Math. Ann. 318, 453–516 (2000) **Mathematische Annalen**

Digital Object Identifier (DOI) 10.1007/s002080000127

Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups

Nicola Garofalo · Dimiter Vassilev

Received April 15, 1999 / Revised November 29, 1999 /
Published online October 11, 2000 –

1. Introduction

This paper constitutes the first part of a project devoted to the study of a class of nonlinear sub-elliptic problems which arise in function theory on CR manifolds. The infinitesimal groups naturally associated with these problems are non-commutative Lie groups whose Lie algebra admits a stratification. The fundamental role of such groups in analysis was envisaged by E. M. Stein [72] in his address at the Nice International Congress of Mathematicians in 1970, see also the recent monograph [73]. There has been since a tremendous development in the analysis of the so-called stratified nilpotent Lie groups, nowadays also known as Carnot groups, and in the study of the sub-elliptic partial differential equations, both linear and non-linear, which arise in this connection. Despite all the progress, our understanding of a large number of basic questions is not to present day as substantial as one may desire. Such situation is due primarily to the complexity of the underlying sub-Riemannian geometry, on the one hand, and to the considerable obstacles which are imposed by non-commutativity and by the presence of characteristic points on the other.

To introduce the problems studied in this paper we recall that a Carnot group $\boldsymbol{G}$ is a simply connected nilpotent Lie group such that its Lie algebra $\mathfrak{g}$ admits a stratification $\mathfrak{g} = \bigoplus_{j=1}^{r} V_j$, with $[V_1, V_j] = V_{j+1}$ for $1 \leq j < r$, $[V_1, V_r] = \{0\}$.

N. Garofalo · D. Vassilev
Department of Mathematics, Purdue University, West Lafayette IN 47907–1935
(e-mail: garofalo@math.purdue.edu / dvassil@math.purdue.edu)

The first author was supported in part by NSF Grant No. DMS-9706892.
The second author was supported in part by a grant of the Purdue Research Foundation and also by the first author's NSF Grant No. DMS-9706892

We assume that a scalar product $< \cdot, \cdot >$ is given on $\mathfrak{g}$ for which the $V_j's$ are mutually orthogonal. Every Carnot group is naturally equipped with a family of non-isotropic dilations defined by

$$\delta_\lambda(g) = exp \circ \Delta_\lambda \circ exp^{-1}(g), \quad g \in \boldsymbol{G},$$

where $exp : \mathfrak{g} \to \boldsymbol{G}$ is the exponential map and $\Delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ is defined by $\Delta_\lambda(X_1 + ... + X_r) = \lambda X_1 + ... + \lambda^r X_r$. The topological dimension of $\boldsymbol{G}$ is $N = \sum_{j=1}^{r} dim\, V_j$, whereas the *homogeneous dimension* of $\boldsymbol{G}$, attached to the automorphisms $\{\delta_\lambda\}_{\lambda>0}$, is given by $Q = \sum_{j=1}^{r} j\ dim\, V_j$. We denote by $dH = dH(g)$ a fixed Haar measure on $\boldsymbol{G}$. One has $dH(\delta_\lambda(g)) = \lambda^Q dH(g)$, so that the number Q plays the role of a dimension with respect to the group dilations. Let $X = \{X_1, \ldots, X_m\}$ be a basis of V_1 and continue to denote by X the corresponding system of sections on $\boldsymbol{G}$. The *sub-Laplacian* associated with X is the second-order partial differential operator on $\boldsymbol{G}$ given by

$$\mathcal{L} = -\sum_{j=1}^{m} X_j^* X_j = \sum_{j=1}^{m} X_j^2$$

(we recall that in a Carnot group one has $X_j^* = -X_j$, see [26]). By the assumption on the Lie algebra one immediately sees that the system X satisfies the well-known finite rank condition, therefore thanks to Hörmander's theorem [39] the operator $\mathcal{L}$ is hypoelliptic. However, it fails to be elliptic, and the loss of regularity is measured by the step r of the stratification of $\mathfrak{g}$. For a function u on $\boldsymbol{G}$ we let $|Xu| = (\sum_{j=1}^{m}(X_j u)^2)^{1/2}$. For $1 \leq p < Q$ we set

$$\overset{o}{\mathcal{D}}{}^{1,p}(\Omega) = \overline{C_o^\infty(\Omega)}^{||\cdot||_{\mathcal{D}^{1,p}(\Omega)}},$$

where $\mathcal{D}^{1,p}(\Omega)$ indicates the space of functions $u \in L^{p^*}(\Omega)$ having distributional horizontal gradient $Xu = (X_1 u, ..., X_m u) \in L^p(\Omega)$. The space $\mathcal{D}^{1,p}(\Omega)$ is endowed with the obvious norm

$$||u||_{\mathcal{D}^{1,p}(\Omega)} = ||u||_{L^{p^*}(\Omega)} + ||Xu||_{L^p(\Omega)}.$$

Here, $p^* = \frac{pQ}{Q-p}$ is the Sobolev exponent relative to p. The relevance of such number is emphasized by the following embedding due to Folland and Stein [26], [27].

Theorem (Folland and Stein). *Let $\Omega \subset \boldsymbol{G}$ be an open set. For any $1 < p < Q$ there exists $S_p = S_p(\boldsymbol{G}) > 0$ such that for $u \in C_o^\infty(\Omega)$*

$$\left(\int_\Omega |u|^{p^*} dH\right)^{1/p^*} \leq S_p \left(\int_\Omega |Xu|^p dH\right)^{1/p}. \tag{1.1}$$

The purpose of the present paper is to study the non-linear Dirichlet problem

$$(1.2) \qquad \begin{cases} \mathcal{L}u = -u^{\frac{Q+2}{Q-2}} \\ u \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega), \quad u \geq 0. \end{cases}$$

The exponent $\frac{Q+2}{Q-2} = 2^* - 1$ is critical for the case $p = 2$ of the embedding (1.1). To motivate our results we recall that in the classical Riemannian setting the equation $\Delta u = -u^{(n+2)/(n-2)}$ is connected to the compact Yamabe problem [76], [3], [69], see also the book [4] and the survey article [59]. There exists an analogue of such problem in CR geometry, namely: *Given a compact, strictly pseudo-convex CR manifold M of real dimension $2n+1$, with contact form θ, find a choice of contact form in the conformal class of θ for which the Webster- Tanaka pseudo-hermitian scalar curvature R is constant*. Denoting with $\theta^* = u^{\frac{4}{Q-2}}\,\theta$ a conformal change of θ, one obtains for the corresponding scalar curvature

$$R^* = u^{-\frac{Q+2}{Q-2}} \left(\frac{2Q}{Q-2}\, \mathcal{L}u \, + R\, u \right),$$

where we have let $Q = 2n + 2$. It is then clear that in the flat case $R = 0$ the pde associated with the CR Yamabe problem is the one that appears in (1.2). Although on the formal level this problem has many similarities with its Riemannian predecessor, the analysis is considerably harder since, as we mentioned, the sub-Laplacian $\mathcal{L}$ fails to be elliptic everywhere. In 1984-88 D. Jerison and J. Lee in a series of important papers [44], [45], [46], [47] gave a complete solution to the CR Yamabe problem when the CR manifold M has dimension ≥ 5 and M is not locally CR equivalent to the sphere in $\mathbb{C}^{n+1}$. They proved first that the CR Yamabe problem can be solved on any compact CR manifold M provided that the CR Yamabe invariant of M is strictly less than that of the sphere in $\mathbb{C}^{n+1}$. Similarly to Aubin's approach in the Riemannian case, in order to determine when the problem can be solved they then proved that the Yamabe functional is minimized by the standard Levi form on the sphere and its images under CR automorphisms. A crucial step in this analysis is the explicit computation of the extremal functions in the special case when $p = 2$ and $\boldsymbol{G}$ is the Heisenberg group in the above stated Folland-Stein embedding . Jerison and Lee made the deep discovery that, up to group translations and dilations, a suitable multiple of the function

$$(1.3) \qquad u(z,t) = ((1 + |z|^2)^2 + t^2)^{-(Q-2)/4},$$

is the only positive solution of (1.2) when $\Omega = \mathbb{H}^n$. Here, we have denoted with (z,t), $z \in \mathbb{C}^n$, $t \in \mathbb{R}$, the variable point in $\mathbb{H}^n$.

In 1980 A. Kaplan [48] introduced a class of Carnot groups of step two in connection with hypoellipticity questions. Such groups, which are called of *Heisenberg type*, constitute a direct generalization of the Heisenberg group, as they include, in particular, the nilpotent component in the Iwasawa decomposition of simple groups of rank one. Since their introduction there has been a considerable amount of work in the study of such groups and of their geometry, we refer the reader to the papers [49], [17], [50], [51], [16], [54], [18], [19], [20], [15], [9], and to the references therein. From our perspective groups of Heisenberg type display a crucial feature: Their conformal invariances can be revealed. This leads to the construction of some beautiful solutions to various problems. In this connection, in his first work on the subject Kaplan [48] constructed an explicit fundamental solution for the sub-Laplacian, thus extending Folland's result for the Heisenberg group [25], see (1.5). In [9] Capogna, Danielli and one of us found explicit formulas for the fundamental solution of the p-sub-Laplacian in any group of Heisenberg type, and for the horizontal p-capacity of rings.

When $\Omega = \boldsymbol{G}$ is a group of Heisenberg type we have discovered that problem (1.2) possesses a one- parameter family of explicit entire solutions.

Theorem 1.1. *Let $\boldsymbol{G}$ be a group of Heisenberg type. For every $\epsilon > 0$ the function*

$$K_\epsilon(g) = \left(\frac{m\,(Q-2)\,\epsilon^2}{(\epsilon^2 \ + |x(g)|^2)^2 + 16\,|y(g)|^2}\right)^{\frac{Q-2}{4}}, \qquad g \in \boldsymbol{G} \tag{1.4}$$

is a positive, entire solution of the Yamabe equation (1.2).

The symbols $x(g)$, $y(g)$ in (1.4) respectively denote the projection of the exponential coordinates of the point $g \in \boldsymbol{G}$ onto the first and second layer of the Lie algebra $\mathfrak{g}$, whereas m indicates the dimension of the first layer. The reader should compare (1.4) with the Jerison-Lee minimizer (1.3). To give a glimpse of the complexity of the present situation with respect to the classical one we recall Folland's mentioned fundamental solution for the Kohn sub- Laplacian on $\mathbb{H}^n$

$$\Gamma(z,t) = C_Q(|z|^4 + t^2)^{-(Q-2)/4}, \tag{1.5}$$

where C_Q is a suitable constant. Whereas Γ is a function of the natural homogeneous gauge $N = N(z,t) = (|z|^4 + t^2)^{1/4}$, the Jerison-Lee minimizer in (1.3) is not. This is in strong contrast with the famous results of Aubin [1], [2] and Talenti [75] who proved that for every value of p the minimizers in the Sobolev embedding are functions with spherical symmetry.

When $\Omega = \boldsymbol{G}$ problem (1.2) is invariant with respect to the group left-translations and also with respect to the scaling

$$u \to u_\lambda = \lambda^{(Q-2)/2}\, u \circ \delta_\lambda, \qquad \lambda > 0.$$

Now it is immediate to verify that $(K_\epsilon)_\lambda = K_{\epsilon/\lambda}$, i.e., one obtains a function of the same type. We were thus naturally led to conjecture that, up to left- translations and dilations, the function K_1 in (1.4) is the only non-negative entire solution to (1.2). Such conjecture, if true, would generalize Jerison and Lee's cited result to groups of Heisenberg type. This problem turns out to be considerably harder than its already difficult Heisenberg group predecessor. Our objective is to come back to it in a subsequent study and prove the conjecture.

We next describe the plan of the paper. Section two is devoted to collect several basic results which will play a role in the following sections. In section three we establish some integral identities for Carnot groups which are reminiscent of those originally discovered for the standard Laplacian by Rellich [68], and subsequently by Pohožaev [67]. The implementation of such identities, whose existence is an interesting fact in its own right, is one of the principal motivations behind this paper. To understand this point the reader should glance at Theorem 3.7 which is the main result of section three. It states that, when the ground domain Ω is starlike with respect to one of its points, the problem

$$\begin{cases} \mathcal{L}u = -f(u) \\ u \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega), \quad u \geq 0, \end{cases} \tag{1.6}$$

admits no non-trivial solution such that $Xu, Zu \in L^\infty(\Omega)$, provided that the following analogue of the famous Pohožaev condition is fulfilled

$$2QF(u) - (Q-2)uf(u) \ \leq \ 0, \tag{1.7}$$

where $F(u) = \int_0^u f(s)ds$. In particular, when $f(u) = u^p$ condition (1.7) reduces to $p \geq \frac{Q+2}{Q-2}$, so that problem (1.2) has no non-trivial solution u such that $Xu \in L^\infty(\Omega)$ and $Zu \in L^\infty(\Omega)$. Here, the notion of starlikeness is expressed by means of the infinitesimal generator Z of the group dilations $\{\delta_\lambda\}_{\lambda>0}$. We remark that such vector field is neither left-invariant, nor it is *sub-unitary* according to C. Fefferman and D.H. Phong [24]. One easily sees that, in exponential coordinates, the vector field Z involves commutators up to maximum length. In the classical case the boundary regularity of the relevant solution which is necessary to apply the Rellich-Pohožaev identity is guaranteed, via standard elliptic theory, by suitable smoothness assumptions on the ground domain Ω, see, e.g., [67]. The situation is drastically different in the sub-elliptic setting even if the domain Ω is C^∞, due to the presence of characteristic points on the boundary of Ω. We recall that the characteristic set of a smooth domain $\Omega \subset \boldsymbol{G}$ with respect to the system X is

$$\Sigma = \Sigma_{\Omega,X} = \{g \in \partial\Omega \mid X_j(g) \in T_g(\partial\Omega),\, j = 1, ..., m\}.$$

A bounded domain with trivial topology in a group of Heisenberg type with odd-dimensional center always has a non-empty characteristic set. On the one hand the L^∞ estimates in the Appendix of this paper, combined with the local regularity theory of Folland and Stein [27], [26], allow to conclude that a weak solution to (1.2) belongs in fact to $C^\infty(\Omega)$. On the other hand near a characteristic point u can experience a sudden loss of regularity. For $\mathcal{L}$-harmonic functions on the Heisenberg group $\mathbb{H}^n$, i.e., solutions of the Kohn sub-Laplacian $\mathcal{L}$, this phenomenon was discovered by D. Jerison [43]. He constructed an explicit solution for the smooth domain $\{(z,t) \in \mathbb{H}^n \mid t > -M|z|^2\}$, with $M > 0$ suitably fixed, which vanishes on the boundary and which is at most in a Hölder class $\Gamma^{0,\alpha}$ near the isolated characteristic point $e = (0,0)$. As a consequence, such $\mathcal{L}$-harmonic function fails to satisfy the condition $Xu \in L^\infty(\Omega)$, as well as $Zu \in L^\infty(\Omega)$. This example should alert the reader about the difficulties that can occur at characteristic points. At this point it should be clear that assuming *a priori*, as we do in Theorem 3.7, the boundedness near the characteristic set of Xu and Zu for a solution of (1.2) constitutes a serious obstacle to overcome. This is even more so for Zu, since, as we have observed, the Z-derivative involves commutators up to maximum order. In connection with the results in section three we mention that for the Heisenberg group $\mathbb{H}^n$ integral identities of Rellich- Pohožaev type were first discovered in [30], [31]. In [31], however, the relevant solutions were a priori assumed to be globally smooth and the basic question of regularity at characteristic points was not addressed.

Section four is devoted to the study of the regularity properties of a weak solution to (1.2) near the characteristic set. This is the central section of the paper. One of the key ingredients of our approach are some sub-elliptic barriers constructed in Theorem 4.3. The existence of these barriers is established under natural geometric assumptions on the domain near the characteristic set, such as uniform starlikeness with respect to the generator of group dilations, see (4.10), plus a suitable condition of “convexity", see (4.12). The latter can be stated as follows. Let $\rho \in C^\infty(\boldsymbol{G})$ be a defining function for Ω. By this we mean that

$$\Omega = \{g \in \boldsymbol{G} \mid \rho(g) < R\}$$

for some $R \in \mathbb{R}$. Denote by ψ the smooth function on $\boldsymbol{G}$ defined by $\psi(g) = |x(g)|^2$, where $x(g)$ indicates the projection onto the first layer V_1 of $\mathfrak{g}$ of the exponential coordinates of $g \in \boldsymbol{G}$. We assume the existence of a neighborhood U of the characteristic set Σ and of a constant $C > 0$ such that for every $g \in \Omega \cap U$

$$\mathcal{L}\rho(g) \ \geq \ C \ < X\rho(g), X\psi(g) > . \tag{1.8}$$

We emphasize that, since $X\rho(g) = 0$ for every $g \in \Sigma$, (1.8) implies in particular that $\mathcal{L}\rho \geq 0$ on Σ. On the other hand it should be clear that a sufficient condition for (1.8) to hold is

$$\mathcal{L}\rho(g) \ \geq \ C \ > \ 0, \qquad \text{for every} \quad g \in \Sigma,$$

i.e., the strict $\mathcal{L}$-sub-harmonicity of the defining function ρ on the characteristic set (we recall that the latter is compact). This latter property is fulfilled, for instance, by those bounded sets which play a key role in the analysis of the Heisenberg group $\mathbb{H}^n$, namely the level sets of the Jerison-Lee minimizers (1.3), and those of the Folland fundamental solution (1.5) above, i.e., the gauge balls in $\mathbb{H}^n$. However, as we show after the proof of Theorem 9.5, (1.8) fails for the non-convex “cone" $\{(z,t) \in \mathbb{H}^n \mid t > -M|z|^2\}$, and the existence of the above mentioned Jerison’s negative example provides a strong reason for this failure. In Theorems 4.6 and 4.7 we prove that a weak solution of (1.2) does possess the properties $Xu,\ Zu \in L^\infty(\Omega)$, provided that the domain Ω is strictly starlike at the characteristic set, and the defining function of Ω satisfies the condition (1.8). This implies that Theorem 3.7 applies and therefore domains having these properties do not support solutions to (1.2), other than the trivial one. We stress that, unlike Theorem 4.6, in which we make no restriction on the step of the group $\boldsymbol{G}$, in Theorem 4.7 to prove the boundedness of Zu near the characteristic set we need to assume that $\boldsymbol{G}$ be of step two. We do not presently know whether the result continues to hold for groups of higher step. In connection with the results of section four we mention that Capogna, Nhieu and one of us [11], [12] have recently obtained a complete solution of the L^p Dirichlet problem for a general class of sub-Laplacians which includes those treated in this paper. The class of domains which is introduced in [12] is however somewhat different from that considered in the present paper and a direct comparison is not immediate. For instance, our assumption (1.8) seems stronger than the *outer $\mathcal{L}$- ball condition* introduced in [12] and it would be interesting to know whether this is really the case. We recall that a uniform outer $\mathcal{L}$-ball condition has been proved to imply the boundedness of the horizontal gradient of the Green function near the boundary, see [57] (for the case of $\mathbb{H}^n$) and [11], [12] for general Hörmander operators. On one hand, by adapting the ideas in Theorem 4.6 we can prove, independently from [12], an analogous result in the context of this paper. On the other hand, we also obtain in Theorem 4.7 the boundedness of Zu, and such result does not seem to follow from the general theory developed in [12].

In section five we continue the study, initiated in section four, of the harmonicity and sub-harmonicity properties of the components $x(g)$ and $y(g)$ of the exponential coordinates in the first and second layer of the Lie algebra of a Carnot group. Using the results in Lemmas 4.2 and 5.2 we prove that the level sets of the function

$$f_\epsilon(g) = \left((\epsilon^2 + |x(g)|^2)^2 + 16|y(g)|^2\right)^{1/4}, \qquad \epsilon \in \mathbb{R},$$

fulfill the geometric assumptions in Theorems 4.6, 4.7. As a consequence, we obtain the following non- existence result.

Theorem 1.2. *Let $\boldsymbol{G}$ be a Carnot group of step two. Given any $R > 0$, and $\epsilon \in \mathbb{R}$ with $\epsilon^2 < R^2$, the function $u \equiv 0$ is the only non-negative weak solution of (1.2) in*

$$\Omega_{R,\epsilon} = \{g \in \boldsymbol{G} \mid (\epsilon^2 + |x(g)|^2)^2 + 16|y(g)|^2 < R^4\}.$$

From Theorem 1.2 we infer, in particular, that in any group of Heisenberg type the gauge balls, and the level sets of the entire solutions in Theorem 1.1 support no solution to problem (1.2) other than the trivial one.

In section six we state without proof (for the latter we refer the reader to [79]) the main result about the existence of global minimizers in every Carnot group. Theorem 6.1 guarantees the existence of an entire non-negative solution to (1.9) below when Ω is the whole group $\boldsymbol{G}$. The proof of such result is based on a suitable adaptation of P. L. Lions' method of concentration of compactness [60] - [63].

In the subsequent sections of the paper we study groups of Heisenberg type. Section seven is devoted to proving Theorem 1.1. In section eight we consider the CR inversion and Kelvin transform, introduced by Korányi [53] for the Heisenberg group, and later generalized to groups of Heisenberg type in [16], [15]. In Proposition 8.2 we show that the inversion preserves starlikeness with respect to the generator of the group dilations. In [15] it was proved that if the ambient group is of Iwasawa type, then the CR Kelvin transform possesses several very useful properties. The ones which are particularly relevant for us are collected in Theorem 8.4. We exploit these results to establish new properties. In Theorem 8.6, we show that the Kelvin transform is an isometry between the Sobolev spaces $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega)$ and $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega^*)$, where Ω^* denotes the image of Ω under the CR inversion. This result, which plays a key role in the next section, when we study equations on unbounded domains, reflects the conformal invariance of the Yamabe equation in (1.2). In Definition 8.12 we introduce the notion of characteristic cones and half-spaces in a Carnot group $\boldsymbol{G}$ of step two. Let $\mathbb{R}^k_+$ denote the cone $\{(y_1, ..., y_k) \in \mathbb{R}^k \mid y_i \geq 0, i = 1, ..., k\}$. Given $M, b \in \mathbb{R}$, and $\mathbf{a} \in \mathbb{R}^k_+ \setminus \{0\}$, we call the open sets

$$C^+_{M,b,\mathbf{a}} = \{g \in \boldsymbol{G} \mid < y(g), \mathbf{a} > > \ M|x(g)|^2 + b\}$$

and

$$C^-_{M,b,\mathbf{a}} = \{g \in \boldsymbol{G} \mid < y(g), \mathbf{a} > < \ -M|x(g)|^2 + b\}$$

characteristic cones. The cone will be said *convex* if $M \geq 0$, *concave* if $M < 0$. When $M = 0$ we use the notation $H^{\pm}_{b,\mathbf{a}}$ to introduce the *characteristic half-spaces*

$$H^+_{b,\mathbf{a}} = C^+_{0,b,\mathbf{a}} = \{g \in \boldsymbol{G} \mid < y(g), \mathbf{a} > > \ b\},$$
$$C^-_{0,b,\mathbf{a}} = \{g \in \boldsymbol{G} \mid < y(g), \mathbf{a} > < \ b\}.$$

The boundaries of such half-spaces are called *characteristic hyperplanes*. It is an interesting fact that the image through the CR inversion of a convex characteristic cone is a left-translation along the center of the group of the bounded domains $\Omega_{R,\epsilon}$ in Theorem 1.2, i.e., the level sets of the entire solutions K_ϵ in Theorem 1.1. We prove this in Proposition 8.13, see also Proposition 8.8. By means of the Kelvin transform we obtain explicit formulas for the Poisson kernel with singularity on the characteristic set for the gauge balls in Iwasawa groups. We have also found explicit formulas for the Poisson kernel for the bounded regions which are the conformal images of the non-characteristic “hyperplanes" in the group $\boldsymbol{G}$. These formulas display a new phemonenon. The behavior of a non-negative $\mathcal{L}$-harmonic function near a singular boundary point changes drastically depending on whether such point is characteristic or not, see Theorems 8.10, 8.14, Remark 8.11 and (8.10).

In section nine we exploit the conformal invariance of (1.2) to establish Lemma 9.1. The latter allows to transplant, via the CR Kelvin transform, problem (1.2) from a set Ω to its conformal image Ω^* under the inversion. Combining Lemma 9.1 with Theorem 1.2 and Proposition 8.13 we obtain the following non-existence result.

Theorem 1.3. *Consider a group of Iwasawa type $\boldsymbol{G}$. Let $C^{\pm}_{M,b,\boldsymbol{a}} \subset \boldsymbol{G}$ be a convex characteristic cone. There exists no solution to* (1.2) *in $\Omega = C^{+}_{M,b,\boldsymbol{a}}$, other than $u \equiv 0$. In particular, there exists no non-trivial solution for the characteristic half-spaces $H^{\pm}_{b,\boldsymbol{a}}$.*

Theorem 1.3 should be viewed as conformally dual to Theorem 1.2. An interesting open question is whether the concave cones support non-trivial solutions to the Yamabe problem (1.2). At the moment we ignore the answer. As we explain at the end of section nine, interestingly, our approach does not work for these regions since, as we previously mentioned, their bounded images through the CR inversion fail to satisfy the convexity assumption (1.8) near the characteristic set. The above stated open problem is closely related to another one. Consider the bounded domain $\Omega_{R,\epsilon}$ in Theorem 1.2, with $\epsilon > 0$, and let $\Omega^* = \boldsymbol{G} \setminus \overline{\Omega}_{R,\epsilon}$. Does Ω^* support non-trivial solutions to (1.2)? We only know the answer when $\epsilon = 0$, i.e., for the complement of a gauge ball, and it is negative.

Theorem 1.4. *Let $\boldsymbol{G}$ be a group of Iwasawa type and consider the unbounded domain $\Omega^* = \{g \in \boldsymbol{G} \mid N(gg_o^{-1}) > R\}$, where N is the gauge in (8.1), $g_o \in \boldsymbol{G}$ and $R > 0$ are fixed. There exist no non-trivial solution to (1.2) in Ω^*.*

Theorems 1.3 and 1.4 are proved in section nine. In connection with Theorem 1.3 we mention that Lanconelli and Uguzzoni [58] have recently obtained in the special case of the Heisenberg group $\mathbb{H}^n$ an interesting non-existence result for the non-characteristic hyperplanes, i.e., those hyperplanes which are parallel to the group center (the t-axis). Their analysis is essentially different from ours since, given the absence of characteristic points on the boundary, their focus is on the asymptotic behavior of a solution to (1.2) at infinity. In a note added in proof in [58] it is said that in the forthcoming article [77] Uguzzoni has been able to obtain, for the characteristic hyperplanes H_a in the Heisenberg group, a uniqueness result similar to the second part of our Theorem 1.3.

It may be appropriate to mention in closing that one of the ultimate goals of our project is to understand as explicitly as possible the extremal functions when $\Omega = \boldsymbol{G}$ in (1.1), for the full range $1 \leq p < Q$. More precisely, when $\boldsymbol{G}$ is a group of Heisenberg type we would like to obtain an appropriate sub-elliptic version of the cited results of Aubin and Talenti. In this connection, given a C^∞, connected open set $\Omega \subset \boldsymbol{G}$ it is interesting to consider the following non- linear Dirichlet problem with critical growth

$$\begin{cases} \mathcal{L}_p u = -u^{p^*-1} \\ u \in \overset{o}{\mathcal{D}}{}^{1,p}(\Omega), \quad u \geq 0, \end{cases} \tag{1.9}$$

where $\mathcal{L}_p u = -\sum_{i=1}^m X_j^*(|Xu|^{p-2}X_j u) = \sum_{i=1}^m X_j(|Xu|^{p-2}X_j u)$ is what we call the *p- sub-Laplacian* of u. Standard variational arguments show that when $\Omega = \boldsymbol{G}$ the problem of characterizing the extremals in (1.1) is equivalent to determining all solutions to (1.9). This is a very difficult task and at the moment we only have some partial progress. We hope to come back to this and related questions in a future study.

The results in this paper were presented at the Conference in memory of Filippo Chiarenza, held in Catania, November 12-14 1998, see [34].

2. Preliminaries

In this section we introduce the relevant definitions and state some results which will be needed in the sequel. Consider the Lie algebra $\mathfrak{g} = \oplus_{j=1}^r V_j$ of $\boldsymbol{G}$. We assume that on $\mathfrak{g}$ there is a scalar product with respect to which the V_j's are mutually orthogonal. The exponential mapping $exp : \mathfrak{g} \to \boldsymbol{G}$ is an analytic diffeomorphism. We use it to define analytic maps $\xi_i : \boldsymbol{G} \to V_i, i = 1, ..., r$, through the equation $g = exp(\xi_1(g) + \xi_2(g) + ... + \xi_r(g))$, if $\xi(g) = \xi_1(g) + ... + \xi_r(g)$ is such that $g = exp(\xi(g))$. With $m = dim(V_1)$, the coordinates of the projection ξ_1 in the basis $X_1, \ldots, X_m$ will be denoted by $x_1 = x_1(g), \ldots, x_m = x_m(g)$, i.e.,

$$x_j(g) =< \xi(g), X_j > \qquad\qquad j = 1, ..., m, \tag{2.1}$$

and we set $x = x(g) = (x_1, \ldots, x_m) \in \mathbb{R}^m$. The Euclidean distance to the origin $|\cdot|$ on $\mathfrak{g}$ induces a homogeneous norm $|\cdot|_\mathfrak{g}$ on $\mathfrak{g}$ and (via the exponential map) one on the group $\boldsymbol{G}$ in the following way (see also [26]). For $\xi \in \mathfrak{g}$, with $\xi = \xi_1 + ... + \xi_r, \xi_i \in V_i$, we let

$$|\xi|_\mathfrak{g} = \left(\sum_{i=1}^r |\xi_i|^{2r!/i} \right)^{2r!}, \tag{2.2}$$

and then define $|g|_{\boldsymbol{G}} = |\xi|_{\mathfrak{g}}$ if $g = exp\ \xi$. Such homogeneous norm on $\boldsymbol{G}$ can be used to define a pseudo- distance on $\boldsymbol{G}$:

$$\rho(g,h) = |h^{-1}g|_{\boldsymbol{G}}. \tag{2.3}$$

The pseudo-distance (2.3) is equivalent to the Carnot- Carathéodory distance $d(\cdot,\cdot)$ generated by the system X, i.e., there exists a constant $C = C(\boldsymbol{G}) > 0$ such that

$$C\ \rho(g,h)\ \leq d(g,h)\ \leq\ C^{-1}\ \rho(g,h), \qquad g,h \in \boldsymbol{G}, \tag{2.4}$$

see [66]. We will almost exclusively work with the distance d, except in few situations where we will find more convenient to use (2.3).

If $B(x,R) = \{y \in \boldsymbol{G}\ \mid\ d(x,y) < R\}$, then by left- translation and dilation it is easy to see that the Haar measure of $B(x,R)$ is proportional to R^Q, where $Q = \sum_{i=1}^{r} i\ \dim V_i$ is the homogeneous dimension of $\boldsymbol{G}$. One has for every $f, g, h \in \boldsymbol{G}$ and for any $\lambda > 0$

$$d(gf, gh) = d(f,h), \qquad d(\delta_\lambda(g), \delta_\lambda(h)) = \lambda\ d(g,h).$$

Further on, we will need to exploit the properties of the exponential coordinates in the second layer of the stratification of $\mathfrak{g}$. We thus fix an orthonormal basis $Y_1, \ldots, Y_k$ of V_2 and, similarly to (2.1), we define the exponential coordinates in the second layer V_2 of a point $g \in \boldsymbol{G}$ by letting

$$y_i(g)\ = <\xi(g), Y_i>, \qquad i = 1\ldots k, \tag{2.5}$$

and $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$. We next recall the Baker-Campbell-Hausdorff formula, see, e.g., [39]

$$exp\,\xi\ \exp\ \eta = exp\,(\xi + \eta + 1/2[\xi,\eta] + \ldots), \qquad \xi,\eta \in \mathfrak{g}, \tag{2.6}$$

where the dots indicate a linear combination of terms of order three and higher which is finite due to the nilpotency of $\boldsymbol{G}$. By definition the order of an element in V_j is j.

We next list some known results. To state the former we recall that given a bounded open set $D \subset \boldsymbol{G}$, and a function $\phi \in C(\partial D)$, the Dirichlet problem for a sub-Laplacian $\mathcal{L}$ and D consists in finding a solution to $\mathcal{L}w = 0$ in D such that $u = \phi$ on ∂D.

Theorem 2.1 (Bony's maximum principle [5]). *Let $D \subset \boldsymbol{G}$ be a connected, bounded open set, and $\phi \in C(\partial D)$. There exists a unique $\mathcal{L}$-harmonic function H^D_ϕ which solves the Dirichlet problem in the sense of Perron-Wiener-Brelot. Moreover, H^D_ϕ satisfies*

$$\sup_D\ |H^D_\phi| \leq \sup_{\partial D}\ |\phi|.$$

Theorem 2.2 (Schauder type interior estimates [78]). *Let $D \subset \boldsymbol{G}$ be an open set and suppose that w is $\mathcal{L}$-harmonic in D. For every $g \in D$ and $r > 0$ for which $\overline{B}(g,r) \subset D$, one has for $s \in \mathbb{N}$*

$$|X_{j_1} X_{j_2} ... X_{j_s} w(g)| \leq \frac{C}{r^s} \max_{\overline{B}(g,r)} |w|,$$

for $j_i \in \{1, ..., m\}$, $i = 1, ..., s$, and for some constant $C = C(\boldsymbol{G}, s) > 0$.

To state the next result we introduce a definition. Given an open set $D \subset \boldsymbol{G}$ we denote with $\mathcal{L}^{1,\infty}(D)$ the space of those distributions $u \in L^\infty(D)$ such that $Xu \in L^\infty(D)$, endowed with the natural norm.

Theorem 2.3 (L^∞ Poincaré inequality [33]). *Given a Carnot group $\boldsymbol{G}$ there exists $C = C(\boldsymbol{G}) > 0$, such that if $u \in \mathcal{L}^{1,\infty}(B(g_o, 3R))$, then u can be modified on a set of measure zero in $\bar{B}(g_o, R)$ so to satisfy*

$$|u(g) - u(h)| \leq C\, d(g,h) \|u\|_{\mathcal{L}^{1,\infty}(B(g_o,3R))}$$

for every $g, h \in \bar{B}(g_o, R)$. If, furthermore, $u \in C^\infty(B(g_o, 3R))$, then only the L^∞ norm of Xu suffices in the right hand side of the previous inequality.

We note explicitly that the theorem asserts that every function $u \in \mathcal{L}^{1,\infty}(B(g_o, 3R))$ has a representative which is Lipschitz continuous in $B(g_o, R)$ with respect to the Carnot- Carathéodory distance d. The reverse implication also holds, see [33].

Let $1 \leq p < \infty$. The notion of *horizontal p- capacity* associated with a system X was introduced in [9], see also [56] for a different, yet equivalent, definition in the case $p = 2$ for the Heisenberg group $\mathbb{H}^n$. We will need the following result which is contained in Theorem 8.1 in [9].

Theorem 2.4 (Capacitary estimates of rings [9]). *Let $\boldsymbol{G}$ be a Carnot group. Given $1 \leq p < Q$ there exist constants $C_1, C_2 > 0$, depending on $\boldsymbol{G}$ and p, such that for every $g \in \boldsymbol{G}$, $0 < r < R$ one has*

$$C_1\, r^{Q-p} \;\leq\; cap_p(B(g,r), B(g,R)) \;\leq\; C_2\, r^{Q-p}.$$

In particular,

$$C_1\, r^{Q-p} \;\leq\; cap_p(B(g,r)) \;\leq\; C_2\, r^{Q-p}.$$

The latter estimates gives

$$cap_p(\{g\}) = \lim_{r \to 0} cap_p(B(g,r)) = 0, \qquad 1 \leq p < Q.$$

We will also need the following special case of Proposition 6.1 from [9].

Proposition 2.5. *Let $\Omega \subset \boldsymbol{G}$ be a bounded open set, and fix $1 < p < Q$. For every relatively closed subdomain $E \subset \Omega$, with $cap_p(E) = 0$, there exists a sequence $\zeta_k \in C_o^\infty(\Omega \setminus E)$ such that $0 \le \zeta_k \le 1$, $\zeta_k \to 1$ in $\Omega \setminus E$ and*

$$\int_\Omega |X\zeta_k|^p dH \to 0$$

as $k \to \infty$.

In the next result we indicate with $N = \sum_{j=1}^{r} dim V_j$ the topological dimension of $\boldsymbol{G}$. The symbol H_{N-1} denotes $(N-1)$-dimensional Hausdorff measure constructed using the Riemannian distance on $\boldsymbol{G}$.

Theorem 2.6. *Let $D \subset \boldsymbol{G}$ be a C^∞ domain and denote by $\Sigma \equiv \Sigma_{D,X} = \{g \in \partial D \mid X_j(g) \in T_g(\partial D),\ j = 1, ..., m\}$ its characteristic set with respect to the system X. One has*

$$H_{N-1}(\Sigma_{D,X}) = 0.$$

Theorem 2.6 is due to Derridj [22], [23]. In the sequel we will denote with $\Gamma^{k,\alpha}$, $\Gamma^{k,\alpha}_{loc}$ the Folland-Stein Hölder classes, see [25].

Theorem 2.7. *Let D be a bounded C^∞ domain in the Heisenberg group $\mathbb{H}^n$ and let $\phi \in C_o^\infty(\mathbb{H}^n)$ be supported in a small neighborhood of a non-characteristic point $g_o \in \partial D$. Given $f \in \Gamma^{k,\alpha}(\overline{D})$, $k \in \mathbb{N} \cup \{0\}$, $0 < \alpha < 1$, then for the unique solution u to the Dirichlet problem for the Kohn sub-Laplacian*

$$\mathcal{L}u = f \quad in \quad D\ , \quad u = 0 \quad on \quad \partial D\ , \tag{2.7}$$

one has $\phi u \in \Gamma^{k+2,\alpha}(\overline{D})$.

Theorem 2.7 is a special case of the results of Jerison in [42]. It is quite natural to conjecture that Theorem 2.7 is in fact valid for arbitrary groups. However, for an arbitrary Carnot group $\boldsymbol{G}$ a corresponding Schauder theory at non-characteristic points is presently lacking, and this is why we now introduce the following hypothesis which will be assumed valid throughout the paper:

(2.8) *Let $D \subset \boldsymbol{G}$ be a bounded C^∞ domain and consider $f \in \Gamma^{k,\alpha}(\overline{D})$, $k \in \mathbb{N} \cup \{0\}$, $0 < \alpha < 1$. For every $g_o \in \partial D \setminus \Sigma$ there exists a neighborhood U of g_o such that the solution u to (2.7) belongs to $\Gamma^{k+2,\alpha}(\overline{D} \cap U)$.*

We plan to return to this point in a future study.

3. Some integral identities and their consequences

In this section we establish some integral identities for solutions of the following problem

$$\begin{cases} \mathcal{L}u = -f(u) \\ u \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega), \quad u \geq 0. \end{cases} \tag{3.1}$$

Such identities are reminiscent of those originally discovered by Rellich [68] and subsequently by Pohožaev [67] for Laplace equation. Unlike what happens in the classical case, however, the presence of characteristic points on the boundary of Ω causes weak solutions of (3.1) to lack the amount of regularity which is necessary to implement such integral identities. The subsequent section will be devoted to overcoming this serious obstacle.

Theorem 3.1. *Let $\boldsymbol{G}$ be a Carnot group and let $D \subset \boldsymbol{G}$ be a C^1 bounded open set with outer unit normal η. For $u \in C^2(\overline{D})$ one has*

$$2\sum_{i=1}^{m} \int_{\partial D} Yu\ X_i u\ < X_i, \eta >\ dH_{N-1} + \int_D div_{\boldsymbol{G}} Y\ |Xu|^2\, dH$$
$$- 2\sum_{i=1}^{m} \int_D X_i u\ [X_i, Y]u\, dH\ - 2\int_D Yu\ \mathcal{L}u\, dH$$
$$= \int_{\partial D} |Xu|^2\ < Y, \eta >\ dH_{N-1},$$

where Y is any smooth vector field in $\boldsymbol{G}$.

Proof. The divergence theorem gives

$$\int_{\partial D} |Xu|^2\ < Y, \eta >\ dH_{N-1} = \int_D div_{\boldsymbol{G}} Y\ |Xu|^2\, dH$$
$$- 2\sum_{i=1}^{m} \int_D X_i u\ [X_i, Y]u\, dH + 2\int_D < X(Yu), Xu >\ dH.$$

In the above we have denoted by $div_{\boldsymbol{G}}$ the Riemannian divergence in $\boldsymbol{G}$. Another application of the divergence theorem and the observation that $div_{\boldsymbol{G}}\, X_i = 0$ allow to obtain

$$\int_D < X(Yu), Xu >\ dH = \sum_{i=1}^{m} \int_{\partial D} Yu\ X_i u\ < X_i, \eta$$
$$>\ dH_{N-1} - \int_D Yu\ \mathcal{L}u\, dH.$$

The latter two identities imply the conclusion. □

We next make a special choice of the vector field Y in Theorem 3.1, namely we let $Y = Z$, where Z is the infinitesimal generator of the one-parameter group of non- isotropic dilations $\{\delta_\lambda\}_{\lambda>0}$. Such vector field is characterized by the property that a function $u : \boldsymbol{G} \to \mathbb{R}$ is homogeneous of degree s with respect to $\{\delta_\lambda\}_{\lambda>0}$, i.e., $u(\delta_\lambda(x)) = \lambda^s u(x)$ for every $x \in \boldsymbol{G}$, if and only if $Zu = su$. We will need the following

Lemma 3.2. *In a Carnot group* $\boldsymbol{G}$ *the infinitesimal generator of group dilations* Z *enjoys the following properties:*

(i) $div_{\boldsymbol{G}} Z \equiv Q$.
(ii) For any $X_i \in X = \{X_1, ..., X_m\}$ *one has* $[X_i, Z] = X_i$.
(iii) $\mathcal{L}\,(Zu) = Z(\,\mathcal{L}\,u) + 2\,\mathcal{L}\,u$*, for any* $u \in C^\infty(\boldsymbol{G})$.
(iv) In particular, Zu *is* $\mathcal{L}$*-harmonic if such is* u.

Proof. Properties $(ii) - (iv)$ where established in [21], so we only need to prove (i). This follows from the fact that $dH(\delta_\lambda(x)) = \lambda^Q dH(x)$ and thus taking the Lie derivative of the volume form in the direction of Z gives $div_{\boldsymbol{G}} Z = Q$, i.e., $Z^* = -Z + Q$. □

Property (iv) is useful in obtaining higher regularity for a solution of problem (3.1) at characteristic points. We notice explicitly that the vector field Z is not sub-unitary according to the definition of C. Fefferman and D. H. Phong [24] and that its expression in exponential coordinates involves derivation along the vector fields X_j, $j = 1, ..., m$, and their commutators up to maximum length.

Corollary 3.3. *Under the assumptions of Theorem 3.1, let* Z *be the generator of the group dilations. For any* $u \in C^2(\overline{D})$ *we have*

$$2\sum_{i=1}^{m}\int_{\partial D} Zu\ X_i u\ < X_i, \eta >\ dH_{N-1} + (Q-2)\int_D\ |Xu|^2\,dH$$
$$-\,2\int_D Zu\,\mathcal{L}u\,dH = \int_{\partial D}\ |Xu|^2\ < Z, \eta >\ dH_{N-1}$$

Proof. It follows immediately from Theorem 3.1 and from Lemma 3.2. □

We next state our basic sub-elliptic Rellich-Pohožaev identity.

Theorem 3.4. *Let* $D \subset \boldsymbol{G}$ *be a* C^1 *domain and* $u \in C^2(\overline{D})$ *be a solution of*

(3.2) $$\mathcal{L}u = -f(u), \qquad in \quad D,$$

for some function $f \in C(\mathbb{R})$ *such that* $f(0) = 0$*. Setting* $F(s) = \int_0^s f(t)dt$*, the following identity holds*

$$\begin{aligned}
&\int_D [2QF(u) - (Q-2)uf(u)]\, dH \\
&\quad = 2\sum_{j=1}^{m} \int\limits_{\partial D} Zu\, X_j u < X_j, \eta > dH_{N-1} - \int\limits_{\partial D} |Xu|^2 < Z, \eta > dH_{N-1} \\
&\quad + 2 \int\limits_{\partial D} F(u) < Z, \eta > dH_{N-1} \; + (Q-2) \\
&\quad \times \sum_{j=1}^{m} \int\limits_{\partial D} u\, X_j u < X_j, \eta > dH_{N-1}.
\end{aligned}$$

Proof. We obtain from (3.2) and from the divergence theorem

$$\begin{aligned}
&-2 \int_D Zu\mathcal{L}u\, dH = 2\int_D Z(F(u))\, dH \\
&\quad = -2\int_D div_{\boldsymbol{G}} Z\; F(u)\, dH + 2\int_{\partial D} F(u) < Z, \eta > \; dH_{N-1} \\
&\quad - 2Q \int_D F(u)\, dH + 2\int_{\partial D} F(u) < Z, \eta > \; dH_{N-1},
\end{aligned}$$

where in the latter equality we have used (i) of Lemma 3.2. Next, we use the equation

$$|Xu|^2 = \frac{1}{2}\mathcal{L}(u^2) - u\mathcal{L}u$$

in combination with (2.2) and the divergence theorem to obtain

$$\int_D |Xu|^2\, dH = \int_D uf(u)dH + \sum_{j=1}^{m} \int\limits_{\partial D} u\, X_j u < X_j, \eta > dH_{N-1}.$$

Substitution in Corollary 3.3 completes the proof. □

Lemma 3.5. *Let* $D \subset \boldsymbol{G}$ *be an open set and suppose that* $u \in \Gamma^{0,\alpha}_{loc}(D)$ *be a solution to (2.2) for some function* $f \in C^{\infty}(\mathbb{R})$*. Then* $u \in C^{\infty}(D)$*.*

Proof. By the assumptions we have $\mathcal{L}u \in \Gamma^{0,\alpha}_{loc}(D)$. The local regularity theory developed by Folland and Stein, see [26], allows to infer $u \in \Gamma^{2,\alpha}_{loc}$. At this point the conclusion follows from the smoothnes of f by a standard iteration argument. □

To state our next result we introduce a definition.

Definition 3.6. *Let $D \subset \boldsymbol{G}$ be a connected open set of class C^1 containing the group identity e. We say that D is starlike with respect to the identity e (or simply starlike) along a subset $M \subset \partial D$, if*

$$< Z, \eta > (g) \ \geq \ 0$$

at every $g \in M$. D is called starlike with respect to the identity e if it is starlike along $M = \partial D$. We say that D is uniformly starlike with respect to e along M if there exists a constant $\alpha = \alpha_D > 0$ such that for every $g \in M$

$$< Z, \eta > (g) \ \geq \ \alpha.$$

A domain as above is called starlike (uniformly starlike) with respect to one of its points g along $M \subset \partial D$, if $g^{-1}D$ is starlike (uniformly starlike) along $g^{-1}M$ with respect to e.

Theorem 3.7. *Let $D \subset \boldsymbol{G}$ be C^∞, bounded and starlike with respect to $g_o \in D$. Suppose that $u \in \Gamma^{0,\alpha}(\overline{D})$ is a non-negative solution of (2.1) with $f \in C^\infty(\mathbb{R})$, such that $u = 0$ on ∂D. Assume in addition that $Xu \in L^\infty(D)$ and $Zu \in L^\infty(D)$. If*

$$2QF(u) - (Q-2)uf(u) \ \leq \ 0, \tag{3.3}$$

then $u \equiv 0$. In particular, if $\boldsymbol{G}$ is of step two (2.1) has no non-trivial such solution when $f(u) = u^q$, if $q \geq \frac{Q+2}{Q-2}$.

Remark 3.8. The inequality (3.3) is the analogue of the famous Pohožaev condition for Laplace equation, see [67].

Proof. By invariance with respect to left-translation we can assume that $g_o = e$. According to Lemma 3.5 we have $u \in C^\infty(D)$. Recall that Σ is a compact set. By Theorem 1 in [81] (see also Theorem 4 in [52] and the results in [22]) for every bounded open neighborhood U of Σ one has $u \in C^\infty(\overline{D} \setminus U)$. Using this observation and Theorem 2.6 we can choose an exhaustion of D with a family of C^∞, connected, open sets $D_\epsilon \nearrow D$, as $\epsilon \to 0$, such that $u \in C^\infty(\overline{D_\epsilon})$, and for which $\partial D_\epsilon = \Gamma_\epsilon^1 \cup \Gamma_\epsilon^2$, with $\Gamma_\epsilon^1 \subset \partial D \setminus \Sigma$, $\Gamma_\epsilon^1 \nearrow \partial D \setminus \Sigma$, $H_{N-1}(\Gamma_\epsilon^2) \to 0$. We apply Theorem 3.4 to the sets D_ϵ to obtain

$$\int_{D_\epsilon}[2QF(u)-(Q-2)uf(u)]\,dH = 2\sum_{j=1}^m \int_{\partial D_\epsilon} Zu\, X_ju < X_j,\eta > dH_{N-1}$$

$$-\int_{\partial D_\epsilon}|Xu|^2 < Z,\eta > dH_{N-1} + 2\int_{\partial D_\epsilon} F(u) < Z,\eta > dH_{N-1} + (Q-2)$$

$$\times \sum_{j=1}^m \int_{\partial D_\epsilon} u\, X_ju < X_j,\eta > dH_{N-1} = 2\sum_{j=1}^m \int_{\Gamma_\epsilon^1} Zu\, X_ju < X_j,\eta > dH_{N-1}$$

$$-\int_{\Gamma_\epsilon^1}|Xu|^2 < Z,\eta > dH_{N-1} + 2\int_{\Gamma_\epsilon^1} F(u) < Z,\eta > dH_{N-1} + (Q-2)$$

$$\times \sum_{j=1}^m \int_{\Gamma_\epsilon^1} u\, X_ju < X_j,\eta > dH_{N-1} + 2\sum_{j=1}^m \int_{\Gamma_\epsilon^2} Zu\, X_ju < X_j,\eta > dH_{N-1}$$

$$-\int_{\Gamma_\epsilon^2}|Xu|^2 < Z,\eta > dH_{N-1} + 2\int_{\Gamma_\epsilon^2} F(u) < Z,\eta > dH_{N-1} + (Q-2)$$

$$\times \sum_{j=1}^m \int_{\Gamma_\epsilon^2} u\, X_ju < X_j,\eta > dH_{N-1}.$$

Since $u = 0$ on Γ_ϵ^1 and $u > 0$ inside D_ϵ, one has $Du(g) = k(g)\eta(g)$ for every $g \in \Gamma_\epsilon^1$, for a function $k \le 0$. This implies $F(u) = 0$ on Γ_ϵ^1 and also

$$Zu \sum_j X_ju < X_j,\eta >= k < Z,\eta > \sum_j X_ju < X_j,\eta >= |Xu|^2 < Z,\eta >,$$

and the above identity gives

$$\int_{D_\epsilon}[2QF(u)-(Q-2)uf(u)]\,dH \; - \int_{\Gamma_\epsilon^1}|Xu|^2 < Z,\eta > dH_{N-1}$$

$$= 2\sum_{j=1}^m \int_{\Gamma_\epsilon^2} Zu\, X_ju < X_j,\eta > dH_{N-1} - \int_{\Gamma_\epsilon^2}|Xu|^2 < Z,\eta > dH_{N-1}$$

$$+2\int_{\Gamma_\epsilon^2} F(u) < Z,\eta > dH_{N-1} + (Q-2)\sum_{j=1}^m \int_{\Gamma_\epsilon^2} u\, X_ju < X_j,\eta > dH_{N-1}.$$

(3.4)

By the assumption $Xu, Zu \in L^\infty(D)$, and from the fact that $H_{N-1}(\Gamma_\epsilon^2) \to 0$, we infer that the boundary integrals in the right-hand side of (3.4) tend to zero.

On the other hand, in view of the starlikeness of D we have $< Z, \eta > \geq 0$, so that we obtain from the monotone convergence theorem

$$\int_{\Gamma_\epsilon^1} |Xu|^2 < Z, \eta > dH_{N-1} \rightarrow \int_{\partial D} |Xu|^2 < Z, \eta > dH_{N-1},$$

whereas

$$\int_{D_\epsilon} [2QF(u) - (Q-2)uf(u)]\, dH \rightarrow \int_D [2QF(u) - (Q-2)uf(u)]\, dH,$$

thanks to the assumption $u \in \Gamma^{0,\alpha}(\overline{D})$. These considerations allow to conclude

$$\int_D [2QF(u) - (Q-2)uf(u)]\, dH - \int_{\partial D} |Xu|^2 < Z, \eta > dH_{N-1} = 0. \tag{3.5}$$

Using (3.3) we finally obtain

$$\int_{\partial D} |Xu|^2 \ < Z, \eta > \ dH_{N-1} = 0. \tag{3.6}$$

The divergence theorem and (i) of Lemma 3.2 give

$$\int_{\partial D} < Z, \eta > dH_{N-1} \ = \ Q\ |D|.$$

We must thus have $< Z, \ \eta > > \ 0$ on some subset of ∂D of positive H_{N-1} measure. From the smoothness of D we infer the existence of an open set $V \subset \boldsymbol{G}$ such that $< Z, \ \eta > \geq \ \alpha \ > \ 0$ on $\Delta = V \cap \partial D$. Since the characteristic set Σ is compact, we can assume without loss of generality that $\Delta \cap \Sigma \ = \Omega$. In conclusion we see that the horizontal gradient vanishes on Δ. By extending u across the boundary by setting it equal to zero outside of D we obtain a weak solution to (3.1) in V which vanishes in the open set $V^+ = V \cap (\boldsymbol{G} \setminus \overline{D})$. By Theorem 10.6 in the Appendix we conclude that must be $u \ \equiv \ 0$ in D. This proves the first part of the theorem.

Suppose now that $\boldsymbol{G}$ is of step two. The non-linearity $f(u) = u^{\frac{Q+2}{Q-2}}$, for $u \geq 0$, $f(u) \equiv 0$, when $u \leq 0$, is not in $C^\infty(\mathbb{R})$, since $1 < \frac{Q+2}{Q-2} < 2$ when $Q > 6$, but only belongs to a class $C^{1,\delta}_{loc}(\mathbb{R})$. Since $f \in C^\infty(0,\infty)$, using the local regularity theory in [26] we conclude as before that $u \in C^\infty(D)$. The assumption (2.8) guarantees the regularity $\Gamma^{3,\beta}$ of u up to the boundary away from the characteristic set Σ. Since for a group of step two the generator of dilations Z involves only the X_j's and their first commutators, we infer that Zu is in $\Gamma^{1,\beta}$ up to the boundary in a neighborhood of a non-characteristic point. We can thus apply the Rellich type identity to the exaustion domains D_ϵ and argue as in the first part. $\square$

4. Regularity at the characteristic set of the horizontal and radial derivatives

In the previous section we have obtained a uniqueness result for non-negative solutions of a Yamabe type equation by making the strong a priori assumptions that $Xu, Zu \in L^\infty(D)$. In practice, the existence of characteristic points on the boundary of the ground domain imposes serious restrictions to the regularity of the solution, see [43], [12], and it is not clear that Theorem 3.7 has any content at all. The purpose of this section is to prove that it does indeed, at least if the domain D satisfies some very natural and simple geometric requirements. We start by considering a weak solution to the non-linear Dirichlet problem with critical exponent in a connected, bounded open set $\Omega \subset \boldsymbol{G}$

$$\begin{cases} \mathcal{L}u = -u^{\frac{Q+2}{Q-2}} \\ u \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega), \quad u \geq 0. \end{cases} \tag{4.1}$$

By Theorem 10.1 in the appendix we know that $u \in L^\infty(\Omega)$. This crucial information allows to implement the local regularity theory of Folland and Stein, [27], [26], as in the proof of Lemma 3.5, to conclude $u \in C^\infty(\Omega)$. We next suppose that Ω satisfies in addition the following natural condition: There exist $A, r_o > 0$ such that for every $Q \in \partial\Omega$ and every $0 < r < r_0$

$$|(\boldsymbol{G} \setminus \Omega) \cap B(Q,r)| \geq A|B(Q,r)|. \tag{4.2}$$

Such geometric assumption is fulfilled if, e.g., Ω satisfies the uniform corkscrew condition, see [10], [12]. These papers contain an extensive study of examples of domains which, in particular, satisfy (4.2). What counts for us is that (4.2) allows to adapt to the present setting the classical arguments that lead, via Moser's iteration, to obtain $u \in \Gamma^{0,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$, see, e.g., [35], Section 8.10. Extending u with zero outside Ω, we can assume henceforth that

$$u \in \Gamma^{0,\alpha}(\boldsymbol{G}). \tag{4.3}$$

If we suppose further that Ω is a C^∞ domain, and denote by $\Sigma = \Sigma_{\Omega,X}$ the characteristic set of Ω, then thanks to the assumption (2.8) at every non-characteristic point $g_o \in \partial\Omega$ one has in fact $u \in \Gamma^{3,\delta}(\overline{\Omega} \cap U)$ for a suitable neighborhood U of g_o. To see this observe that the non-linearity $f(u) = u^{\frac{Q+2}{Q-2}}$, for $u \geq 0$, $f(u) \equiv 0$, when $u \leq 0$, belongs in general to a class $C^{1,\delta}_{loc}(\mathbb{R})$ since $1 < \frac{Q+2}{Q-2} < 2$ when $Q > 6$. We note explicitly that thanks to Theorem 2.7 this regularity is amply fulfilled in the important case of the Heisenberg group $\mathbb{H}^n$. From these considerations it is clear that the main new obstacle to overcome is the regularity of a weak solution to (4.1) near the characteristic set Σ. To this task we now turn.

Our first objective is the construction of suitable sub-elliptic barriers for Carnot groups. The existence of such barriers has far reaching implications. We begin with a simple, yet useful result.

Lemma 4.1. *Let $\boldsymbol{G}$ be a Carnot group of step r. For every bounded set $V \subset \boldsymbol{G}$ there exists a constant $C(r, V) > 0$ such that for every $g \in V$ and $0 \leq \lambda \leq 1$*

$$d(\delta_\lambda(g), g) \ \leq \ C\ (1-\lambda)^{1/r}.$$

Proof. In view of (2.4) it suffices to prove the above inequality for the pseudo-distance ρ. We illustrate the proof in the case $r = 2$, and then indicate the changes necessary when r is arbitrary. By (2.3) we have for every $g \in \boldsymbol{G}$

$$\rho(\delta_\lambda(g), g) = |g^{-1}\delta_\lambda(g)|_{\boldsymbol{G}} = |Y|_{\mathfrak{g}},$$

where $Y \in \mathfrak{g}$ is such that $exp(Y) = g^{-1}\delta_\lambda g$. If $g = exp(\xi)$, with $\xi = \xi_1 + \xi_2$, we obtain from the Baker-Campbell-Hausdorff formula

$$exp(Y) = exp(-\xi + \Delta_\lambda(\xi) - 1/2[\xi, \Delta_\lambda(\xi)]).$$

It is now easy to see that $[\xi, \Delta_\lambda(\xi)] = [\xi_1 + \xi_2, \lambda\xi_1 + \lambda^2\xi_2] = 0$. We thus have $Y = -\xi + \Delta_\lambda(\xi) = (\lambda - 1)\xi_1 + (\lambda^2 - 1)\xi_2$. Applying (2.2) for $|Y|_g$ one easily obtains the conclusion if g belongs to a bounded set V, with a constant $C = C(V) > 0$. This proves the lemma when $r = 2$. In a group of step r one has $\xi = \xi_1 + ... + \xi_r$ and the Baker-Campbell-Hausdorff formula contains, besides, $[\xi, \Delta_\lambda(\xi)]$, commutators of higher order. However, one sees easily that

$$[\xi, \Delta_\lambda(\xi)] = (\lambda - 1)\sum_{i<j} p_{i,j}(\lambda)[\xi_i, \xi_j],$$

where $p_{i,j}(\lambda)$ is a polynomial. Using this fact one reaches the conclusion similarly to the case $r = 2$. □

Given a point $g \in \boldsymbol{G}$ we let $x(g) = (x_1(g), ..., x_m(g))$, where $x_j(g)$ is defined as in (2.1). The following lemma will play a crucial role in the sequel.

Lemma 4.2. *The function $\psi(g) \overset{def}{=} |x(g)|^2$ enjoys the following properties:*

$$\mathcal{L}\psi = 2m, \tag{4.4}$$

$$|X\psi|^2 = 4\psi. \tag{4.5}$$

Furthermore, for any $M_o > 0$ one has

$$\mathcal{L}(\psi/M_o) \ \geq \ m/M_o + |X(\psi/M_o)|^2, \tag{4.6}$$

at all points of the set $\{g \in \boldsymbol{G} \mid |x(g)|^2 \leq \frac{mM_o}{4}\}$.

Proof. We define the functions

$$\psi_j(t) = |\xi_1(g\ exp(tX_j))|^2 \qquad\qquad j = 1, ..., m.$$

The Baker-Campbell-Hausdorff formula implies

$$\begin{aligned} g\ exp(tX_j) &= exp(\xi_1(g) + tX_j + \xi_2(g) \\ &+ ... + \xi_r(g) + \frac{1}{2}[\xi_1(g) + ... + \xi_r(g), tX_j] + ...). \end{aligned}$$

From this one immediately sees that

$$\psi_j(t) = |\xi_1(g)|^2 + 2t < \xi_1(g), X_j > +t^2. \tag{4.7}$$

One obtains from (4.7)

$$\psi_j'(0) = 2 < \xi_1(g), X_j >= 2x_j(g), \qquad\qquad \psi_j''(0) = 2. \tag{4.8}$$

The equation (4.8) gives

$$\mathcal{L}\psi = \sum_{j=1}^{m} \psi_j''(0) = 2m,$$

$$|X\psi|^2 = \sum_{j=1}^{m} \psi_j'(0)^2 = 4\sum_{j=1}^{m} < \xi(g), X_j >^2 = 4\sum_{j=1}^{m} |x_j(g)|^2 = 4\psi,$$

which proves the first part of the lemma. The second part follows from the first by elementary considerations. □

Henceforth, we denote with ψ the function in Lemma 4.2. We next consider a C^∞, connected, bounded open set $\Omega \subset \boldsymbol{G}$. Since our assumptions on Ω are of a local nature, and they involve the geometry of the domain near its characteristic set Σ, there is no restriction in assuming the existence of $\rho \in C^\infty(\boldsymbol{G})$ and of $\gamma_\Omega > 0$ such that for some $R \in \mathbb{R}$

$$\Omega = \{g \in \boldsymbol{G} \mid \rho(g) < R\}, \tag{4.9}$$

and for which one has $|D\rho(g)| \geq \gamma_\Omega > 0$, for every g in some relatively compact neighborhood K of ∂D. The outward pointing unit normal to $\partial\Omega$ is $\eta = \frac{D\rho}{|D\rho|}$. In the next theorem we prove that if Ω satisfies two natural geometric assumptions at the charateristic set, then one can construct some sub-elliptic barriers near Σ. One such hypothesis is that Ω be uniformly starlike along Σ, see Definition 3.6, with respect to one of its points, which by performing a left-translation we can take to be the group identity e. We explicitly remark that when this is the case, then by the compactness of Σ we can find a bounded open set U and a constant $\delta > 0$ such that $\Sigma \subset U$ and for which, setting $\Delta = \partial\Omega \cap U$, one has

$$Z\rho(g_o) \geq \delta \ > \ 0, \qquad \text{for } g_o \in \Delta. \tag{4.10}$$

We note that the uniform transversality condition (4.10) implies that the trajectories of Z starting from points of Δ fill a full open set ω interior to Ω. This can be seen by locally "straightening out" Z and taking a finite cover. In each of the neighborhoods where Z is constant the trajectories are straight lines transversal to Δ. By possibly shrinking the set U we can assume that $\omega = \Omega \cap U$. To fix the notation we suppose that there exists $\lambda_o > 0$ such that

$$\delta_\lambda g_o \in \omega \qquad \text{for} \quad \lambda_o < \lambda < 1.$$

Henceforth, we assume that the parameter $M_o > 0$ in (4.6) in Lemma 4.2 has been chosen sufficiently large. Precisely, given the domain Ω, and an open neighborhood U of the characteristic set fixed as in the preceding discussion, we assume that $M_o > 0$ has been so chosen that it fulfills the condition

$$\overline{U} \subset \{g \in \boldsymbol{G} \mid |x(g)|^2 \leq \frac{mM_o}{4}\}. \tag{4.11}$$

Having made this choice, we will henceforth assume that the inequality (4.6) in Lemma 4.2 is valid in the whole U, and therefore in ω. This being said, we will continue to use the symbols $\Delta, \omega, \lambda_o$ and M_o with the same meaning as above throughout the rest of the section.

Theorem 4.3. *Let $\Omega \subset \boldsymbol{G}$ be a smooth, connected bounded open set as in (4.9) which is uniformly starlike along Σ with respect to one of its points. We assume in addition that there exists $M_1 > 0$ such that the definining function ρ of Ω satisfies the differential inequality*

$$\mathcal{L}\rho \ \geq \ \frac{2}{M_1} \ < X\rho, X\psi > \qquad in \quad \omega. \tag{4.12}$$

Let $M \geq max\{M_o, M_1\}$. For $0 < \alpha \leq 1$ we define

$$\Psi_\alpha = (R - \rho)^\alpha e^{-\psi/M}.$$

Under the stated hypothesis we obtain for every $g \in \omega$

$$\mathcal{L}\Psi_\alpha(g) \ \leq -\frac{m}{M} \ \Psi_\alpha(g). \tag{4.13}$$

Furthermore, there exist $C_1, C_2 > 0$ such that for every $g_o \in \Delta$ and $\lambda_o \leq \lambda \leq 1$ one has

$$C_1(1-\lambda)^\alpha \ \leq \ \Psi_\alpha(\delta_\lambda g_o) \ \leq \ C_2(1-\lambda)^\alpha. \tag{4.14}$$

Proof. We note that for any function ϕ on G and another function f on the real line one has

$$\mathcal{L}(f(\phi)) = f''(\phi)|X\phi|^2 + f'(\phi)\mathcal{L}\phi.$$

This observation implies

$$\begin{aligned}
\mathcal{L}\Psi_\alpha &= (R-\rho)^\alpha \mathcal{L}(e^{-\psi/M}) + e^{-\psi/M}\mathcal{L}((R-\rho)^\alpha) \\
&\quad + 2 < X((R-\rho)^\alpha), X(e^{-\psi/M}) > \\
&= \left[|X(\psi/M)|^2 - \mathcal{L}(\psi/M)\right] \Psi_\alpha \\
&\quad + \left[\frac{\alpha(\alpha-1)}{(R-\rho)^2}\,|X\rho|^2 - \frac{\alpha}{(R-\rho)}\mathcal{L}\rho\right] \Psi_\alpha \\
&\quad + \frac{2\alpha}{M(R-\rho)} < X\rho, X\psi > \Psi_\alpha \\
&= \left[-\frac{\alpha(1-\alpha)}{(R-\rho)^2}|X\rho|^2 - \frac{\alpha}{R-\rho}\mathcal{L}\rho \right.\\
&\quad \left. + \frac{2\alpha M^{-1}}{R-\rho} < X\rho, X\psi > + |X(\psi/M)|^2 - \mathcal{L}(\psi/M)\right] \Psi_\alpha .
\end{aligned}$$

The first term above is negative and (4.12) gives

$$-\frac{\alpha}{R-\rho}\mathcal{L}\rho + \frac{2\alpha M^{-1}}{R-\rho} < X\rho, X\psi > \le 0.$$

From this it is clear that (4.13) would follow provided that one has on ω

$$|X(\psi/M)|^2 - \mathcal{L}(\psi/M) \le -m/M.$$

The latter inequality is a consequence of the fact that (4.6) holds for all points in $\overline{U}$, with M_o replaced by M, thanks to (4.11) and to the trivial inclusion $\{g \in \boldsymbol{G} \mid |\xi_1(g)|^2 \le \frac{mM_o}{4}\} \subset \{g \in \boldsymbol{G} \mid |\xi_1(g)|^2 \le \frac{mM}{4}\}$.

The proof of (4.14) is obtained as follows. We consider for a fixed $g_o \in \Delta$ the smooth function $\phi(\lambda) = \rho(\delta_\lambda(g_o))$. By taking a Taylor expansion about the point $\lambda = 1$, and keeping in mind that $\phi(1) = \rho(g_o) = R$, we find for every $g_o \in \Delta$ and $\lambda_o \le \lambda \le 1$

$$R - \rho(\delta_\lambda(g_o)) = Z\rho(g_o)(1-\lambda)[1 + O(1-\lambda)], \tag{4.15}$$

where $O(1-\lambda)$ denotes a function which is bounded by $C(1-\lambda)$, uniformly in $g_o \in \Delta$ and $\lambda_o \le \lambda \le 1$. It is clear that from the latter identity (4.14) follows using (4.10), the smoothness of ρ and the boundedness of Ω. □

Remark 4.4. We mention that for the Heisenberg group $\mathbb{H}^n$ sub-elliptic barriers related to those in Theorem 4.3 were found by one of us in [29].

Theorem 4.5. *Consider a C^∞ domain Ω in a Carnot group $\boldsymbol{G}$ satisfying* (4.2) *and all the hypothesis in Theorem 4.3, including (4.12). Let u be a weak solution of (4.1), then there exists a constant $C = C(\boldsymbol{G}, \Omega, u) > 0$ such that for every $g_o \in \Delta$ and $0 \le \lambda \le 1$ one has*

$$u(\delta_\lambda(g_o)) \le C\ (1-\lambda) \tag{4.16}$$

Proof. We begin by observing that thanks to (4.3) and to the fact that $u = 0$ on $\partial\Omega$, we have for any $g_o \in \partial\Omega$

$$u(\delta_\lambda g_o) \le C\ d(\delta_\lambda g_o, g_o)^\alpha, \tag{4.17}$$

where d is as in (2.3). Lemma 4.1 now gives for every $g_o \in \Delta$ and $0 \le \lambda \le 1$

$$d(\delta_\lambda g_o, g_o) \le C\ (1-\lambda)^{1/r}, \tag{4.18}$$

for some constant $C = C(\Omega) > 0$. Using (4.17), (4.18) and setting $\alpha_1 = \alpha/r$ we infer

$$u(\delta_\lambda g_o) \le C\ (1-\lambda)^{\alpha_1} \tag{4.19}$$

for every $g_o \in \Delta$, $\lambda_o < \lambda < 1$. Clearly, $0 < \alpha_1 < 1$. We now let $\sigma = 2^* - 1 = (Q+2)/(Q-2)$ and notice that $\sigma > 1$. Choose $n \in \mathbb{N}$ such that $\sigma^{-n} \le \alpha_1$ and let $\alpha_o = \sigma^{-n}$ so that

$$\sigma^n\ \alpha_o = 1. \tag{4.20}$$

Observe that (4.19) implies trivially

$$u(\delta_\lambda g_o) \le C\ (1-\lambda)^{\alpha_o} \tag{4.21}$$

for every $g_o \in \Delta$, $\lambda_o < \lambda < 1$. We next use the barriers constructed in Theorem 4.3. For any point $\delta_\lambda g_o \in \omega$ we have from (4.1), (4.21), (4.14) and from (4.13)

$$\begin{aligned} &-\mathcal{L}u(\delta_\lambda g_o) = u(\delta_\lambda g_o)^\sigma \\ &\quad \le C(1-\lambda)^{\sigma\alpha_o} \le CC_1^{-1}\Psi_{\sigma\alpha_o}(\delta_\lambda g_o) \le -CC_1^{-1}Mm^{-1}\mathcal{L}\Psi_{\sigma\alpha_o}(\delta_\lambda g_o) \\ &\quad = -\mathcal{L}(C^*\Psi_{\sigma\alpha_o})(\delta_\lambda g_o). \end{aligned}$$

Keeping in mind that as g_o varies in Δ and λ in the interval $(\lambda_o, 1)$, the point $\delta_\lambda g_o$ covers ω, we have proved

$$\mathcal{L}(C^*\Psi_{\sigma\alpha_o} - u) \le 0 \qquad \text{in} \quad \omega.$$

At this point we observe that (possibly using a constant larger than C^*) we also have the estimate

$$C^*\Psi_{\sigma\alpha_o} \ge u \qquad \text{on} \quad \partial\omega. \tag{4.22}$$

To see that (4.22) holds we argue as follows. It is clear that (4.22) holds on $\partial\omega \cap \partial\Omega$, since both u and $\Psi_{\sigma\alpha_o}$ vanish there. On the other hand, the set $\Lambda = \left(\partial\omega \cap \overline{\Omega}\right) \setminus \Delta$ is at a fixed distance away from the characteristic set Σ, therefore for every $g_o \in \Lambda$ there exists $j \in \{1, ..., m\}$ such that $X_j\rho(g_o) \neq 0$. By continuity, the trajectories of X_j fill a (sufficiently small) full neighborhood V_{g_o} of g_o. This means that there exists $t_o = t_o(g_o) > 0$ such that every $g \in \Omega \cap V_{g_o}$ can be written as $g_1 \exp tX_j$ for some $g_1 \in \partial\Omega \cap V_{g_o}$ and some $0 < t < t_o$. Using the uniform transversality of X_j to $\partial\Omega$ in $\Omega \cap V_{g_o}$ and Taylor's formula we infer the existence of $C = C(g_o) > 0$ such that

$$|R - \rho(g_1 \exp tX_j)| \geq C\,|t|, \tag{4.23}$$

for every $g_1 \in \partial\Omega \cap V_{g_o}$ and $0 < t < t_o$. We now use the assumption (2.8) to conclude the existence of a constant $C^* = C^*(u, g_o) > 0$ such that

$$u(g_1 \exp tX_j) \leq C^*\,|t| \leq C^*\,|t|^{\alpha\sigma_o}$$

for every $g_1 \in \partial\Omega \cap V_{g_o}$ and $0 < t < t_o$. The latter inequality and (4.23) allow to conclude that (4.22) does hold, for a constant depending on u and g_o, in the set $\Omega \cap V_{g_o}$. By a finite covering we see that (4.22) continues to hold in the intersection of a small neighborhood of $\partial\Omega$ with Λ. We can thus detach from $\partial\Omega$. Once inside Ω we can use the C^∞ smoothness of u to conclude that (4.22) holds on t he remaining portion of $\partial\omega \cap \Omega$ as well. This completes the proof of (4.22). We can now apply Bony's maximum principle Theorem 2.1 to ω to infer that a similar estimate also holds in ω. From this result and from the right-hand side of (4.14) we conclude for every $g_o \in \omega$ and $\lambda_o < \lambda < 1$

$$u(\delta_\lambda g_o) \leq C\,(1-\lambda)^{\sigma\alpha_o}, \tag{4.24}$$

which shows that we have improved on (4.21). It is now clear that repeating the above arguments n times, where n is as in (4.20), we reach the desired conclusion (4.16). □

We are now ready to state the two main results of this section. We start with the boundedness of the horizontal gradient of a solution of (4.1) at characteristic points.

Theorem 4.6. *Consider a C^∞ domain Ω in a Carnot group $\boldsymbol{G}$ satisfying* (4.2) *and all the hypothesis in Theorem 4.3, including (4.12). Let u be a weak solution of (4.1), then*

$$Xu \in L^\infty(\Omega).$$

Proof. Due to the left-invariance of the problem (4.1) we can assume that Ω is uniformly starlike along Σ with respect to e. Since by the results in [52], [22] we know that u is smooth away from Σ, in order to prove the theorem it will be enough to show that

$$Xu \in L^{\infty}(\omega), \tag{4.25}$$

where ω is fixed as before. We begin by introducing $v = u^{2^*-1} * \Gamma$, where Γ is the positive fundamental solution of $\mathcal{L}$, i.e., $\mathcal{L}\Gamma = -\delta$. According to Corollary 2.8 in [26], v satisfies the equation $\mathcal{L}v = -u^{2^*-1}$ in $\boldsymbol{G}$. Since by (4.3) u^{2^*-1} is in $\Gamma^{0,\beta}(\boldsymbol{G})$ for some $0 < \beta < 1$ (and u is compactly supported in $\boldsymbol{G}$), we have

$$v \in \Gamma^{2,\beta}_{loc}(\boldsymbol{G}) \tag{4.26}$$

from [26], Theorem 6.1. Therefore, if we let $w \overset{def}{=} u - v$, in order to prove (4.25) it is enough to show it for w, i.e., that $Xw \in L^{\infty}(\omega)$. We notice that w is $\mathcal{L}$-harmonic, i.e., $\mathcal{L}w = 0$ in Ω.

Let $g \in \omega$, then there exist $g_o \in \Delta$ and $\lambda \in (\lambda_o, 1)$ such that $g = \delta_\lambda(g_o)$. We now note that, as in the proof of (4.15), assumption (4.10) implies for every $g_o \in \Delta$, $\lambda_o < \lambda < 1$, and every $g_1 \in \partial\Omega$

$$(1-\lambda) \le C\,(\rho(g_o) - \rho(\delta_\lambda g_o)) = C\,(\rho(g_1) - \rho(\delta_\lambda g_o)). \tag{4.27}$$

This allows to obtain, in view of (4.16) in Theorem 4.5,

$$u(\delta_\lambda g_o) \le C\,(\rho(g_1) - \rho(\delta_\lambda g_o)). \tag{4.28}$$

At this point we apply Theorem 2.3 to the defining function ρ of Ω to obtain for every $g_1, g_2 \in \overline{\Omega}$

$$|\rho(g_1) - \rho(g_2)| \le C\,||X\rho||_{L^{\infty}(\overline{\Omega})}\, d(g_1, g_2).$$

Using the latter inequality with $g_2 = \delta_\lambda g_o$ in (4.28) one finds for every $g_o \in \Delta$, $\lambda_o < \lambda < 1$, and every $g_1 \in \partial\Omega$

$$u(\delta_\lambda g_o) \le C^*\, d(g_1, \delta_\lambda g_o). \tag{4.29}$$

For $g = \delta_\lambda g_o \in \omega$ we now choose $g_1 \in \partial\Omega$ in (4.29) in such a way that $d(g_1, \delta_\lambda g_o) = dist(g, \partial\Omega)$. This gives

$$u(g) \le C\, d(g, \partial\Omega) \qquad \text{for every} \quad g \in \omega.$$

Since we know that $u \in C^{\infty}(\overline{\Omega} \setminus \omega)$ we can use a similar argument based on the use of Theorem 2.3 to conclude

$$u(g) \le C\, d(g, \partial\Omega) \qquad \text{for every} \quad g \in \Omega. \tag{4.30}$$

We now fix a point $g \in \omega$ and, with $r = dist(g, \partial\Omega)/2$, consider the ball $B(g,r) \subset \overline{B}(g,r) \subset \Omega$. Applying the interior Schauder estimates in Theorem 2.2 to the $\mathcal{L}$- harmonic function $w - w(g)$ one has

$$|Xw(g)| \leq \frac{C}{r} \sup_{B(g,r)} |w - w(g)|. \tag{4.31}$$

Note that (4.30) gives for $g' \in B(g,r)$

$$u(g') \;\leq\; C\, dist(g', \partial\Omega) \;\leq\; C\, [d(g',g) + dist(g, \partial\Omega)] \;\leq\; C\, r. \tag{4.32}$$

Since $w = u - v$, one has for $g' \in B(g,r)$ in view of (4.32), (4.30)

$$\begin{aligned} |w(g') - w(g)| \;&\leq\; [u(g') \\ &+u(g)] + |v(g') - v(g)| \;\leq\; C\, [r + |v(g') - v(g)|]. \end{aligned} \tag{4.33}$$

Finally, we observe that (4.26) implies $v \in \mathcal{L}^{1,\infty}(\Omega)$, and therefore applying Theorem 2.3 once more we conclude for $g, g' \in \Omega$

$$|v(g) - v(g')| \;\leq\; C\, d(g, g').$$

Substitution of this information in (4.33) gives

$$\sup_{B(g,r)} |w - w(g)| \;\leq\; C\, r.$$

Combining the latter inequality with (4.31) brings the sought for conclusion $Xw \in L^\infty(\omega)$. This finishes the proof of Theorem 4.6. □

In the next result we establish the boundedness of the $Z-$derivative of the solution of (4.1) near the characteristic set. We stress that such derivative involves commutators of the vector fields X_j up to maximum order. Although we suspect the result to be true for groups of any step, we have been able to establish it only for groups of step two.

Theorem 4.7. *Let **G** be a Carnot group of step two. Consider a C^∞ connected, bounded open set $\Omega \subset$ **G** satisfying* (4.2) *and all the hypothesis in Theorem 4.3, including (4.12). Under these assumptions, if u is a weak solution of (4.1) one has*

$$Zu \;\in\; L^\infty(\Omega). \tag{4.34}$$

Proof.
We proceed as in the proof of Theorem 4.6 and note that in order to prove the theorem it is enough to show that

$$Zw \;\in\; L^\infty(\omega), \tag{4.35}$$

where ω is fixed as before and v, w have the same meaning as in Theorem 4.6.

For λ very close to 1, we define $\omega_\lambda = \delta_\lambda(\omega) \cap \omega$, $\Delta_\lambda = \delta_\lambda(\Delta)$ and consider the difference quotient

$$\phi_\lambda(g) = \frac{w(g) - w(\delta_{\lambda^{-1}} g)}{1 - \lambda^{-1}}, \qquad g \in \omega_\lambda. \tag{4.36}$$

We claim that there exists a constant $C > 0$ such that for all λ sufficiently close to 1 one has for $g \in \omega_\lambda$

$$|\phi_\lambda(g)| \le C. \tag{4.37}$$

Suppose the claim (4.37) true, then passing to the limit as $\lambda \to 1$ we conclude $|Zw(g)| \le C$ for every $g \in \omega$, which proves (4.35), thus establishing the theorem. We then turn to the proof of (4.37). The key observation is that ϕ_λ is $\mathcal{L}$-harmonic in ω_λ since

$$\mathcal{L}\phi_\lambda(g) = \frac{\mathcal{L}w(g) - \mathcal{L}\big(w(\delta_{\lambda^{-1}} g)\big)}{1 - \lambda^{-1}} = \frac{\mathcal{L}w(g) - \lambda^{-2}\mathcal{L}w(\delta_{\lambda^{-1}} g)}{1 - \lambda^{-1}} = 0. \tag{4.38}$$

From Theorem 2.1 it is therefore enough to prove that (4.37) holds for $g \in \partial\omega_\lambda$ and $\lambda \in (\lambda_1, 1)$, for some λ_1 close to 1. We note that $\partial\omega_\lambda = \Delta_\lambda \cup (\partial\omega_\lambda \setminus \Delta_\lambda)$. We analyze the two portions separately. Since any point $g \in \Delta_\lambda$ can be written as $g = \delta_\lambda g_o$ for some $g_o \in \Delta$ we have

$$\phi_\lambda(g) = -\lambda \frac{w(\delta_\lambda g_o) - w(g_o)}{1 - \lambda}. \tag{4.39}$$

Recalling that $w = u - v$ we find

$$\begin{aligned} |\phi_\lambda(g)| &= |-\lambda \frac{u(\delta_\lambda g_o) - u(g_o) - v(\delta_\lambda g_o) + v(g_o)}{1 - \lambda}| \\ &\le \frac{u(\delta_\lambda g_o)}{1 - \lambda} + |\frac{v(\delta_\lambda g_o) - v(g_o)}{1 - \lambda}|, \end{aligned} \tag{4.40}$$

since $u = 0$ on Δ. At this point we use Theorem 4.5 to conclude

$$|\phi_\lambda(\delta_\lambda g_o)| \le C + |\frac{v(\delta_\lambda g_o) - v(g_o)}{1 - \lambda}|, \qquad g_o \in \Delta, \lambda_o < \lambda < 1. \tag{4.41}$$

Next, we remember that (4.26) holds. The embedding Theorem 5.25 in [26] implies that

$$\Gamma^{2,\beta}_{loc}(\boldsymbol{G}) \subset \Lambda^{1,\frac{\beta}{2}}_{loc}(\boldsymbol{G}) = C^{1,\frac{\beta}{2}}_{loc}(\boldsymbol{G}), \tag{4.42}$$

where the latter space denotes the standard Hölder class with respect to the Riemannian distance $d_R(\cdot,\cdot)$ on $\boldsymbol{G}$. We conclude, in particular, that v is locally Lipschitz continuous with respect to $d_R(\cdot,\cdot)$, and therefore

$$|v(g) - v(h)| \le d_R(g,h), \qquad g, h \in \overline{\Omega}.$$

One easily sees that

$$d_R(\delta_\lambda g_o, g_o) \ \leq \ C\ (1-\lambda).$$

The latter two inequalities imply the uniform boundedness of the difference quotient of v in (4.41) for $g_o \in \Delta$ and λ between λ_o and 1. This shows that (4.37) holds on Δ_λ. Finally, to obtain the same inequality on $\partial\omega_\lambda \setminus \Delta_\lambda$ it is enough to observe that for λ close to 1 such sets are uniformly away from the characteristic set Σ, so that the desired conclusion follows from the $\Gamma^{2,\alpha}$ regularity of u in a uniform neighborhood of such sets. In conclusion, we have proved the claim (4.37), and therefore the theorem. □

Remark 4.8. We emphasize that the step two hypothesis on $\boldsymbol{G}$ has been used only in the embedding in (4.42).

5. Non-existence of positive solutions to the Yamabe equation in bounded domains

In this section we apply the results of sections four and five to obtain non-existence theorems for a class of bounded domains which play a basic role in the analysis of Carnot groups. Such class contains the gauge pseudo-balls defined via (2.3), as well as, when the group is of Heisenberg type, the level sets of the entire solutions (1.4) to the CR Yamabe problem (1.2) in $\Omega = \boldsymbol{G}$. We begin by stating a corollary of Theorems 4.6, 4.7 and 3.7.

Theorem 5.1. *Let $\boldsymbol{G}$ be a Carnot group of step two. Let $\Omega \subset \boldsymbol{G}$ be a C^∞ bounded domain, starlike with respect to $g_o \in \Omega$ and uniformly starlike with respect to g_o along the characteristic set Σ. Suppose in addition that condition* (4.2) *holds and that a defining function ρ of Ω fulfills* (4.12) *for some constant $M_1 > 0$ in a neighborhood of Σ. Under these hypothesis, $u \equiv 0$ is the only weak solution of* (4.1).

In order to produce interesting geometric examples to which Theorem 5.1 can be applied we establish a useful lemma, which is a simple consequence of the Baker-Campbell- Hausdorff formula. This result shows in particular that in a Carnot group the coordinates in the first and second layers of the Lie algebra $\mathfrak{g}$ are $\mathcal{L}$-harmonic. We will use the notations of section two, see the definitions (2.1), (2.5).

Lemma 5.2. *Let $\boldsymbol{G}$ be a Carnot group. One has*

$$\mathcal{L}x_j = 0, \quad j = 1, \ldots, m, \qquad \mathcal{L}y_i = 0, \quad i = 1, \ldots, k.$$

From the latter equation we infer, in particular, that the function $g \to |y(g)|^2$ is $\mathcal{L}$-subharmonic and in fact

$$\mathcal{L}(|y|^2) = 2 \sum_{i=1}^{k} |X(y_i)|^2 \ \geq \ 0.$$

There exists a constant $C = C(\boldsymbol{G}) > 0$ such that

$$|X(|y|^2)|^2 \ \leq \ C \ |x|^2 \ |y|^2.$$

Proof. Let $g = exp(\xi)$ with $\xi = \xi_1 + ... + \xi_r$. For $t \in \mathbb{R}$ the Baker-Campbell-Hausdorff formula and the stratification of $\mathfrak{g}$ give for $l = 1, ..., m$

$$\begin{aligned} x_j(g \exp tX_l) &= x_j(g) + t\delta_{jl}, \\ y_i(g \exp tX_l) &= y_i(g) + \frac{t}{2} < [\xi_1, X_l], Y_i > . \end{aligned} \tag{5.1}$$

From (5.1) the $\mathcal{L}$-harmonicity of $x_j(g)$ and $y_i(g)$ is obvious. Using (5.1) we now define for $l = 1, ..., m$

$$\begin{aligned} \phi_l(t) = |y(g \exp tX_l)|^2 = \sum_{i=1}^{k} \Big(y_i^2 \\ +t < [\xi_1, X_l], Y_i > y_i + \frac{t^2}{4} < [\xi_1, X_l], Y_i >^2 \Big). \end{aligned} \tag{5.2}$$

Differentiating with respect to t we find

$$\phi_l'(0) = \sum_{i=1}^{k} < [\xi_1, X_l], Y_i > y_i, \tag{5.3}$$

hence

$$\phi_l'(0)^2 \ \leq \ |y|^2 \ \sum_{i=1}^{k} (< [\xi_1, X_l], Y_i >)^2.$$

Keeping in mind that $\xi_1 = \sum_{j=1}^{m} x_j X_j$ we easily obtain

$$\sum_{i=1}^{k} (< [\xi_1, X_l], Y_i >)^2 \ \leq \ |x|^2 \ \sum_{j=1}^{m} \ \sum_{i=1}^{k} (< [X_j, X_l], Y_i >)^2.$$

In conclusion

$$|X(|y|^2)|^2 = \sum_{l=1}^{m} \phi_l(0)^2 \ \leq \ C \ |x|^2 \ |y|^2,$$

where

$$C = \sum_{j,l=1}^{m} \sum_{i=1}^{k} (< [X_j, X_l], Y_i >)^2.$$

□

Let now $\boldsymbol{G}$ be a Carnot group of step two. We define the function

$$f_\epsilon(g) = \left((\epsilon^2 + |x(g)|^2)^2 + 16|y(g)|^2\right)^{1/4}, \qquad \epsilon \in \mathbb{R}. \tag{5.4}$$

For $R > 0$ and $\epsilon \in \mathbb{R}$, with $\epsilon^2 < R^2$, consider the C^∞ bounded open set

$$\Omega_{R,\epsilon} = \{g \in \boldsymbol{G} \mid f_\epsilon(g) < R\}. \tag{5.5}$$

When $\epsilon = 0$ it is clear that $\Omega_{R,\epsilon}$ is nothing but a gauge pseudo-ball centered at the group identity e, except that the natural gauge was defined in (2.2) without the factor 16. Here we have introduced such factor for the purpose of keeping a consistent definition with the case of groups of Heisenberg type, studied in the next sections. For all practical purposes the reader can neglect it and identify f_0 in (5.4) with (2.2). For $g \in \boldsymbol{G}$, we let $\Omega_{R,\epsilon}(g) = \{h \in \boldsymbol{G} \mid f_\epsilon(g^{-1}h) < R\} = g\ \Omega_{R,\epsilon}$.

Theorem 5.3. *Let $\boldsymbol{G}$ be a Carnot group of step two. Given any $g \in \boldsymbol{G}$, $R \in \mathbb{R}$ and $\epsilon \in \mathbb{R}$ with $\epsilon^2 < R^2$, the function $u \equiv 0$ is the only non-negative weak solution of (4.1) in $\Omega_{R,\epsilon}(g)$.*

Proof. We need to show that the set $\Omega_{R,\epsilon}(g)$ fulfills the conditions of Theorem 5.1. By left translation it is enough to consider $\Omega_{R,\epsilon}$. Since $\rho = f_\epsilon^4$ is also a defining function for the domain, we will work with this function. Noting that $|x|^2$ is homogeneous of degree two and that $|y|^2$ is homogeneous of degree four, we find

$$Z\rho(g) = 4[(\epsilon^2 + |x(g)|^2)|x(g)|^2 + 16|y(g)|^2]. \tag{5.6}$$

On $\partial\Omega_{R,\epsilon}$ we have $(\epsilon^2 + |x(g)|^2)^2 + 16|y(g)|^2 = R^4$, it is thus easy to recognize from (5.6) that $\Omega_{R,\epsilon}$ is uniformly starlike. Furthermore, according to the Definition 7 in section 5.2 of [10], the domain is $CS - \Sigma$, i.e., it has cylindrical symmetry near the characteristic set Σ and therefore as a consequence of Theorem 15 in [10], section 5.2, $\Omega_{R,\epsilon}$ is a X-NTA domain. In particular, the existence of a uniform exterior corkscrew guarantees that (4.2) is satisfied. Let $\psi = |x(g)|^2$ and $\psi_2 = |y(g)|^2$, so that $\rho = \psi^2 + 16\psi_2$. One has

$$\mathcal{L}\rho = 2|X\psi|^2 + 2\psi\mathcal{L}\psi + 16\mathcal{L}\psi_2,$$

$$X\rho = 2\psi X\psi + 16X\psi_2.$$

Using Lemmas 4.2 and 5.2 we obtain

$$\mathcal{L}\rho \;\geq\; 4(m+2)\psi.$$

On the other hand

$$< X\rho, X\psi >= 2\psi^2 + 16 < X\psi, X\psi_2 >, \tag{5.7}$$

and using again Lemmas 4.2 and 5.2 we see that

$$< X\psi, X\psi_2 > \leq\; C\;\psi\;|y|, \tag{5.8}$$

where $C > 0$ is a universal constant. This shows that

$$< X\rho, X\psi > \leq\; 2\psi^2 + C^*\psi \;\leq\; \frac{M_1}{2}\psi \;\leq\; \mathcal{L}\rho,$$

for a sufficiently large constant M_1, which exists since $\Omega_{R,\epsilon}$ is a bounded domain. We have proved that all the hypothesis in Theorem 5.1 are satisfied. This completes the proof. □

6. Existence of global minimizers

In their paper [44] on the CR Yamabe problem D. Jerison and J. Lee observed that using the concentration compactness method of P. L. Lions, see [60], [61] and also [74], one can see that the best constant in the Folland- Stein embedding (1.1) for the Heisenberg group in the case $p = 2$ is achieved and thus (1.2) admits an entire non-negative solution. P. L. Lions' method is very powerful and general and can, in fact, be suitably adapted to the homogeneous setting of a Carnot group $\boldsymbol{G}$ to prove that for any $1 < p < Q$ the best constant in (1.1) is achieved. Consequently, for any such p the quasi-linear equation with critical exponent

$$\mathcal{L}_p u = \sum_{j=1}^{m} X_j(|Xu|^{p-2}X_j u) = -u^{p^*-1} \qquad \text{in} \quad \boldsymbol{G} \tag{6.1}$$

possesses an entire non-negative solution. The purpose of this section is to record these basic results without presenting the proofs, which will appear elsewhere [79].

Two crucial aspects of the equation (6.1) are its invariance with respect to the group translations and dilations. The former is obvious, since the vector fields X_j are left- invariant. The latter must be suitably interpreted and follows from the observation that for every $c > 0$ one has $\mathcal{L}_p(cu) = c^{p-1}u$, and that furthermore

$$\mathcal{L}_p(u \circ \delta_\lambda) = \lambda^p\;\delta_\lambda \circ \mathcal{L}_p u. \tag{6.2}$$

If we thus define, for a solution u of (6.1) and for $\lambda > 0$, the rescaled function $u_\lambda = \lambda^\alpha u \circ \delta_\lambda$, then it is clear that u_λ satisfies (6.1) if and only if $\alpha = Q/p^* = (Q-p)/p$. These consideration lead to introduce for $u \in C_o^\infty(\boldsymbol{G})$ two new functions

$$\tau_h u \stackrel{def}{=} u \circ \tau_h, \qquad h \in \boldsymbol{G}, \tag{6.3}$$

where $\tau_h : \boldsymbol{G} \to \boldsymbol{G}$ is the operator of left-translation $\tau_h(g) = hg$,

$$u_\lambda \stackrel{def}{=} \lambda^{Q/p^*} u \circ \delta_\lambda, \qquad \lambda > 0. \tag{6.4}$$

It is easy to see that the norms in (1.1) are invariant under (6.3) and (6.4).

The problem of finding the best constant in the Folland-Stein embedding (1.1) leads to the following variational problem

$$I \equiv I_1 \stackrel{def}{=} \inf \left\{ \int_{\boldsymbol{G}} |Xu|^p \;\mid\; u \in C_o^\infty(\boldsymbol{G}),\ \int_{\boldsymbol{G}} |u|^{p^*} = 1 \right\}. \tag{6.5}$$

A minimizing sequence $\{u_m\} \in C_o^\infty(\boldsymbol{G})$ is thus characterized by the properties

$$\int_{\boldsymbol{G}} |u_m|^{p^*} = 1 \qquad \text{and} \qquad \int_{\boldsymbol{G}} |Xu_m|^p \underset{m\to\infty}{\to} I. \tag{6.6}$$

The following is the main result about existence of global minimizers.

Theorem 6.1. *Let $\boldsymbol{G}$ be a Carnot group and consider the minimization problem (6.5). Every minimizing sequence $\{u_m\}$ of (6.5) is relatively compact in $\overset{o}{\mathcal{D}}{}^{1,p}(\boldsymbol{G})$, after possibly translating and dilating each of its elements using* (6.3) *and* (6.4). *In particular, there exists a minimum of (6.5) and the equation*

$$\mathcal{L}_p u = -u^{p^*-1} \tag{6.7}$$

admits a non-trivial, non-negative solution $u \in \overset{o}{\mathcal{D}}{}^{1,p}(\boldsymbol{G})$.

The proof of Theorem 6.1 is based on an adaptation of the method of concentration of compactness of P. L. Lions [60], [61], [62], [63]. In such adaptation the Euclidean space $\mathbb{R}^n$ is replaced by a Carnot group $\boldsymbol{G}$ with its homogeneous structure and Carnot-Carathéodory distance. Similarly to Lions' cited works, in particular [62] and [63], the crucial ingredients are the following lemmas.

Lemma 6.2. *Suppose ν_m is a sequence of probability measures on $\boldsymbol{G}$. There exists a subsequence, which we denote by $d\nu_m$, such that exactly one of the following three conditions holds:*

(1)(compactness) There is a sequence $(g_m) \in \boldsymbol{G}$ *such that for every* $\epsilon > 0$ *there exists* $R > 0$ *for which, for every* m,

$$\int_{B(g_m,R)} d\nu_m \geq 1 - \epsilon.$$

(2)(vanishing) For all $R > 0$ *we have*

$$\lim_{m\to\infty}\Big(\sup_{g\in\boldsymbol{G}} \int_{B(g,R)} d\nu_m\Big) = 0.$$

(3)(dichotomy) There exists $\lambda, 0 < \lambda < 1$ *such that for every* $\epsilon > 0$ *there exist* $R > 0$ *and a sequence* (g_m) *with the following property: Given* $R' > R$ *there exist non-negative measures* ν_m^1 *and* ν_m^2 *for which*

$$0 \leq \nu_m^1 + \nu_m^2 \leq \nu_m \tag{6.8}$$

$$supp\, \nu_m^1 \subset B(g_m, R), \quad supp\, \nu_m^2 \subset \boldsymbol{G} \setminus B(g_m, R') \tag{6.9}$$

$$\Big|\lambda - \int \nu_m^1\Big| + \Big|(1-\lambda) - \int \nu_m^2\Big| \leq \epsilon. \tag{6.10}$$

Lemma 6.3. *Suppose* $u_m \rightharpoonup u$ *in* $\overset{o}{\mathcal{D}}{}^{1,p}(\boldsymbol{G})$, $\mu_m = |Xu_m|^p dH \rightharpoonup \mu$, *and* $\nu_m = |u_m|^{p^*} dH \rightharpoonup \nu$ *weak-$*$ in measure, where* μ *and* ν *are bounded, non-negative measures on* $\boldsymbol{G}$. *There exist at most countable points* $g_j \in \boldsymbol{G}$ *and real numbers* $d_j > 0, e_j > 0$, *such that*

$$\nu = |u|^{p^*} + \sum_j d_j \delta_{g_j} \tag{6.11}$$

$$\mu \geq |Xu|^p dH + \sum_j e_j \delta_{g_j} \tag{6.12}$$

$$I {d_j}^{p/p^*} \leq e_j, \tag{6.13}$$

where I *is the constant in (6.5). In particular,*

$$\sum {d_j}^{p/p^*} < \infty. \tag{6.14}$$

We mention that the implementation of Lions' program relies, among other things, on the Rellich-Kondrachov compact embedding. In the sub-elliptic setting the proof of this result requires a substantial amount of work. A general version of it was proved in [32]. It states that if Ω denotes a bounded X-PS domain (Poincaré-Sobolev domain) in a Carnot-Carathéodory space, then the embedding

$$\mathcal{L}^{1,p}(\Omega) \subset L^q(\Omega)$$

is compact provided that $1 \leq q < p^* = pQ/(Q-p)$. Here, $\mathcal{L}^{1,p}(\Omega)$ indicates the Sobolev space of those functions $f \in L^p(\Omega)$ such that $Xf \in L^p(\Omega)$, endowed with the natural norm. Carnot groups are the basic models of Carnot-Carathéodory spaces. We need to apply such result to an increasing sequence of bounded domains $\Omega_k \subset \Omega_{k+1} \subset \boldsymbol{G}$, such that $\Omega_k \nearrow \boldsymbol{G}$. We can take as Ω_k the Carnot-Carathéodory ball centered at the identity $e \in \boldsymbol{G}$ with radius k, since it was proved in [28], [32] that such sets are X-PS domains in any Carnot-Carathéodory space.

For the proof of Lemmas 6.2 and 6.3, and that of Theorem 6.1, we refer the reader to [79].

7. Existence of explicit entire solutions to the Yamabe equation on groups of Heisenberg type

In this section we consider a special class of Carnot groups, those so-called of Heisenberg type. Such groups were introduced by Kaplan [48] and have been subsequently intensively studied by several authors, see the references cited in the introduction. We list only some of the basic properties of groups of Heisenberg type and refer the reader to the cited references for further details.

Let $\boldsymbol{G}$ be a Carnot group of step two whose Lie algebra $\mathfrak{g} = V_1 \oplus V_2$. Consider the map $J : V_2 \to End(V_1)$ defined by

$$< J(\xi_2)\xi_1', \xi_1'' >=< \xi_2, [\xi_1', \xi_1''] >, \qquad \text{for} \quad \xi_2 \in V_2 \quad \text{and} \quad \xi_1', \xi_1'' \in V_1. \tag{7.1}$$

$\boldsymbol{G}$ is said of *Heisenberg type* if for every $\xi_2 \in V_2$, with $|\xi_2| = 1$, the map $J(\xi_2) : V_1 \to V_1$ is orthogonal. The definition of J and the orthogonality assumption respectively imply

$$< J(\xi_2)\xi_1, \xi_1 >= 0, \qquad |J(\xi_2)\xi_1| = |\xi_2|\,|\xi_1|. \tag{7.2}$$

The next properties of groups of Heisenberg type can be found in [48], [9]. For the reader's convenience we have collected them in a lemma, whose proof has been included for completeness.

Lemma 7.1. *Let $\boldsymbol{G}$ be a group of Heisenberg type. The following formulas hold*

$$\mathcal{L}(|y(g)|^2) = \frac{k}{2}\,|x(g)|^2 \tag{7.3}$$

$$|X(|y|^2)|^2 = |x|^2\,|y|^2 \tag{7.4}$$

$$< X(|x(g)|^2), X(|y(g)|^2) >= 0. \tag{7.5}$$

Proof. Recalling (5.2) one sees

$$\phi_l''(0) = \frac{1}{2}\sum_{i=1}^{k}(< Y_i, [\xi_1, X_l] >)^2 = \frac{1}{2}\sum_{i=1}^{k}(< J(Y_i)\xi_1, X_l >)^2.$$

This implies in view of (7.1), (7.2)

$$\begin{aligned} \mathcal{L}(|y|^2) &= \sum_{l=1}^{m}\phi_l''(0) = \frac{1}{2}\sum_{l=1}^{m}\sum_{i=1}^{k}(< J(Y_i)\xi_1, X_l >)^2 \\ &= \frac{1}{2}\sum_{i=1}^{k}|J(Y_i)\xi_1|^2 = \frac{k}{2}|x|^2. \end{aligned} \tag{7.6}$$

From (5.3) one has

$$\phi_l'(0) = \sum_{i=1}^{k} < [\xi_1, X_l], Y_i >< \xi_2, Y_i >=< \xi_2, [\xi_1, X_l] > . \tag{7.7}$$

Using (7.2) we obtain from the latter equality

$$|X(|y|^2)|^2 = \sum_{l=1}^{m}\phi_l'(0)^2 = \sum_{l=1}^{m}(< J(\xi_2)\xi_1, X_l >)^2 = |J(\xi_2)\xi_1|^2 = |x|^2\,|y|^2.$$

Finally, (4.7), (7.7), (7.1) and (7.2) imply

$$\begin{aligned} < X(|x|^2), X(|y|^2) > &= 2\sum_{l=1}^{m} < \xi_1, X_l >< \xi_2, [\xi_1, X_l] > \\ &= 2 < J(\xi_2)\xi_1, \xi_1 >= 0. \end{aligned} \tag{7.8}$$

This completes the proof. □

We consider next the function introduced in (5.4)

$$f_\epsilon(g) = \left((\epsilon^2 + |x(g)|^2)^2 + 16|y(g)|^2\right)^{1/4}, \qquad \epsilon \in \mathbb{R}.$$

Lemma 7.2. *Let $\mathbf{G}$ be a group of Heisenberg type, then for $g \in \mathbf{G}$ one has*

$$|Xf_\epsilon(g)|^2 = \frac{|x(g)|^2}{f_\epsilon(g)^2},$$

$$\mathcal{L}f_\epsilon(g) = \frac{Q-1}{f_\epsilon(g)}|Xf_\epsilon(g)|^2 + \frac{m\epsilon^2}{f_\epsilon(g)^3}.$$

Proof. For ease of notation we let $f = f_\epsilon$. Setting $\rho = f^4$ as in the proof of Theorem 5.3, one easily finds

$$|Xf|^2 = \frac{1}{16f^6}|X\rho|^2, \tag{7.9}$$

$$\mathcal{L}f = \frac{1}{4f^3}\left[\mathcal{L}\rho - \frac{3}{4f^4}|X\rho|^2\right]. \tag{7.10}$$

Since

$$X\rho = 2(\epsilon^2 + |x|^2)X(|x|^2) + 16X(|y|^2),$$

using Lemmas 4.2 and 7.1 we obtain

$$\begin{aligned} |X\rho|^2 &= 4(\epsilon^2 + |x|^2)^2|X(|x|^2)|^2 + 16^2|X(|y|^2)|^2 \\ &\quad + 64(\epsilon^2 + |x|^2) < X(|x|^2), X(|y|^2) > \\ &= 16(\epsilon^2 + |x|^2)^2|x|^2 + 16^2|x|^2|y|^2 \\ &= 16|x|^2\left[(\epsilon^2 + |x|^2)^2 + 16|y|^2\right] = 16|x|^2 f^4 \end{aligned} \tag{7.11}$$

Substitution in (7.9) gives the first part of the lemma. We compute next $\mathcal{L}\rho$. Applying Lemma 7.1 again one finds

$$\mathcal{L}\rho = \mathcal{L}\left((\epsilon^2 + |x|^2)^2\right) + 16\mathcal{L}(|y|^2) = \mathcal{L}\left((\epsilon^2 + |x|^2)^2\right) + 8k|x|^2.$$

On the other hand, Lemma 4.2 gives

$$\begin{aligned} \mathcal{L}\left((\epsilon^2 + |x|^2)^2\right) &= 2|X(|x|^2)|^2 + 2(\epsilon^2 + |x|^2)\mathcal{L}(|x|^2) \\ &= 4(m+2)|x|^2 + 4m\epsilon^2. \end{aligned}$$

Recalling that the homogeneous dimension of $\boldsymbol{G}$ is $Q = m + 2k$, we conclude

$$\mathcal{L}\rho = 4(Q+2)|x|^2 + 4m\epsilon^2. \tag{7.12}$$

Finally, replacing (7.11) and (7.12) in (7.10) we obtain the second part of the lemma. $\square$

We can now give the

Proof of Theorem 1.1. With $f = f_\epsilon$ as above and $\epsilon > 0$, we consider the function $w = h(f)$, where $h \in C^2(\mathbb{R})$, and look for conditions on h for which w satisfies the Yamabe type equation

$$\mathcal{L}u = -u^{\frac{Q+2}{Q-2}}. \tag{7.13}$$

Using Lemma 7.2 we find

$$\begin{aligned}\mathcal{L}w &= h''(f)|Xf|^2 + h'(f)\mathcal{L}f \\ &= h''(f)\frac{|x|^2}{f^2} + h'(f)\left[\frac{Q-1}{f}\,|Xf|^2 + \frac{m\epsilon^2}{f^3}\right] \\ &= \left[h''(f) + \frac{Q-1}{f}h'(f)\right]|Xf|^2 + \frac{m\epsilon^2}{f^3}h'(f)\end{aligned} \tag{7.14}$$

Formula (7.14) suggests that we choose h such that

$$h''(t) + \frac{Q-1}{t}h'(t) = 0,$$

for each $t \in \mathbb{R}$. The choice $h(t) = \lambda t^{2-Q}$, $\lambda \in \mathbb{R}$, accomplishes this. Having taken $w = \lambda f^{2-Q}$ we must now wonder whether we can satisfy (7.13) for some value of λ. In view of (7.14) this amounts to satisfy the equation

$$\frac{m\epsilon^2}{f^3}h'(f) = -\frac{\lambda^{(Q+2)/(Q-2)}}{f^{(Q+2)}},$$

which reduces to

$$\lambda = \left(m(Q-2)\epsilon^2\right)^{(Q-2)/4}.$$

This completes the proof. □

8. The Kelvin transform

In [53] Korányi introduced an inversion on the Heisenberg group and used it to define an analogue of the Kelvin transform in such setting. Subsequently, such inversion formula, as well as the Kelvin transform, were generalized in [16] and [15] to all groups of Heisenberg type. The purpose of this section is to recall the relevant definitions and establish some more properties of the CR Kelvin transform. Such properties are particularly far reaching in the context of Iwasawa groups, where we show that the Kelvin transform is an isometry between the spaces $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega)$ and $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega^*)$, where Ω^* denotes the image of Ω under the CR inversion. This will be a useful fact in the next section when we study equations on unbounded domains. Using the Kelvin transform, we have obtained explicit formulas for the Poisson kernel with singularity on the characteristic set for the gauge balls. We have also found explicit formulas for the Poisson kernel for the bounded regions which are the conformal images of the non-characteristic “hyperplanes" in the group $\boldsymbol{G}$. An interesting observation is the different asymptotic behavior of the Poisson kernel with pole at a point on the characteristic set and at a point outside of it.

Recall that Iwasawa type groups arise naturally as the nilpotent component in the Iwasawa decomposition KAN of any simple group of rank one. Every

Iwasawa group is a group of Heisenberg type. We refer the reader to [16] and [15] for more details.

Definition 8.1. *Let $\boldsymbol{G}$ be a group of Heisenberg type with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$. For $g = exp(\xi) \in \boldsymbol{G}$, with $\xi = \xi_1 + \xi_2$, the inversion $\sigma : \boldsymbol{G}^* \to \boldsymbol{G}^*$, where $\boldsymbol{G}^* = \boldsymbol{G} \setminus \{e\}$ is defined by*

$$\sigma(g) = \Big(-\big(|x(g)|^2 \ I + 4J(\xi_2)\big)^{-1}\xi_1, -\frac{\xi_2}{|x(g)|^4 + 16|y(g)|^2} \Big),$$

where the map J is as in (7.1), and I denotes the identity map on V_1. One easily verifies that

$$\sigma^2(g) = g, \qquad\qquad g \in \boldsymbol{G}^*.$$

As in the previous two sections, in the sequel we will use, instead of (2.2), the renormalized gauge

$$N(g) = \big(|x(g)|^4 + 16|y(g)|^2\big)^{1/4}, \tag{8.1}$$

Kaplan proved in [48] that in a group of Heisenberg type the fundamental solution Γ of the sub-Laplacian $\mathcal{L}$ is given by the formula

$$\Gamma(g,h) = C_Q \ N(h^{-1}g)^{-(Q-2)}, \qquad\qquad g, h \in \boldsymbol{G}, g \neq h, \tag{8.2}$$

where C_Q is a suitable constant. Equation (8.2) will play a key role in Definition 8.3 below. Writing $\sigma(g) = \exp(\eta)$, with $\eta = \eta_1 + \eta_2$, for the image of g, we easily obtain from Definition 8.1 and (7.2) that

$$|\eta_1| = \frac{|\xi_1|}{N(g)^2}, \qquad \text{and} \quad |\eta_2| = \frac{|\xi_2|}{N(g)^4} \tag{8.3}$$

An immediate consequence of (8.3) is that

$$N(\sigma(g)) = N(g)^{-1}, \qquad\qquad g \in \boldsymbol{G}^*. \tag{8.4}$$

A direct verification, using (7.1) and the definition of the group dilations, shows that the inversion anticommutes with the group dilations, i.e.,

$$\sigma(\delta_\lambda(g)) = \delta_{\lambda^{-1}}(\sigma(g)), \qquad\qquad g \in \boldsymbol{G}^*. \tag{8.5}$$

A corollary of (8.5) is that starlikeness behaves well under inversion. This is contained in the following result.

Proposition 8.2. *Let $\rho \in C^\infty(\boldsymbol{G})$. The following formula holds*

$$Z(\rho \circ \sigma) = -(Z\rho) \circ \sigma.$$

Proof. Set $h(g) = \rho(\sigma(g))$. Applying (8.5) we obtain

$$\frac{h(\delta_\lambda g) - h(g)}{\lambda - 1} = -\lambda \frac{\rho(\delta_{\lambda^{-1}}\sigma(g)) - \rho(\sigma(g))}{\lambda^{-1} - 1}, \qquad g \in \boldsymbol{G}^*,$$

and taking the limit $\lambda \to 1$ finishes the proof. □

Definition 8.3. *Let* $\boldsymbol{G}$ *be a group of Heisenberg type, and consider a function* u *on* $\boldsymbol{G}$. *The CR Kelvin transform of* u *is defined by the equation*

$$u^*(g) = N(g)^{-(Q-2)}\, u(\sigma(g)), \qquad g \in \boldsymbol{G}^*.$$

When $\boldsymbol{G}$ is a group of Iwasawa type, then it was proved in [15] that the inversion and the Kelvin transform possess various basic properties. In the following theorem we collect the two which will be needed in the sequel.

Theorem 8.4 (see [15]).
Let $\boldsymbol{G}$ *be a group of Iwasawa type. The Jacobian of the inversion is given by*

$$d(H \circ \sigma)(g) = N(g)^{-2Q}\, dH(g), \qquad g \in \boldsymbol{G}^*.$$

The Kelvin transform u^* *of a function satisfies the equation*

$$\mathcal{L}u^*(g) = N(g)^{-(Q+2)}(\mathcal{L}u)(\sigma(g)), \qquad g \in \boldsymbol{G}^*.$$

Remark 8.5. In the sequel we denote by Ω^* the image of a generic domain Ω under the inversion σ. We stress that, since we have chosen not to define the inversion of the point at infinity, in the case in which Ω is a neighborhood of ∞, by which we mean that there exists a ball $B(e, R)$ such that $(\boldsymbol{G} \setminus \overline{B}(e, R)) \subset \Omega$, then Ω^* is a punctured neighborhood of the identity, i.e., $\Omega^* = D \setminus \{e\}$, for an open set D such that $e \in D$. The reader should keep this point in mind for the proof of the next result, as well as for the results in section nine. The following theorem is a consequence of the conformal properties of the inversion and of the Kelvin transform. Such result will be used in the next section in combination with the conformal invariance of the Yamabe type equation expressed by Lemma 9.1.

Theorem 8.6. *The Kelvin transform is an isometry between* $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega)$ *and* $\overset{o}{\mathcal{D}}{}^{1,2}()$ Ω^*.

Proof. Let $u,\ v \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega)$ and $u^*, v^* \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega^*)$ be their Kelvin transforms. We begin by observing that thanks to Theorem 8.4 and (8.4)

$$\begin{aligned}
&\int_{\Omega^*} (u^*(g'))^{2^*} dH(g') \\
&\quad = \int_{\Omega^*} \left[N(g')^{-(Q-2)} u(\sigma(g'))\right]^{2Q/(Q-2)} dH(g') \\
&\quad = \int_{\Omega} \left[N(\sigma(g))^{-(Q-2)} u(g)\right]^{2Q/(Q-2)} N(g)^{-2Q} dH(g) = \int_{\Omega} u(g)^{2^*} dH(g).
\end{aligned}$$

We want to show next that

$$\int_{\Omega} < Xu(g), Xv(g) > \ dH(g) = \int_{\Omega^*} < Xu^*(g'), Xv^*(g') > \ dH(g'). \tag{8.6}$$

By an easy density argument it suffices to assume that $u, v \in C_o^\infty(\Omega)$. An integration by parts shows that (8.6) is equivalent to

$$\int_{\Omega} u(g)\ \mathcal{L}v(g)\ dH(g) = \int_{\Omega^*} u^*(g')\ \mathcal{L}v^*(g')\ dH(g').$$

Using again Theorem 8.4 and (8.4) we obtain

$$\begin{aligned} &\int_{\Omega^*} u^*(g')\ \mathcal{L}v^*(g')\ dH(g') \\ &\quad = \int_{\Omega^*} N(g')^{-(Q-2)}\ u(\sigma(g'))\ N(g')^{-(Q+2)}\ (\mathcal{L}v)(\sigma(g'))\ dH(g') \\ &\quad = \int_{\Omega} u(g)\ \mathcal{L}v(g)\ N(g)^{(Q-2)}\ N(g)^{(Q+2)}\ N(g)^{-2Q}\ dH(g) \\ &\quad = \int_{\Omega} u(g)\ \mathcal{L}v(g)\ dH(g). \end{aligned}$$

This completes the proof. □

Our next task is to investigate how the CR inversion acts on various domains which play a basic role in the geometry of Carnot groups of Heisenberg type. Given the ubiquitous role of the Heisenberg group $\mathbb{H}^n$ in analysis, we begin by deriving various explicit formulas in this special setting and then generalize them to groups of Heisenberg type. We recall that $\mathbb{H}^n$ is the Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ with group law

$$(z,t)\ (z',t') = (z+z', t+t'+2Im(z\cdot \overline{z}')), \tag{8.7}$$

where for $z, z' \in \mathbb{C}^n$ we have let $z\cdot z' = \sum_{j=1}^n z_j z_j'$. In real coordinates a basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^n$ is given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad X_{n+j} = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, ..., n, \qquad \frac{\partial}{\partial t}.$$

Here, we have identified $z = x + iy \in \mathbb{C}^n$, with the real vector $(x, y) \in \mathbb{R}^{2n}$. Since $[X_j, X_{n+k}] = -4\delta_{jk}\frac{\partial}{\partial t}$, the Lie algebra is generated by the system $X = \{X_1, ..., X_{2n}\}$. The relative sub- Laplacian $\mathcal{L} = \sum_{j=1}^{2n} X_j^2$ is the real part of the Kohn complex Laplacian. We recall that the exponential map is the identity and that, as for any group of step two, the dilations are the parabolic ones $\delta_\lambda(z,t) = (\lambda z, \lambda^2 t)$. The corresponding homogeneous dimension is $Q = 2n + 2$. The natural gauge for $\mathbb{H}^n$ is obtained by specializing (2.2)

$$|(z,t)| = (|z|^4 + t^2)^{1/4}.$$

We next write the formulas for the inversion in $\mathbb{H}^n$, see [53]. Let $A = |z|^2 + it$ so that $A\bar{A} = |(z,t)|^4$. The inversion of a point $(z,\ t)$ is given by

$$(w,\ \tau) = \sigma(z,\ t) \overset{def}{=} \Big(-\frac{z}{A},\ -\frac{t}{A\bar{A}}\Big).$$

The expression of the inversion in the real variables is

$$u = -\frac{|z|^2 x - ty}{|z|^4 + t^2}, \qquad v = -\frac{|z|^2 y + tx}{|z|^4 + t^2}, \qquad \tau = -\frac{t}{|z|^4 + t^2}, \tag{8.8}$$

where $w = u + iv$. We now introduce some unbounded regions which play an interesting role in the analysis of $\mathbb{H}^n$.

Definition 8.7. *Given $M, b \in \mathbb{R}$, we call the open sets*

$$C^+_{M,b} = \{(z,t) \in \mathbb{H}^n \mid t > M|z|^2 + b\}$$

and

$$C^-_{M,b} = \{(z,t) \in \mathbb{H}^n \mid t < -M|z|^2 + b\}$$

characteristic cones. The cone is called convex if $M \geq 0$, concave if $M < 0$. When $M = 0$ we use the notation $H_b^\pm$ to indicate the characteristic half-spaces

$$C^+_{0,b} = \{(z,t) \in \mathbb{H}^n \mid t > b\}, \qquad C^-_{0,b} = \{(z,t) \in \mathbb{H}^n \mid t < b\}.$$

The boundaries of such half-spaces are called characteristic hyperplanes.

A simple computation shows that a characteristic cone $C^\pm_{M,b}$ has the isolated characteristic point $(0,b)$ on the group center. Using the left-translations (8.7) one sees that $C^\pm_{M,b} = (0,b)\ C^\pm_{M,0}$. It is worth mentioning that the concave cone $C^\pm_{M,0}$, with $M < 0$ suitably chosen, is precisely the region for which D. Jerison [43] produced the counterexample to the boundedness of the horizontal gradient referred to in the introduction. We will further comment on this example subsequently, when we prove Theorem 9.5. Our next task is to compute the images of the convex characteristic cones in $\mathbb{H}^n$ under the inversion. We only consider the cones $C^+_{M,b}$, the obvious modifications for $C^-_{M,b}$ being left to the reader.

Proposition 8.8. *Let σ be the inversion on $\mathbb{H}^n$ defined by (8.8). For every $M \geq 0$, $b > 0$, define $\epsilon = \sqrt{M/2b}$, $R = ((M^2+1)/4b^2)^{1/4}$, and consider the set*

$$\Omega_{R,\epsilon} = \{(z,t) \in \mathbb{H}^n \mid (|z|^2+\epsilon^2)^2 + t^2 \ < \ R^4\}.$$

One has

$$\sigma(C^+_{M,b}) \ = (0, -\frac{1}{2b})\ \Omega_{R,\epsilon} = \{(z,t) \in \mathbb{H}^n \mid (|z|^2+\epsilon^2)^2 + (t+\frac{1}{2b})^2 \ < \ R^4\}.$$

In particular, the image through the inversion of the characteristic half-space $H_b^+ = \{(z,t) \in \mathbb{H}^n \mid t \ > \ b\}$ is the gauge ball $B((0,-\frac{1}{2b})) = \{(z,t) \in \mathbb{H}^n \mid |z|^4 + (t+\frac{1}{2b})^2 \ < \ R^4\}$.

Remark 8.9. Several comments are in order. First, it is obvious from the definition that $\epsilon^2 \ < \ R^2$. Secondly, the sets $\Omega_{R,\epsilon}$ are precisely the level sets of the Jerison-Lee minimizers (1.3) and of their generalizations introduced in (5.5), if in the latter we neglect the immaterial factor 16. In the statement of the proposition we deliberately did not consider the case $b \leq 0$. The reason for this is that when $b \leq 0$, then the group identity is either contained in $\Omega_{R,\epsilon}$ ($b \ < \ 0$), or it belongs to its boundary ($b = 0$). The image through the inversion would not be a bounded domain and we are not interested in such situation.

Proof. It follows from straightforward computations using (8.8). □

We next use the Kelvin transform and Proposition 8.8 to derive an explicit formula for the Poisson kernel of the gauge ball in $\mathbb{H}^n$ with singularity at one of the two characteristic points on the boundary. We thank A. Koranýi for bringing to our attention that a similar formula already appeared in [40] and that related (but different) formulas are contained in [56] and also in the unpublished notes [55].

Theorem 8.10. *Consider that gauge ball $B(e,R) \subset \mathbb{H}^n$ centered at $e = (0,0)$ with radius $R > 0$ and denote by $g_o^{\pm} = (0, \pm R^2)$ the only two characteristic points. The Poisson kernel for the Kohn sub-Laplacian with singularity at the point $g_o^{\pm}$ is given by*

$$P(g, g_o^{\pm}) = R^{Q-2}\ \frac{R^4 - |g|^4}{|g^{-1}\ g_o^{\pm}|^{(Q+2)}}, \qquad \text{where} \quad g = (z,t).$$

Proof. We only prove the formula for the point $g_o^+ = (0, R^2)$, the other case being treated similarly. Consider the gauge ball $B((0,-R^2), R)$. In view of Proposition 8.8 such ball is the image through the inversion of the characteristic half-space $H_b^+ = \{(z,t) \in \mathbb{H}^n \mid t \ > \ b\}$ with $b = \frac{1}{2R^2}$. The function $w(z,t) = t - b$ is a non- negative $\mathcal{L}$-harmonic function in H_b^+ which vanishes on the boundary. Since the point at infinity in H_b^+ is mapped into the characteristic point $e \in \partial B((0,-R^2),R)$, in virtue of Theorem 8.4 we conclude that the Kelvin transform w^* of w is non-negative and $\mathcal{L}$-harmonic in $B((0,-R^2),R)$. A simple calculation gives

$$w^*(z,t) = \frac{1}{2R^2}\,\frac{R^4 - (|z|^4 + (t+R^2)^2)}{|(z,t)|^{Q+2}}, \qquad (z,t) \in B((0,-R^2),R).$$

Left-translating by $(0,-R^2)$ we obtain a non-negative $\mathcal{L}$-harmonic function in $B(e,R)$

$$u(z,t) = \frac{1}{2R^2}\,\frac{R^4 - (|z|^4 + t^2)}{|(z,(t-R^2))|^{Q+2}},$$

which vanishes everywhere on $\partial B(e,R)$ except at g_o^+. Using the notation $g = (z,t)$ we consider the normalized function

$$P(g,g_o^+) = R^{Q-2}\,\frac{R^4 - |g|^4}{|g^{-1}g_o^+|^{Q+2}}$$

which has the same properties of u plus the additional one $P(e,g_o^+) = 1$. According to the definition introduced in [10], $P(g,g_o^+)$ is thus a kernel function for $\mathcal{L}$ and $B(e,R)$, normalized at e and with pole at g_o^+. On the other hand it was proved in [10], Corollary 1.1, that the gauge balls are X-NTA domains, and that moreover, Theorem 4.11, for such domains there is uniqueness of the normalized kernel function. This concludes the proof of the theorem. □

Remark 8.11. The explicit representation of $P(g,g_o^+)$ sheds light on a striking new phenomenon. As it is well known, the standard Poisson kernel $P(x,x_o)$ for the unit ball $B \subset \mathbb{R}^n$ has the following property: If one considers a non-tangential region (a Euclidean cone) with vertex at $x_o \in \partial B$, $\Gamma_\alpha(x_o)$, then there exist constants $C_1, C_2 > 0$, depending on n, α, such that for $x \in \Gamma_\alpha(x_o)$

$$\frac{C_1}{|x-x_o|^{n-1}} \;\le\; P(x,x_o) \;\le\; \frac{C_2}{|x-x_o|^{n-1}}.$$

Guided by these considerations one might be led to conjecture that an analogous asymptotic behavior should hold, where $n-1$ be replaced by $Q-1$ and the Euclidean distance by the Carnot- Carathéodory one. Such intuition, however, is only correct away from the characteristic set. As Theorem 8.10 shows, there exist non-negative $\mathcal{L}$-harmonic functions in the gauge ball $B(e,R)$ which vanish everywhere on the boundary but at one single characteristic point g_o^+, and whose non-tangential behavior near g_o^+ is

$$\frac{C_1}{d(g,g_o^+)^Q} \;\le\; P(g,g_o^+) \;\le\; \frac{C_2}{d(g,g_o^+)^Q}. \tag{8.9}$$

This shows that the presence of characteristic points causes a sudden jump in the rate of blow-up near a singular boundary point for $\mathcal{L}$-harmonic functions. We plan to return to these and related questions in a forthcoming paper.

We next consider the Poisson kernel for another family of domains of interest, namely the *non-characteristic half-spaces*. Similarly to what was done in Definition 8.7, these domains are introduced in a natural way in the Heisenberg group. Their definition can then be extended to any Carnot group of step two. Since in this paper we are not concerned with non- characteristic domains, in what follows we confine ourselves to describe one particular, yet significant, example. We consider in the Heisenberg group the half-space parallel to the center

$$\Pi_{\mathbf{a},\mathbf{b}} = \{(x, y, t) \in \mathbb{H}^n \mid < x, \mathbf{a} > + < y, \mathbf{b} > > 1\},$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ are fixed so that $|\mathbf{a}|^2 + |\mathbf{b}|^2 \neq 0$. Denoting with $\rho(x, y, t) = 1- < x, \mathbf{a} > + < y, \mathbf{b} >$ the defining function of $\Pi_{\mathbf{a},\mathbf{b}}$ one easily sees that $|X\rho(x, y, t)|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 \neq 0$, hence this half-space has no characteristic points on its boundary. We indicate with $\Pi^*_{\mathbf{a},\mathbf{b}}$ the image of $\Pi_{\mathbf{a},\mathbf{b}}$ through the inversion (8.8). Since the point at infinity is on $\partial\Pi_{\mathbf{a},\mathbf{b}}$, the group identity e belongs to $\partial\Pi^*_{\mathbf{a},\mathbf{b}}$. If we consider the function

$$w(x, y, t) =< x, \mathbf{a} > + < y, \mathbf{b} > -1,$$

then by Lemma 5.2 w is $\mathcal{L}$-harmonic in $\Pi_{\mathbf{a},\mathbf{b}}$, and non-negative. As a consequence of Theorem 8.4 the Kelvin transform of w, w^*, has the same properties. Furthermore, it vanishes everywhere on $\partial\Pi^*_{\mathbf{a},\mathbf{b}}$ except at e. An elementary computation gives

$$w^*(x, y, t) = -\frac{|z|^4 + t^2 + |z|^2(< x, \mathbf{a} > + < y, \mathbf{b} >) + t(< x, \mathbf{b} > - < y, \mathbf{a} >)}{|(z, t)|^{Q+2}}.$$

From this formula one easily obtains for $g = (z, t) = (x, y, t)$ for a constant $C = C(\mathbf{a}, \mathbf{b}) > 0$

$$w^*(g) \leq C \frac{d(g, e)^3(1 + O(d(g, e))}{d(g, e)^{Q+2}} \leq \frac{C}{d(g, e)^{Q-1}}, \tag{8.10}$$

as $d(g, e) \to 0$, $g \in \Pi^*_{\mathbf{a},\mathbf{b}}$. A comparison of this estimates with (8.9) underlines the strikingly different behavior of non-negative $\mathcal{L}-$harmonic functions near a singular boundary point, depending on whether the latter is characteristic or not.

After this excursion into the Heisenberg group we return to the setting of groups of Heisenberg type. Our first objective is to introduce an appropriate notion of cones and half-spaces in a Carnot group. This can be done in a natural way by means of the exponential map, or instead working directly on the group by exploiting its homogeneous structure. This latter approach was fully developed in [10]. Below, we will use the former approach. Given a point $g \in \boldsymbol{G}$ we will continue to denote with $x(g) = (x_1(g), ..., x_m(g))$ and $y(g) = (y_1(g), ..., y_k(g))$ the projection of the exponential coordinates of g on the first and second layer of the Lie algebra $\mathfrak{g}$. We indicate with $\mathbb{R}^k_+$ the cone $\{(y_1, ..., y_k) \in \mathbb{R}^k \mid y_i \geq 0, i = 1, ..., k\}$.

Definition 8.12. *Let $\boldsymbol{G}$ be a Carnot group of step two. Given $M, b \in \mathbb{R}$, and $\boldsymbol{a} \in \mathbb{R}^k \setminus \{0\}$, we call the open sets*

$$C^+_{M,b,\boldsymbol{a}} = \{g \in \boldsymbol{G} \mid < y(g), \boldsymbol{a} > > M|x(g)|^2 + b\}$$

and

$$C^-_{M,b,\boldsymbol{a}} = \{g \in \boldsymbol{G} \mid < y(g), \boldsymbol{a} > < -M|x(g)|^2 + b\}$$

characteristic cones. In the case in which $\boldsymbol{a} \in \mathbb{R}^k_+ \setminus \{0\}$, then we call the cone convex if $M \geq 0$, concave if $M < 0$. When $M = 0$ we use the notation $H^{\pm}_{b,\boldsymbol{a}}$ to indicate the characteristic half-spaces

$$C^+_{0,b,\boldsymbol{a}} = \{g \in \boldsymbol{G} \mid < y(g), \boldsymbol{a} > > b\}, \qquad C^-_{0,b,\boldsymbol{a}} = \{g \in \boldsymbol{G} \mid < y(g), \boldsymbol{a} > < b\}.$$

The boundaries of such half-spaces are called characteristic hyperplanes.

We next consider the analogue of Proposition 8.8 for convex cones in groups of Heisenberg type. Again, we only describe the images of $C^+_{M,b,\mathbf{a}}$.

Proposition 8.13. *Let $\boldsymbol{G}$ be a group of Heisenberg type with the inversion as in Definition 8.1. For every $M \geq 0$, $b > 0$, $\boldsymbol{a} \in \mathbb{R}^k_+ \setminus \{0\}$, define $\epsilon = \sqrt{M/2b}$, $R^2 = \sqrt{16M^2 + |\boldsymbol{a}|^2}/8b$, and consider the set*

$$\Omega_{R,\epsilon} = \{g \in \boldsymbol{G} \mid (|x(g)|^2 + \epsilon^2)^2 + 16|y(g)|^2 < R^4\}.$$

One has

$$\begin{aligned} \sigma(C^+_{M,b,\boldsymbol{a}}) &= (0, -\frac{\boldsymbol{a}}{32b})\ \Omega_{R,\epsilon} \\ &= \left\{g \in \boldsymbol{G} \mid \ (|x(g)|^2 + \epsilon^2)^2 + 16|y(g) + \frac{\boldsymbol{a}}{32b}|^2 < R^4\right\}. \end{aligned}$$

In particular, the image through the inversion of the characteristic half-space $H^+_{b,\boldsymbol{a}} = \{g \in \boldsymbol{G} \mid < y(g), \boldsymbol{a} > > b\}$ is the gauge ball $B((0, -\frac{\boldsymbol{a}}{32b}), R) = \{g \in \boldsymbol{G} \mid |x(g)|^4 + \left|y(g) + \frac{\boldsymbol{a}}{32b}\right|^2 < R^4\}$.

Proof. We begin by observing that from their definitions one immediately sees that $\epsilon^2 < R^2$. Let $g = (x, y)$ and $h = \sigma(g) = (x^*, y^*)$. If $h \in C^+_{M,b,\mathbf{a}}$, from Definition 8.1 and (8.3) we see that the coordinates of g must satisfy the inequality

$$|x|^4 + 16|y|^2 + \frac{M}{b}|x|^2 + \frac{1}{b} < y, \mathbf{a} > < 0.$$

By elementary calculations the latter is easily seen equivalent to

$$\left(|x|^2+\frac{M}{2b}\right)^2+16\left|y+\frac{\mathbf{a}}{32b}\right|^2 < \frac{16M^2+|\mathbf{a}|^2}{64b^2}.$$

From this the theorem easily follows. One only needs to remember that the Baker-Campbell-Hausdorff formula gives for the group law in exponential coordinates $g=(x,y)$, $g'=(x',y')$

$$g\,g'=(x+x',y+y')+1/2[(x,y),(x',y')].$$

This gives in particular

$$(0,y)\,g'=(x',y+y').$$

□

We next obtain an analogue of Theorem 8.10 for groups of Iwasawa type.

Theorem 8.14. *Let $\boldsymbol{G}$ be a group of Iwasawa type and consider a gauge ball $B(e,R)$. The characteristic set Σ of $B(e,R)$ is a sphere contained in the submanifold $\{g\in\boldsymbol{G}\mid x(g)=0\}$. Furthermore, for any $g_o=(0,\boldsymbol{a})\in\Sigma$, $\boldsymbol{a}\in\mathbb{R}^k$, the Poisson kernel with singularity at g_o is given by*

$$P(g,g_o)=R^{Q-2}\,\frac{R^4-N(g)^4}{N(g^{-1}g_o)^{(Q+2)}} \tag{8.11}$$

Proof. Let $\rho(g)=|x(g)|^4+16|y(g)|^2$ so that $B(e,r)=\{g\in\boldsymbol{G}\mid\rho(g)<R^4\}$. Setting $\epsilon=0$ in (7.11) we obtain

$$|X\rho(g)|^2=16|x(g)|^2\,\rho(g)=16R^4\,|x(g)|^2.$$

The latter equation implies the claim about Σ since

$$\Sigma=\{g\in\partial B(e,R)\mid|X\rho(g)|=0\}=\{g\in\boldsymbol{G}\mid x(g)=0,\ |y(g)|^2=\frac{R^4}{16}\}.$$

Consider now $g_o\in\Sigma$, so that $g_o=(0,\mathbf{a})$ with $|\mathbf{a}|^2=\frac{R^4}{16}$. The left-translation by $(0,-\mathbf{a})$ sends $B(e,R)$ to the gauge ball $B((0,-\mathbf{a}),R)$, and the point g_o to the group identity e. According to Proposition 8.13 the ball $B((0,-\mathbf{a}),R)$ is the image through the inversion of the characteristic hyperplane $H^+_{b,\mathbf{a}}$, with $b=\frac{1}{32}$. Thanks to Lemma 5.2 the supporting function

$$w(g)=<y(g),\mathbf{a}>-b$$

of $H^+_{b,\mathbf{a}}$ is a non-negative $\mathcal{L}$-harmonic function which vanishes on the boundary. Consider the Kelvin transform of w,

$$\begin{aligned}w^*(g)&=-N(g)^{-(Q+2)}\,[<y(g),\mathbf{a}>+bN(g)^4]\\&=\frac{1}{32}\,\frac{R^4-(|x(g)|^4+16|y(g)+\mathbf{a}|^2)}{N(g)^{Q+2}},\end{aligned}$$

see Definition 8.3. Due to Theorem 8.4 w^* is $\mathcal{L}$-harmonic and non-negative in $B((0, -\mathbf{a}), R)$. We left-translate this function to $B(e, R)$ using g_o and normalize the resulting function, denoted by $P(g, g_o)$, so that it has value one at e. One easily finds

$$P(g, g_o) = R^{Q-2} \frac{R^4 - (|x(g)|^4 + 16|y(g)|^2)}{(|x(g)|^4 + 16|y(g) - \mathbf{a}|^2)^{Q+2}}.$$

At this point the conclusion of the proof follows the same argument of the proof of Theorem 8.10. □

9. Non-existence of positive solutions to the Yamabe equation in unbounded domains

We are interested in deriving some non-existence results for

$$\begin{cases} \mathcal{L}u^* = -(u^*)^{(Q+2)/(Q-2)} \\ u^* \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega^*), \quad u^* \geq 0, \end{cases} \tag{9.1}$$

with Ω^* an unbounded open set in an Iwasawa type group $\boldsymbol{G}$. As in the case of bounded domains, in order to show non-existence of positive solutions some assumption on the domain are needed.

Let $\boldsymbol{G}$ be an Iwasawa group and Ω^* be an unbounded open set. Using the Kelvin transform we will reduce the problem to one on a bounded domain, where we can apply Theorem 5.1. By Ω we denote the image of the open set Ω^* under the inversion with center at the identity e. We recall Remark 8.5. We also note that since problem (9.1) is translation invariant we can by left- translation send Ω^* to another conveniently chosen unbounded domain. If the complement of Ω^* contains a ball we can thus suppose from the beginning that such ball is centered at the group identity e. Furthermore, by a simple rescaling we can without restriction assume that the radius of the ball be one. We start with a simple, yet crucial, lemma.

Lemma 9.1. *Let u be a solution of*

$$\begin{cases} \mathcal{L}u = -u^p \\ u \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega), \quad u \geq 0, \end{cases} \tag{9.2}$$

and denote by u^* its Kelvin transform. Then u^* satisfies

$$\mathcal{L}u^*(g) = -N(g)^{p(Q-2)-(Q+2)}\, u^*(g)^p \qquad g \in \Omega^*. \tag{9.3}$$

In particular, when $p = \frac{Q+2}{Q-2}$ we conclude that if u satisfies problem (4.1), then u^* is a solution of (9.1) in Ω^*.

Proof. Let u be a solution to (9.2). From Theorem 8.6 we know $u^* \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega^*)$. Consider an arbitrary function $\psi \in C_o^\infty(\Omega^*)$, then we can write $\psi = \phi^*$, for some $\phi \in C_o^\infty(\Omega)$. Integrating by parts and applying Theorem 8.4 gives

$$\begin{aligned}
\int\limits_{\Omega^*} < Xu^*(g'), X\psi(g') > dH(g') &= \int\limits_{\Omega^*} < Xu^*(g'), X\phi^*(g') > dH(g') \\
&= -\int\limits_{\Omega^*} u^*(g')\mathcal{L}\phi^*(g')dH(g') \\
&= -\int\limits_{\Omega^*} N(g')^{-2Q} u(\sigma(g'))\mathcal{L}\phi^*(\sigma(g'))dH(g') \\
&= -\int\limits_{\Omega} u(g)\mathcal{L}\phi(g)dH(g) = \int\limits_{\Omega} < Xu(g), X\phi(g) > dH(g) \\
&= \int_\Omega u(g)^p \phi(g)dH(g),
\end{aligned}$$

where in the last equality we have used the fact that u is a solution to (9.2). We now make the change of variable $g = \sigma(g')$, $g' \in \Omega^*$, and use Theorem 8.4 again to obtain

$$\int_\Omega u(g)^p \phi(g)dH(g) = \int_{\Omega^*} u(\sigma(g'))^p \phi(\sigma(g'))N(g')^{-2Q} dH(g')$$
$$\int_{\Omega^*} u^*(\sigma(g'))^p \phi^*(\sigma(g'))N(g')^{(Q-2)p-(Q+2)} dH(g').$$

In conclusion we have found

$$\begin{aligned}
&\int\limits_{\Omega^*} < Xu^*(g'), X\psi(g') > dH(g') \\
&\quad = \int_{\Omega^*} u^*(\sigma(g'))^p \phi^*(\sigma(g'))N(g')^{(Q-2)p-(Q+2)} dH(g').
\end{aligned}$$

By the arbitrariness of $\psi \in C_o^\infty(\Omega^*)$, (9.3) follows. □

In the following theorem we show that if u^* is a solution to (9.1) in a neighborhood of infinity (see Remark 8.5), then the Kelvin transform of u^* has a removable singularity at the group identity e.

Theorem 9.2. *Let **G** be an Iwasawa group. Suppose that u^* is a solution of (9.1) in Ω^*, with Ω^* a neighborhood of infinity. Let u be the Kelvin transform of u^* defined in Ω, then the group identity e is a removable singularity, i.e., u can be extended as a smooth function in a neighborhood of e where the equation is satisfied.*

Proof. Due to the assumptions on Ω^* we can write $\Omega = D \setminus \{e\}$, where D is a bounded open set containing e. Theorem 8.6 implies that $u \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega)$, moreover from Lemma 9.1 (with the roles of u and u^* reversed) we know that u satisfies (4.1) in Ω, hence for every $\psi \in C_o^\infty(D \setminus \{e\})$ one has

$$\int_D < Xu, X\psi > dH = \int_D u^{2^*-1} \psi dH. \tag{9.4}$$

According to Theorem 2.4 we have $cap_2(\{e\}) = 0$, therefore thanks to Proposition 2.5 we can find a sequence of functions $\zeta_k \in C_o^\infty(D \setminus \{e\})$ such that $0 \le \zeta_k \le 1$, $\zeta_k(g) \to 1$ for every $g \in D \setminus \{e\}$, and for which

$$\int_D |X\zeta_k|^2 dH \to 0, \tag{9.5}$$

as $k \to \infty$. We fix $\phi \in C_o^\infty(D)$ arbitrarily. For every $k \in \mathbb{N}$ one has $\phi\zeta_k \in C_o^\infty(D \setminus \{e\})$, and therefore we obtain from (9.4)

$$\begin{aligned} \int_D u^{2^*-1} \phi\zeta_k dH &= \int_D < Xu, X(\phi\zeta_k) > dH \\ &= \int_D \zeta_k < Xu, X\phi > dH + \int_D \phi < Xu, X\zeta_k > dH. \end{aligned}$$

Since $u \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega) \subset \mathcal{D}^{1,2}(D)$ we can apply Lebesgue dominated convergence theorem to conclude, using (9.5),

$$\int_D < Xu, X\phi > dH = \int_D u^{2^*-1} \phi dH. \tag{9.6}$$

The arbitrariness of $\phi \in C_o^\infty(D)$ shows that the identity is a removable singularity. This completes the proof. □

The above results imply non-existence of positive solutions for unbounded domains Ω^* whose image through the inversion, Ω, is a bounded punctured domain which fulfills the geometric assumptions in Theorem 3.7.

Theorem 9.3. *Let* $\boldsymbol{G}$ *be a group of Iwasawa type. Consider a* C^∞ *unbounded open set* $\Omega^* \subset \boldsymbol{G}$ *and denote by* Ω *its image through the inversion. Suppose that* $\Omega = D \setminus \{e\}$*, where* D *is a bounded open set, containing the identity, which satisfies all the hypothesis in Theorem 5.1. In this situation there exists no solution to problem (9.1) in* Ω^**, other than* $u^* \equiv 0$.

Proof. We argue by contradiction and suppose the existence of a non- trivial solution u^* to (9.1) in Ω^*. Consider the Kelvin transform u defined in Ω. By Theorem 8.6 we know that $u \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega)$. Moreover, Lemma 9.1 guarantees that u is a non-trivial solution to problem (4.1) in Ω. At this point we invoke Theorem 9.2 to conclude that u has a removable singularity at e. We can thus extend u to a non-trivial solution to (4.1) in the whole D. But this is in contradiction with Theorem 5.1, therefore we must have $u^* \equiv 0$. □

To illustrate the scope of Theorem 9.3 we present an interesting application of it.

Corollary 9.4. *Let $\boldsymbol{G}$ be a group of Iwasawa type and consider the unbounded domain $\Omega^* = \{g \in \boldsymbol{G} \mid N(gg_o^{-1}) > R\}$, where N is the gauge in (8.1), $g_o \in \boldsymbol{G}$ and $R > 0$ are fixed. There exist no non-trivial solution to (9.1) in Ω^*.*

Proof. By left-translation and rescaling we can suppose that $g_o = e$, $R = 1$. In this situation, it is easy to verify Ω^* is mapped by the inversion in $D = \Omega \setminus \{e\}$, where $\Omega = \{g \in \boldsymbol{G} \mid N(g) < 1\}$. To complete the proof it is enough to observe that, as it was proved in Theorem 5.3 (case $\epsilon = 0$), the domain Ω fulfills the assumptions in Theorem 5.1. □

We finally consider a notable class of unbounded domains with non- compact boundary, the convex characteristic cones, and prove that these sets do not support non-trivial solutions to the Yamabe problem (9.1).

Theorem 9.5. *Consider a group of Iwasawa type $\boldsymbol{G}$. Let $C^{\pm}_{M,b,\boldsymbol{a}} \subset \boldsymbol{G}$ be a convex characteristic cone as in Definition 8.12. There exists no solution to* (9.1) *in $\Omega^* = C^{+}_{M,b,\boldsymbol{a}}$, other than $u \equiv 0$. In particular, there exist no non-trivial solutions for the characteristic half-spaces $H^{\pm}_{b,\boldsymbol{a}}$.*

Proof. Suppose u^* is a non-trivial solution to (9.1) in $C^{+}_{M,b,\mathbf{a}}$ and denote by u its Kelvin transform. In view of Proposition 8.13, u is defined in $(0, -\frac{\mathbf{a}}{32b})\ \Omega_{R,\epsilon}$, where $\Omega_{R,\epsilon}$ is the domain in (5.5), with R and ϵ specified as in Proposition 8.13. By left- translation we obtain a new non-trivial function, which for simplicity we continue to denote with u, defined in the bounded open set $\Omega_{R,\epsilon}$. From Theorem 8.6 we infer that $u \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega_{R,\epsilon})$. Thanks to Lemma 9.1 we know that u is a non-trivial solution to problem (4.1) in $\Omega_{R,\epsilon}$. At this point we invoke Theorem 5.3 to reach a contradiction. The proof is thus completed. □

An open problem. In closing, we remark an interesting open question connected with Theorem 9.5. We do not know whether the concave characteristic cones introduced in Definition 8.12 admit non-trivial solutions to the problem (9.1). We emphasize that, interestingly, the approach taken in this paper brakes down for these unbounded regions. To illustrate why, we focus on the model case of the Heisenberg group $\mathbb{H}^n$, and consider the concave characteristic cone

$$C^{-}_{M,1} = \{(z,t) \in \mathbb{H}^n \mid t\ >\ -M|z|^2 + 1\},$$

with $M > 0$. Using (8.8) one easily sees that $C^-_{M,1}$ is mapped through the inversion into the bounded region

$$\Lambda_M = \{(z,t) \in \mathbb{H}^n \mid (|z|^2 - \frac{M}{2})^2 + (t + \frac{1}{2})^2 \ < \ \frac{M^2+1}{4}\}.$$

Letting $\epsilon^2 = \frac{M}{2}$, $R^4 = \frac{M^2+1}{4}$, it is then clear that

$$\Lambda_M = (0, -\frac{1}{2})\ \Lambda_{R,\epsilon},$$

where $\Lambda_{R,\epsilon}$ denotes the C^∞ bounded open set

$$\Lambda_{R,\epsilon} = \{(z,t) \in \mathbb{H}^n \mid (|z|^2 - \epsilon^2)^2 + t^2 \ < \ R^4\}, \qquad 0 < \epsilon < R.$$

Performing a left-translation we are thus reduced to analyzing $\Lambda_{R,\epsilon}$. Such set differs from $\Omega_{R,\epsilon}$ defined in (5.5) since its defining function contains the term $-\epsilon^2$, instead of $+\epsilon^2$. Elementary calculations prove that, letting $\rho(z,t) = (|z|^2 - \epsilon^2)^2 + t^2$, then

$$|X\rho(z,t)|^2 = 16R^4|z|^2,$$

on $\partial\Lambda_{R,\epsilon}$, so that the characteristic set of $\Lambda_{R,\epsilon}$ is given by $\Sigma = \{(0, \pm\sqrt{R^4 - \epsilon^4})\}$. Since $\Lambda_{R,\epsilon}$ has cylindrical symmetry by Theorem 5 in [10] we infer that it is a X-NTA domain and therefore, in particular, condition (4.2) is satisfied. Denoting by Z the infinitesimal generator of the group dilations one easily finds for $(z,t) \in \partial\Lambda_{R,\epsilon}$

$$Z\rho(z,t) = 4(R^4 - \epsilon^4) + 4\epsilon^2|x|^2 \ \geq \ 4(R^4 - \epsilon^4) \ > \ 0,$$

which proves that $\Lambda_{R,\epsilon}$ is uniformly starlike according to Definition 3.6. At this point suppose for a moment that $\Lambda_{R,\epsilon}$ satisfy the condition (4.12) in a neighborhood of the two isolated characteristic points. We could then apply the theory developed in section four and conclude in view of Theorem 5.1 that there exists no non-trivial solution to (4.1) in $\Lambda_{R,\epsilon}$. Via inversion and Kelvin transform, such result would imply an analogous non-existence result for the concave characteristic cone $C^-_{M,1}$. However, as we will now show, the assumption (4.12) cannot be fulfilled for the domain $\Lambda_{R,\epsilon}$. This amounts to say that we cannot prove, following Theorems 4.6 and 4.7, that for a solution u to (4.1) in $\Lambda_{R,\epsilon}$ one has Xu and Zu bounded near a characteristic point. The counterexample of D. Jerison referred to in the introduction provides evidence that this obstruction is not merely a fault of our method, but rather is deeply connected with the lack of “convexity" near the characteristic set. Our condition (4.12) thus rule out the non-convex cones.

To complete our discussion we then turn to proving that (4.12) fails. The following calculations are a special case of those in the proof of Theorem 5.3 and we thus omit all details. With ρ as above we find

$$\mathcal{L}\rho(z,t) = (8n + 12)|z|^2 - 8n\epsilon^2.$$

On the other hand we have with $\psi(z,t) = |z|^2$

$$< X\rho, X\psi > (z,t) = 8|z|^2(|z|^2 - \epsilon^2).$$

Since $\epsilon > 0$ from these formulas one easily verifies that for no $M_1 > 0$ and $\delta > 0$ it is possible to satisfy the inequality

$$\mathcal{L}\rho \ \geq \frac{2}{M_1} \ < X\rho, X\psi >$$

in a δ-neighborhood of one of the two characteristic points of $\Lambda_{R,\epsilon}$.

10. Appendix: A priori estimates in Lebesgue spaces

In this section we establish those basic regularity results on which the work of the previous sections rests. Although we could have worked in a far more general setting we have chosen to confine the attention to the model problem

$$\begin{cases} \mathcal{L}_p u = \sum_{i=1}^m X_j(|Xu|^{p-2}X_j u) = -u^{p^*-1} \\ u \in \overset{o}{\mathcal{D}}{}^{1,p}(\Omega), \quad u \geq 0. \end{cases} \tag{10.1}$$

since the latter is particularly interesting from the geometric viewpoint. Hereafter, Ω indicates an open set (not necessarily bounded) in a Carnot group $\boldsymbol{G}$. By a weak solution to (10.1) we mean a function $u \in \overset{o}{\mathcal{D}}{}^{1,p}(\Omega)$ such that for every $\phi \in C_o^\infty(\Omega)$

$$\int_\Omega |Xu|^{p-2} < Xu, X\phi > dH = \int_\Omega u^{p^*-1}\phi dH. \tag{10.2}$$

Thanks to the embedding (1.1), in definition (10.2) it is possible to replace the request $\phi \in C_o^\infty(\Omega)$ with the weaker one $\phi \in \overset{o}{\mathcal{D}}{}^{1,p}(\Omega)$. We want to establish the following regularity result.

Theorem 10.1. *Let $1 < p < Q$ and suppose that u be a weak solution to the problem (10.1), then $u \in L^\infty(\Omega)$.*

The proof of Theorem 10.1 is based on the following global result which extends the local L^∞ estimates in [7].

Lemma 10.2. *Let $u \in \overset{o}{\mathcal{D}}{}^{1,p}(\Omega)$ be a weak solution to the equation*

$$\sum_{i=1}^m X_i(|Xu|^{p-2}X_i u) = -V\ |u|^{p-2}u \qquad in \quad \Omega, \tag{10.3}$$

where $V \in L^{Q/p}(\Omega) \cap L^t(\Omega)$ for some $t > \frac{Q}{p}$. Then $u \in L^\infty(\Omega)$.

Remark 10.3. We note explicitly that thanks to the embedding (1.1), even when $V \in L^{\frac{Q}{p}}(\Omega)$ in Lemma 10.2, it is still possible to allow test functions $\phi \in \overset{o}{\mathcal{D}}{}^{1,p}(\Omega)$ in the weak formulation (10.2). It is clear that when Ω is bounded the hypothesis $V \in L^{Q/p}(\Omega)$ in Lemma 10.2 is superfluous.

We do not present the proof of Lemma 10.2 since it consists essentially in a suitable modification of the test function and truncation ideas introduced in Serrin's seminal paper [70], and subsequently generalized to the subelliptic setting in [7], and also in different forms in [37], [38], [64], [80]. The reader can consult the paper [36] for the proof of a related result in the Euclidean setting. With Lemma 10.2 in hands we turn to the

Proof of Theorem 10.1. We rewrite the equation in the form (10.3), with $V = u^{p^*-p}$. Since by (1.1) one has $V = u^{p^*-p} \in L^{Q/p}(\Omega)$, in order to apply Lemma 10.2 it suffices to prove that there exists $t > \frac{Q}{p}$ such that $V \in L^t(\Omega)$, which is equivalent to proving that $u \in L^s(\Omega)$ for some $s > p^*$. To establish this we let $\alpha = \frac{p^*}{p'}$ and notice that $\frac{\alpha}{p-1} = \frac{p^*}{p} > 1$. For every $j \in \mathbb{N}$ we consider the function

$$H_j(t) = \begin{cases} sign\ t\ |t|^{\frac{\alpha}{p-1}} & \text{if } |t| \le j, \\ \frac{\alpha}{p-1} j^{\frac{\alpha}{p-1}-1} t + (1 - \frac{\alpha}{p-1}) j^{\frac{\alpha}{p-1}} & \text{if } |t| > j, \end{cases}$$

and define

$$\phi_j(t) = \int_0^t H_j'(s)^p ds.$$

It is easy to verify that $\phi_j \in C^1(\mathbb{R})$, and that there exists a constant $\Phi > 0$, independent of j, such that for every $t \in \mathbb{R}$

$$0 \le \phi_j'(t) \le \Phi, \qquad ||t|^{p-1}\phi_j(t)| \le \Phi |H_j(t)|^p. \tag{10.4}$$

Since by assumption $u \in L^{p^*}(\Omega)$, there exists a number $M > 0$ such that

$$\Phi S_p \left(\int_{\Omega_M} u^{p^*} dH \right)^{\frac{p^*-p}{p^*}} \le \frac{1}{2}, \tag{10.5}$$

where $\Omega_M = \{x \in \Omega \mid u(x) > M\}$. The chain rule in [32] gives $H_j(u), \phi_j(u) \in \overset{o}{\mathcal{D}}{}^{1,p}(\Omega)$. We thus obtain from (1.1) and (10.2)

$$\begin{aligned} \left(\int_\Omega |H_j(u)|^{p^*} dH \right)^{\frac{p}{p^*}} &\le S_p \int_\Omega |X(H_j(u))|^p dH \\ &= S_p \int_\Omega |Xu|^{p-2} < Xu, X(\phi_j(u)) > dH = S_p \int_\Omega u^{p^*-1} \phi_j(u) dH \\ &\le S_p \Phi \int_{\Omega_M} u^{p^*-p} |H_j(u)|^p dH + S_p \Phi M^{p^*-p} \int_{\Omega \setminus \Omega_M} |H_j(u)|^p dH, \end{aligned}$$

where in the last inequality we have used (10.4). We next exploit (10.5) to find

$$
\begin{aligned}
& S_p \Phi \int_{\Omega_M} u^{p^*-p} |H_j(u)|^p dH \\
& \le \; S_p \Phi \left(\int_{\Omega_M} u^{p^*} dH \right)^{\frac{p^*-p}{p^*}} \left(\int_{\Omega} |H_j(u)|^{p^*} dH \right)^{\frac{p}{p^*}} \\
& \le \frac{1}{2} \left(\int_{\Omega} |H_j(u)|^{p^*} dH \right)^{\frac{p}{p^*}} .
\end{aligned}
$$

Substitution in the previous inequality finally gives

$$
\left(\int_{\Omega} |H_j(u)|^{p^*} dH \right)^{\frac{p}{p^*}} \le 2 S_p \Phi M^{p^*-p} \left(\int_{\Omega} |H_j(u)|^{p} dH \right)^{\frac{1}{p}}
$$

Since $H_j(u)$ converges increasingly and $a.e.$ to $u^{\frac{\alpha}{p-1}}$, by Fatou and Lebesgue dominated convergence theorems we infer letting $j \to \infty$

$$
\left(\int_{\Omega} u^{\frac{\alpha p^*}{p-1}} dH \right)^{\frac{p}{p^*}} \le 2 S_p \Phi M^{p^*-p} \left(\int_{\Omega} u^{\alpha p'} dH \right)^{\frac{1}{p}}
$$

By our choice, $\alpha p' = p^*$, so that the right-hand side of the latter inequality is finite. We have thus proved that $u \in L^s(\Omega)$ with $s = \frac{\alpha p^*}{p-1} = \frac{(p^*)^2}{p} > p^*$. This completes the proof. □

The next result is an interesting consequence of Theorem 10.1. It provides a delicate L^∞ estimate on metric balls for weak solutions to (10.1). Such estimate is achieved by combining several ideas which, in the case $p = 2$ are present in the works of Moser [65] and Trudinger [76]. Although we do not make any direct use of Theorem 10.4 in this paper, we have nonetheless decided to insert such result in this appendix because of its independent interest.

Theorem 10.4. *Let u be a nonnegative solution to the problem (10.1). We assume that u has been extended with zero outside Ω. Suppose that $s \ge p$ is an exponent such that $u \in L^s(\Omega)$, with $||u||_{L^s(\Omega)}$ depending only on $||u||_{\mathcal{D}^{1,p}(\Omega)}$. There exists $C = C(\boldsymbol{G}, p, ||u||_{\mathcal{D}^{1,p}(\Omega)}) > 0$ such that for every $x \in \boldsymbol{G}$*

$$
\operatorname*{ess\,sup}_{B(x,1)} u \;\le\; C \left(\frac{1}{|B(x,2)|} \int_{B(x,2)} u^s dH \right)^{\frac{1}{s}} . \tag{10.6}
$$

In particular, we can take $s = p^*$ in the above inequality.

Proof. Given a function $\alpha \in C_o^\infty(\boldsymbol{G})$, $\alpha \geq 0$, for $\gamma \geq 1$ we consider the function $\phi = \alpha^p u^\gamma \in \overset{o}{\mathcal{D}}{}^{1,p}(\Omega)$. Using ϕ as a test function in (10.2) we find

$$\int_\Omega \alpha^p u^{\gamma+p^*-1} dH = \gamma \int_\Omega \alpha^p u^{\gamma-1}|Xu|^p dH \\ +p\int_\Omega \alpha^{p-1}u^\gamma |Xu|^{p-2} < Xu, X\alpha > dH.$$

At this point we choose $\gamma = s - p + 1$ to obtain

$$\begin{aligned}\int_\Omega \alpha^p u^{s+p^*-p} dH &= (s-p+1)\int_\Omega \alpha^p u^{s-p}|Xu|^p dH \\ &\quad +p\int_\Omega \alpha^{p-1}u^{s-p+1}|Xu|^{p-2} < Xu, X\alpha > dH \\ &\geq \int_\Omega \alpha^p |Xu|^{p-2} < Xu, X(u^{s-p+1}) > dH \\ &\quad -p\int_\Omega \alpha^{p-1}u^{s-p+1}|Xu|^{p-1}|X\alpha|\, dH. \end{aligned} \tag{10.7}$$

Now Young's inequality gives

$$\begin{aligned}&\int_\Omega \alpha^{p-1}u^{s-p+1}|Xu|^{p-1}|X\alpha|\, dH \\ &= \int_\Omega \alpha^{p-1}u^{\frac{s-p}{p'}}|Xu|^{p-1}u^{\frac{s}{p}}|X\alpha|\, dH \\ &\leq \left(\int_\Omega u^{s-p}\alpha^p|Xu|^p dH\right)^{\frac{1}{p'}}\left(\int_\Omega u^s|X\alpha|^p dH\right)^{\frac{1}{p}} \\ &\leq \frac{\epsilon}{p'}\int_\Omega \alpha^p u^{s-p}|Xu|^p dH + \frac{1}{\epsilon p}\int_\Omega u^s|X\alpha|^p dH, \end{aligned} \tag{10.8}$$

where $\epsilon > 0$ is arbitrary. Substituting (10.8) in (10.7), and choosing $\epsilon = \frac{1}{p}$, we find

$$\begin{aligned}\int_\Omega \alpha^p u^{s+p^*-p} dH \ &\geq \ (s-p+1)\int_\Omega \alpha^p u^{s-p}|Xu|^p dH \\ &\quad -\epsilon(p-1)\int_\Omega \alpha^p u^{s-p}|Xu|^p dH - \frac{1}{\epsilon}\int_\Omega u^s|X\alpha|^p dH \\ &\geq [s-p+\frac{1}{p}]\int_\Omega \alpha^p u^{s-p}|Xu|^p dH - p\int_\Omega u^s|X\alpha|^p dH \\ &\geq \frac{1}{p}\int_\Omega \alpha^p u^{s-p}|Xu|^p dH - p\int_\Omega u^s|X\alpha|^p dH. \end{aligned} \tag{10.9}$$

Let now $\psi = u^{\frac{s}{p}}$, so that $|X\psi|^p = \left(\frac{s}{p}\right)^p u^{s-p}|Xu|^p$. We obtain from (10.9)

$$\int_\Omega \alpha^p |X\psi|^p dH \ \le \ p \int_\Omega \alpha^p u^{p^*-p+s} dH + p^2 \int_\Omega u^s |X\alpha|^p dH. \tag{10.10}$$

For $x \in \boldsymbol{G}$ we now consider $1 \le r < R \le 2$ and let $\alpha \in C_o^\infty(B(x, R))$ be such that $\alpha \equiv 1$ in $B(x, r)$ and $|X\alpha| \le \frac{C}{R-r}$. Applying (1.1) one has with $\theta = \frac{p^*}{p} > 1$

(10.11)

$$\begin{aligned}
\left(\int_{B(x,r)} u^{\theta s} dH\right)^{\frac{1}{\theta}} &= \left(\int_{B(x,r)} \psi^{p^*} dH\right)^{\frac{p}{p^*}} \le S_p \int_{B(x,R)} |X(\alpha\psi)|^p dH \\
&\le 2^p \left(\int_{B(x,R)} \alpha^p |X\psi|^p dH \ + \int_{B(x,R)} \psi^p |X\alpha|^p dH\right) \\
&\le 2^p \left(p \int_\Omega \alpha^p u^{p^*-p+s} dH + (p^2+1) \int_\Omega u^s |X\alpha|^p dH\right) \\
&\le 2^p \left(p ||u||_{L^\infty(\Omega)}^{p^*-p} \int_\Omega \alpha^p u^s dH + (p^2+1) \int_\Omega u^s |X\alpha|^p dH\right)
\end{aligned}$$

In the second to the last inequality we have inserted (10.10), whereas Theorem 10.1 has been used in the last. From (10.11) we conclude the existence of $K = K(G, p, ||u||_{L^\infty(\Omega)}) > 0$ such that

$$\left(\int_{B(x,r)} u^{\theta s} dH\right)^{\frac{1}{\theta}} \ \le \ \frac{K}{(R-r)^p} \int_{B(x,R)} u^s dH.$$

Assuming the finiteness of the integral in the right-hand side of the latter inequality, Moser's iteration procedure finally gives (10.6). □

Theorem 10.4 has several interesting consequences. We only list the most direct one.

Corollary 10.5. *Let $\Omega \subset \boldsymbol{G}$ be an unbounded open set. If u is a solution to (10.1), then*

$$\lim_{g \in \boldsymbol{G},\ d(g,e) \to \infty} u(g) = 0.$$

Proof. Immediate consequence of Theorem 10.4 and of the assumption $u \in L^{p^*}(\Omega)$. □

The next result is a theorem of unique continuation for non- negative weak solutions to equations with critical growth. In the Euclidean setting and for linear equations a result of this kind was first observed in [13]. A version for the Heisenberg group was obtained in [31].

Theorem 10.6. *Let $\Omega \in \boldsymbol{G}$ be an open set and for $1 < p < Q$ let $u \in \mathcal{L}^{1,p}_{loc}(\Omega)$ be a nonnegative weak solution of the equation*

$$\sum_{i=1}^{m} X_i(|Xu|^{p-2}X_iu) = -V\ u^{p-1} \qquad in \quad \Omega,$$

with $V \in L^{Q/p}(\Omega)$. There exist $\delta = \delta(\boldsymbol{G}, ||V||_{L^{Q/p}(\Omega)}) > 0$ and $C = C(\boldsymbol{G}, ||V||_{L^{Q/p}(\Omega)}) > 0$ such that for every $\overline{B}(g, 2r) \subset \Omega$ one has

$$\int_{B(g,2r)} u^{\delta} dH \ \leq \ C \int_{B(g,r)} u^{\delta} dH.$$

Proof. By assumption we have for every $\phi \in \overset{o}{S}{}^{1,p}(\Omega)$

$$\int_{\Omega} |Xu|^{p-2} < Xu, X\phi > dH = \int_{\Omega} Vu^{p-1}\phi dH.$$

We consider the test function $\phi = \alpha^p(u+\epsilon)^{-p+1}$, $\epsilon > 0$, with $\alpha \in C^{\infty}_{o}(\Omega)$, $0 \leq \alpha \leq 1, \alpha \equiv 1$ in $B(g, r)$, $\alpha \equiv 0$ outside $B(g, 2r)$, $|X\alpha| \leq C/r$. Substituting ϕ in the equation we find

$$\begin{aligned}(p-1)&\int_{\Omega} \alpha^p(u+\epsilon)^{-p}|Xu|^p dH \\ &\leq p\int_{\Omega} \alpha^{p-1}(u+\epsilon)^{-p+1}|Xu||X\alpha|\, dH + \int_{\Omega} |V|\alpha^p dH.\end{aligned}$$

We obtain

$$\begin{aligned}(p-1)&\int_{\Omega} \alpha^p(u+\epsilon)^{-p}|Xu|^p dH \\ &\leq \ p\int_{\Omega} \alpha^{p-1}(u+\epsilon)^{-p+1}|Xu|^{p-1}|X\alpha| dH \ + \int_{\Omega} |V|\alpha^p dH,\end{aligned}$$

which, letting $v = log(u+\epsilon)$, we rewrite as follows

$$(p-1)\int_{\Omega} \alpha^p|Xv|^p dH \ \leq \ p\int_{\Omega} \alpha^{p-1}|Xv|^{p-1}|X\alpha| dH + \int_{\Omega} |V|\alpha^p dH.$$

At this point a standard application of Hölder and Young inequalities allows to infer for every $\sigma > 0$

$$\begin{aligned}&\int_{\Omega} \alpha^p|Xv|^p dH \ \leq \ \sigma\int_{\Omega} \alpha^p|Xv|^p dH \\ &+\sigma^{-1}\frac{1}{p-1}\int_{\Omega} |X\alpha|^p dH \ + \frac{|B(g,2r)|}{p-1}||V||_{L^{Q/p}(B(g,2r))}.\end{aligned}$$

Choosing $0 < \sigma < 1$ we find

$$\int_{\Omega} \alpha^p |Xv|^p dH \leq \ C \frac{|B(g,2r)|}{r^p} \ \left(1 + ||V||_{L^{Q/p}(B(g,2r))}\right).$$

Invoking D. Jerison's Poincaré inequality [41] one concludes

$$\frac{1}{|B(g,r)|} \int_{B(g,r)} |v - v_{B(g,r)}|^p dH \ \leq \ C \left(1 + ||V||_{L^{Q/p}(B(g,2r))}\right).$$

The latter inequality implies that $v \in BMO$ with respect to the homogeneous structure of $(\boldsymbol{G}, d, dH)$. By the results in [6] there exists $\delta > 0$ such that $(u + \epsilon)^{\delta} \in A_2$, i. e.,

$$\left(\frac{1}{|B(g,r)|} \int_{B(g,r)} (u+\epsilon)^{\delta} dH\right) \left(\frac{1}{|B(g,r)|} \int_{B(g,r)} (u+\epsilon)^{-\delta} dH\right) \ \leq \ C,$$

for every $B(g,r)$ such that $\overline{B}(g,2r) \subset \Omega$. By Fatou theorem the latter inequality continues to hold replacing $(u+\epsilon)$ by u. Finally, one obtains the doubling inequality in the statement of the lemma by a by now standard argument [14] . □

Corollary 10.7. *Let u be a non-negative solution to (10.1) in a connected, open set Ω. If u vanishes to infinite order at one point $g \in \Omega$, then $u \equiv 0$ in Ω.*

Proof. We can rewrite (1.3) in the form

$$\sum_{i=1}^{m} X_i(|Xu|^{p-2} X_i u) = -V \ u^{p-1},$$

with $V = u^{p^*-p}$, so that $V \in L^{Q/p}(\Omega)$ if and only if $u \in L^{p^*}(\Omega)$. Since this is true as a consequence of the Folland-Stein embedding (1.1), we can thus apply Theorem 10.6 to reach the conclusion. □

Acknowledgements. We thank Bill Beckner, Adam Korányi and Giorgio Talenti for useful discussions.

References

1. T. Aubin, Problèms isopérimétriques et espaces de Sobolev, J. Diff. Geometry, **11** (1976), 573-598.
2. T. Aubin, Espaces de Sobolev sur les variétés Riemanniennes, Bull. Sc. Math., **100** (1976), 149-173.
3. T. Aubin, Equations différentielles non linéaires et Problème de Yamabe concernant la courbure scalaire, J. Math. Pures et appl., **55** (1976), 269-296.
4. T. Aubin, Nonlinear analysis on manifolds. Monge-Ampère equations, Springer-Verlag, New York, 1982.

5. J. M. Bony, Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les operateurs elliptique degeneres, Ann. Inst. Fourier, Grenoble, 1, **119** (1969), 277-304.
6. N. Burger, Espaces des fonctions à variation moyenne bornée sur une espace de nature homogène, C. R. Acad. Sci. PAris Ser. A-B, 3, **286** (1978), 139-142.
7. L. Capogna, D. Danielli, N. Garofalo, An imbedding theorem and the Harnack inequality for nonlinear subelliptic equations, Comm. Part. Diff. Eq., **18** (1993), 1765-1794.
8. L. Capogna, D. Danielli, N. Garofalo, The geometric Sobolev embedding for vector fields and the isoperimetric inequality, Comm. Anal. and Geom., **2** (1994), 203-215.
9. L. Capogna, D. Danielli, N. Garofalo, Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations, Amer. J. of Math., **118** (1997), 1153-1196.
10. L. Capogna, N. Garofalo, NTA domains for Carnot-Carathéodory metrics and a Fatou type theorem, J. Fourier Anal. Appl., 4, **4-5** (1998), 403-432.
11. L. Capogna, N. Garofalo, Duy-Minh Nhieu, A version of a theorem of Dahlberg for the sub-elliptic Dirichlet problem, Math. Res. Letters, **5** (1998), 541-549.
12. L. Capogna, Properties of harmonic measure in the Dirichlet problem for certain nilpotent Lie groups, preprint.
13. F. Chiarenza, N. Garofalo, Unique continuation for non-negative solutions of Schrödinger operators, Univ. Of Minnesota, IMA Preprint series no.122, 1984.
14. Coifman, C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Math. **51** (1974), 241–250.
15. M. Cowling, A. H. Dooley, A. Korányi, F. Ricci, H-type groups and Iwasawa decompositions, Adv. Math., **87** (1991), 1-41.
16. M. Cowling, A. Korányi, Harmonic analysis on Heisenberg type groups from a geometric viewpoint, in “Lie Group Representation III", pp.60-100, Lec. Notes in Math., **1077** (1984), Springer-Verlag.
17. J. Cygan, Subadditivity of homogeneous norms on certain nilpotent Lie groups, Proc. Amer. Math. Soc., **83** (1981), 69-70.
18. E. Damek, Harmonic functions on semidirect extensions of type H nilpotent groups, Trans. Amer. Math. Soc., 1, **290** (1985), 375-384.
19. E. Damek, A Poisson kernel on Heisenberg type nilpotent groups, Coll. Math., **53** (1987), 255-268.
20. E. Damek, The geometry of semi-direct extensions of a Heisenberg type nilpotent group, Coll. Math., **53** (1987), 255-268.
21. D. Danielli, N. Garofalo, Geometric properties of solutions to subelliptic equations in nilpotent Lie groups, Lect. Notes in Pure and Appl. Math., “Reaction Diffusion Systems", Trieste, October 1995, Ed. G. Caristi Invernizzi, E. Mitidieri, Marcel Dekker, **194** (1998).
22. M. Derridj, Un probléme aux limites pour une classe d’opérateurs du second ordre hypoelliptiques, Ann. Inst. Fourier, Grenoble, **21**, 4 (1971), 99-148.
23. M. Derridj, Sur un théorème de traces, Ann. Inst. Fourier, Grenoble, **22**, 2 (1972), 73-83.
24. C. Fefferman, D.H. Phong, Subelliptic eigenvalue problems, Proceedings of the Conference in Harmonic Analysis in Honor of A. Zygmund, Wadsworth Math. Ser., Belmont, CA, (1981), 530-606.
25. G. Folland, A fundamental solution for a subelliptic operator, Bull. Amer. Math. Soc., **79** (1973), 373-376.
26. G. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups, Ark. Math., **13** (1975), 161-207.
27. G. B. Folland, E. M. Stein, Estimates for the $\bar{\partial}_b$ Complex and Analysis on the Heisenberg Group, Comm. Pure Appl. Math., **27** (1974), 429-522.
28. B. Franchi, C. Gutierrez, R. L. Wheeden, Weighted Sobolev-Poincaré inequalities for Grushin type operators, Comm. PDE, **19** (1994), 523-604.

29. N. Garofalo, The Brezis-Nirenberg phenomenon on the Heisenberg group, unpublished notes, 1990.
30. N. Garofalo, E. Lanconelli, Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation, Ann. Inst. Fourier (Grenoble), **40** (1990), no. 2, 313–356.
31. N. Garofalo, E. Lanconelli , Existence and non-existence results for semilinear elliptic problems in unbounded domains, Indiana Univ. Math. J., **41** (1992) 71-98.
32. N. Garofalo, D. M. Nhieu, Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces, Comm. Pure Appl. Math., **49** (1996), 1081-1144.
33. N. Garofalo, D. M. Nhieu, Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot-Carathéodory spaces, J. d'Analyse Math., **74** (1998), 67-97.
34. N. Garofalo, D. Vassilev, The non-linear Dirichlet problem and the CR Yamabe problem, Proc. of the Conference in memory of Filippo Chiarenza, Catania, November 12-14 1998. To appear in Le Matematiche.
35. D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Second ed., Revised third printing, Vol. 244, Springer-Verlag, Grundlehren der mathematischen Wissenschaften, 1998.
36. T. Hashimoto, M. Otani, Nonexistence of weak solutions of nonlinear elliptic equations in exterior domains, Houston J. Math., 2, **23** (1997), 267-290.
37. I. Holopainen, Positive solutions of quasilinear elliptic equations on Riemannian manifolds, Proc. London Math. Soc., (3) **65** (1992), no. 3, 651-672.
38. I. Holopainen, S. Rickman, Quasiregular mappings of the Heisenberg group, Math. Ann., **294** (1992), no. 4, 625-643.
39. H. Hörmander, Hypoelliptic second-order differential equations, Acta Math., **119** (1967), 147-171.
40. H. Hueber, The Poisson space of the Korányi ball, Math. Ann., **268** (1984), no. 2, 223-232.
41. D. Jerison, The Poincaré inequality for vector fields satisfying Hörmander's condition, Duke Math. J., **53** (1986), 503-523.
42. D. Jerison, The Dirichlet problem for the Kohn Laplacian on the Heisenberg group, I, J. of Funct. Anal. **43** (1981), 97-141.
43. D. Jerison, The Dirichlet problem for the Kohn Laplacian on the Heisenberg group, II, J. of Funct. Anal. **43** (1981), 224-257.
44. D. Jerison, J. Lee, A subelliptic, nonlinear eigenvalue problem and scalar curvature on CR manifolds, Contemporary Math., **27** (1984), 57-63.
45. D. Jerison, J. Lee, The Yamabe problem on CR manifolds, J. Diff. Eq., **25** (1987), 167-197.
46. D. Jerison, J. Lee, Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem, J. of the Amer. Math. Soc., 1, **1** (1988), 1-13.
47. D. Jerison, J. Lee, Intrinsic CR normal coordinates and the CR Yamabe problem, J. Diff. Geom., **29** (1989), no. 2, 303–343.
48. A. Kaplan, Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms, Trans. Amer. Math. Soc., **258** (1980), 147-153.
49. A. Kaplan, Riemannian nilmanifolds attached to Clifford modules,Geom. Dedicata, **11** (1981), 127-136.
50. A. Kaplan, On the geometry of groups of Heisenberg type, Bull. London Math. Soc., **15** (1983), 35-42.
51. A. Kaplan, F. Ricci, Harmonic ananlysis on groups of Heisenberg type, in "Harmonic Analysis", pp.416-435, Lec. Notes in Math., **992** (1983), Springer-Verlag.
52. J. J. Kohn, L. Nirenberg, Non-coercive boundary value problems, Comm. Pure and Appl. Math., 18, **18** (1965), 443-492.

53. A. Korányi, Kelvin transform and harmonic polynomials on the Heisenberg group, Adv.Math. **56** (1985), 28-38.
54. A. Korányi, Geometric properties of Heisenberg-type groups, Adv.Math. **56** (1985), 28-38.
55. A. Korányi, Radial Poisson kernel for $\{t > 0\}$ in the Heisenberg group, unpublished notes. Personal communication.
56. A. Korányi, H. M. Reimann, Horizontal normal vectors and conformal capacity of spherical rings in the Heisenberg group, Bull. Sc. Math., **111** (1987), 3-21.
57. E. Lanconelli, F. Uguzzoni, On the Poisson kernel for the Kohn Laplacian, Rend. Mat. Appl., 7 **17** (1998), no.4, 659-677.
58. E. Lanconelli, F. Uguzzoni , Asymptotic behavior and non-existence theorems for semilinear Dirichlet problems involving critical exponent on unbounded domains of the Heisenberg group, Boll. Un. Mat. Ital., (8) **1-B** (1998), 139-168.
59. J. M. Lee, T. H. Parker, The Yamabe problem, Bull. Am. Math. Soc., 1, **17** (1987), 37-91.
60. P. L. Lions, The concentration compactness principle in the calculus of variations. The locally compact case. Part 1, Ann. Inst. H. Poincaré, **1** (1984), 109-145
61. P. L. Lions, The concentration compactness principle in the calculus of variations. The locally compact case. Part 2, Ann. Inst. H. Poincaré, **1** (1984), 223-283
62. P. L. Lions, The concentration compactness principle in the calculus of variations. The limit case. Part 1, Rev. Mat. Iberoamericana **1.1** (1985), 145-201
63. P. L. Lions, The concentration compactness principle in the calculus of variations. The limit case. Part 2, Rev. Mat. Iberoamericana **1.2** (1985), 45-121
64. G. Lu, Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications, Rev. Mat. Iberoamericana, **8** (1992), no. 3, 367-439.
65. J. Moser, On Harnack's theorem for elliptic differential equations, Comm. Pure Appl. Math.**14** (1961), 577-591.
66. A. Nagel, E.M. Stein, S. Wainger, Balls and metrics defined by vector fields I: Basic properties, Acta Math. **155** (1985), 103-147.
67. S. Pohožaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl., **6** (1965), 1408-1411.
68. F. Rellich, Darstellung der Eigenwerte von $\Delta u + \lambda u$ durch ein Randintegral, Math. Zeit., **46** (1940), 635-646.
69. R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Differential Geom. **20** (1984), no. 2, 479–495.
70. J. Serrin, Local behavior of solutions of quasi-linear equations, Acta Math., **111** (1964), 247-302.
71. J. Serrin, Isolated singularities of solutions of quasi-linear equations, Acta Math., **113** (1965), 219-240.
72. E. M. Stein, Some problems in harmonic analysis suggested by symmetric spaces and semisimple groups, Proc. Int. Congr. Math., Nice I, 1970, Gauthier-Villars, Paris, 1971, 173-179.
73. E. M. Stein, Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals, Princeton Univ. Press., 1993.
74. M. Struwe, Variational Methods, Aplications to Nonlinear Partial Differential Equations and Hamilton Systems, Springer-Verlag, 1990.
75. G. Talenti, Best constant in the Sobolev inequality, Ann. Mat. Pura e Appl., **110** (1976), 353-372.
76. N. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Pisa, **22** (1968), 265-274.
77. F. Uguzzoni, A non-existence theorem for a semilinear Dirichlet problem involving critical exponent on half spaces of the Heisenberg group, preprint.

78. N. Th. Varopoulos, L. Saloff-Coste, T. Coulhon, Analysis and Geometry on Groups, Cambridge U. Press, 1992.
79. D. Vassilev, Ph. D. Dissertation, Purdue University.
80. C. J. Xu, Subelliptic variational problems, Bull. Soc. Math. France, **118** (1990), 147-169.
81. C. J. Xu, The Dirichlet problems for a class of semilinear sub-elliptic equations, Nonlinear Anal., **37** (1999), 1039-1049.