h)\right], \end{aligned} \tag{3.10}$$

where $T(\xi, X, Y) = g(T_\xi X, Y)$ for a vertical vector ξ and horizontal vectors X and Y. With the help of Proposition 2.3 we decompose the torsions into symmetric and anti-symmetric part $T_{\xi_i} = T^0_{\xi_i} + I_i U$, $i = 1,\ 2,\ 3$, and then express the symmetric parts of the torsion terms in the form $T^0_{\xi_1} = (T^0_{\xi_1})^{--+} + (T^0_{\xi_1})^{-+-}$, $T^0_{\xi_2} = (T^0_{\xi_2})^{--+} + (T^0_{\xi_2})^{+--}$. Hence, using $T^{0^{--+}} = 2(T^0_{\xi_2})^{+--} I_2 = 2(T^0_{\xi_1})^{-+-} I_1$ etc., which follows again from Proposition 2.3, the sum of the torsion terms in (3.10) can be seen to equal $2T^{0^{--+}}(X,\, \nabla h) \;-\; 4\, U(X, \nabla h)$. This allows us to rewrite (3.10) in the form

$$\begin{aligned} 4A_3(X) \;=\;& (-4h^{-1} \;+\; h^{-3}\|\nabla h\|^2)\, dh(X) - \; 2\, h^{-1}\left[\nabla dh\,(I_1 X, \xi_1) \;+\; \nabla dh\,(I_2 X, \xi_2)\right] \\ & +\; 2\, h^{-2}\left[\, dh(\xi_1)\, dh(I_1 X) \;+ dh(\xi_2)\, dh(I_2 X) \;+\; 2\, dh(\xi_3)\, dh(I_3 X)\right] \\ & -\; h^{-2}\left[\,\nabla dh\,(I_1 X, I_1\nabla h) \;+\; \nabla dh\,(I_2 X, I_2\nabla h)\right] \;+\; 4h^{-1}\left[\,(T^{0^{--+}}(\nabla h, X) \;-\; 2U(\nabla h, X)\right]. \end{aligned} \tag{3.11}$$

Using (3.2) the $T^{0^{--+}}$ component of the torsion can be expressed by h as follows, see (2.1) and (2.9),

$$\begin{aligned} 4T^{0^{--+}}(\nabla h, X) \;=&\; T^0(\nabla h, X) \;-\; T^0(I_1\nabla h, I_1 X) \;-\; T^0(I_2\nabla h, I_2 X) \;+\; T^0(I_3\nabla h, I_3 X) \\ =&\; -h^{-1}\Big\{[\nabla dh]_{[-1]}(\nabla h, X) - [\nabla dh]_{[-1]}(I_1\nabla h, I_1 X) - [\nabla dh]_{[-1]}(I_2\nabla h, I_2 X) + [\nabla dh]_{[-1]}(I_3\nabla h, I_3 X)\Big\} \\ & - h^{-1}\sum_{s=1}^{3}\Big\{dh(\xi_s)\Big[g(I_s\nabla h, X) \;-\; g(I_s I_1\nabla h, I_1 X) - \; g(I_s I_2\nabla h, I_2 X) \;+\; g(I_s I_3\nabla h, I_3 X)\Big]\Big\} \\ =&\; -h^{-1}\Big\{\nabla dh\,(\nabla h, X) \;-\; \nabla dh\,(I_1\nabla h, I_1 X) \;-\; \nabla dh\,(I_2\nabla h, I_2 X) \;+\; \nabla dh\,(I_3\nabla h, I_3 X)\Big\} \\ & +\; 4h^{-1}\, dh(\xi_3)\, dh(I_3 X). \end{aligned}$$

Invoking equation (3.9) we can put ∇h in second place in the Hessian terms, thus, proving the formula

$$\begin{aligned} 4T^{0^{--+}}(\nabla h, X) \;=&\; -\; 4h^{-1}\, dh(\xi_3)\, dh(I_3 X) \\ & - h^{-1}\Big\{\nabla dh\,(X, \nabla h) \;-\; \nabla dh\,(I_1 X, I_1\nabla h) \;-\; \nabla dh\,(I_2 X, I_2\nabla h) \;+\; \nabla dh\,(I_3 X, I_3\nabla h)\Big\}. \end{aligned} \tag{3.12}$$

On the other hand, (2.10), (3.3) and the Yamabe equation (3.4) give

$$
\begin{aligned}
(3.13)\quad 8\,U(\nabla h, X) \;&=\; -h^{-1}\Big\{\nabla dh\,(\nabla h, X) \;+\; \sum_{s=1}^{3}\nabla dh\,(I_s\nabla h, I_sX)\\
&\qquad -\,2h^{-1}\|\nabla h\|^2\,dh(X) \;-\; \frac{\triangle h}{n}\,dh(X) \;+\; 2h^{-1}\frac{\|\nabla h\|^2}{n}\,dh(X)\Big\}\\
&=\; -h^{-1}\Big\{\nabla dh\,(\nabla h, X) \;+\; \sum_{s=1}^{3}\nabla dh\,(I_s\nabla h, I_sX)\Big\}\\
&\quad -\; h^{-1}\Big\{-2h^{-1}\|\nabla h\|^2\,dh(X) \;-\; \frac{2n-4nh+(n+2)h^{-1}\|\nabla h\|^2}{n}\,dh(X) \;+\; 2h^{-1}\frac{\|\nabla h\|^2}{n}\,dh(X)\Big\}\\
&=\; -h^{-1}\Big\{\nabla dh\,(X,\nabla h) \;+\; \sum_{s=1}^{3}\nabla dh\,(I_sX, I_s\nabla h)\Big\} -\; h^{-1}\big(-3h^{-1}\|\nabla h\|^2 \;-\; 2 \;+\; 4h\big)dh(X).
\end{aligned}
$$

Substituting the last two formulas in (3.11) gives A_3 in the form of (3.6) written for A_1, cf. the paragraph after (3.6). □

4. Divergence formulas

We shall need the divergences of various vector/forms through the almost complex structures, so we start with a general formula valid for any horizontal vector/form A. Let $\{e_1,\dots,e_{4n}\}$ be an orthonormal basis of H. The divergence of I_1A is

$$\nabla^*(I_1A) \;\equiv\; (\nabla_{e_a}(I_1A))(e_a) \;=\; -(\nabla_{e_a}A)(I_1e_a) \;-\; A((\nabla_{e_a}I_1)e_a),$$

recalling $I_1A(X) = -A(I_1X)$.

We say that an orthonormal frame

$$\{e_1, e_2 = I_1e_1, e_3 = I_2e_1, e_4 = I_3e_1, \dots, e_{4n} = I_3e_{4n-3}, \xi_1, \xi_2, \xi_3\}$$

is a qc-normal frame (at a point) if the connection 1-forms of the Biquard connection vanish (at that point). Lemma 4.5 in [IMV] asserts that a qc-normal frame exists at each point of a QC manifold. With respect to a qc-normal frame the above divergence reduces to

$$\nabla^*(I_1A) \;\;=\;\; -(\nabla_{e_a}A)(I_1e_a).$$

Lemma 4.1. *Suppose $(M,\eta,\mathbb{Q})$ is a quaternionic contact manifold with constant qc-scalar curvature. For any function h we have the following formulas*

$$
\begin{aligned}
\nabla^*\Big(\sum_{s=1}^{3} dh(\xi_s)I_sA_s\Big) \;&=\; \sum_{s=1}^{3}\;\nabla dh\,(I_se_a,\xi_s)A_s(e_a)\\
\nabla^*\Big(\sum_{s=1}^{3} dh(\xi_s)I_sA\Big) \;&=\; \sum_{s=1}^{3}\;\nabla dh\,(I_se_a,\xi_s)A(e_a).
\end{aligned}
$$

Proof. Using the identification of the 3-dimensional vector spaces spanned by $\{\xi_1,\xi_2,\xi_3\}$ and $\{I_1,I_2,I_3\}$ with $\mathbb{R}^3$, the restriction of the action of $Sp(n)Sp(1)$ to this spaces can be identified with the action of the group $SO(3)$, i.e., $\xi_i = \sum_{t=1}^{3}\Psi_{it}\bar\xi_t$ and $I_i = \sum_{t=1}^{3}\Psi_{it}\bar I_t$, $i=1,2,3$ with $\Psi\in SO(3)$. One verifies easily that the vectors A, $\sum_{s=1}^{3} dh(\xi_s)I_sA_s = -\sum_{i=1}^{3} dh(\xi_i)[\xi_j,\xi_k]$ and $\sum_{s=1}^{3} dh(\xi_s)I_sA$ are $Sp(n)Sp(1)$ invariant on $\mathbb{H}$, for example $\bar A = (det\Psi)\,A$. Thus, it is sufficient to compute their divergences in a qc-normal frame. To avoid the introduction of new variables, in this proof, we shall assume that $\{e_1,\dots,e_{4n},\xi_1,\xi_2,\xi_3\}$ is a qc-normal frame.

We apply (2.14). Using that the Biquard connection preserves the splitting of TM, we find

$$
\begin{aligned}
\nabla^*[\xi_1,\xi_2] \;&=\; -g(\nabla_{e_a}\big(T(\xi_1,\xi_2)\big), e_a)\\
&=\; -g(\,(\nabla_{e_a}T)\,(\xi_1,\xi_2), e_a) \;-\; g(\,T(\nabla_{e_a}\xi_1,\xi_2), e_a) \;-\; g(\,T(\xi_1,\nabla_{e_a}\xi_2), e_a).
\end{aligned}
$$

From Bianchi's identity we have ($\sigma_{A,B,C}$ means a cyclic sum over (A,B,C))

$$\begin{aligned} g((\nabla_{e_a}T)(\xi_1,\xi_2),e_a) &= -g((\nabla_{\xi_1}T)(\xi_2,e_a),e_a) - g((\nabla_{\xi_2}T)(e_a,\xi_1),e_a) \\ &\quad - g(\sigma_{e_a,\xi_1,\xi_2}\Big\{T(T(e_a,\xi_1),\xi_2)\Big\},e_a) + g(\sigma_{e_a,\xi_1,\xi_2}\Big\{R(e_a,\xi_1)\xi_2\Big\},e_a) \\ &= -g(T(T(e_a,\xi_1),\xi_2),e_a) - g(T(T(\xi_1,\xi_2),e_a),e_a) - g(T(T(\xi_2,e_a),\xi_1),e_a) \\ &= g(T(T(\xi_1,e_a),\xi_2),e_a) - g(T(T(\xi_2,e_a),\xi_1),e_a) - g(T(T(\xi_1,\xi_2),e_a),e_a), \end{aligned}$$

taking into account that as mappings on H the torsion tensors $T(\xi_i,X)$ and the curvature tensor $R(\xi_1,\xi_2)$ are traceless, so $g((\nabla_{\xi_1}T)(\xi_2,e_a),e_a)$ and $g(R(\xi_1,\xi_2)e_a,e_a) = 0$, while the connection preserves the splitting, to obtain the next to last line. The last term is equal to zero as

$$g(T(T(\xi_1,\xi_2),e_a),e_a) = g\big(T(-\frac{\mathrm{Scal}}{8n(n+2)}\xi_3 - [\xi_1,\xi_2]_H,e_a),e_a\big) = -\frac{\mathrm{Scal}}{8n(n+2)}\,g(T(\xi_3,e_a),e_a) = 0,$$

taking into account that the torsion T_{ξ_3} is traceless and $T([\xi_1,\xi_2]_H,e_a)$ is a vertical vector. On the other hand,

$$\begin{aligned} & g\,(T(T_{\xi_1}e_a,\xi_2),e_a) - g\,(T(T_{\xi_2}e_a,\xi_1),e_a) \\ &\quad = -\Big[g(T(e_b,\xi_2),e_a)\,g(T(\xi_1,e_a),e_b) - g(T(e_b,\xi_1),e_a)\,g(T(\xi_2,e_a),e_b)\Big] \\ &\quad = \Big[g(T(\xi_2,e_b),e_a)\,g(T(\xi_1,e_a),e_b) - g(T(\xi_1,e_b),e_a)\,g(T(\xi_2,e_a),e_b)\Big] = 0. \end{aligned}$$

The equalities $\nabla^*(I_1A_1) = \nabla^*(I_2A_2) = 0$ with respect to a qc-normal frame can be obtained similarly. Hence, the first formula in Lemma 4.1 follows.

We are left with proving the second divergence formula. Since the scalar curvature is constant (2.12) implies

$$A(X) = -2\sum_{s=1}^{3}\rho_s(X,\xi_s). \tag{4.1}$$

Fix an $s\in\{1,2,3\}$. Working again in a qc-normal frame we have

$$(\nabla_{e_a}A)(I_se_a) = -2\sum_{t=1}^{3}(\nabla_{e_a}\rho_t)(I_se_a,\xi_t).$$

A calculation involving the expressions (2.11) and the properties of the torsion shows that

$$\mathrm{tr}\,(\rho_t\circ I_s) = -\frac{1}{2(n+2)}\,\delta_{st}\,\mathrm{Scal}. \tag{4.2}$$

The second Bianchi identity

$$\begin{aligned} 0 &= g((\nabla_{e_a}R)(I_se_a,\xi_t)e_b,I_te_b) + g((\nabla_{I_se_a}R)(\xi_t,e_a)e_b,I_te_b) + g((\nabla_{\xi_t}R)(e_a,I_se_a)e_b,I_te_b) \\ &\quad + g(R(T(e_a,I_se_a),\xi_t)e_b,I_te_b) + g(R(T(I_se_a,\xi_t),e_a)e_b,I_te_b) + g(R(T(\xi_t,e_a),I_se_a)e_b,I_te_b), \end{aligned}$$

together with the constancy of the qc-scalar curvature and (4.2) show that the third term on the right is zero and thus

$$\sum_{t=1}^{3}\Big\{2\,(\nabla_{e_a}\rho_t)(I_se_a,\xi_t) - 2\,\rho_t(T(\xi_t,I_se_a),e_a) + \rho_t(T(e_a,I_se_a),\xi_t)\Big\} = 0.$$

Substituting (2.5) in the above equality we come to the equation

$$\sum_{t=1}^{3}(\nabla_{e_a}\rho_t)(I_se_a,\xi_t) = \sum_{t=1}^{3}\rho_t(T(\xi_t,I_se_a),e_a) - 4n\sum_{t=1}^{3}\rho_t(\xi_s,\xi_t) = 0, \tag{4.3}$$

where the vanishing of the second term follows from (2.13), while the vanishing of the first term is seen as follows. Using the standard inner product on $End(H)$

$$g(C,B)=tr(B^*C)=\sum_{a=1}^{4n}g(C(e_a),B(e_a)),$$

where $C,B\in End(H)$, $\{e_1,...,e_{4n}\}$ is a g-orthonormal basis of H, the definition of $T^0_{\xi_s}$, the formulas in Theorem 2.5 and Proposition 2.3 imply

$$\begin{aligned}
&\sum_{s=1}^3\rho_s(T(\xi_s,I_1e_a),e_a)\\
&\quad= g(\rho_1,T^0_{\xi_1}I_1)\ +\ g(\rho_2,T^0_{\xi_2}I_1)\ +\ g(\rho_3,T^0_{\xi_3}I_1)\ -\ g(\rho_1,u)\ -\ g(\rho_2,I_3u)\ +\ g(\rho_3,I_2u)\\
&\quad= g(\rho_1,T^0_{\xi_1}I_1)\ +\ g(\rho_2,T^0_{\xi_2}I_1)\ +\ g(\rho_3,T^0_{\xi_3}I_1)\\
&\quad= g(2(T^0_{\xi_2})^{--+}I_3\ -\ 2I_1u\ -\ \frac{Scal}{8n(n+2)}\,I_1,\,T^0_{\xi_1}I_1)\\
&+\ g(2(T^0_{\xi_3})^{+--}I_1\ -\ 2I_2u\ -\ \frac{Scal}{8n(n+2)}\,I_2,\,T^0_{\xi_2}I_1)+\ g(2(T^0_{\xi_1})^{-+-}I_2\ -\ 2I_3u\ -\ \frac{Scal}{8n(n+2)}\,I_3,\,T^0_{\xi_3}I_1)\\
&\quad= -2g((T^0_{\xi_2})^{--+}I_2,T^0_{\xi_1})\ +\ 2g((T^0_{\xi_3})^{+--},T^0_{\xi_2})\ +\ 2g((T^0_{\xi_1})^{-+-}I_3,T^0_{\xi_3})\\
&\quad= 2g((T^0_{\xi_3})^{+--},(T^0_{\xi_2})^{+--})\ +\ 2g((T^0_{\xi_1})^{-+-},I_3(T^0_{\xi_3})^{+--})\\
&\quad= 2g(I_2(T^0_{\xi_3})^{+--},I_2(T^0_{\xi_2})^{+--})\ -\ 2g(I_1(T^0_{\xi_1})^{-+-},I_2(T^0_{\xi_3})^{+--})\ =\ 0.
\end{aligned}$$

Renaming the almost complex structures shows that the same conclusion is true when we replace I_1 with I_2 or I_3 in the above calculation.

Finally, the second formula in Lemma 4.1 follows from (4.1) and (4.3). □

We shall also need the following one-forms

$$\begin{aligned}
D_1(X)\ &=\ -h^{-1}T^{0^{+--}}(X,\nabla h)\\
D_2(X)\ &=\ -h^{-1}T^{0^{-+-}}(X,\nabla h)\\
D_3(X)\ &=\ -h^{-1}T^{0^{--+}}(X,\nabla h)
\end{aligned}\tag{4.4}$$

For simplicity, using the musical isomorphism, we will denote with D_1, D_2, D_3 the corresponding (horizontal) vector fields, for example $g(D_1,X)=D_1(X)\quad \forall X\in H.$ Finally, we set

$$D\ =\ D_1\ +\ D_2\ +\ D_3\ =\ -h^{-1}\,T^0(X,\nabla h).\tag{4.5}$$

Lemma 4.2. *Suppose (M,η) is a quaternionic contact manifold with constant qc-scalar curvature $Scal=16n(n+2)$. Suppose $\bar\eta=\frac{1}{2h}\eta$ has vanishing $[-1]$-torsion component $\overline{T}^0=0$. We have*

$$D(X)\ =\ \frac14h^{-2}\Big(3\ \nabla dh(X,\nabla h)\ -\ \sum_{s=1}^3\ \nabla dh(I_sX,I_s\nabla h)\Big)\ +\ h^{-2}\sum_{s=1}^3dh(\xi_s)\,dh(I_sX).$$

and the divergence of D satisfies

$$\nabla^*D\ =\ |T^0|^2\ -h^{-1}g(dh,D)\ -\ h^{-1}(n+2)\,g(dh,A).$$

Proof. a) The formula for D follows immediately from (3.2).

b) We work in a qc-normal frame. Since the scalar curvature is assumed to be constant we use (2.15) to find

$$\begin{aligned}
\nabla^*D\ &=\ -h^{-1}\,dh(e_a)D(e_a)\ -\ h^{-1}\nabla^*T^0(\nabla h)\ -\ h^{-1}T^0(e_a,e_b)\ \nabla dh(e_a,e_b)\\
&=\ -h^{-1}\,dh(e_a)D(e_a)\ -\ h^{-1}(n+2)\,dh(e_a)A(e_a)\ -\ g(T^0,\,h^{-1}\ \nabla dh)\\
&=\ |T^0|^2\ -h^{-1}dh(e_a)D(e_a)\ -\ h^{-1}(n+2)\,dh(e_a)A(e_a),
\end{aligned}$$

using (3.2) in the last equality. □

Let us also consider the following one-forms (and corresponding vectors)

$$F_s(X) \;=\; -h^{-1}\,T^0(X, I_s\nabla h), \quad X\in H \quad s=1,2,3.$$

From the definition of F_1 and (4.4) we find

$$\begin{aligned} F_1(X) \;&=\; -h^{-1}T^0(X,I_1\nabla h)\\ &=\; -h^{-1}T^{0^{+--}}(X,I_1\nabla h)\;-\;h^{-1}T^{0^{-+-}}(X,I_1\nabla h)\;-h^{-1}T^{0^{--+}}(X,I_1\nabla h)\\ &=\; h^{-1}T^{0^{+--}}(I_1X,\nabla h)\;-\;h^{-1}T^{0^{-+-}}(I_1X,\nabla h)\;-h^{-1}T^{0^{--+}}(I_1X,\nabla h)\\ &=\; -D_1(I_1X)\;+\;D_2(I_1X)\;+\;D_3(I_1X). \end{aligned}$$

Thus, the forms F_s can be expressed by the forms D_s as follows

$$\begin{aligned} F_1(X) \;&=\; -D_1(I_1X)\;+\;D_2(I_1X)\;+\;D_3(I_1X)\\ F_2(X) \;&=\; D_1(I_2X)\;-\;D_2(I_2X)\;+\;D_3(I_2X)\\ F_3(X) \;&=\; D_1(I_3X)\;+\;D_2(I_3X)\;-\;D_3(I_3X). \end{aligned}\tag{4.6}$$

Lemma 4.3. *Suppose (M,η) is a quaternionic contact manifold with constant qc-scalar curvature $Scal=16n(n+2)$. Suppose $\bar\eta=\frac{1}{2h}\eta$ has vanishing $[-1]$-torsion component, $\overline{T}^0=0$. We have*

$$\begin{aligned} \nabla^*\Big(\sum_{s=1}^3 dh(\xi_s)F_s\Big) \;&=\; \sum_{s=1}^3\Big[\;\nabla dh\,(I_se_a,\xi_s)F_s(I_se_a)\Big]\\ &\quad+\; h^{-1}\sum_{s=1}^3\Big[dh(\xi_s)dh(I_se_a)D(e_a)\;+(n+2)\,dh(\xi_s)dh(I_se_a)\,A(e_a)\Big]. \end{aligned}$$

Proof. We note that the vector $\sum_{s=1}^3 dh(\xi_s)F_s$ is an $Sp(n)Sp(1)$ invariant vector, hence, we may assume that $\{e_1,\dots,e_{4n},\xi_1,\xi_2,\xi_3\}$ is a qc-normal frame. Since the scalar curvature is assumed to be constant we can apply Theorem 2.6, thus $\nabla^*T^0 \;=\; (n+2)A$. Turning to the divergence, we compute

$$\begin{aligned} \nabla^*\Big(\sum_{s=1}^3 dh(\xi_s)F_s\Big) \;&=\; \sum_{s=1}^3\Big[\;\nabla dh\,(e_a,\xi_s)F_s(e_a)\Big]\;-\;\sum_{s=1}^3 h^{-1}\,dh(\xi_s)\,\nabla^*T^0(I_s\nabla h)\\ &\quad+\sum_{s=1}^3\Big[h^{-2}\,dh(\xi_s)\,dh(e_a)T^0(e_a,I_se_b)\,dh(e_b)\;-\;h^{-1}\,dh(\xi_s)\,T^0(e_a,I_se_b)\;\nabla dh\,(e_a,e_b)\Big]\\ &=\;\sum_{s=1}^3\Big[\;\nabla dh\,(e_a,\xi_s)F_s(e_a)\Big]\;-\;\sum_{s=1}^3 h^{-1}\,dh(\xi_s)\,\nabla^*T^0(I_s\nabla h)\\ &\quad+\sum_{s=1}^3\Big[\;h^{-1}\,dh(\xi_s)\,dh(I_se_a)\,D(e_a)\Big]\\ &=\;\sum_{s=1}^3\Big[\;\nabla dh\,(e_a,\xi_s)F_s(e_a)\;+\;h^{-1}\,dh(\xi_s)\,dh(I_se_a)\,D(e_a)\;+\;h^{-1}(n+2)\,dh(\xi_s)\,dh(I_se_a)\,A(e_a)\Big], \end{aligned}\tag{4.7}$$

using the symmetry of T^0 in the next to last equality and the fact $T^0(e_a,I_1e_b)\;\;\nabla dh\,(e_a,e_b)=0$. The latter can be seen, for example, by first using (3.2) and the formula for the symmetric part of

∇dh given after (3.3) from which we have

$$T^0(e_a,I_1e_b)\ \nabla dh\,(e_a,e_b)\ =-\ h^{-1}\nabla dh_{[sym][-1]}(e_a,I_1e_b)\Big[\nabla dh_{[sym]}(e_a,e_b)-\sum_{s=1}^{3}dh(\xi_s)\omega_s(e_a,e_b)\Big]$$
$$=-\ h^{-1}\nabla dh_{[sym][-1]}(e_a,I_1e_b)\nabla dh_{[sym][-1]}(e_a,e_b)-\ h^{-1}\nabla dh_{[sym][-1]}(e_a,I_1e_b)\nabla dh_{[sym][3]}(e_a,e_b)$$
$$+\ h^{-1}\nabla dh_{[sym][-1]}(e_a,I_1e_b)\sum_{s=1}^{3}dh(\xi_s)\omega_s(e_a,e_b)=0,$$

using the zero traces of the [-1]-component to justify the vanishing of the third term in the last equality. Switching to the basis $\{I_se_a : a=1,\dots,4n\}$ in the first term of the right-hand-side of (4.7) completes the proof. □

At this point we restrict our considerations to the 7-dimensional case, i.e. $n=1$. Following is our main technical result. As mentioned in the introduction, we were motivated to seek a divergence formula of this type based on the Riemannian and CR cases of the considered problem. The main difficulty was to find a suitable vector field with non-negative divergence containing the norm of the torsion. The fulfilment of this task was facilitated by the results of [IMV], which in particular showed that similarly to the CR case, but unlike the Riemannian case, we were not able to achieve a proof based purely on the Bianchi identities, see [IMV, Theorem 4.8].

Theorem 4.4. *Suppose (M^7,η) is a quaternionic contact structure conformal to a 3-Sasakian structure $(M^7,\bar\eta)$, $\tilde\eta=\frac{1}{2h}\,\eta$. If $Scal_\eta=Scal_{\tilde\eta}=16n(n+2)$, then with f given by*

$$f\ =\ \frac12\ +\ h\ +\ \frac14 h^{-1}|\nabla h|^2,$$

the following identity holds

$$\nabla^*\Big(fD\ +\ \sum_{s=1}^{3}dh(\xi_s)\,F_s\ +\ 4\sum_{s=1}^{3}dh(\xi_s)I_sA_s\ -\ \frac{10}{3}\sum_{s=1}^{3}dh(\xi_s)\,I_sA\Big)=\ f|T^0|^2\ +\ h\,\langle QV,\,V\rangle.$$

Here, Q is a positive semi-definite matrix and $V=(D_1,D_2,D_3,A_1,A_2,A_3)$ with A_s, D_s defined, correspondingly, in (3.5) and (4.4).

Proof. Using the formulas for the divergences of D, $\sum_{s=1}^3 dh(\xi_s)\, F_s$, $\sum_{s=1}^3 dh(\xi_s) I_s A_s$ and $\sum_{s=1}^3 dh(\xi_s)\, I_s A$ given correspondingly in Lemmas 4.2, 4.3 and 4.1 we have the identity ($n=1$ here)

$$
\begin{aligned}
(4.8)\quad & \nabla^*\Big(fD \,+\, \sum_{s=1}^3 dh(\xi_s)\, F_s \,+\, 4\sum_{s=1}^3 dh(\xi_s) I_s A_s \,-\, \frac{10}{3}\sum_{s=1}^3 dh(\xi_s)\, I_s A\Big) \\
&= \Big(dh(e_a) \,-\, \frac{1}{4}h^{-2}dh(e_a)|\nabla h|^2 \,+\, \frac{1}{2}h^{-1}\,\nabla dh(e_a,\nabla h)\Big)\, D(e_a) \\
&\quad + f\,\Big(|T^0|^2 \,-h^{-1}dh(e_a)D(e_a) \,-\, h^{-1}(n+2)\, dh(e_a)A(e_a)\Big) \\
&+ \sum_{s=1}^3 \nabla dh(I_s e_a,\xi_s)F_s(I_s e_a) \,+\, h^{-1}\sum_{s=1}^3\Big[dh(\xi_s)\, dh(I_s e_a)\, D(e_a) \,+\, (n+2)\, dh(\xi_s)\, dh(I_s e_a)\, A(e_a)\Big] \\
&\quad + 4\sum_{s=1}^3 \nabla dh(I_s e_a,\xi_s)\, A_s(e_a) \,-\frac{10}{3}\sum_{s=1}^3 \nabla dh(I_s e_a,\xi_s)\, A(e_a) \\
&= \Big(dh(e_a) \,-\, \frac{1}{4}h^{-2}dh(e_a)|\nabla h|^2 \,+\, \frac{1}{2}h^{-1}\,\nabla dh(e_a,\nabla h)\Big)\sum_{t=1}^3 D_t(e_a) \\
&\quad + f\,\Big(|T^0|^2 \,-h^{-1}dh(e_a)\Big)\big(\sum_{t=1}^3 D_t(e_a)\big) \,-\, fh^{-1}(n+2)\, dh(e_a)\big(\sum_{t=1}^3 A_t(e_a)\big) \\
&+ \nabla dh(I_1 e_a,\xi_1)\,(D_1(e_a) \,-\, D_2(e_a) \,-\, D_3(e_a)) \,+\, \nabla dh(I_2 e_a,\xi_2)\,(-D_1(e_a) \,+D_2(e_a) \,-\, D_3(e_a)) \\
&\quad + \nabla dh(I_3 e_a,\xi_3)\,(-D_1(e_a) \,-\, D_2(e_a) \,+D_3(e_a)) \\
&+ h^{-1}\big(\sum_{s=1}^3 dh(\xi_s)\, dh(I_s e_a)\big)\,\big(\sum_{t=1}^3 D_t(e_a)\big) \,+\, h^{-1}(n+2)\big(\sum_{s=1}^3 dh(\xi_s)\, dh(I_s e_a)\big)\,\big(\sum_{t=1}^3 A_t(e_a)\big) \\
&\quad + 4\sum_{s=1}^3 \nabla dh(I_s e_a,\xi_s)\, A_s(e_a) \,-\frac{10}{3}\,\big(\sum_{s=1}^3 \nabla dh(I_s e_a,\xi_s)\big)\,\big(\sum_{t=1}^3 A_t(e_a)\big),
\end{aligned}
$$

where the last equality uses (4.6) to express the vectors F_s by D_s, and the expansions of the vectors A and D according to (3.5) and (4.5). Since the dimension of M is seven it follows $U = \bar U = [\,\nabla dh - 2h^{-1} dh\otimes dh]_{[3][0]} = 0$. This, together with the Yamabe equation (3.4) , which when $n=1$ becomes $\triangle h = 2-4h+3h^{-1}|\nabla h|^2$, yield the formula, cf. (3.13),

$$
(4.9)\qquad \nabla dh\,(X,\nabla h) \,+\, \sum_{s=1}^3 \nabla dh\,(I_s X, I_s\nabla h) \,-\, \big(\,2 \,-\, 4h \,+3h^{-1}|\nabla h|^2\big)\, dh(X) \,=\, 0.
$$

From equations (4.4) and (3.12) we have

$$
\begin{aligned}
D_1(X) &= h^{-2}\, dh(\xi_1)\, dh(I_1 X) \,+\, \frac{1}{4}h^{-2}\big[\,\nabla dh\,(X,\nabla h) + \nabla dh\,(I_1 X, I_1\nabla h) \\
&\qquad - \nabla dh\,(I_2 X, I_2\nabla h) \,-\, \nabla dh\,(I_3 X, I_3\nabla h)\big], \\
D_2(X) &= h^{-2}\, dh(\xi_2)\, dh(I_2 X) \,+\, \frac{1}{4}h^{-2}\big[\nabla dh\,(X,\nabla h) \,-\, \nabla dh\,(I_1 X, I_1\nabla h) \\
&\qquad + \nabla dh\,(I_2 X, I_2\nabla h) \,-\, \nabla dh\,(I_3 X, I_3\nabla h)\big], \\
D_3(X) &= h^{-2}\, dh(\xi_3)\, dh(I_3 X) \,+\, \frac{1}{4}h^{-2}\big[\nabla dh(X,\nabla h) - \nabla dh\,(I_1 X, I_1\nabla h) \\
&\qquad - \nabla dh\,(I_2 X, I_2\nabla h) \,+\, \nabla dh\,(I_3 X, I_3\nabla h)\big].
\end{aligned}
$$

Expressing the first term in (4.9) by the rest and substituting with the result in the above equations we come to

$$(4.10)\quad D_i(e_a) \;=\frac{1}{4}h^{-2}\left(2\;-\;4h\;+3h^{-1}|\nabla h|^2\right)dh(e_a)+h^{-2}\,dh(\xi_i)\,dh(I_ie_a)+ \frac{1}{2}h^{-2}\left[-\;\nabla dh\,(I_je_a,I_j\nabla h)\;-\nabla dh\,(I_ke_a,I_k\nabla h)\right].$$

At this point, by a purely algebraic calculation, using Lemma 3.1 and (4.10) we find:

$$\begin{aligned}\frac{22}{3}\,A_1\;-\;\frac{2}{3}\,A_2\;-\;\frac{2}{3}\,A_3\;+\;\frac{11}{3}\;D_1\;-\;\frac{1}{3}\;D_2\;-\;\frac{1}{3}\;D_3 \\ = -3h^{-1}\left(1+\frac{1}{2}h^{-1}dh(e_a)+\frac{1}{4}h^{-2}|\nabla h|^2\right)dh(e_a)+3h^{-2}\left(\sum_{s=1}^{3}dh(\xi_s)\,dh(I_se_a)\right) \\ +\frac{2}{3}h^{-1}\nabla dh(I_1e_a,\xi_1)-\frac{10}{3}h^{-1}\nabla dh(I_2e_a,\xi_2)-\frac{10}{3}h^{-1}\nabla dh(I_3e_a,\xi_3).\end{aligned}$$

Similarly,

$$\begin{aligned}3\,A_1-A_2-A_3+2D_1=\left(-2h^{-1}+\frac{1}{2}h^{-2}+h^{-3}|\nabla h|^2\right)dh(e_a)-\frac{1}{2}h^{-2}\sum_{s=1}^{3}\nabla dh(I_se_a,I_s\nabla h) \\ +h^{-1}\;\nabla dh(I_1e_a,\xi_1)-h^{-1}\;\nabla dh(I_2e_a,\xi_2)-h^{-1}\;\nabla dh(I_3e_a,\xi_3)+h^{-2}\sum_{s=1}^{3}dh(\xi_s)\,dh(I_se_a).\end{aligned}$$

On the other hand, the coefficient of $A_1(e_a)$ in (4.8) is found to be, after setting $n=1$,

$$\begin{aligned}h\Big[-\;3\left(\;1\;+\frac{1}{2}h^{-1}\;+\;\frac{1}{4}h^{-2}|\nabla h|^2\right)h^{-1}\,dh(e_a)+\;3h^{-2}\left(\sum_{s=1}^{3}dh(\xi_s)\,dh(I_se_a)\right) \\ +\;\frac{2}{3}h^{-1}\nabla dh(I_1e_a,\xi_1)-\frac{10}{3}h^{-1}\nabla dh(I_2e_a,\xi_2)-\frac{10}{3}h^{-1}\nabla dh(I_3e_a,\xi_3)\Big],\end{aligned}$$

while the coefficient of $D_1(e_a)$ in (4.8) is

$$\begin{aligned}(4.11)\quad dh(e_a)\;-\;\frac{1}{4}h^{-2}dh(e_a)|\nabla h|^2\;+\;\frac{1}{2}h^{-1}\;\nabla dh(e_a,\nabla h)-\;f\,h^{-1}dh(e_a) \\ +\;\nabla dh(I_1e_a,\xi_1)\;-\;\nabla dh(I_2e_a,\xi_2)-\;\nabla dh(I_3e_a,\xi_3)D_1(e_a) \\ +\;h^{-1}\big(\sum_{s=1}^{3}dh(\xi_s)\,dh(I_se_a)\big).\end{aligned}$$

Substituting $\nabla dh(e_a,\nabla h)$ according to (4.9), i.e., $\nabla dh\,(e_a,\nabla h)\;=-\;\sum_{s=1}^{3}\nabla dh\,(I_se_a,I_s\nabla h)\;+\left(\,2\;-\;4h\;+3h^{-1}|\nabla h|^2\right)dh(e_a)$ and using the definition of f transforms the above expression into

$$\begin{aligned}dh(e_a)\;-\;\frac{1}{4}h^{-2}dh(e_a)|\nabla h|^2\;-\;\left(\frac{1}{2}\;+\;h\;+\;\frac{1}{4}h^{-1}|\nabla h|^2\right)h^{-1}dh(e_a) \\ +\;\frac{1}{2}h^{-1}\;\left(-\sum_{s=1}^{3}\nabla dh\,(I_se_a,I_s\nabla h)\;+\;\left(\,2\;-\;4h\;+3h^{-1}|\nabla h|^2\right)dh(e_a)\right) \\ +\;\nabla dh(I_1e_a,\xi_1)\;-\;\nabla dh(I_2e_a,\xi_2)-\;\nabla dh(I_3e_a,\xi_3)D_1(e_a)+\;h^{-1}\left(\sum_{s=1}^{3}dh(\xi_s)\,dh(I_se_a)\right).\end{aligned}$$

Simplifying the above expression shows that the coefficient of $D_1(e_a)$ in (4.8) is

$$\Big(-2 + \frac{1}{2}h^{-1} + h^{-2}|\nabla h|^2\Big)\, dh(e_a) - \frac{1}{2}h^{-1}\left(\sum_{s=1}^{3}\nabla dh\,(I_s e_a, I_s\nabla h)\right)$$
$$+ \nabla dh(I_1 e_a,\xi_1) - \nabla dh(I_2 e_a,\xi_2) - \nabla dh(I_3 e_a,\xi_3) + h^{-1}\left(\sum_{s=1}^{3} dh(\xi_s)\, dh(I_s e_a)\right)$$

Hence, we proved that the coefficient of $D_1(e_a)$ in (4.8) is $h\,(3\,A_1 - A_2 - A_3 + 2D_1)\,(e_a)$, while those of $A_1(e_a)$ is $h\big(\frac{22}{3}\,A_1 - \frac{2}{3}\,A_2 - \frac{2}{3}\,A_3 + \frac{11}{3}\,D_1 - \frac{1}{3}\,D_2 - \frac{1}{3}\,D_3\big)(e_a)$. A cyclic permutation gives the rest of the coefficients in (4.8). With this, the divergence (4.8) can be written in the form

$$\nabla^*\Big(fD + \sum_{s=1}^{3} dh(\xi_s)\,F_s + 4\sum_{s=1}^{3} dh(\xi_s) I_s A_s - \frac{10}{3}\sum_{s=1}^{3} dh(\xi_s)\,I_s A\Big)$$
$$= f|T^0|^2 + h\,\sigma_{1,2,3}\Big\{ g\,\big(D_1,\, 3A_1 - A_2 - A_3 + 2\,D_1\big)$$
$$+ g\,\Big(A_1,\, \frac{22}{3}\,A_1 - \frac{2}{3}\,A_2 - \frac{2}{3}\,A_3 + \frac{11}{3}\,D_1 - \frac{1}{3}\,D_2 - \frac{1}{3}\,D_3\Big)\Big\},$$

where $\sigma_{1,2,3}$ denotes the sum over all positive permutations of $(1,2,3)$. Let Q be equal to

$$Q := \begin{bmatrix} 2 & 0 & 0 & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} \\ 0 & 2 & 0 & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} \\ 0 & 0 & 2 & -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} \\ \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} \end{bmatrix}$$

so that

$$\nabla^*\Big(fD + \sum_{s=1}^{3} dh(\xi_s)\,F_s + 4dh(\xi_s)\,I_s A_s - \frac{10}{3}\sum_{s=1}^{3} dh(\xi_s)\,I_s A\Big) = f\,|T^0|^2 + h\,\langle QV,\, V\rangle,$$

with $V = (D_1, D_2, D_3, A_1, A_2, A_3)$. It is not hard to see that the eigenvalues of Q are given by

$$\{0,\quad 0,\quad 2\,(2+\sqrt{2}),\quad 2\,(2-\sqrt{2}),\quad 10,\quad 10\},$$

which shows that Q is a non-negative matrix. □

5. Proofs of the main theorems

The proofs rely on Theorem 4.4 and the following characterization of all qc-Einstein structures conformal to the standard qc structures on the Heisenberg group.

Theorem 5.1. [IMV, Theorem 1.2] *Let $\Theta = \frac{1}{2h}\tilde\Theta$ be a conformal deformation of the standard qc-structure $\tilde\Theta$ on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$. Then Θ is qc-Einstein if and only if, up to a left translation the function h is given by*

$$h = c\,\Big[\big(1 + \nu\,|q|^2\big)^2 + \nu^2\,(x^2 + y^2 + z^2)\Big], \tag{5.1}$$

where c and ν are any positive constants.

Consider first the case of the (seven dimensional) sphere.

5.1. **Proof of Theorem 1.1.** Integrating the divergence formula of Theorem 4.4 we see that according to the divergence theorem established in [IMV, Proposition 8.1] the integral of the left-hand side is zero. Thus, the right-hand side vanishes as well, which shows that the quaternionic contact structure η has vanishing torsion, i.e., it is also qc-Einstein according to Theorem 2.4.

Next we bring into consideration the 7-dimensional quaternionic Heisenberg group and the quaternionic Cayley transform as described in [IMV, Section 5.2]. The quaternionic Heisenberg group of dimension 7 is $\boldsymbol{G}\,(\mathbb{H}) \ = \mathbb{H} \times \operatorname{Im}\mathbb{H}$. The group law is given by $(q', \omega') \ = \ (q_o, \omega_o) \circ (q, \omega) \ = \ (q_o \ + \ q, \omega \ + \ \omega_o \ + \ 2 \ \operatorname{Im} q_o \, \bar{q})$, where $q, \ q_o \in \mathbb{H}$ and $\omega, \omega_o \in \operatorname{Im}\mathbb{H}$. The left-invariant orthonormal basis of the horizontal space is

$$T_1 \ = \ \frac{\partial}{\partial t_1} \ + 2x^1 \frac{\partial}{\partial x} + 2y^1 \frac{\partial}{\partial y} + 2z^1 \frac{\partial}{\partial z}\,, \quad X_1 \ = \ \frac{\partial}{\partial x_1} \ - 2t^1 \frac{\partial}{\partial x} - 2z^1 \frac{\partial}{\partial y} + 2y^1 \frac{\partial}{\partial z}$$
$$Y_1 \ = \ \frac{\partial}{\partial y_1} \ + 2z^1 \frac{\partial}{\partial x} - 2t^1 \frac{\partial}{\partial y} - 2x^1 \frac{\partial}{\partial z}\,, \quad Z_1 \ = \ \frac{\partial}{\partial z_1} \ - 2y^1 \frac{\partial}{\partial x} + 2x^1 \frac{\partial}{\partial y} - 2t^1 \frac{\partial}{\partial z}$$

using $q \ = \ t_1 \ + \ i\,x_1 \ + \ j\,y_1 \ + \ k\,z_1$ and $\omega \ = \ i\,x \ + \ j\,y \ + \ k\,z$. The central (vertical) orthonormal vector fields ξ_1, ξ_2, ξ_3 are described as follows

$$\xi_1 = 2\frac{\partial}{\partial x} \qquad \xi_2 = 2\frac{\partial}{\partial y} \qquad \xi_3 = 2\frac{\partial}{\partial z}\,.$$

Let us identify the (seven dimensional) group $\boldsymbol{G}\,(\mathbb{H})$ with the boundary Σ of a Siegel domain in $\mathbb{H} \times \mathbb{H}$,

$$\Sigma \ = \ \{(q', p') \in \mathbb{H} \times \mathbb{H} \ : \ \Re\, p' \ = \ |q'^2\}.$$

Σ carries a natural group structure and the map $(q, \omega) \ \mapsto \ (q, |q|^2 - \omega) \ \in \ \Sigma$ is an isomorphism between $\boldsymbol{G}\,(\mathbb{H})$ and Σ.

The standard contact form, written as a purely imaginary quaternion valued form, on $\boldsymbol{G}\,(\mathbb{H})$ is given by $2\tilde{\Theta} \ = \ (d\omega \ - \ q \cdot d\bar{q} \ + \ dq \cdot \bar{q})$, where $\cdot$ denotes the quaternion multiplication. Since $dp \ = \ q \cdot d\bar{q} \ + \ dq \cdot \bar{q} \ - \ d\omega$, under the identification of $\boldsymbol{G}\,(\mathbb{H})$ with Σ we also have $2\tilde{\Theta} \ = \ -dp' \ + \ 2dq' \cdot \bar{q}'$. Taking into account that $\tilde{\Theta}$ is purely imaginary, the last equation can be written also in the following form

$$4\,\tilde{\Theta} \ = \ (d\bar{p}' \ - \ dp') \ + \ 2dq' \cdot \bar{q}' \ - \ 2q' \cdot d\bar{q}'.$$

The (quaternionic) Cayley transform is the map $\mathcal{C} \ : \ S \setminus \{(-1, 0)\} \ \mapsto \ \Sigma$ from the sphere $S \ = \ \{(q, p) \in \mathbb{H} \times \mathbb{H} \ : \ |q|^2 + |p|^2 = 1\} \subset \mathbb{H} \times \mathbb{H}$ minus a point to the Heisenberg group $\Sigma = \{(q_1, p_1) \in \mathbb{H} \times \mathbb{H} \ : \ \Re\, p_1 \ = \ |q_1|^2\}$, with $\mathcal{C}$ defined by

$$(q_1, p_1) \ = \ \mathcal{C}\,\Big((q, p)\Big), \qquad q_1 \ = \ (1 + p)^{-1}\, q, \quad p_1 \ = \ (1 + p)^{-1}\,(1 - p). \tag{5.2}$$

with an inverse $(q, p) \ = \ \mathcal{C}^{-1}\Big((q_1, p_1)\Big)$ given by

$$q \ = \ 2\,(1 + p_1)^{-1}\, q_1, \quad p \ = \ (1 - p_1)\,(1 + p_1)^{-1}. \tag{5.3}$$

The Cayley transform is a conformal quaternionic contact diffeomorphism between the quaternionic Heisenberg group with its standard quaternionic contact structure $\tilde{\Theta}$ and $S \setminus \{(-1, 0)\}$ with its standard structure $\tilde{\eta}$, see [IMV],

$$\lambda \ \cdot (\mathcal{C}_*\, \tilde{\eta}) \ \cdot \bar{\lambda} \ = \ \frac{8}{|1 + p_1\,|^2}\, \tilde{\Theta}, \tag{5.4}$$

where $\lambda \ = \ \frac{1+p_1}{|1+p_1\,|}$ is a unit quaternion and $\tilde{\eta}$ is the standard quaternionic contact form on the sphere, $\tilde{\eta} \ = \ dq \cdot \bar{q} \ + \ dp \cdot \bar{p} \ - \ q \cdot d\bar{q} - \ p \cdot d\bar{p}$. Hence, up to a constant multiplicative factor and a quaternionic contact automorphism the forms $\mathcal{C}_*\tilde{\eta}$ and $\tilde{\Theta}$ are conformal to each other. It follows that the same is true for $\mathcal{C}_*\eta$ and $\tilde{\Theta}$. In addition, $\tilde{\Theta}$ is qc-Einstein by definition, while η and hence also $\mathcal{C}_*\eta$ are qc-Einstein as we observed at the beginning of the proof. According to Theorem 5.1, up to a multiplicative constant factor, the forms $\mathcal{C}_*\tilde{\eta}$ and $\mathcal{C}_*\eta$ are related by a translation or dilation on the Heisenberg group. Hence, we conclude that up to a multiplicative constant, η is obtained from

$\tilde\eta$ by a conformal quaternionic contact automorpism which proves the first claim of Theorem 1.1. From the conformal properties of the Cayley transform and [Va1, Va] it follows that the minimum $\lambda(S^{4n+3})$ is achieved by a smooth 3-contact form, which due to the Yamabe equation is of constant qc-scalar curvature. This shows the second claim of Theorem 1.1.

5.2. **Proof of Theorem 1.3.** Let $\mathcal{D}^{1,2}$ be the space of functions $u \in L^{2^*}(\boldsymbol{G}\,(\mathbb{H}))$ having distributional horizontal gradient $|\nabla u|^2 = |T_1 u|^2 + |X_1 u|^2 + |Y_1 u|^2 + |Z_1 u|^2 \in L^2(\boldsymbol{G}\,(\mathbb{H}))$ with respect to the Lebesgue measure dH on $\mathbb{R}^7$, which is the Haar measure on the group. Let us define the constant ($2^* = 5/2$ here)

$$\Lambda \overset{def}{=} inf \left\{ \int\limits_{\boldsymbol{G}\,(\mathbb{H})} |\nabla v|^2 \; dH \; : \; v \in \mathcal{D}^{1,2}, \; v \geq 0, \; \int\limits_{\boldsymbol{G}\,(\mathbb{H})} |v|^{2^*} \; dH \; = \; 1 \right\}.$$

Let v be a function for which the infimum is achieved. Note that such function exists by [Va1] or [Va]. Furthermore, $\Lambda = S_2^{-2}$, where S_2 is the best constant in the L^2 Folland-Stein inequality (1.1), since $v \in \mathcal{D}^{1,2}$ implies $|v| \in \mathcal{D}^{1,2}$ and the gradient is the same a.e.. From the choice of v we have

$$\Lambda \; = \int\limits_{\boldsymbol{G}\,(\mathbb{H})} |\nabla v|^2 \; dH, \qquad \int\limits_{\boldsymbol{G}\,(\mathbb{H})} v^{2^*} \; dH \; = \; 1.$$

Writing the Euler-Lagrange equation of the constrained problem we see that v is a non-negative entire solution of $\big(T_1^2 \; + \; X_1^2 \; + \; Y_1^2 \; + \; Z_1^2\big)v = -\Lambda \; v^{3/2}$. By [GV2, Lemma 10.2] (see [Va] or [Va1, Theorem 10.3] for further details) v is a bounded function. Similarly to [FSt, Theorem 16.7] it follows v is a Lipschits continuous function in the sense of non-isotropic Lipschits spaces [F]. Iterating this argument and using [F, Theorem 5.25] we see that v is a C^∞ smooth function on the set where it is positive, while being of class $\Gamma^{2,\beta}_{loc}$, the non-isotropic Lipschits space, for some $\beta > 0$. In particular v is continuously differentiable function by [F, Theorem 5.25]. Applying the Hopf lemma [GV1, Theorem 2.13] on the set where v is positive shows that v cannot vanish, i.e., it is a positive entire solution to the Yamabe equation. The positivity can also be seen by the Harnack inequality, see [W] for example. Let $u \overset{def}{=} \Lambda^{\frac{1}{2^*-2}} \, v$, then u is a positive entire solution of the Yamabe equation

$$\big(T_1^2 \; + \; X_1^2 \; + \; Y_1^2 \; + \; Z_1^2\big)\, u \; = \; -u^{3/2} \tag{5.5}$$

¿From the definition of u, we have

$$\Lambda \; = \; \Big(\int\limits_{\boldsymbol{G}\,(\mathbb{H})} |\nabla u|^2 \; dH \Big)^{\frac{1}{5}} \; = \; \Big(\int\limits_{\boldsymbol{G}\,(\mathbb{H})} u^{5/2} \; dH \Big)^{\frac{1}{5}}.$$

We shall compute the last integral by determining u with the help of the divergence formula.

As before, let $\tilde\Theta$ be the standard contact form on $\boldsymbol{G}\,(\mathbb{H})$ identified with Σ. Using the inversion and the Kelvin transform on $\boldsymbol{G}\,(\mathbb{H})$, cf. [GV2, Sections 8 and 9], we can see that if $\Theta = \frac{1}{2h}\tilde\Theta$ has constant scalar curvature, then the Cayley transform lifts the qc structure defined by Θ to a qc structure of constant qc-scalar curvature on the sphere, which is conformal to the standard. The details are as follows. Let us define two contact forms Θ_1 and Θ_2 on Σ setting

$$\Theta_1 \; = \; u^{4/(Q-2)}\tilde\Theta, \quad \text{and} \quad \Theta_2 \; = \; (\mathcal{K}u)^{4/(Q-2)} \; \frac{\bar p'}{|p'|} \; \tilde\Theta \; \frac{p'}{|p'|},$$

where u is as in (5.5), $\mathcal{K}u$ is its Kelvin transform, see (5.8) for the exact formula, and Q is the homogeneous dimension of the group. Notice that $\frac{\bar p'}{|p'|} \; \tilde\Theta \; \frac{p'}{|p'|}$ defines the same qc structure on the Heisenberg group as $\tilde\Theta$ and $\mathcal{K}u$ is a smooth function on the whole group according to [GV2, Theorem 9.2]. We are going to see that using the Cayley transform these two contact forms define a contact form on the sphere, which is conformal to the standard and has constant qc-scalar curvature.

Let $P_1 = (-1, 0)$ and $P_2 = (1, 0)$ be correspondingly the 'south' and 'north' poles of the unit sphere $S = \{|q|^2 + |p|^2 = 1\}$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the corresponding Cayley transforms defined, respectively, on $S \setminus \{P_1\}$ and $S \setminus \{P_2\}$. Note that $\mathcal{C}_1$ was defined in (5.2), while $\mathcal{C}_2$ is given by

$$(q_2, p_2) = \mathcal{C}_2 \Big((q, p)\Big), \qquad q_2 = -(1-p)^{-1}\, q, \quad p_2 = (1-p)^{-1}\,(1+p). \tag{5.6}$$

In order that Θ_1 and Θ_2 define a contact form η on the sphere it is enough to see that

$$\Theta_1(p, q) = \Theta_2 \circ \mathcal{C}_2 \circ \mathcal{C}_1^{-1}(p, q), \quad \text{i.e.,} \quad \Theta_1 = (\mathcal{C}_2 \circ \mathcal{C}_1^{-1})^* \Theta_2. \tag{5.7}$$

A calculation shows that $\mathcal{C}_2 \circ \mathcal{C}_1^{-1} : \Sigma \to \Sigma$ is given by

$$q_2 = -p_1^{-1}\, q_1, \qquad p_2 = p_1^{-1},$$

or, equivalently, in the model $\boldsymbol{G}\,(\mathbb{H})$

$$q_2 = -(|q_1|^2 - \omega_1)^{-1}\, q_1, \qquad \omega_2 = -\frac{\omega_1}{|q_1|^4 + |\omega_1|^2}.$$

Hence, $\sigma = \mathcal{C}_2 \circ \mathcal{C}_1^{-1}$ is an involution on the group. Furthermore, with the help of (5.4) we calculate

$$\mathcal{C}_{1*} \circ \mathcal{C}_2{}^*\ \Theta = \frac{1}{|p_1|^2}\, \bar{\mu}\, \Theta\, \mu, \qquad \mu = \frac{p_1}{|p_1|},$$

which proves the identity (5.7). Using the properties of the Kelvin transform, [GV2, Sections 8 and 9],

$$(\mathcal{K}u)\,(q', p') \overset{def}{=} |p'^{-(Q-2)/2}\, u\big(\sigma(q', p')\big), \tag{5.8}$$

we see that u and $\mathcal{K}u$ are solutions of the Yamabe equation (5.5). This implies that the contact form η has constant qc-scalar curvature, equal to $\frac{4(Q+2)}{Q-2}$.

Notice that η is conformal to the standard form $\tilde{\eta}$ and the arguments in the preceding proof imply then that η is qc-Einstein. A small calculation shows that this is equivalent to the fact that if we set

$$\bar{u} = 2^{10}\ [(1 + |q|^2)^2\ +\ |\omega|^2]^{-2}, \tag{5.9}$$

then $\bar{u}$ satisfies the Yamabe equation (5.5) and all other nonnegative solutions of (5.5) in the space $\mathcal{D}^{1,2}$ are obtained from $\bar{u}$ by translations and dilations,

$$\tau_{(q_o, \omega_o)} \bar{u}\ (q, \omega) \overset{def}{=} \bar{u}(q_o + q, \omega + \omega_o), \tag{5.10}$$

$$\bar{u}_\lambda\,(q) \overset{def}{=} \lambda^4 \bar{u}(\lambda q, \lambda^2 \omega), \qquad \lambda > 0. \tag{5.11}$$

Thus, u which was defined in the beginning of the proof is given by equation (5.9) up to translations and dilations. This allows the calculation of the best constant in the Folland-Stein inequality, see [GV1, (4.52)],

$$\Lambda^5 = \int\limits_{\boldsymbol{G}\,(\mathbb{H})} \frac{2^{25}}{[(1+|q|^2)^2\ +\ |\omega|^2)]^5}\, dH = 2^{25}\, \pi^{7/2} \frac{\Gamma(\frac{7}{2})}{\Gamma(7)} = \frac{\pi^{12/10}}{12},$$

where Γ is the Gamma function. Hence

$$S_2 = \Lambda^{-1/2} = \frac{2\sqrt{3}}{\pi^{3/5}}.$$

Recalling the relation between u and v we find that the extremals in the Folland-Stein embedding are given by

$$v = \frac{2^{11}\sqrt{3}}{\pi^{3/5}} [(1+|q|^2)^2\ +\ |\omega|^2]^{-2}$$

and its translations and dilations. The proof of Theorem 1.3 is complete.

References

[AK] Alekseevsky, D. & Kamishima, Y., *Pseudo-conformal quaternionic CR structure on* $(4n+3)$*-dimensional manifold*, math.GT/0502531. 2

[Biq1] Biquard, O., *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **265** (2000). 1, 2, 4, 5, 6, 7, 9

[Biq2] Biquard, O., *Quaternionic contact structures*, Quaternionic structures in mathematics and physics (Rome, 1999), 23–30 (electronic), Univ. Studi Roma "La Sapienza", Roma, 1999. 1, 2

[CSal] Capria, M. & Salamon, S., *Yang-Mills fields on quaternionic spaces* Nonlinearity **1** (1988), no. 4, 517–530. 5

[D] Duchemin, D., *Quaternionic contact structures in dimension 7*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 4, 851–885. 5, 6

[D1] ———, *Quaternionic contact hypersurfaces*, math.DG/0604147. 2

[F] Folland, G., *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Math., **13** (1975), 161–207. 20

[FSt] Folland, G. B. & Stein, E. M., *Estimates for the* $\bar\partial_b$ *Complex and Analysis on the Heisenberg Group*, Comm. Pure Appl. Math., **27** (1974), 429–522. 3, 20

[GV1] Garofalo, N. & Vassilev, D., *Symmetry properties of positive entire solutions of Yamabe type equations on groups of Heisenberg type*, Duke Math J, **106** (2001), no. 3, 411–449. 3, 4, 20, 21

[GV2] ———, *Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups*, Math Ann., **318** (2000), no. 3, 453–516. 20, 21

[IMV] Ivanov, St., Minchev, I., & Vassilev, D., *Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem*, preprint, math.DG/0611658. 2, 3, 4, 6, 7, 8, 9, 11, 15, 18, 19

[IV] Ivanov, St. & Vassilev, D., *Conformal quaternionic contact curvature and the local sphere theorem*, preprint, MPIM2007-79, arXiv:0707.1289, 2007. 2

[JL1] Jerison, D., & Lee, J., *The Yamabe problem on CR manifolds*, J. Diff. Geom., **25** (1987), 167–197. 2

[JL2] ———, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc., **1** (1988), no. 1, 1–13. 3

[LP] Lee, J. M. & Parker, T., *The Yamabe Problem*, Bull Am. Math. Soc. **17** (1987), no. 1, 37–91. 3

[M] Mostow, G. D., *Strong rigidity of locally symmetric spaces*, Annals of Mathematics Studies, No. 78. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973. v+195 pp. 1

[Ob] Obata, M., *The conjecture of conformal transformations in Riemannian manifolds*, J. Diff. Geom., **6** (1971), 247–258. 3

[P] Pansu, P., *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. (2) 129 (1989), no. 1, 1–60. 1

[Va] Vassilev, D, *Yamabe type equations on Carnot groups*, Ph. D. thesis Purdue University, 2000. 3, 20

[Va1] Vassilev, D., *Regularity near the characteristic boundary for sub-laplacian operators*, Pacific J Math, **227** (2006), no. 2, 361–397. 3, 20

[W] Wang, W., *The Yamabe problem on quaternionic contact manifolds*, Ann. Mat. Pura Appl., **186** (2007), no. 2, 359–380.

2, 20

(Stefan Ivanov) University of Sofia, Faculty of Mathematics and Informatics, blvd. James Bourchier 5, 1164, Sofia, Bulgaria

E-mail address: ivanovsp@fmi.uni-sofia.bg

(Ivan Minchev) University of Sofia, Sofia, Bulgaria, and Institut für Mathematik, Humboldt Universität zu Berlin, Unter den Linden 6, Berlin D-10099, Germany

E-mail address: minchevim@yahoo.com

E-mail address: minchev@fmi.uni-sofia.bg

(Dimiter Vassilev) Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87131-0001, and, University of California, Riverside, Riverside, CA 92521

E-mail address: vassilev@math.unm.edu