December 2010 Problems

Please send your solutions or questions to Janet Vassilev (jvassil@math.unm.edu) or Dimiter Vassilev (vassilev@math.unm.edu). We are looking forward to hearing from you.

Goal: Show that there are only five possible regular polyhedrons. Reference: Rademacher H., Toeplitz O., *The enjoyment of mathematics*, Princeton University Press.

A polyhedron is a solid figure bounded by portions of planes called faces. According to Euclid, a polyhedron is "regular" if all its faces are congruent regular polygons. We shall use a more general definition, namely, a polyhedron is "*regular*" if all its faces have the same number of sides (edges), and if the same number of faces come together at each vertex. Notation: given a regular polyhedron, we define the following *polyhedron numbers*: i) V – the number of vertices of each face; ii) F – the number of faces that meet at each vertex; iii) v – the number of vertices of the polyhedron; iv) e – the number of edges of the polyhedron; v) f – the number of faces of the polyhedron.

1) Compute the polyhedron numbers for a regular polyhedron formed by
   a) triangles; b) quadrilaterals; c) show that in either case we have the relations

(*) $$f - e + v = 2, \qquad fV = 2e, \qquad vF = 2e.$$

2) Show that for any regular polyhedron $F \geq 3$ and $V \geq 3$.

3) Show that the relations (*) are valid for any regular polyhedron.
   Hint: In order to prove Euler's formula $f - e + v = 2$ you might want to imagine that the polyhedron is hollow and made out of rubber which you stretch until it turns into a spherical map on which each original face represents a country and each original edge turns into a border line between two countries. Notice that the map is fairly "regular", i.e., all countries have the same number of different borders and the same number of countries meet at each corner.

4) Show that the Euler numbers of a regular polyhedron are
$$f = \frac{4F}{2V + 2F - FV}, \quad e = \frac{2FV}{2V + 2F - FV}, \quad v = \frac{4V}{2V + 2F - FV}.$$

5) Show that for any regular polyhedron

(**) $$(V - 2)(F - 2) < 4.$$

   Hint: First prove, using the polyhedron numbers, that $VF < 2(V + F)$.

6) Prove that the only possible positive integer solutions of (**) are

V	3	3	4	3	5
F	3	4	3	5	3

   Thus, the only possible regular polyhedrons are those with faces being only triangles, quadrilaterals or pentagons.

7) Prove that the regular polyhedrons are the ones in the following table (*the Platonic solids*)

V	F	f	e	v	name
3	3	4	6	4	*tetrahedron*
3	4	8	12	6	*octahedron*
4	3	6	12	8	*hexahedron*
3	5	20	30	12	*icosahedron*
5	3	12	30	20	*dodecahedron*