COMPLEX VARIABLES

SYLLABUS FOR MASTER'S-PHD QUALIFYING EXAM

July 1996

Partial list of topics:

1. Complex numbers
2. Limits, continuity and analyticity of complex functions
3. Exponential and trigonometric functions; logarithms
4. Elementary conformal mappings
5. Analytic continuation
6. Power series; Taylor and Laurent expansions
7. Complex integration (Cauchy theorems)
8. Residue theory
9. Singularities
10. Harmonic functions
11. Maximum principle
12. Reflection principle
13. Meromorphic functions
14. Mittag-Leffler theorem
15. Canonical products and Weierstrass theorem

SUGGESTED REFERENCES

*1. L.V. Ahlfors, *Complex Analysis,* 3rd edition.

*2. K. Knopp, *Theory of Functions,* Vols. 1 and 2.

3. E. Hille, *Analytic Function Theory,* Vol.1.

4. G.F. Carrier, M. Krook and C.E. Pearson, *Functions of a Complex Variable.*

Complex Analysis Qualifying Examination

January 2009

Instructions: Please do the **eight** problems listed below. You may choose to answer the problems in any order. However, to help us in grading your exam please make sure to:
i. Start each question on a new sheet of paper.
ii. Write **only on one side** of each sheet of paper.
iii. Number each page and write the last four digits of your **UNM ID #** on each page.

1. (a) Show that $f(z) = -\dfrac{i}{2\cos z}$ and $g(z) = \dfrac{\sin z}{1 + e^{i2z}}$ have the same poles and principal parts.
   (b) Find an entire function $\varphi(z)$ such that $\varphi(z) = -\left[\dfrac{i}{2\cos z} + \dfrac{\sin z}{1 + e^{i2z}}\right]$.

2. Find the Laurent expansion for $f(z) = \dfrac{z^3 + 2z - 1}{z^2 - 1}$ valid in the domain D, $D = \{z \in \mathbb{C} : |z| > 1\}$.

3. (a) If f is analytic on $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, show that f is constant.
   (b) If f is analytic on $\mathbb{C}$ and satisfies $\max\{|f(z)| : |z| = r\} \leq Mr^n$ for a fixed $M > 0$, $n > 0$, and a sequence of values $r = r_k$, with $r_k \to \infty$ as $k \to \infty$, show that f is a polynomial of degree less or equal to n.

4. (a) State Rouche's theorem.
   (b) If $f(z) = 4z^4 + 13z^2 + 3$, find the number of zeros of $f(z)$ inside the circle $\{z : |z| = 1\}$ and the number of zeros inside the annulus $\{z : 1 < |z| < 2\}$.

5. Use the residue theorem to evaluate
$$\int_0^{2\pi} \frac{d\theta}{4 + 3\cos\theta}.$$

6. Prove in complete detail: If f is analytic on an open set containing $\{z : \text{Im } z \leq 0\}$ except for a finite number of singularities (none on the real axis), and $\lim_{z\to\infty} f(z) = 0$ (with $\text{Im } z \leq 0$), and if $m < 0$, then
$$\lim_{R\to\infty} \int_{-R}^{R} f(x)e^{imx}\, dx = -2\pi i \sum_k \text{res } (f(z)e^{imz}, z = z_k)$$
   where z_k are the singularities of f in the lower half-plane.

7. (a) Where is the function $w = \cos z$ conformal?
   (b) Find the image of the domain $D = \{z = x + iy : -\pi/2 < x < \pi/2, 0 < y < \infty\}$ under the map $w = \cos z$.

8. Assume that $\{f_n(z)\}_{n=1}^{\infty}$ is a sequence of entire functions that converges to $f(z)$ uniformly on compact sets. Prove that $f(z)$ is entire and that for any $z_0 \in \mathbb{C}$ and $k \in \mathbb{N}$, $f_n^{(k)}(z_0) \to f^{(k)}(z_0)$ as $n \to \infty$

Complex Variables Fall 2008
MS/PhD Qualifying Examination

Instruction: Complete all problems.

1a) Calculate $w = (\sqrt{3}+i)^6$.
1b) Calculate the principle value of $w = (1+i)^{4i}$ and write it in the form

$$w = r(\cos\alpha + i\sin\alpha), \quad r > 0, \quad \alpha \in \mathbb{R} \ .$$

2) Let

$$f(z) = \frac{e^z - 1}{z^4}, \quad z \neq 0 \ .$$

a) Evaluate

$$\int_\Gamma f(z)\, dz$$

when Γ is the positively oriented circle $|z| = 2$.
b) Determine the Laurent expansion of $f(z)$ valid for $z \neq 0$.

3) Find the Laurent expansion of

$$f(z) = \frac{5z}{(z+2)(z-3)}$$

which is valid for

$$2 < |z| < 3 \ .$$

4) Let $f(z)$ denote a function which is holomorphic in an open connected set U and for which $|f(z)|$ is constant in U. Can one conclude that $f(z)$ is constant in U? Justify your answer.

5) State Rouché's theorem and use it to prove that the equation

$$az^n = e^z$$

has n zeros (counted according to multiplicity) in $|z| < 1$ if

$$a \in \mathbb{C}, \quad |a| > e, \quad n \in \{1, 2, \ldots\} \ .$$

6) Let Γ denote the positively oriented unit circle. Evaluate

$$\int_\Gamma \frac{e^z}{z}\, dz$$

and use the result to evaluate

$$\int_0^{2\pi} e^{\cos t} \cos(\sin t)\, dt \quad \text{and} \quad \int_0^{2\pi} e^{\cos t} \sin(\sin t)\, dt \ .$$

7) Let $f(z)$ denote an entire function and assume that $f''(z)$ is bounded. Can you prove that $f(z)$ has the form

$$f(z) = az^2 + bz + c \ ?$$

8) Recall that the Γ–function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

for suitable complex numbers z. a) In which z–region does the above formula define $\Gamma(z)$ as an analytic function? Justify your answer.

b) In which region of the complex plane can one continue Γ as an analytic function? Justify your answer and construct an analytic continuation of the function defined by the integral.

Complex Variables Spring 2008
MS/PhD Qualifying Examination

Instruction: Complete all problems.

1) Let $f(z) = \log(z)$ denote the main branch of the complex logarithm, which is defined and holomorphic in

$$\mathbb{C} \setminus (-\infty, 0]\ .$$

In which domain is the function

$$g(z) = \log(\log(z))$$

holomorphic?

2) Let

$$U = \{z \in \mathbb{C}\ :\ 0 < |z| < 1\}$$

denote the open unit disk with the origin removed and let f denote a holomorphic function on U which has a pole of order three at the origin. Prove or disprove the following statements:

a) There is a constant $C > 0$ with

$$|f(z)| \le \frac{C}{|z|^3} \quad \text{for} \quad 0 < |z| \le \frac{1}{2}\ .$$

b) There is a constant $c > 0$ with

$$|f(z)| \ge \frac{c}{|z|^3} \quad \text{for} \quad 0 < |z| \le \frac{1}{2}\ .$$

c) There is a constant $C > 0$ with

$$|f(z)| \le \frac{C}{|z|^3} \quad \text{for} \quad 0 < |z| < 1\ .$$

3) Let f and g be holomorphic functions defined for all z with $|z - c| < r$. Assume that

$$g(c) = g'(c) = 0, \quad g''(c) \neq 0\ .$$

Show that the residue of the function

$$h(z) = \frac{f(z)}{g(z)}, \quad 0 < |z - c| < \varepsilon\ ,$$

is given by

$$\frac{n_1 f'(c) g''(c) + n_2 f(c) g'''(c)}{n_3 (g''(c))^2}$$

where n_1, n_2, n_3 are integers. Determine the integers n_j.

4) Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 2x + 2}\, dx \; .$$

5) Show that for any positive integer n all roots of

$$(1+z)^n + z^n = 0$$

lie on the line $x = -\frac{1}{2}$.

6) Let

$$f(0) = 0 \quad \text{and} \quad f(x+iy) = u(x,y) + iv(x,y) \quad \text{for} \quad z = x + iy \neq 0$$

where

$$u(x,y) = \frac{x^3 - y^3}{x^2 + y^2}, \quad v(x,y) = \frac{x^3 + y^3}{x^2 + y^2} \; .$$

a) Are the Cauchy–Riemann equations satisfied at $z = 0$?
b) Does the complex derivative $f'(0)$ exist?

7) Evaluate

$$\int_0^{2\pi} \frac{dt}{5 + 4\sin t} \; .$$

8) Determine the number of roots of the equation

$$z^3 - z^2 + 3z + 5 = 0$$

in the open right half–plane ($\mathrm{Re}\, z > 0$).

Complex Analysis Qualifying Exam

January 2006

Directions: Do all of the following problems. Show all of your work, and justify all of your calculations.

1. Let

$$f(z) := \frac{\mathrm{e}^z}{(z-1)^4}.$$

(a) Classify *all* of the singularities and find the associated residues.

(b) Determine the Laurent expansion of f centered at $z = 1$.

(c) If $\mathcal{C}$ denotes the positively oriented circle of radius 2 centered at $z = 0$, evaluate

$$\oint_{\mathcal{C}} f(z)\,\mathrm{d}z.$$

2. Let

$$\Pi_{\mathrm{u}} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

Find a conformal mapping $f : \Pi_{\mathrm{u}} \mapsto D(0,3)$ such that $f(3+2\mathrm{i}) = 0$.

3. Let

$$f(z) := \frac{1}{(z-1)(z-2)}.$$

Write $f(z)$ as a Laurent series centered at $z = 0$ which converges on the annulus $1 < |z| < 2$.

4. Let $f : D(0,1) \mapsto D(0,1)$ be holomorphic and satisfy $f(0) = 0$. What does Schwarz' lemma say about f? Prove it.

5. For each $n \in \mathbb{N}$ set

$$p_n(z) := \sum_{j=0}^{n} (-1)^j \frac{z^{-2j}}{j!}.$$

(a) For each fixed $n \in \mathbb{N}$, show that $p_n(z) = 0$ has precisely $2n$ solutions.

(b) For a given $\rho > 0$, show that there is an $N(\rho)$ such that if $n > N(\rho)$, then all of the zeros of $p_n(z)$ lie within $D(0,\rho)$.

6. Show that

$$\int_{-\infty}^{+\infty} \operatorname{sech}^2(x)\cos(2x)\,\mathrm{d}x = \frac{2\pi}{\sinh(\pi)}.$$

7. Let

$$f(z) := \frac{\sin(z^{1/2})}{z^{1/2}}.$$

(a) Show that $f(z)$ is entire.

(b) Let $p_n(z)$ be a polynomial of order $n \geq 1$. For each $A \in \mathbb{C}$ show that there exist an infinite number of distinct solutions to

$$p_n(z)f(z) = A.$$

8. Consider

$$f(z) := \prod_{j=1}^{\infty} \left(1 - \frac{z}{j^3}\right).$$

(a) Show that $f(z)$ is entire.

(b) What is the order of $f(z)$?

Complex Variables Fall 2006
MS/PhD Qualifying Examination

Instruction: Complete all problems.

1a) State Rouché's theorem and use it to show that all zeros of the polynomial

$$p(z) = z^4 + 6z + 3$$

lie in the circle $|z| < 2$.

b) How many zeros of $p(z)$ lie in the annulus $1 < |z| < 2$?

2) Classify the singularities in $\mathbb{C}$ of the functions

$$f(z) = \frac{z - \sin z}{z^4}$$

and

$$g(z) = \frac{1}{z^2(z+1)} + \sin\left(\frac{1}{z}\right) .$$

3) Let

$$f(z) = \frac{1}{z^2(e^z - e^{-z})}, \quad 0 < |z| < \pi .$$

Compute the first three non–zero terms of the Laurent expansion of $f(z)$ in $0 < |z| < \pi$.

4) Let $f(z)$ and $g(z)$ be entire functions satisfying

$$|f(z)| \leq 10|g(z)| \quad \text{for all} \quad z \in \mathbb{C} .$$

Does it follow that there exists $\lambda \in \mathbb{C}$ with

$$f(z) = \lambda g(z) \quad \text{for all} \quad z \in \mathbb{C} \ ?$$

Give a proof or a counterexample.

5) Let $f(z)$ be an entire function for which the real part

$$\operatorname{Re} f(x + iy) = u(x, y)$$

is a bounded function. Does it follow that $f(z)$ is a constant function? Give a proof or a counterexample.

6) Evaluate

$$\int_0^{\pi} \frac{dt}{5 + 4\cos t} .$$

7) Evaluate

$$\int_0^{\infty} \frac{x \sin x}{(x^2+1)(x^2+4)}\, dx .$$

8) Let $a \in \mathbb{C}$, $|a| > 1$, and let $f(t)$ denote the 2π–periodic function

$$f(t) = \frac{1}{a + e^{it}}, \quad t \in \mathbb{R} .$$

Write $f(t)$ as a Fourier series,

$$f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikt} .$$

Determine the Fourier coefficients.

Hint: Use the geometric sum formula.

Complex Analysis Qualifying Exam

August 2005

Directions: Do all of the following problems. Show all of your work, and justify all of your calculations.

1. Classify *all* of the singularities and find the associated residues for each of the following functions:

(a) $\dfrac{(z+3)^2}{z}$

(b) $\dfrac{\mathrm{e}^{-z}}{(z-1)(z+2)^2}$.

2. Consider

$$f(z) := \sqrt{(z-x_1)(z-x_2)},$$

where $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$. Upon writing

$$z - x_j = r_j \mathrm{e}^{\mathrm{i}\theta_j},$$

if one supposes that

$$0 \le \theta_1 < 2\pi, \quad -\pi \le \theta_2 < \pi,$$

determine the branch points and branch cuts for $f(z)$.

3. Show that

$$\int_{-\infty}^{+\infty} \frac{\cos x - \cos a}{x^2 - a^2}\,\mathrm{d}x = -\pi\frac{\sin a}{a}, \quad a \in \mathbb{R}^+.$$

4. Recall that a linear fractional transformation (LFT) is of the form

$$\ell(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0.$$

(a) Find a LFT which maps the upper half-plane onto itself and which satisfies $\ell(0) = 1$, $\ell(\mathrm{i}) = 2\mathrm{i}$.

(b) Suppose that an LFT $\ell(z)$ has two distinct and finite fixed points α and β. Show that there is a constant $C \in \mathbb{C}$ such that

$$\frac{\ell(z)-\alpha}{\ell(z)-\beta} = C\frac{z-\alpha}{z-\beta}.$$

5. Let $f : D(P,r)\backslash\{P\} \mapsto \mathbb{C}$ be holomorphic, and suppose that f has an essential singularity at $z = P$. Show that there exists a sequence $\{z_j\} \subset D(P,r)\backslash\{P\}$ with $z_j \to P$ such that for each $j \in \mathbb{N}$,

$$|(z_j - P)^j f(z_j)| \ge j.$$

6. State some version of Rouche's theorem, and then use it to show that all of the zeros for

$$f(z) := z^8 - 4z^3 + 10$$

lie in the annulus $D(0,2)\backslash\overline{D}(0,1)$.

7. Set

$$h(z) := \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \mathrm{e}^{-z/n}.$$

(a) Show that $h(z)$ is an entire function.

(b) The gamma function, $\Gamma(z)$, is nonzero and has simple poles at $z = 0, -1, -2, \ldots$. Show that there exists an entire function $g(z)$ such that

$$\frac{1}{\Gamma(z)} = z\mathrm{e}^{g(z)} h(z).$$

8. For each $n \in \mathbb{N}$ consider the polynomial

$$P_n(z) := 1 + z + z^2 + \cdots + z^n.$$

(a) For any given $0 < \rho < 1$, show that $P_n(z)$ has no zeros in $D(0, \rho)$ for n sufficiently large.

(b) Show that all of the zeros of $P_n(z)$ lie on $\partial D(0, 1)$.

Complex Analysis Qualifying Exam

January 2005

Directions: Do all of the following problems. Show all of your work, and justify all of your calculations.

1. Show that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

2. Classify *all* of the singularities and find the associated residues for each of the following functions:

(a) $\dfrac{\sin^2 z}{z^4}$

(b) $\dfrac{z^2 + 2z - 1}{(z-1)(z+3)^2}$.

3. Let $p(z)$ be a polynomial. Show that that there exists an infinite sequence $\{z_j\} \subset \mathbb{C}$ such that $p(z_j) = \mathrm{e}^{z_j}$ for each j.

4. Let $f : D(0,1) \mapsto D(0,1)$ be a conformal map.

(a) If $f(0) = 0$, show that $f(z) = \omega z$ for some $\omega \in \partial D(0,1)$.

(b) If $f(0) \neq 0$, show that there exists an $a \in D(0,1)$ and $\omega \in \partial D(0,1)$ such that

$$f(z) = \omega \frac{z - a}{1 - \bar{a} z}.$$

5. State some version of Rouche's theorem, and then use it to show that

$$f(z) := z\mathrm{e}^{3-z} - 1$$

has only one real zero in $D(0,1)$.

6. Consider

$$f(z) := \sum_{n=1}^{\infty} \mathrm{e}^{-n} \sin(nz).$$

It is clear that $f(0) = 0$. Determine the largest subset $S \subset \mathbb{C}$ for which $\{0\} \subset S$ and $f(z)$ is analytic for all $z \in S$.

7. Show that there exists an entire function $g(z)$ such that

$$\sin \pi z = z\mathrm{e}^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

8. Let $\gamma \subset \mathbb{C}$ be the square centered at $z = 0$ with vertices at $z = \pm 2 \pm 2\mathrm{i}$. Compute

$$\oint_{\gamma} \frac{z}{z^3 + 1} \, \mathrm{d}z,$$

where γ is traversed once in the counterclockwise direction.

Complex Analysis Qualifying Exam

August 2004

Directions: Do all of the following problems. Show all of your work, and justify all of your calculations.

1. Evaluate

(a) $\displaystyle\int_0^{\infty} \mathrm{e}^{-x^2} \cos \lambda x \, \mathrm{d}x$

(b) $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{1+\cos^2\theta}$

It may be helpful to know that

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2} \, \mathrm{d}x = \sqrt{\pi}.$$

2. Classify *all* of the singularities and find the associated residues for each of the following functions:

(a) $z \cos 2z$

(b) $\displaystyle\frac{z^3 - z^2 + 2}{z-1}$.

3. Let $f : \mathbb{C} \mapsto \mathbb{C}$ be entire, and set $g(z) := f(1/z)$. Prove that f is a polynomial if and only if $g(z)$ has a pole at $z = 0$.

4. Let $U \subset \mathbb{C}$ be open, and let $f \in C^0(U)$. For each $r \in \mathbb{R}^+$ set $D(0,r) := \{z \in \mathbb{C} : |z| < r\}$. Suppose that for every $\overline{D}(z_0, r) \subset U$ and for all $z \in \overline{D}(z_0, r)$ one has that

$$f(z) = \oint_{\partial D(z_0,r)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

(the curve $\partial D(z_0, r)$ is followed in the counterclockwise direction). Prove that f is analytic.

5. Set

$$f(z) := \ln(z + (z^2-1)^{1/2}).$$

The branch cut for $(z^2-1)^{1/2}$ is to be on the axis $\operatorname{Im} z = 0$ with $-1 \le \operatorname{Re} z \le 1$.

(a) Show that $(z^2-1)^{1/2}$ is analytic at all values of $z \in \mathbb{C}$ not on the branch cut.

(b) For the transformation $w := z + (z^2-1)^{1/2}$ determine the branch cut for $f(w)$.

6. Let $D := D(0,1)$, and suppose that $f : D \mapsto \mathbb{C}$ is analytic, and further suppose that f is continuous on $\overline{D}$. Assume that $f(z) \neq 0$ for all $z \in \overline{D}$, and that $|f(z)| = 1$ for $z \in \partial D$. Show that $f(z) = \mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in [0, 2\pi)$.

7. For each $n \in \mathbb{N}$ set

$$f_n(z) := \sum_{j=1}^{n} \frac{z^{-j}}{j!}.$$

For a given $\rho > 0$, show that there is an $N(\rho)$ such that if $n > N(\rho)$, then all of the zeros of $f_n(z)$ lie within $D(0, \rho)$.

8. Let

$$f(z) := \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

(a) Compute the order of $f(z)$.

(b) Write the Hadamard product expansion of $f(z)$.

Complex Analysis Qualifying Examination

January 2004 SS#: ____________

Instructions: Please do **any eight** out of the nine problems listed below. You may choose to answer the problems in any order. However, to help us in grading your exam please make sure to:
i. Start each question on a new sheet of paper.
ii. Write **only on one side** of each sheet of paper.
iii. Number each page and write your SS# in each page.
Good luck!

1. If $f(z)$ is analytic in the unit disc $D = \{z \in C \mid |z| < 1\}$, prove that there is an analytic function $F(z)$ in D with $F'(z) = f(z)$.
   You can quote the Cauchy-Goursat theorem.

2. (a) Find the Möbius transformation that sends the points $z = 0, \infty, i$ into $w = -1, 1, i$ respectively.
   (b) Find the image of the first quadrant under the transformation found in part (a).

3. Find the Laurent expansion of $f(z) = (1 - z^2)e^{1/z}$ around $z = 0$. Determine its annulus of convergence and the residue of $f(z)$ at $z = 0$.

4. Compute the integral $I = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\, dx$. Carefully justify all your steps.

5. Assume that $f(z)$ is an entire function with $\displaystyle\lim_{|z| \to \infty} \frac{f(z)}{z^2} = 0$. Prove that $f(z)$ must be linear, that is $f(z) = a + bz$, with $a, b \in C$. Please provide all the details.

6. Assume that $\sqrt{z}$ is given in terms of its principal branch, $0 \le \arg(z) < 2\pi$. Find the image of the half-disc $D^+ = \{(x, y) \in C \mid x^2 + y^2 < 1,\ 0 < y < \infty\}$ under the map $w = \dfrac{1}{2}\left(\sqrt{z} + \dfrac{1}{\sqrt{z}}\right)$.

7. Show that if $f(z)$ is an entire function that never vanishes, then there is an entire function $g(z)$ so that $f(z) = e^{g(z)}$. Please provide all the details.

8. (a) Show that $u(x, y) = (x + 1)y$ is harmonic in the entire plane.
   (b) Find a harmonic conjugate $v(x, y)$ of $u(x, y)$.
   (c) Give explicitly an analytic function $w = f(z)$ with $u = \text{Re} f$ and $v = \text{Im} f$.

9. Prove the following version of the Schwarz reflection principle: if $f(z)$ is analytic in the right plane $\text{Re} z > 0$, continuous in $\text{Re} z \ge 0$, and $f(z)$ is purely imaginary on the imaginary axis, $\text{Re} f(iy) = 0$, then $f(z)$ can be extended to an analytic function on the entire plane. Please provide all the details.

Complex Analysis
Qualifying Examination

Summer 2003

Instructions: There are eight (8) questions on this examination, and each question is worth 25 points. A maximum score of 200 points is possible.

1. Determine the location and type of singularities of the following functions:
   (i) $\cosh^2(\frac{1}{z-\pi})$
   (ii) $z + z^3$
   (iii) $(cosz - sinz)^{-1}$
   (iv) $\tan(\frac{1}{z})$

2. Expand in Laurent series in region indicated:
   $\exp(\frac{1}{z-1})$, $|z| > 1$

3. Let $f(z) = z^{10} + \frac{1}{2}z^6 + \frac{1}{100}\exp(z^5)$
   (i) State Rouche's theorem.
   (ii) Show that $f(z)$ has no zeros on $|z| = 1$.
   (iii) How many zeros does $f(z)$ have inside $|z| = 1$? Justify your answer.

4. Assume $f(z)$ is analytic inside and on a contour C and b is a complex number such that $f(z) = b$ for some z on C. What is the significance of

$$\frac{1}{2\pi i}\int_C \frac{f' dz}{f - b}$$

5. State and prove Liouville's theorem.

√6. Given

$$f(z) = \frac{1}{z} - \frac{2}{z^2},$$

find

$$\int_C z^2 \exp(\frac{1}{z}) f(z) dz$$

where C is the unit circle traversed counterclockwise.

7. Evaluate

$$\int_0^\infty \frac{\sqrt{(x)}}{(1+x)^3}\,dx$$

by contour integration.

8. Evaluate

$$\int_{-\infty}^\infty \frac{\exp(ikx)}{(1+x^2)}\,dx$$

by contour integration (assume $k > 0$, real).

Complex Analysis Qualifying Exam
January 16, 2003

Instructions: *There are nine problems, please do eight of them. Start each problem on a new sheet of paper and write on one side of each sheet of paper. Remember to write the last four digits of your Social Security number in all pages and to clearly number them. Good luck!!*

1. (a) Let the roots of $(z-1)^n + z^n = 0$ be denoted by z_k, $k = 0, 1, \ldots, n-1$. Show that all the roots z_k lie on the line $x = 1/2$.

(b) Let w be any n-th root of the unity not equal to 1. Find the sum

$$1 + 2w + 3w^2 + 4w^3 \cdots + nw^{n-1}.$$

2. (a) Show that the only conformal maps from the complex plane onto itself (bijections!) are the non-constant linear maps, i.e. maps of the form $f(z) = az + b$, $a \neq 0$.

(b) Show that the only conformal maps from the unit disc onto itself (bijections!) are the Moebius transformations of the form

$$Tz = e^{i\alpha}\left(\frac{z-a}{1-\bar{a}z}\right),$$

where $\alpha \in \mathbf{R}$ and $|a| < 1$.

3. (a) Classify the singularities of the function

$$f(z) = \frac{z}{\sin z}.$$

Include the point at infinity.

(b) Find a Laurent expansion, valid in the region $|z+1| > 3$, for

$$f(z) = \frac{7z-2}{z^3 - z^2 - 2z}.$$

Find the residue of $f(z)$ at $z = 0$.

4. Find the sum of the series

$$f(z) = \sum_{n=0}^{\infty} e^{-n}\sin(nz)$$

and indicate the domain of convergence. Find the domain of analyticity of $f(z)$, and calculate $f'(z)$.

5. Let f be analytic in $\mathbf{C}$. Assume $\max\{|f(z)| : |z| = r\} \leq Mr^n$ for a fixed constant $M > 0$, and a sequence of values of r going to infinity. Show that f is a polynomial of degree less than or equal to n.

6. Evaluate the following integrals:

(a) $$\oint_\gamma \frac{e^z}{(z+1)(z-2i+1)} dz,$$

where γ is the ellipse given by $\frac{x^2}{4} + y^2 = 1$, with positive orientation (counterclockwise!).

(b) $$\int_0^\infty \frac{\ln x}{4+x^2}\, dx.$$

7. Give two distinct harmonic functions on $\mathbf{C}$ that vanish on the entire real axis. Why is this not possible for analytic functions?

8. Prove that if f is analytic on the region U (open and simply connected), $z_0 \in U$, and $f'(z_0) = 0$, then f is not one-to-one in any neighborhood of z_0.

9. (a) State the Mittag-Leffler Theorem.

(b) Show that

$$\pi \tan(\pi z) = \sum_{n=1}^{\infty} \frac{2\,z}{\left(\frac{2n-1}{2}\right)^2 - z^2}.$$

Complex Analysis Qualifying Examination

August 2002 SS # : ____________________

Directions:

Do the following 8 problems. You may choose to answer the problems in any order. Please start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages and *write your SS #* in each page. Please show all your work and explain all steps in a proof or derivation.

Questions:

1. a) Three points, z_1, z_2, and z_3 satisfy the conditions

$$z_1 + z_2 + z_3 = 0, \text{ and } |z_i| = 1, \quad 1 \leq i \leq 3.$$

Show that these points lie at the vertices of an equilateral triangle inscribed in the unit circle.

b) Give a generalization (without proof) of the result in part (a) for the case of n points.

2. a) Give a precise statement of the Cauchy Integral Formula.
b) Give a proof of this formula.

3. a) State Rouche's theorem.
b) Find the number of zeros of the function

$$f(z) = 2z^5 + 7z^3 + z^2 - 3$$

in the annulus $1 < |z| < 2$

4. Let $f(z)$ be a complex valued continuous function on a simple contour γ and define a function $g(z)$ by

$$g(z) = \int_\gamma \frac{f(\xi) \cdot d\xi}{\xi - z}$$

a) Show that $g(z)$ is analytic in any domain containing no points of γ.
b) Find an expression for $g'(z)$.

5. a) Find a bound for the modulus of the integral shown below:

$$\int_\gamma \sin^2(z)\, dz$$

where γ is the simple contour $\gamma(t) = (1-t) + i\pi t$ and $0 \leq t \leq 1$.

b) Evaluate exactly the modulus of the integral in (a).

6. Classify the singularities and find residues for each of the following functions. Include points at infinity.

a) $z^2 + 2z^5$

b) $\dfrac{1}{\sin(z) - \cos(z)}$

c) $\cot(z)$

d) $\dfrac{1}{z\{\exp(z) - 1\}}$

7. Evaluate the integral

$$\int_0^{\pi} \frac{d\theta}{a + \sin^2(\theta)} \qquad a > 0$$

8. Evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos(x)\, dx}{1 + x^4}$$

Include justifications for all steps in your calculation.

Complex Analysis Qualifying Examination

August 2000 SS # : ____________________

Directions:

Do the following 8 problems. You may choose to answer the problems in any order. Please start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages and *write your SS #* in each page. Please show all your work and explain all steps in a proof or derivation.

Questions:

1. (a) Find the linear fractional transformation $w = f(z)$ for which $f(0) = 0$, $f(2) = 4$, $f(\mathrm{i}) = 1 - \mathrm{i}$, and its inverse.
   (b) Obviously $z = 0$ is a fixed point; are there any other fixed points?
   (c) Describe the image of the region $1 \leq y$ in the w-plane.

2. Classify the singularities (including the point at ∞) and find the residues for

   a) $f(z) = \sin\left(\frac{1}{z}\right)$ b) $f(z) = \frac{\sin(z^2)}{z^7}$ c) $f(z) = \frac{1}{z^2}\cot z$.

3. Expand the function $f(z) = \frac{z^3 + 2z - 4}{z}$ in power series around $z = 1$ and give its radius of convergence.

4. Evaluate the real integral (and justify all steps)

$$\int_0^\infty \frac{x^2}{(x^2+1)^2}\, dx$$

5. (a) Show that $u(x, y) = x^3 - 3xy^2 + y^2 - x^2$ is harmonic in the entire plane.
   (b) Find a harmonic conjugate $v(x, y)$.
   (c) Give explicitely an analytic function $w = f(z)$ with $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$.

6. State and prove the Cauchy integral formula.

7. Assume that $w = f(z) = u(z) + \mathrm{i}v(z)$ is an analytic function mapping a domain D in the z-plane onto a domain D' in the w-plane. If $\phi(u, v)$ is a harmonic function in D', show that the function

$$\Phi(x, y) = \phi(u(x, y), v(x, y))$$

   is harmonic in D.

8. (a) State Rouche's theorem.
   (b) Find the number of zeros of $f(z) = 2z^5 + 7z^3 + z^2 - 3$ in the annulus $1 < |z| < 2$.

Complex Variables
Master's Examination

Spring 2000

Instructions: There are nine (9) questions on this examination, and each question is worth 25 points. Work any 8 problems. A maximum score of 200 points is possible.

1. Find a conformal mapping of the strip $0 < \Re z < 1$ onto the unit disk in such a way that $z = 1/2$ goes to $w = 0$ and $z = \infty$ goes to $w = 1$.

2. According to the Weirstrass factorization theorem, $f(z) = \cos\sqrt{z}$ can be written as an infinite product
$$f(z) = Ce^{g(z)}z^m \prod_1^\infty \left(1 - \frac{z}{a_n}\right) e^{h_n(z)} \quad ,$$
where $g(z)$ is entire and
$$h_n = \begin{cases} 0, & k = 0 \\ \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{k}\left(\frac{z}{a_n}\right)^k \ , & k \in \mathbb{Z}^+ \end{cases}$$
Determine a_n, k and m

3. According to the Mittag-Leffler theorem, the meromorphic function
$$f(z) = \frac{\pi^2}{\sin^2 \pi z} \quad ,$$
can be expressed by the infinite series
$$f(z) = \sum_k \left[P_k\left(\frac{1}{z - b_k}\right) - p_k(z)\right] + g(z) \quad ,$$
where $P_k(z)$, $p_k(z)$ are appropriately chosen polynomials, b_k are appropriately chosen complex numbers and $g(z)$ is analytic in the entire complex plane. Determine b_k, P_k and p_k.

4. For $a, b > 0$, evaluate the integral
$$\int_0^\infty \frac{\cos ax}{x^2 + b^2} dx \ .$$
Carefully justify any estimate you make.

5. Evaluate the integral
$$\int_0^\infty \frac{x^p}{1 + x^2} dx \ ,$$
with $-1 < p < 1$ by contour integration. As in the previous problem, carefully justify all your estimates.

6. Let $B(0;1) = \{z \in \mathbb{C} | \; |z| < 1\}$ be the unit disk. If $a > e$, and n is a positive integer, prove that the equation $e^z = az^n$ has n distinct roots in $B(0;1)$ (counted with multiplicity).

7. Let $\Omega \subset \mathbb{C}$ be a simply connected region, and $u : \Omega \to \mathbb{R}$ be a harmonic function. Prove that there exists $v : \Omega \to \mathbb{R}$ such that $u + iv$ is analytic on Ω.
Hint: Consider
$$g(z) = \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right) \quad .$$

8. Assume that $f(z)$ is analytic on $\mathbb{C} \setminus \{0\}$ and
$$|f(z)| \le |z|^2 + \frac{1}{|z|^{1/2}}$$
for all $z \in \mathbb{C} \setminus \{0\}$. Prove that f is a polynomial of degree at most 2.

9. Let $f(z)$ be continuous on the closed right half-plane $\bar{\mathcal{H}} = \{z \in \mathbb{C} | \Re z \ge 0\}$ and analytic on the open right half-plane $\mathcal{H} = \{z \in \mathbb{C} | \Re z > 0\}$. Suppose there exist constants C, $M \in \mathbb{R}$ and a positive integer n such that
(a) $|f(iy)| \le M$ for all $y \in \mathbb{R}$,
(b) $|f(z)| \le C(1 + |z|^n)$ for all $z \in \mathcal{H}$.
Prove that $|f(z)| \le M$ for all $z \in \mathcal{H}$.
Hint: For $\epsilon > 0$, consider
$$f_\epsilon(z) := \frac{f(z)}{(1 + \epsilon z)^{n+1}}$$
and apply the maximum principle.

Complex Analysis Qualifying Exam

August 1999

Do the following 8 problems. Show all your work and explain all steps in a proof or derivation.

1. Let f be analytic on $\mathbb{C}$ and real-valued on the circle $|z| = 1$. Show tht f is a constant on $\mathbb{C}$.

2. Classify the singularities at $z = 0$ of the following functions $f(z)$ (including the point at ∞).

a) $f(z) = \dfrac{\sin^2 z}{z^4}$.

b) $f(z) = \sin(\frac{1}{z}) + \dfrac{1}{z^2(z-1)}$.

c) $f(z) = \csc z - \dfrac{1}{z}$.

3. Show the map $f(z) = \dfrac{z-i}{z+i}$ is a bijection of the upper half plane $H = \{z \in \mathbb{C} | \text{Im } z > 0\}$ onto the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.

4. Let f be a continuous map of a connected open subset $U \subset \mathbb{C}$ into $\mathbb{C}$.

a) Show that f has a primitive on U if and only if $\int_\gamma f(z)dz = 0$ for every simple closed curve γ contained in U.

b) Consider $f(z) = \frac{1}{z}$ on the punctured unit disc $D(0,1) - \{0\}$. Does f have a primitive on $D(0,1) - \{0\}$? Explain.

5. Use the theory of residues to evaluate the integral $\displaystyle\int_0^\infty \frac{\ln x \, dx}{x^2 + a^2}$.

6. State the Argument Principle and use it to prove the Open Mapping Theorem: Let f be analytic on some region Ω. Then the image $f(U)$ is open in $\mathbb{C}$ for every open set $U \subset \Omega$. Hint. Apply the Argument Principle to the function $f(z) - w$.

7. Use the Casorati-Weierstrass Theorem to prove that if the composition $f \circ g$ of two entire functions f and g is a polynomial, then both f and g are polynomials.

8. Let f be a bounded analytic function on the disc $D(0, R)$. Suppose that f also satisfies $f^{(i)}(0) = 0$ for all $i = 0, \cdots, k$. Show that f satisfies the inequality $|f(z)| \leq \dfrac{M}{R^{k+1}}|z|^{k+1}$ on $D(0, R)$ where $M = \sup_{z \in D(0,R)} |f(z)|$.

Complex Analysis Qualifying Exam

January 1999

Do the following 8 problems. Show all your work and explain all steps in a proof or derivation.

1. State the Cauchy-Goursat Integral Theorem and give an outline of its proof.

2. Classify the singularities at $z = 0$ of the following functions $f(z)$ and find their residue.

   a) $f(z) = \frac{1}{z}$.

   b) $f(z) = z\cos(\frac{1}{z})$.

   c) $f(z) = z^{-3}\csc(z^2)$.

3. Consider the following meromorphic functions.

   a) Expand $f(z) = \frac{1}{2z - z^2}$ in a power series about $z = 1$.

   b) Find a Laurent expansion of $f(z) = \frac{1}{z} + \frac{1}{z+2} + \frac{1}{(z-1)^2}$ which is valid in the annulus $1 < |z| < 2$.

4. Show that $\tan z = z$ has no complex solutions of the form $z = x + iy$ with $x \neq 0, y \neq 0$.

5. Use the theory of residues to evaluate the integral $\int_0^\infty \frac{\sqrt{x}dx}{x^2+4}$.

6. State Rouche's theorem and use it to determine how many roots of the polynomial $z^4 + 5z + 3$ lie inside:

   a) the unit disc.

   b) the annulus $1 < |z| < 2$.

7. Prove the Casorati-Weierstrass Theorem: Let z_0 be an isolated essential singularity of a function $f(z)$. Then the image of a punctured neighborhood $U \backslash \{z_0\}$ under f is dense in the complex plane $\mathbb{C}$.

8. State the Mittag-Leffler Theorem and use it to prove that $\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

Complex Analysis Qualifying Examination

August 1998 SS # : ______________________

Directions:

1. You are trying to convince the reader that you know what you are doing. To that end we suggest your presentation be *clear, consise* and *complete.*

2. Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages and *write your name* in each page.

Questions:

Part A: Answer *four* out of the five questions below.

1. Expand $f(z) = \dfrac{z+6}{z^2-2z-3}$ in

   (a) Taylor series around $z = 0$.

   (b) Laurent series in the annulus $1 < |z| < 3$.

   (c) Laurent series in the region $3 < |z| < \infty$.

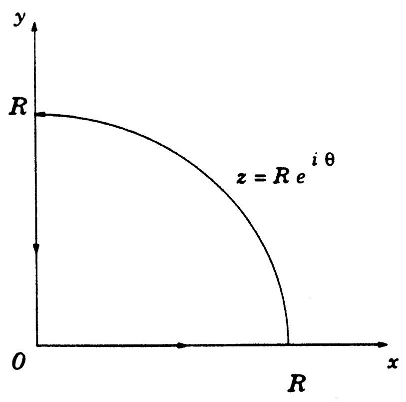
2. Find the bilinear transformation which takes $z_1 = 0$, $z_2 = 1$, and $z_3 = 2$ to $w_1 = 2i$, $w_2 = -2$, and $w_3 = -2i$.

3. Evaluate

   (a) $\displaystyle\int_{|z|=2} \frac{z+6}{z^2-2z-3}\, dz$

   (b) $\displaystyle\int_0^{\pi} \frac{d\theta}{6-3\cos\theta}$

4. Consider $I = \displaystyle\int_\gamma \frac{z^2\, dz}{1+z^4}$, where γ is the contour shown below



   (a) Evaluate I when $R < 1$.

   (b) Evaluate I when $R > 1$.

   (c) Discuss the results obtained when $R \to \infty$.

5. (a) State and prove Liouville's theorem.
   (b) Deduce the fundamental theorem of algebra from part (a).

Part B: Answer *four* out of the six questions below.

1. Let $f(z) = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$

   (a) Show that $f(z)$ is analytic in the complex plane minus the non-zero integers.

   (b) Show that if $f(z) = 0$, then z is real.

2. Recall that a polynomial $p(z)$ for which $p(0) > 0$ and which has only negative real zeros has positive coefficients. Now consider the function

$$f(z) = \frac{1}{z\Gamma(z)}$$

   which has only negative real zeros ($\Gamma(z)$ is the Gamma function). Are all the Taylor coefficients of $f(z)$ positive? Explain.

3. Let $w = f(z) = \sum_{0}^{\infty} a_n z^n$ be analytic and univalent in a disc D of radius r, centered at $z = 0$. Show that the area A of the image $f(D)$ is given by

$$A = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}$$

4. (a) State and prove a form of the maximum principle.
   (b) State Schwarz's lemma and give a proof.

5. (a) Explain why there is an entire function $f(z)$ such that

$$e^{f(z)} = \frac{e^{2z} + 1}{\cos(iz)}.$$

   (b) Find an entire function $f(z)$ satisfying the above relation.

6. (a) State and prove Rouche's theorem.
   (b) Use Rouche's theorem to show that there is $\epsilon_0 > 0$ so that for $0 < \epsilon < \epsilon_0$ the equation $z^3 - \epsilon z - 1 = 0$ has three distinct roots.

Complex Analysis Qualifying Examination

August 1997 **Name:** ______________________

Directions:

1. You are trying to convince the reader that you know what you are doing. To that end we suggest your presentation be *clear, consise* and *complete*.

2. Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages and *write your name* in each page.

Questions:

Part A: Answer *four* out of the six questions below.

1. Find the image of the quarter disc $\Omega = \{z \in C | \, |z| \leq 1, \ \Re z \geq 0, \ \Im z \geq 0\}$ under the map $w = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)$.

2. Assume that $w = f(z)$ is an entire function, and that a and b are two positive constants so that $f(z)$ satisfyes $|f(z)| \leq a + b|z|^2$ for all $z \in C$. Prove that $f(z)$ is a polynomial of degree no larger than two.

3. Answer *only one* of the following questions:
   (a) Evaluate the integral
   $$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx, \text{ where } 0 < \alpha < 1$$
   (b) Evaluate the integral
   $$\int_{-\infty}^{\infty} \frac{\cos kx}{(1+x^2)^2}\,dx, \text{ where } 0 < k < \infty$$

4. Assume that $\{f_n(z)\}_{n=1}^{\infty}$ is a sequence of analytic functions defined on the region Ω, such that $\lim_{n\to\infty} f_n(z) = f(z)$ uniformly on compact subsets of Ω. Show that $f(z)$ is analytic in Ω and that $\lim_{n\to\infty} f_n'(z) = f'(z)$ uniformly on compact subsets of Ω.

5. Let $u(x,y)$ be the bounded harmonic function in the upper half-plane $\{z = x + iy \in C \mid y > 0\}$ that has the boundary value
   $$u(x,0) = \ \mathrm{sgn} x = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$$
   Find a harmonic conjugate $v(x,y)$ of $u(x,y)$.

6. Find the residue of the following functions at the indicated singularities:
   (a) $\dfrac{\sin^2 z}{z^5}$ at $z = 0$
   (b) $\dfrac{\sqrt{1-z}}{z^2}$ at $z = 0$
   (c) $\dfrac{1}{e^z - 1}$ at $z = 2\pi i$

Part B: Answer *four* out of the six questions below.

1. Answer *only one* of the following questions:
   (a) Find a meromorphic function $f(z)$ that has simple poles at $z = \sqrt{n}$, $n = 1, 2, 3, \ldots$, with residue $res(f(z), z = n) = 1$.
   (b) Find an entire function $f(z)$ that has double roots at $z = \sqrt{n}$, where $n = 1, 2, 3, \ldots$

2. Answer *only one* of the following questions:
   (a) State and prove Rouche's theorem.
   (b) State and prove the argument principle.

3. Show that the equation $(z-2)^2 = e^{-z}$ has two distinct roots in the disc $|z-2| \leq 1$.

4. Prove that if $f(z)$ is an entire function without any roots, then there is an entire function $g(z)$ so that $f(z) = e^{g(z)}$.

5. Answer *only one* of the following questions:
   (a) Prove that if $u(z)$ is a non-constant harmonic function in the domain Ω, then $u(z)$ does not have local maximum in Ω.
   (b) Let $u(x,y)$ be a harmonic function on the entire plane, and let $v(x,y)$ be a harmonic conjugate of $u(x,y)$. Assume that $u(x,y) \leq v^2(x,y)$ for all $(x,y) \in C$. Prove that both $u(x,y)$ and $v(x,y)$ must be constant.

6. Assume that $f(z)$ is analytic in the unit disc $D = \{z \in C \mid |z| < 1\}$, $f(0) = 0$, and $|f(z)| \leq 1$ for $|z| < 1$. Prove that $|f(z)| \leq |z|$ for $|z| < 1$.

Complex Analysis Qualifying Examination

January 1997 **Name:** ____________________

Directions:

1. You are trying to convince the reader that you know what you are doing. To that end we suggest your presentation be *clear, consise* and *complete.*

2. Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages and *write your name* in each page.

Questions:

Part A: Answer *four* out of the five questions below.

1. Find the Möbius transformation that maps the left half-plane $\{z \in C \mid \Re z < 1\}$ to the unit disc $\{w \in C \mid |w| < 1\}$ and has $z = 0$ and $z = 1$ as fixed points.

2. Prove that if $w = f(z)$ is holomorphic in the disc $D(0,2)$ then $w = f(z)$ has a Taylor expansion centered at $z_0 = 0$ which converges for $|z| \leq 1$.

3. Answer *only one* of the following questions:
(a) Evaluate the integral
$$\int_0^{2\pi} \frac{d\theta}{1 + \sin^2\theta}$$
(b) Evaluate the integral
$$\int_0^{\infty} \frac{x \sin x}{1 + x^2} dx$$

4. Characterize all the entire functions that satisfy $f^{(n)}(0) = 1/n!$, for $n = 0, 1, 2, \ldots$.

5. Classify the singularities of the following functions in the *extended* complex plane $C \cup \{\infty\}$, and find the singular part at each of the isolated singular points:

(a) $\dfrac{1 - \cos z}{z^4}$;

(b) $\sqrt{1 - \sin z}$;

(c) $\dfrac{1 - z^3}{1 - z^2}$;

Part B: Answer *four* out of the six questions below.

1. Answer *only one* of the following questions:
(a) Characterize all the meromorphic functions $f(z)$ that have simple poles at $z = n$, $n = 1, 2, 3, \ldots$, with residue $res(f(z), z = n) = n^2$.
(b) Characterize all the entire functions $f(z)$ that have simple roots at $z = \sqrt{n}$, where $n = 1, 2, 3, \ldots$

2. Answer *only one* of the following questions:
   (a) State and prove Liouville's theorem.
   (b) State and prove the Casorati-Weierstrass theorem.

3. (a) Prove the argument principle.
   (b) How many roots does the polynomial $p(z) = z^4 + z + 1$ have in the first quadrant?

4. Find the image of the region $\{z \in C \mid 0 < \Re z < \pi,\ 0 < \Im z\}$ under the mapping $w = \cos z$.

5. If u is harmonic and bounded in $0 < |z| < \rho$, show that the origin is a removable singularity in the sense that u becomes harmonic in $|z| < \rho$ when $u(0)$ is properly defined.

6. Prove the reflection principle: if $f(z)$ is a continuous function in upper half-plane $\Omega = \{z \in C \mid \Im z \geq 0\}$, analytic in the interior of Ω, and $f(z)$ takes real values on the real axis, then $f(z)$ can be extended to an entire function.

Complex Analysis Qualifying Examination

August 1996

Directions:

1. You are trying to convince the reader that you know what you are doing. To that end we suggest your presentation be *clear, concise* and *complete.*

2. Solve any eight problems.

3. Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages.

Terminology:

1. A *domain* is a non-empty open connected set in the complex plane.

2. A sequence or series of functions is said to be *locally uniformly convergent* on a domain Ω if it converges uniformly on every compact subset of Ω.

Notation:

1. $D(a, r) = \{z \in \mathbb{C}\,; |z-a| < r\}$ $(r > 0)$.

2. $f(z) \in H(\Omega)$ means that $f(z)$ is analytic on the domain Ω.

---

1. Suppose $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$ has the property that the series $\sum_{n=0}^{\infty} f^{(n)}(c)$ converges. Show that $f(z)$ is an entire function.

2. Show that an entire function that takes real values on the real axis and purely imaginary values on the imaginary axis must be an odd function:

$$f(-z) = -f(z) \quad \text{for all } z \in \mathbb{C}.$$

3. Let $f(z)$ be analytic in the unit disc. Define $g(w) = f(z)$ where $w = Tz$ is a Möbius transformation mapping the unit disc conformally onto itself. Show that

$$(1-|w|^2)\left|\frac{dg}{dw}\right| = (1-|z|^2)\left|\frac{df}{dz}\right|.$$

4. Let $f_1(z), f_2(z), \cdots, f_n(z) \in H(\Omega)$, and

$$\varphi(z) = |f_1(z)|^2 + |f_2(z)|^2 + \cdots + |f_n(z)|^2.$$

   (a) Show that $\varphi(z)$ is harmonic on the domain Ω only if all the functions $f_k(z)$ $(k = 1, 2, \cdots, n)$ reduce to constant functions.

August

(b) Show that $\varphi(z)$ has no local maximum in Ω unless all the functions $f_k(z)$ ($k = 1, 2, \cdots, n$) reduce to constant functions.

5. Find all entire functions that satisfies the Lipschitz condition on $\mathbf{C}$. A function $f(z)$ is said to satisfy the *Lipschitz condition* on $\mathbf{C}$ if there exists a positive constant M such that

$$|f(z_1) - f(z_2)| \leq M \cdot |z_1 - z_2| \quad \text{for all } z_1,\, z_2 \in \mathbf{C}.$$

6. Find the *Fourier transform* of the function $f(x) = e^{-x^2/2}$; i.e., find the function $\hat{f}(t)$ defined for all $t \in \mathbf{R}$ by

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cdot e^{-itx} dx.$$

7. Show that $z^5 - 15z + 1 = 0$ has one root in the disc $|z| < \frac{1}{8}$ and four roots in the annulus $\frac{3}{2} < |z| < 2$.

8. If $f(z)$ is analytic in the unit disc and $f(0) = 0$, show that

$$f(z) + f(z^2) + \cdots + f(z^n) + \cdots$$

converges locally uniformly to an analytic function in the unit disc.

9. Construct an entire function $f(z)$ such that

$$f(n) = n! \quad (n = 0, 1, 2, \cdots).$$

10. Find the conformal mapping $w = f(z)$ of a convex lens $D(\sqrt{3}, 2) \cap D(-\sqrt{3}, 2)$ onto the unit disc satisfying $f(0) = 0$, $f'(0) > 0$.

Complex Variables Master's Qualifying Examination
January 1996

1) (a) Determine the region of absolute convergence of the products:

(i) $\prod_{n=1}^{\infty}(1 - z^n)$

(ii) $\prod_{n=1}^{\infty}(1 + z^{2n})$

(iii) $\prod_{n=0}^{\infty}(1 + c_n z)$ where $\sum_{n=0}^{\infty} |c_n| < \infty$

(iv) $\prod_{n=1}^{\infty}\left(1 - \frac{1}{n^z}\right)$

(b) For the following functions in the <u>extended</u> complex plane give the branch points and the isolated singular points. Say whether the isolated singular points are removable, essential or poles (and, if poles, give the order)

(a) $\sqrt{z^2 - 1}$

(b) $\exp\left(\frac{1}{\sin z}\right)$

(c) $\ln(\sqrt{z^2 + 1})$

(d) $\tan z$

2) Let $D = \{z \in \mathbb{C} \mid |z| \le 1 \text{ or } |z - 2| \le 1\}$. Let $\tilde{D} = (\mathbb{C}\backslash D) \cup \{\infty\}$. Show $\tilde{D}$ is conformally equivalent to the open upper half plane.

3) (a) Let f and g be entire functions satisfying $|f(z)| \le |g(z)|$ for $|z| \ge 100$. Assume g is not identically zero. Show f/g is rational.

(b) Let u be harmonic in $\mathbb{C}$ and $u(x, y) \ge -2$ for all $x + iy \in \mathbb{C}$. Show u is constant in $\mathbb{C}$.

4) Let $f(z) = z^4 - 5z + 1$

(a) How many zeros does $f(z)$ have in the disc $\{z \in \mathbb{C} \mid |z| < 1\}$.
(b) How many zeros does $f(z)$ have in the annulus $1 < |z| < 2$.

5) Compute $F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2 + 1}\, dx$ where $k \in \mathbb{R}$

6) Expand (if possible) in Laurent series in the indicated region

(a) $e^{1/(z-1)}$ $\quad |z| > 1$

(b) $\frac{1}{(z-a)(z-b)}$ $\quad$ (i) $0 < |a| < |z| < |b|$

(ii) $|a| < |b| < |z|$

(c) $\log\left(\frac{1}{1-z}\right) \quad |z| > 1$

7) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\, dx$$

8) Choose a branch of $\sqrt{z^2-1}$ that is analytic on $\mathbf{C}\setminus\{x+0i \mid -1 \le x \le 1\}$ and has the value $\sqrt{3}$ at $z = 2$. Evaluate $\int_\gamma \sqrt{z^2-1}\, dz$ where γ is the circle of radius 2, centered at 0 and oriented counterclockwise.

Complex Variables Master's - Qualifying Examination

August 1995

1. Let $\{a_n\}$ be a sequence of complex numbers. Assume that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for all z satisfying $|z| \le r$. Prove that if $|a_1| > \sum_{n=2}^{\infty} n|a_n|r^{n-1}$, then f is an injective function on the disc $|z| \le r$.

2. Let $f(z) = \frac{1}{z-1} + \frac{1}{(z-2)^2}$. Expand $f(z)$ in a

   i) Taylor series in $|z| < 1$

   ii) Laurent series in $1 < |z| < 2$.

3. A complex-valued function $f = U + iV$ is said to be harmonic on a domain $D \subset \mathbb{C}$ if U and V are harmonic on D. Show that f is holomorphic on D if and only if both f and zf are harmonic on D.

4. (a) Show $\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ is meromorphic in the complex plane $\mathbb{C}$

   (b) Argue that $\frac{\pi^2}{(\sin \pi z)^2} - \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ is holomorphic in $\mathbb{C}$

   (c) Show that this holomorphic function is 0, i.e. $\frac{\pi^2}{(\sin \pi z)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$

5. Let $f : U \longrightarrow \mathbb{C}$ be a holomorphic function defined on an open subset U of $\mathbb{C}$. Let R be a closed rectangle contained in U (assume that the sides of R are parallel to the lines $\mathrm{Re}z = 0$ and $\mathrm{Im}z = 0$). Give a complete proof of the equality

$$\int_{\partial R} f(z)dz = 0$$

6. Let $p_n(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$. Prove that for every $R > 0$ there exists a positive integer $n(R)$ such that all roots of $p_n(z) = 0$ for $n \geq n(R)$ belong to the set $\{z \in \mathbb{C} \,\big|\, |z| > R\}$.

7. Assume that a, b, c are real numbers satisfying $ac - b^2 > 0$. Prove using residues

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2 + 2bx + c} = \frac{\pi}{\sqrt{ac - b^2}}.$$

8. Let $\{a_n\}$ and $\{b_n\}$ be sequences of complex numbers. Assume that $\{a_n\}$ has no accumulation point. Prove that there exists a holomorphic function $f : \mathbb{C} \longrightarrow \mathbb{C}$ such that $f(a_n) = b_n$.

Complex Analysis Qualifying Examination

January 1995

Directions :

1. You are trying to convince the reader that you know what you are doing. To that end we suggest you be *clear, concise* and *complete.*
2. Solve all nine problems.
3. Start each question on a new sheet of paper. Write only on one side of each sheet of paper. Number the pages.

Terminology : A *domain* is a non-empty open connected set in the complex plane.

1. (a) Find the first 3 non-vanishing terms of the Taylor series expansion of $\tan z$ around the origin.

   (b) Also find its radius of convergence.

2. Let $P_1, P_2, \cdots, P_n$ be n arbitrary points of a plane and let $\overline{PP_k}$ denote the distance between P_k and a variable point P. If P is confined to the closure of a bounded domain Ω, show that the product $\prod_{k=1}^{n} \overline{PP_k}$ attains its maximum on the boundary of Ω.

3. Evaluate
$$\int_{|z|=1} \frac{1 - \cos z}{(e^z - 1) \cdot \sin z} dz.$$

4. Show that $z^5 + 15z + 1 = 0$ has one root in the disc $|z| < \frac{3}{2}$, four roots in the annulus $\frac{3}{2} < |z| < 2$.

5. Find the fallacy in the following 'proof':

   Let m and n be two arbitrary integers. Then we have
$$e^{2m\pi i} = e^{2n\pi i}, \quad \therefore \left(e^{2m\pi i}\right)^i = \left(e^{2n\pi i}\right)^i.$$
   It follows that
$$e^{-2m\pi} = e^{-2n\pi}.$$
   Since $-2m\pi$ and $-2n\pi$ are all real, we must have
$$-2m\pi = -2n\pi, \quad \therefore m = n.$$

6. Find a conformal mapping of the domain

$$\left\{ z \in \mathbf{C}\,;\ |z| < 1,\ \ \Im z > \frac{\sqrt{2}}{2} \right\}$$

onto the unit disc.

7. Construct a meromorphic function having simple poles with residue 1 at Gaussian integers $\omega_{mn} = m + in$ $(m,\ n \in \mathbf{Z})$.

8. True or false:

   (a) If $w = f(z)$ maps Ω_z conformally onto Ω_w, then $f'(z)$ never vanishes in Ω_z.

   (b) Any two annuli can be mapped conformally onto each other.

   (c) Any two non-intersecting circles can be mapped to a pair of concentric circles by a Mõbius transformation.

   (d) A bounded (real-valued) harmonic function in the plane $\mathbf{C}$ is a constant.

   (e) If $\{f_k(z)\}_{k=1}^{\infty}$ is a sequence of univalent (one-to-one) analytic functions which converges uniformly on every compact subset of a domain Ω, to a non-constant function $f(z)$ in a domain Ω, then $f(z)$ is a univalent analytic function in Ω.

   (f) There is a non-constant entire function having both 2π and i as its periods; i.e.

$$f(z + 2\pi) = f(z) = f(z + i) \quad \text{for all } \ z \in \mathbf{C}.$$

9. Prove or disprove two of the six propositions in the previous problem.

COMPLEX ANALYSIS QUALIFYING EXAMINATION

August 1994

DIRECTIONS: You are trying to convince the reader that you know what your are doing, so you should give clear, concise and complete answers explaining your work. Note that holomorphic is synonymous with analytic. **Do any 7 of the following 10 problems.**

1. Let $z = x + iy$ and $f(z) = \sqrt[2]{|xy|}$. Prove that $f(z)$ satisfies the Cauchy-Riemann equations at $z = 0$ but that $f(z)$ is not complex differentiable at $z = 0$.

2. Compute the radius of convergence of the following:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} z^n.$

b) $\displaystyle\sum_{n=0}^{\infty} (n + a^n) z^n$, where $a \in \mathbb{C}$.

c) The Taylor series around zero for the function $z \cot z$.

3. Let $T(z) = \dfrac{az+b}{cz+d}$ be a linear fractional transformation with $a, b, c, d \in \mathbb{C}$. Suppose also that T has two fixed points in $\mathbb{C}$. Prove that the product of the derivatives of T at the two fixed points equals one, i.e. that $T'(z_1)T'(z_2) = 1$, where z_1, z_2 are the two fixed points.

4. Prove Goursat's Lemma: If $f(z)$ is holomorphic in a simply connected region $\Omega \subset \mathbb{C}$ then for the boundary $\partial\Delta$ of every triangle $\Delta \subset \Omega$ one has $\displaystyle\int_{\partial\Delta} f = 0$.

5. Prove the Schwarz Lemma: If $f(z)$ is a holomorphic function in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ satisfying $f(0) = 0$ and $|f(z)| \le 1$ for all $z \in D$, then $|f(z)| \le |z|$ for all $z \in D$.

6. Use the method of contour integration and the calculus of residues to evaluate the integral $\displaystyle\int_0^{\infty} \frac{x^p}{1+x^2} dx$ where $-1 < p < 1$. Draw the relevant contour and justify all steps.

7. Determine the three Laurent series around 0 of the function $f(z) = \dfrac{1}{(z-1)(z-2)}$ in the three regions $|z| < 1, 1 < |z| < 2$, and $|z| > 2$, respectively.

8. Find the number of zeroes of the equation $e^z - 4z^n + 1 = 0$ in the unit disc $D = \{z \in \mathbf{C} : |z| < 1\}$.

9. Do parts a), b) and c), below.

a) State the Mittag-Leffler Theorem on the existence of a meromorphic function with an (infinite) discrete set of poles with prescribed principal parts.

b) State the Weierstrass Theorem on the existence of an entire function with an (infinite) discrete set of zeros with prescribed locations and orders.

c) Prove either one of the theorems in part a or b. Indicate clearly which theorem you are proving.

10. Suppose $f(z)$ is holomorphic in a neighborhood U of the origin 0 and that $f'(0) \neq 0$. Prove that there are a disc $B \subset U$, a positive integer n, and a holomorphic function $g(z)$ such that $f(z) = f(0) + (g(z))^n$ in B.

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COMPLEX ANALYSIS QUALIFYING EXAMINATION

January 1994 01/94

DIRECTIONS: You are trying to convince the reader that you know what your are doing, so you should give clear, concise and complete answers explaining your work. Note that holomorphic is synonymous with analytic. **Do any 7 of the following 10 problems.**

1. Compute the radius of convergence of the following:

a) $\displaystyle\sum_{1}^{\infty} \frac{(3n)!}{(3n)^{3n}} z^n.$

b) $\displaystyle\sum_{0}^{\infty} [3+(-1)^n]^n z^n.$

c) The Taylor series around zero for the function $\dfrac{z}{e^z - 1}$.

2. Suppose that the linear transformation $T(z) = \dfrac{az + b}{cz + d}$ with $a, b, c, d \in \mathbb{C}$ has three fixed points. Prove that T is the identity transformation.

3. Verify that a suitable branch of the function $f(z) = \log(5 + \sqrt{\dfrac{z+1}{z-1}})$ is a well defined (single-valued) holomorphic function in the z-plane outside of a line joining the points $z = 1$ and $z = -1$. Show, however, that if one enters another sheet of the Riemann surface by crossing this line, there will be a branch point at $z = 13/12$.

4. Let $f(z)$ be a holomorphic function in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0$. Prove that

$$f(z) + f(z^2) + f(z^3) + \cdots + f(z^n) + \cdots$$

converges uniformily on compact subsets of D.

5. Let $f(z)$ and $g(z)$ be entire functions. Show that if $f \circ g(z)$ is a polynomial then both $f(z)$ and $g(z)$ are polynomials.

6. Let $f(z)$ be a holomorphic function on the closed unit disc $\bar{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ that satisfies $|f(z)| < 1$ for $|z| = 1$. Prove that f has exactly one fixed point in the open unit disc D.

7. Define the function

$$F(z) = \int_{|\zeta|=2} \frac{d\zeta}{\zeta(\zeta - z)(\zeta - z + 1)}$$

Determine the limit of $F(z)$ as $z \to 2$ from

1) inside the circle $|z| = 2$,

2) outside the the circle $|z| = 2$.

3) Is $F(z)$ continuous at $z = 2$?

8. Show by the method of complex contour integration that the following identities hold:

a)

$$\int_0^\infty \frac{\sin ax}{x(x^2+1)} dx = \frac{1 - e^{-a}}{2}.$$

b)

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \pi.$$

9. Expand $e^{1/z}$ in a Laurent series about $z = 0$, determining the Laurent coefficients. Show that

$$\frac{1}{n!} = \frac{1}{\pi} \int_0^\pi e^{\cos\theta} \cos(n\theta - \sin\theta) d\theta.$$

10. Construct a meromorphic function that has simple poles at the non-zero integers with residue at $z = n$ equal to n^2.