NON-KAEHLER HETEROTIC STRING SOLUTIONS
WITH NON-ZERO FLUXES AND NON-CONSTANT DILATON

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ABSTRACT. Conformally compact and complete smooth solutions to the Strominger system with non vanishing flux, non-trivial instanton and non-constant dilaton using the first Pontrjagin form of the $(-)$-*connection* on 6-dimensional non-Kähler nilmanifold are presented. In the conformally compact case the dilaton is determined by the real slices of the elliptic Weierstrass function. The dilaton of non-compact complete solutions is given by the fundamental solution of the Laplacian on R^4.

CONTENTS

1. INTRODUCTION

The goal of this paper is the explicit construction of smooth six-dimensional non-Kähler solution to the Strominger system with a non-constant dilaton.

A model for string theory proposed in [15] involves a ten dimensional space $\mathbb{R}^{1,3} \times M^6$ which is the product of a Lorentzian spacetime with a six-dimensional Calabi-Yau manifold M. The latter was equipped with an SU(3) Yang-Mills connection, Donaldson-Uhlenbeck-Yau instanton, of the Calabi-Yau metric. In a key paper Strominger [63] systematically considered a generalization of this construction allowing a background with non-zero torsion, H-fluxes, which is motivated by physical significance. This led to a system of differential equations known as the *Strominger system*, which specifies the geometric inner space M to be a complex (non-Kähler) conformally balanced 6-manifold with holomorphically trivial canonical bundle equipped, in addition, with an instanton bundle E compatible with the Green-Schwarz anomaly cancellation condition. The latter also involves the first Pontrjagin form of a linear connection whose determination is a part of the problem. An important problem considered in the past thirty years is to provide "backgrounds" and solutions of the Strominger system. Several connections have been used in order to satisfy the

anomaly condition, such as, the Levi-Civita connection [63, 34], the Chern connection [63, 56, 30], the (+)-connection [17, 20], and the (−)-connection [46, 11] etc..

A smooth compact solution was first found by Li & Yau [56] and Fu & Yau [30, 31]. In [30], developing the ideas of [37], the authors considered compact non-Kähler 6-manifolds which are $\mathbb{T}^2$-bundles over a Calabi-Yau 4-manifold using the first Pontrjagin form of the Chern connection. Thus, [30] showed existence of a balanced metric while satisfying the Hermitian-Yang-Mills equations and the anomaly equation. In this case, the difficulty to satisfy anomaly cancellation condition turns into a non-linear PDE of Monge-Ampère type for the dilaton function. We note that when E is the tangent bundle of a Kähler manifold M the flux H vanishes and Strominger's system is solved by the Calabi-Yau metric [67] and the Donaldson-Uhlenbeck-Yau instanton [64, 22]. In particular, the non-Kähler case can be considered as a generalization of Calabi's conjecture for the case of non-Kähler Calabi-Yau threefolds. Since the choice of the first Pontrjagin form of the (−)-connection is preferable by physical reasons [11], in [10, 9] the authors considered the smooth compact non-Kähler model of a $\mathbb{T}^2$-bundle over a Calabi-Yau four manifold but with the first Pontrjagin form of the (−)-connection in the anomaly cancellation. It was shown in [10, 9] that the PDE system for the dilaton function gives rise to a single PDE which is of the Laplace type for which a solution exists provided the natural compatibility condition holds.

Non-compact solutions can have different physical interpretation in string theory [14, 13, 23, 62]. They may be a local models of a compact solutions or correspond to the supergravity descriptions of the solitonic objects of the theory [29]. A class of non-compact smooth solutions to the Strominger system on a $\mathbb{T}^2$ bundle over the non-compact Eguchi-Hanson space is considered in [29] where the non-linear equation for the dilaton imposed by the anomaly cancellation with the first Pontrjagin form of the Chern connection is solved.

In this paper we construct smooth solutions with non vanishing flux and non-constant dilaton to the Strominger system using the first Pontrjagin form of the (−)-*connection* on 6-dimensional complete non-compact manifold equipped with conformally balanced Hermitian structures coupled with carefully chosen instanton bundle. The source of the construction is the already constructed smooth compact solutions to the Strominger system with constant dilaton on nilmanifods presented in [27] and the ideas of [34] and [37]. In particular, [34] posed the question of solving the anomaly condition for compact supersymmetric geometries that are two-torus bundles over either conformally $\mathbb{T}^4$ or K3 manifold. Our main results are explicit complete smooth examples of the former case. In particular, we prove

Theorem 1.1. *The conformally compact manifold* $M^6 = (\Gamma \backslash H_5, \bar{g}, J, \nabla^-, A_\lambda)$ *is a Hermitian manifold which solves the Strominger system with non-constant dilaton* f, *non-trivial flux* $H = \bar{T}$, *non-flat instanton* A_λ *using the first Pontrjagin form of* ∇^- *and negative* α'. *Furthermore, the heterotic equations of motion* (2.2) *are satisfied up to first order of* α'.

The precise definition of the background and proof of Theorem 1.1 are given in Section 3.4. Our solutions are complete non-Kähler $\mathbb{T}^2$ bundles over conformally compact asymptotically hyperbolic metric on $\mathbb{T}^4$ with conformal boundary at infinity a flat torus $\mathbb{T}^3$. Using the first Pontrjagin form of the (−)-connection together with the first Pontrjagin form of a carefully chosen instanton we arrived via the anomaly cancellation to a single highly non-linear PDE for the dilaton function. Assuming that the dilaton depends only on one of the independent variables we can reduce the equation to Weiestrass' equation. This allows us to determine the dilaton as a real slice of an elliptic function of order two, $f = \frac{1}{2}\ln(\alpha^2 \mathcal{P})$ where $\mathcal{P}$ is the Weierstrass' elliptic function with pole of order two at the origin, $z = \int^{\mathcal{P}} \frac{d\mathcal{P}}{2\sqrt{\mathcal{P}(\mathcal{P}-a)(\mathcal{P}+a)}}$. The positive parameter a depends on the group H_5 and the magnitude of α'.

This suggests there could be a relation with the F-theory/heterotic duality principle. It is a well known fact that for warped compactification there must be some branes with negative tension [35]. It was argued in [49, 58] (see also [61, 42, 43] for earlier discussions) that a negative tension brane in heterotic string theory could be understood as a T-dual of the Atiyah-Hitchin [4] manifold.

In Section 3.5 we present another smooth non-compact but complete solutions to the Strominger system using the first Pontrjagin form of the $(-)$-connection with positive string tension on certain $\mathbb{T}^2$ bundles over $\mathbb{R}^4$ with non-vanishing torsion, non-trivial instanton and non-constant dilaton. We construct an instanton whose first Pontrjagin form together with the first Pontrjagin form of the $(-)$-connection imposes via the anomaly cancellation a system of two equations of Laplace type on the dilaton. The non-constant dilaton function of our smooth non-compact complete solutions is determined by a harmonic function, the fundamental solution of the Laplacian on $\mathbb{R}^4$. In the special case of a trivial $\mathbb{T}^2$ bundle, i.e., on the product of $\mathbb{T}^2 \times \mathbb{R}^4$ we recover the 'symmetric background solution' from [13] thus strengthening the conjectured existence of a non-compact non-trivial solution satisfying the anomaly cancellation. The precise result is the following

Theorem 1.2. *The non-compact simply connected manifold* $(H_5, \bar{g}, J, \nabla^-, A_{0,d})$ *is a complete Hermitian manifold which solves the Strominger system with non-constant dilaton* f *determined by* (3.25), *non-zero flux* $H = \bar{T}$ *and non-flat instanton* $A_{0,d}$ *using the first Pontrjagin form of* ∇^- *and positive* α'.

The complete manifold $(H_5, \bar{g}, J, \nabla^-, A_{0,d})$ *described above also solves the heterotic equations of motion* (2.2) *up to the first order of* α'.

Our conventions: The connection 1-forms ω_{ji} of a metric connection $\nabla, \nabla g = 0$ with respect to a local orthonormal basis $\{E_1, \ldots, E_d\}$ are given by $\omega_{ji}(E_k) = g(\nabla_{E_k} E_j, E_i)$, since we write $\nabla_X E_j = \omega_j^s(X) E_s$.

The curvature 2-forms Ω_j^i of ∇ are given in terms of the connection 1-forms ω_j^i by

$$\Omega_j^i = d\omega_j^i + \omega_k^i \wedge \omega_j^k, \quad \Omega_{ji} = d\omega_{ji} + \omega_{ki} \wedge \omega_{jk}, \quad R_{ijk}^l = \Omega_k^l(E_i, E_j), \quad R_{ijkl} = R_{ijk}^s g_{ls}.$$

The first Pontrjagin class is represented by the 4-form $8\pi^2 p_1(\nabla) = \sum_{1 \leq i < j \leq d} \Omega_j^i \wedge \Omega_j^i$.

2. Motivation from heterotic string theory

The bosonic fields of the ten-dimensional supergravity which arises as low energy effective theory of the heterotic string are the spacetime metric g, the NS three-form field strength (flux) H, the dilaton ϕ and the gauge connection A with curvature 2-form F^A. The bosonic geometry is of the form $\mathbb{R}^{1,9-d} \times M^d$, where the bosonic fields are non-trivial only on M^d, $d \leq 8$. We consider the two connections $\nabla^\pm = \nabla^g \pm \frac{1}{2} H$, where ∇^g is the Levi-Civita connection of the Riemannian metric g. Both connections preserve the metric, $\nabla^\pm g = 0$ and have totally skew-symmetric torsion $\pm H$, respectively. We denote by $R^g, R^\pm$ the corresponding curvature.

We consider the heterotic supergravity theory with an α' expansion where $1/2\pi\alpha'$ is the heterotic string tension. The bosonic part of the ten-dimensional supergravity action in the string frame is ([47], [11], $R = R^-$)

$$(2.1) \qquad S = \frac{1}{2k^2} \int d^{10}x \sqrt{-g} e^{-2\phi} \Big[Scal^g + 4(\nabla^g \phi)^2 - \frac{1}{2}|H|^2 - \frac{\alpha'}{4}\Big(Tr|F^A|^2) - Tr|R|^2\Big) \Big].$$

The string frame field equations (the equations of motion induced from the action (2.1)) of the heterotic string up to the first order of α' in sigma model perturbation theory are [45, 47] (we use the notations in [36])

$$Ric^g(X,Y) - \frac{1}{4} < i_X H, i_Y H > + 2((\nabla^g)^2 \phi)(X,Y) - \frac{\alpha'}{4}\Big[< i_X F^A, i_Y F^A > - < i_X R, i_Y R > \Big] = 0$$

$$(2.2) \qquad \delta(e^{-2\phi}H) = -Tr(\nabla^g(e^{-2\phi}H)) = 0, \quad \delta^{\nabla^+}(e^{-2\phi}F^A) = -Tr(\nabla^+(e^{-2\phi}F^A)) = 0,$$

where i_X is the interior multiplication of tensors and $< .,. >$ is the corresponding scalar product. The field equation of the dilaton ϕ is implied from the first two equations above.

The Green-Schwarz anomaly cancellation mechanism requires that the three-form Bianchi identity receives an α' correction of the form

$$(2.3) \qquad dH = \frac{\alpha'}{4}8\pi^2(p_1(M^d) - p_1(E)) = \frac{\alpha'}{4}\Big(Tr(R \wedge R) - Tr(F^A \wedge F^A)\Big),$$

where $p_1(M^d)$ and $p_1(E)$ are the first Pontrjagin forms of M^d with respect to a connection ∇ with curvature R and the vector bundle E with connection A, respectively.

A class of heterotic-string backgrounds for which the Bianchi identity of the three-form H receives a correction of type (2.3) are those with (2,0) world-volume supersymmetry. Such models were considered in [48]. The target-space geometry of (2,0)-supersymmetric sigma models has been extensively investigated in [48, 63, 44]. Recently, there is revived interest in these models [21, 32, 17, 33, 34, 36] as string backgrounds and in connection with heterotic-string compactifications with fluxes [16, 5, 6, 7, 56, 30, 31, 8, 41, 40, 39, 38, 59, 2, 3, 12].

Equations (2.3), (2.1) and (2.2) involve a subtlety due to the choice of the connection ∇ on M^d since anomalies can be canceled independently of the choice [46]. Different connections correspond to different regularization schemes in the two-dimensional worldsheet non-linear sigma model. Hence the background fields given for the particular choice of ∇ must be related to those for a different choice by a field redefinition [60]. Connections on M^d proposed to investigate the anomaly cancellation (2.3) are ∇^g [63, 34], ∇^+ [17, 20, 27], ∇^- [46, 11, 16, 36, 51, 54, 55, 57, 58, 49], Chern connection ∇^c when $d = 6$ [63, 56, 30, 31, 8].

A heterotic geometry preserves supersymmetry if and only if, in 10 dimensions, there exists at least one Majorana-Weyl spinor ϵ such that the following Killing-spinor equations hold [63, 11]

$$(2.4) \qquad\qquad \nabla^+\epsilon = 0, \qquad (2d\phi - H) \cdot \epsilon = 0, \qquad F^A \cdot \epsilon = 0,$$

where $\cdot$ means Clifford action of forms on spinors. The system of Killing spinor equations (2.4) together with the anomaly cancellation condition (2.3) is known as the *Strominger system* [63, 56]. The last equation in (2.4) is the instanton condition which means that the curvature F^A is contained in a Lie algebra of a Lie group which is a stabilizer of a non-trivial spinor. In dimension 6 this group is $SU(3)$ and the last equation in (2.4) is the Donaldson-Uhlenbeck-Yau instanton. The $SU(3)$-instanton means that the trace of F^A with respect to the Kähler 2 form as well as the (2,0)+(0,2)-part of F^A vanish simultaneously. The real expression of the $SU(3)$-instanton condition on a six dimensional Hermitian manifold (M, g, J) is given by

$$(2.5) \qquad\qquad (F^A)^i_j(JE_k, JE_l) = (F^A)^i_j(E_k, E_l), \qquad \sum_{k=1}^{6}(F^A)^i_j(E_k, JE_k) = 0.$$

The first compact torsional solutions for the heterotic/type I string were obtained via duality from M-theory compactifications on K3×K3 proposed in [19]. The metric was first written down on the orientifold limit in [19] and such backgrounds have since been studied (see [5, 6] and references therein). The metric and the H-flux are derived by applying a chain of supergravity dualities and the resulting geometry in the heterotic theory is a $\mathbb{T}^2$ bundle over a K3.

Compact smooth examples in dimension six solving (2.4) and (2.3) with non-zero flux H and non-constant dilaton were constructed by Li and Yau [56] for U(4) and U(5) principal bundles taking $R = R^c$-the curvature of the Chern connection in (2.3). Non-Kähler compact solutions of (2.4) and (2.3) on some torus bundles over Calabi-Yau 4-manifold (K3 surfaces or complex torus)

are presented by Yau et al. [30, 31, 8] using the Chern connection in (2.3). Compact solutions, up to two loops, in dimension six with non-zero flux H and non-constant dilaton involving the $(-)$-connection are investigated in [10, 9]. Compact examples solving (2.4) and (2.3) with nonzero field strength, non-trivial instanton, constant dilaton and taking $R = R^+$, were constructed in [17, 20, 27].

In the presence of a curvature term $Tr(R \wedge R)$ the solution of the Strominger system (2.4), (2.3) obey the second and the third equations of motion (the second and the third equations in (2.2)) but do not always satisfy the Einstein equations of motion (see [27] where a sufficient quadratic condition on R is found). It was proved in [50] that (2.4) and (2.3) imply (2.2) if and only if R is an instanton in dimensions 5,6,7,8, (see [57] for higher dimensions). In particular, in dimension 6, R is required to be an SU(3)-instanton.

The physically relevant connection on the tangent bundle to be considered in (2.3), (2.1), (2.2) is the $(-)$-connection [11, 46]. One reason is that the curvature R^- of the $(-)$-connection is an instanton up to the first order of α' which is a consequence of the first equation in (2.4), (2.3) and the well known identity

$$(2.6) \qquad R^+(X,Y,Z,U) - R^-(Z,U,X,Y) = \frac{1}{2}dH(X,Y,Z,U).$$

Indeed, (2.3) together with (2.6) imply $R^+(X,Y,Z,U) - R^-(Z,U,X,Y) = O(\alpha')$ and the first equation in (2.4) yields that the holonomy group of ∇^+ is contained in $SU(n)$, i.e. the curvature 2-form $R^+(X,Y) \subset \mathfrak{su}(n)$ and therefore R^- satisfies the instanton condition (2.5) up to the first order of α'. Hence, a solution to the Strominger system with first Pontrjagin form of the $(-)$-connection always satisfies the heterotic equations of motion (2.2) up to the first order of α' (see e.g.[57] and references therein).

We remark that in the case of compact Hermitian manifold with holomorhically trivial canonical bundle, the vanishing theorem from [52, 53] shows that R^- is an instanton if and only if the manifold is Kähler. Indeed, (2.6) yields that if R^- is an instanton then the trace of dH with respect to the Kähler form vanishes since the holonomy group of ∇^+ is contained in $\mathfrak{su}(3)$. Hence, the function h defined in [52, 53] as the trace of dH vanishes which implies, due to [52, Corollary 4.2], that there are no holomorphic top-forms unless the manifold is Kähler.

Concerning the Chern connection, it is shown in [57] that the curvature of the Chern connection is an instanton up to zeros order of α' if and only if the H-flux vanishes and the manifold is Kähler. The proof in [57] relies on a point-wise identity established in [52] and therefore the result is purely local.

2.1. **The geometric model.** Necessary and sufficient conditions to have a solution to the system of gravitino and dilatino equations (the first two equations in (2.4)) in dimension $2n$ were derived by Strominger in [63] involving the notion of $SU(n)$-structure and then studied by many authors [32, 33, 34, 17, 16, 51, 5, 6, 36, 56, 30, 31, 8].

The gravitino equation, the first equation in (2.4) shows that there exists a parallel spinor with respect to the $(+)$-connection. This reduces the structure group $SO(2n)$ to a subgroup of $SU(n)$ since the holonomy group of ∇^+ reduces to a subgroup of $SU(n)$, i.e., the manifold is an almost Hermitian manifold admitting a linear connection having totally skew-symmetric torsion which preserves both the almost Hermitian structure and a non-vanishing $(n,0)$-form (complex volume form).

The dilatino equation, the second identity in (2.4), yields that the almost complex structure is integrable and the trace of the torsion 3-form with respect to the Kähler form is an exact 1-form. Strominger shows in [63] the existence of a unique Hermitian connection with skew-symmetric

torsion on any Hermitian manifolds writing explicitly the torsion 3-form from the exterior derivative of the Kähler form (∇^+ in our notations). He also shows that the ∇^+-parallel complex volume form supplies a holomorphic complex volume form whose norm determines the dilaton.

Next we detail the model in dimension six which is the focus of the paper. Let (M, J, g) be a Hermitian 6-manifold with Riemannian metric g and a complex structure J. The Kaehler form F and the Lee form θ are defined by $F(\cdot, \cdot) = g(\cdot, J\cdot)$, $\quad \theta(\cdot) = \delta F(J\cdot)$, respectively, where $*$ is the Hodge operator and δ is the co-differential, $\delta = -*d*$. The flux H, i.e., the torsion of the connection ∇^+ preserving the Hermitian structure (J, g) is given by [63]

$$(2.7) \qquad H = T = d^c F, \qquad \text{where} \qquad d^c F(X, Y, Z) = -dF(JX, JY, JZ).$$

Clearly, (2.7) determines the connection ∇^+ uniquely since $\nabla^+ g = 0$.

An SU(3)-structure is determined by an additional non-degenerate (3,0)-form $\Psi = \Psi^+ + \sqrt{-1}\,\Psi^-$, or equivalently by a non-trivial spinor, satisfying the compatibility conditions $F \wedge \Psi^\pm = 0$, $\Psi^+ \wedge \Psi^- = \frac{2}{3} F \wedge F \wedge F$. The subgroup of $SO(6)$ fixing the forms F and Ψ simultaneously is SU(3). The Lie algebra of SU(3) is denoted $\mathfrak{su}(3)$.

The necessary and sufficient condition for the existence of solutions to the first two equations in (2.4) derived by Strominger [63] imply that the 6-manifold should be a complex conformally balanced manifold (the Lee form $\theta = 2d\phi$) with non-vanishing holomorphic volume form Ψ satisfying an additional condition. In terms of the five torsion classes on dimension six, described in [18], the Strominger condition is interpreted in [17] as follows (see [51] for a slightly different expression):

$$(2.8) \qquad 2F \lrcorner dF + \Psi^+ \lrcorner d\Psi^+ = 0,$$

where $\lrcorner$ denotes the interior multiplication. Another very useful interpretation of this condition was proposed in [56]. If the dilaton is constant (the Lee form $\theta = 0$) then the Strominger condition reads

$$(2.9) \qquad dF \wedge F = d\Psi^+ = d\Psi^- = 0.$$

Compact examples of the latter on nilmanifolds were presented in [66, 65] and examples via evolution equations were given in [28].

A very promising geometric model in dimension six was proposed by Goldstein and Prokushkin in [37] as a certain $\mathbb{T}^2$-bundle over a Calabi-Yau surface, which we explain next. Let Γ_i, $1 \leq i \leq 2$, be two closed 2-forms on a Calabi-Yau surface M^4 with anti-self-dual (1,1)-part, which represent integral cohomology classes. Denote by ω_1 and by $\omega_2 + \sqrt{-1}\omega_3$ the (closed) Kähler form and the holomorphic volume form on M^4, respectively. Then, there is a (non-Kähler) 6-dimensional manifold M^6, which is the total space of a $\mathbb{T}^2$-bundle over M^4, and it has an $SU(3)$-structure

$$(2.10) \quad g = g_{CY} + \eta_1^2 + \eta_2^2, \quad F = \omega_1 + \eta_1 \wedge \eta_2, \quad \Psi^+ = \omega_2 \wedge \eta_1 - \omega_3 \wedge \eta_2, \quad \Psi^- = \omega_2 \wedge \eta_2 + \omega_3 \wedge \eta_1,$$

where η_i, $1 \leq i \leq 2$, is a 1-form on M^6 such that $d\eta_i = \Gamma_i$, $1 \leq i \leq 2$. From the construction it is easy to check that the $SU(3)$ structure (2.10) satisfies (2.9) and therefore it solves the first two Killing spinor equations in (2.4) with constant dilaton.

For any smooth function f on M^4, the $SU(3)$-structure on M^6 given by

$$F = e^{2f}\omega_1 + \eta_1 \wedge \eta_2, \quad \Psi^+ = e^{2f}\Big[\omega_2 \wedge \eta_1 - \omega_3 \wedge \eta_2\Big], \quad \Psi^- = e^{2f}\Big[\omega_2 \wedge \eta_2 + \omega_3 \wedge \eta_1\Big]$$

satisfies (2.8) and therefore it solves the first two Killing spinor equations in (2.4) with non-constant dilaton $\phi = 2f$. The metric has the form

$$g_f = e^{2f}g_{cy} + \eta_1^2 + \eta_2^2.$$

This ansatz guaranties solution to the first two equations in (2.4). To achieve a smooth solution to the Strominger system we still have to determine an auxiliary vector bundle with an instanton and a linear connection on M^6 in order to satisfy the anomaly cancellation condition (2.3). Taking the first Pontrjagin form of the Chern connection [56, 30, 31, 8] leads to an equation of Monge-Ampère type for the dilaton function, while it is reduced to a PDE of Laplace type for the dilaton when using the first Pontrjagin form of the $(-)$-connection [10, 9].

The $\mathbb{T}^2$-bundle over a K3 surface construction with connection 1-forms of anti-self-dual curvature was used in [56, 30, 31, 8] to produce the first compact smooth solutions in dimension 6 solving the heterotic supersymmetry equations (2.4) with non-zero flux and non-constant dilaton together with the anomaly cancellation (2.3) with the first Pontrjagin form of the Chern connection.

3. The anomaly cancellation and the non-constant dilaton

We apply the construction from Section 2.1 to special non-Kähler 2-step nilmanifolds which are $\mathbb{T}^2$-bundles over $\mathbb{T}^4$ with connection 1-forms of anti-self-dual curvature on the four torus and using the first Pontrjagin form of the $(-)$-connection in investigating the anomaly cancellation (2.3) with non-constant dilaton.

3.1. Two-step nilmanifolds with Abelian complex structure.

In this subsection we show, due to considerations in [66], that the 2-step nilmanifolds which are $\mathbb{T}^2$ bundles over $\mathbb{T}^4$ with connection 1-forms of anti-self-dual curvature are precisely the balanced Hermitian structures with Abelian complex structure, i.e. $[JX, JY] = [X, Y]$.

The invariant balanced Hermitian structures on compact 6-dimensional nilmanifolds which are a $\mathbb{T}^2$-bundle over a 4-torus, according to [66, Theorem 2.11], are parametrized by one of the following three sets of equations

$$(3.1) \qquad de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = t\,(e^{13} - e^{24}), \quad de^6 = t\,(e^{14} + e^{23}),$$

where $t \in \mathbb{R}^*$;

$$(3.2) \qquad \begin{cases} de^1 = de^2 = de^3 = de^4 = 0, \\ de^5 = \frac{t}{s}(\rho + b^2)e^{13} - \frac{t}{s}(\rho - b^2)e^{24}, \\ de^6 = -2t\,(e^{12} - e^{34}) + \frac{t}{s}(\rho - b^2)e^{14} + \frac{t}{s}(\rho + b^2)e^{23}, \end{cases}$$

where $\rho \in \{0, 1\}$, $b \in \mathbb{R}$ and $s, t \in \mathbb{R}^*$;

$$(3.3) \qquad \begin{cases} de^1 = de^2 = de^3 = de^4 = 0, \\ de^5 = sY\left[2b^2 u_1 |u|\,(e^{12} - e^{34}) - b^2 t u_1 |u|Y\,(e^{13} + e^{24}) + 2\rho s u_1\,(e^{13} - e^{24}) \right. \\ \qquad\qquad \left. +2s u_2\left((\rho - b^2)e^{14} + (\rho + b^2)e^{23}\right)\right], \\ de^6 = sY\left[2(2s^2 - b^2 u_2)|u|\,(e^{12} - e^{34}) + b^2 t u_2 |u|Y\,(e^{13} + e^{24}) - 2\rho s u_2\,(e^{13} - e^{24}) \right. \\ \qquad\qquad \left. +2s u_1\left((\rho - b^2)e^{14} + (\rho + b^2)e^{23}\right)\right], \end{cases}$$

where $\rho \in \{0, 1\}$, $b \in \mathbb{R}$, $t \in \mathbb{R}^*$ and $u \in \mathbb{C}^*$ such that $s^2 > |u|^2 > 0$, and where $Y = \frac{2\sqrt{s^2 - |u|^2}}{|u|t}$.

In all the cases the balanced structure (J, F) is given in the standard form, i.e.

$$(3.4) \qquad Je^1 = -e^2,\ Je^3 = -e^4,\ Je^5 = -e^6, \qquad F = e^{12} + e^{34} + e^{56}.$$

The nilpotent Lie algebras (nilmanifolds) underlying the families (3.1)–(3.3) are $\mathfrak{h}_k$, $2 \leq k \leq 6$ (see [66] for a description) However, in order to apply the construction from Section 2.1 we are led to

Lemma 3.1. *Let (J, F) be an invariant balanced Hermitian structure on a 6-dimensional 2-step nilmanifold M. Then, M is the total space of a $\mathbb{T}^2$-bundle over $\mathbb{T}^4$ of anti-self-dual curvature if and only if the complex structure J is Abelian. Moreover, in such case the Lie algebra underlying M is isomorphic to $\mathfrak{h}_3$ or $\mathfrak{h}_5$.*

Proof. The curvature of the bundle is determined by the 2-forms de^5 and de^6 in the structure equations (3.1), (3.2) and (3.3). Taking into account that $s, t, u, Y \neq 0$ we get that $de^5, de^6 \in \langle e^{12} - e^{34}, e^{13} + e^{24}, e^{14} - e^{23} \rangle$ if and only if the balanced Hermitian structure is given by (3.2) or (3.3) with $\rho = 0$. The latter condition means that J is an Abelian complex structure. Finally, when $\rho = 0$ we can take $b \in \{0, 1\}$ since the corresponding balanced Hermitian structures are isomorphic. The case $b = 0$, resp. $b = 1$, corresponds to structures on the Lie algebra $\mathfrak{h}_3$, resp. $\mathfrak{h}_5$. $\square$

Notice that $\mathfrak{h}_3$ is the Lie algebra underlying the nilmanifold given by the product of the 5-dimensional generalized Heisenberg nilmanifold by S^1, whereas $\mathfrak{h}_5$ is the Lie algebra underlying the Iwasawa manifold. It is important to note that the holonomy of the $(+)$-connection of any balanced structure (J, F) with J Abelian is a subgroup of $SU(2)$, hence inside $SU(3)$, [66].

3.2. **Non-constant dilaton in 6-D.** Here we consider the Lie algebra $\mathfrak{h}_5$, which we describe below. We shall construct a background with non-constant dilaton with non-trivial instanton and flux. By a contraction, this will also give analogous solutions on the Lie algebra $\mathfrak{h}_3$ as we shall explain later in Section 4.

The structure equations of the Lie algebra $\mathfrak{h}_5$ are

$$(3.5) \quad de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = \frac{t}{s}(e^{13} + e^{24}), \quad de^6 = -2t(e^{12} - e^{34}) - \frac{t}{s}(e^{14} - e^{23}),$$

where $s, t \in \mathbb{R}^*$. We note that the structure equations (3.5) are obtained from the family (3.2) taking there $\rho = 0$ and $b = 1$. The corresponding Lie group H_5 can be considered as a $\mathbb{R}^2$-bundle over $\mathbb{R}^4$. Moreover, the balanced structure (J, F) on $\mathfrak{h}_5$ is given in the standard form given by (3.4).

Let f be a smooth function on $\mathbb{R}^4$. Following [37] we consider the metric $\bar{g}$ on $\mathfrak{h}_5$ for which the basis of 1-forms

$$(3.6) \qquad \bar{e}^1 = e^f e^1, \quad \bar{e}^2 = e^f e^2, \quad \bar{e}^3 = e^f e^3, \quad \bar{e}^4 = e^f e^4, \quad \bar{e}^5 = e^5, \quad \bar{e}^6 = e^6$$

is orthonormal. The Kähler form of the new Hermitian structure $(\bar{g}, J)$ is given by

$$\bar{F} = \bar{e}^{12} + \bar{e}^{34} + \bar{e}^{56} = e^{2f}(e^{12} + e^{34}) + e^{56},$$

where $df = \sum_{i=1}^4 f_i e^i$, i.e., in local coordinates $f_i = \frac{\partial f}{\partial x_i}$. Furthermore,

$$d\bar{F} = 2e^{-f} f_3\,\bar{e}^{123} + 2e^{-f} f_4\,\bar{e}^{124} + 2te^{-2f}\bar{e}^{125} + 2e^{-f} f_1\,\bar{e}^{134} + \frac{t}{s}e^{-2f}\bar{e}^{136} + \frac{t}{s}e^{-2f}\bar{e}^{145}$$

$$+ 2e^{-f} f_2\,\bar{e}^{234} - \frac{t}{s}e^{-2f}\bar{e}^{235} + \frac{t}{s}e^{-2f}\bar{e}^{246} - 2te^{-2f}\bar{e}^{345}.$$

According to (2.7), the torsion 3-form $\bar{T}$ is represented by

$$(3.7) \quad \begin{aligned} \bar{T} = Jd\bar{F} &= 2e^{-f} f_4\,\bar{e}^{123} - 2e^{-f} f_3\,\bar{e}^{124} - 2te^{-2f}\bar{e}^{126} + 2e^{-f} f_2\,\bar{e}^{134} + \frac{t}{s}e^{-2f}\bar{e}^{135} \\ &\quad - \frac{t}{s}e^{-2f}\bar{e}^{146} - 2e^{-f} f_1\,\bar{e}^{234} + \frac{t}{s}e^{-2f}\bar{e}^{236} + \frac{t}{s}e^{-2f}\bar{e}^{245} + 2te^{-2f}\bar{e}^{346}. \end{aligned}$$

At this point we define the constant

$$(3.8) \qquad\qquad\qquad\qquad \kappa^2 = 1/2\left(2 + 1/s^2\right).$$

Letting $f_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}$, $i, j = 1, 2, 3, 4$, a short calculation gives

$$(3.9) \qquad d\bar{T} = -e^{-4f} \left[\triangle e^{2f} + 4t^2 \left(2 + \frac{1}{s^2} \right) \right] \bar{e}^{1234} = - \left[\triangle e^{2f} + 8t^2 \kappa^2 \right] e^{1234},$$

where $\triangle e^{2f} = (e^{2f})_{11} + (e^{2f})_{22} + (e^{2f})_{33} + (e^{2f})_{44}$ is the standard Laplacian on $\mathbb{R}^4$.

Corollary 3.2. *The $(-)$-connection is an instanton if and only if the torsion 3-form is closed, $d\bar{T} = 0$, i.e., the dilaton function f satisfies the equality*

$$(3.10) \qquad \triangle e^{2f} + 8t^2 \kappa^2 = 0.$$

Proof. Take the trace in (2.6) and use (3.9) together with the fact that the holonomy of ∇^+ is contained in $SU(3)$ to conclude that R^- satisfies the instanton condition (2.5) if and only if (3.10) holds. $\qquad \square$

3.3. The first Pontrjagin form of the $(-)$-connection.
The $(-)$-connection of the Hermitian structure $(\bar{g}, J)$ is defined by the formula $\nabla^- = \nabla^{\bar{g}} - \frac{1}{2}\bar{T}$, where $\nabla^{\bar{g}}$ is the Levi-Civita connection of the metric $\bar{g}$ and the torsion is determined in (3.7).

Using the metric $\bar{g}$, let $\{\bar{e}_1, \ldots, \bar{e}_6\}$ be the dual to $\{\bar{e}^1, \ldots, \bar{e}^6\}$ orthonormal basis. From Koszul's formula, we have that the Levi-Civita connection 1-forms $(\omega^{\bar{g}})^{\bar{i}}_{\bar{j}}$ are given by

$$(3.11) \quad (\omega^{\bar{g}})^{\bar{i}}_{\bar{j}}(\bar{e}_k) = -\frac{1}{2} \Big(\bar{g}(\bar{e}_i, [\bar{e}_j, \bar{e}_k]) - \bar{g}(\bar{e}_k, [\bar{e}_i, \bar{e}_j]) + \bar{g}(\bar{e}_j, [\bar{e}_k, \bar{e}_i]) \Big)$$

$$= \frac{1}{2} \Big(d\bar{e}^i(\bar{e}_j, \bar{e}_k) - d\bar{e}^k(\bar{e}_i, \bar{e}_j) + d\bar{e}^j(\bar{e}_k, \bar{e}_i) \Big)$$

taking into account $\bar{g}(\bar{e}_i, [\bar{e}_j, \bar{e}_k]) = -d\bar{e}^i(\bar{e}_j, \bar{e}_k)$. With the help of (3.11) we compute the expressions for the connection 1-forms $(\omega^-)^{\bar{i}}_{\bar{j}}$ of the connection ∇^-,

$$(3.12) \qquad (\omega^-)^{\bar{i}}_{\bar{j}} = (\omega^{\bar{g}})^{\bar{i}}_{\bar{j}} - \frac{1}{2}(\bar{T})^{\bar{i}}_{\bar{j}}, \qquad \text{where} \qquad (\bar{T})^{\bar{i}}_{\bar{j}}(\bar{e}_k) = \bar{T}(\bar{e}_i, \bar{e}_j, \bar{e}_k).$$

Now, (3.12), (3.11) and (3.7) show that the non-zero connection 1-forms $(\omega^-)^{\bar{i}}_{\bar{j}}$ are given in terms of the basis $\{\bar{e}^1, \ldots, \bar{e}^6\}$ by

$$(3.13) \quad \begin{aligned}
(\omega^-)^{\bar{1}}_{\bar{2}} &= e^{-f} \left[f_2\, \bar{e}^1 - f_1\, \bar{e}^2 + f_4\, \bar{e}^3 - f_3\, \bar{e}^4 \right], \quad (\omega^-)^{\bar{1}}_{\bar{3}} = e^{-f} \left[f_3\, \bar{e}^1 - f_4\, \bar{e}^2 - f_1\, \bar{e}^3 + f_2\, \bar{e}^4 \right], \\
(\omega^-)^{\bar{1}}_{\bar{4}} &= e^{-f} \left[f_4\, \bar{e}^1 + f_3\, \bar{e}^2 - f_2\, \bar{e}^3 - f_1\, \bar{e}^4 \right], \quad (\omega^-)^{\bar{1}}_{\bar{5}} = -e^{-2f} \tfrac{t}{s}\, \bar{e}^3, \\
(\omega^-)^{\bar{1}}_{\bar{6}} &= e^{-2f} 2t \left[\bar{e}^2 + \tfrac{1}{2s} \bar{e}^4 \right], \quad (\omega^-)^{\bar{2}}_{\bar{3}} = e^{-f} \left[f_4\, \bar{e}^1 + f_3\, \bar{e}^2 - f_2\, \bar{e}^3 - f_1\, \bar{e}^4 \right], \\
(\omega^-)^{\bar{2}}_{\bar{4}} &= e^{-f} \left[-f_3\, \bar{e}^1 + f_4\, \bar{e}^2 + f_1\, \bar{e}^3 - f_2\, \bar{e}^4 \right], \quad (\omega^-)^{\bar{2}}_{\bar{5}} = -e^{-2f} \tfrac{t}{s}\, \bar{e}^4, \\
(\omega^-)^{\bar{2}}_{\bar{6}} &= e^{-2f} 2t \left[-\bar{e}^1 - \tfrac{1}{2s}\, \bar{e}^3 \right], \quad (\omega^-)^{\bar{3}}_{\bar{4}} = e^{-f} \left[f_2\, \bar{e}^1 - f_1\, \bar{e}^2 + f_4\, \bar{e}^3 - f_3\, \bar{e}^4 \right], \\
(\omega^-)^{\bar{3}}_{\bar{5}} &= e^{-2f} \tfrac{t}{s}\, \bar{e}^1, \qquad (\omega^-)^{\bar{3}}_{\bar{6}} = e^{-2f} 2t \left[\tfrac{1}{2s}\, \bar{e}^2 - \bar{e}^4 \right], \\
(\omega^-)^{\bar{4}}_{\bar{5}} &= e^{-2f} \tfrac{t}{s}\, \bar{e}^2, \qquad (\omega^-)^{\bar{4}}_{\bar{6}} = e^{-2f} 2t \left[-\tfrac{1}{2s}\, \bar{e}^1 + \bar{e}^3 \right].
\end{aligned}$$

A long straightforward calculation using (3.13) gives in terms of the basis $\{\bar{e}^1, \ldots, \bar{e}^6\}$ the following formulas for the curvature 2-forms of ∇^-

$$
\begin{aligned}
(\Omega^-)_{\bar 2}^{\bar 1} =\ & -(f_{11}+f_{22}+2f_3^2+2f_4^2+4t^2e^{-2f})\,e^{-2f}\overline{e}^{12}+(f_{14}-f_{23}+2f_2f_3-2f_1f_4)e^{-2f}(\overline{e}^{13}+\overline{e}^{24})\\
& -(f_{13}+f_{24}-2f_1f_3-2f_2f_4+2t^2/se^{-2f})e^{-2f}(\overline{e}^{14}-\overline{e}^{23})\\
& -(2f_1^2+2f_2^2+f_{33}+f_{44}+2t^2/s^2e^{-2f})e^{-2f}\overline{e}^{34},\\[4pt]
(\Omega^-)_{\bar 3}^{\bar 1} =\ & -(f_{14}+f_{23}-2f_2f_3-2f_1f_4)e^{-2f}(\overline{e}^{12}-\overline{e}^{34})-(f_{11}+2f_2^2+f_{33}+2f_4^2+t^2/s^2e^{-2f})e^{-2f}\overline{e}^{13}\\
& +(f_{12}-2f_1f_2-f_{34}+2f_3f_4)e^{-2f}(\overline{e}^{14}-\overline{e}^{23})\\
& +(2f_1^2+f_{22}+2f_3^2+f_{44}+t^2/s^2e^{-2f}+4t^2e^{-2f})e^{-2f}\overline{e}^{24},\\[4pt]
(\Omega^-)_{\bar 4}^{\bar 1} =\ & (f_{13}-f_{24}-2f_1f_3+2f_2f_4-2t^2/se^{-2f})e^{-2f}(\overline{e}^{12}-\overline{e}^{34})-(f_{12}-2f_1f_2+f_{34}-2f_3f_4)e^{-2f}(\overline{e}^{13}+\overline{e}^{24})\\
& -(f_{11}+2f_2^2+2f_3^2+f_{44}+t^2/s^2e^{-2f})\,e^{-2f}\overline{e}^{14}-(2f_1^2+f_{22}+f_{33}+2f_4^2+t^2/s^2e^{-2f}+4t^2e^{-2f})e^{-2f}\overline{e}^{23},\\[4pt]
(\Omega^-)_{\bar 5}^{\bar 1} =\ & 2e^{-3f}t/s\left[f_4(\overline{e}^{12}-\overline{e}^{34})+f_1(e^{13}+e^{24})-f_2(\overline{e}^{14}-\overline{e}^{23})\right],\\[4pt]
(\Omega^-)_{\bar 6}^{\bar 1} =\ & 2e^{-3f}t/s\left[(f_3-2f_1s)(\overline{e}^{12}-\overline{e}^{34})-(f_2-2f_4s)(\overline{e}^{13}+\overline{e}^{24})-(f_1+2f_3s)(\overline{e}^{14}-\overline{e}^{23})\right],\\[4pt]
(\Omega^-)_{\bar 3}^{\bar 2} =\ & (f_{13}-f_{24}-2f_1f_3+2f_2f_4+2t^2/se^{-2f})e^{-2f}(\overline{e}^{12}-\overline{e}^{34})-(f_{12}-2f_1f_2+f_{34}-2f_3f_4)e^{-2f}(\overline{e}^{13}+\overline{e}^{24})\\
& -(f_{11}+2f_2^2+2f_3^2+f_{44}+t^2/s^2e^{-2f}+4t^2e^{-2f})e^{-2f}\overline{e}^{14}-(2f_1^2+f_{22}+f_{33}+2f_4^2+t^2/s^2e^{-2f})e^{-2f}\overline{e}^{23},\\[4pt]
(\Omega^-)_{\bar 4}^{\bar 2} =\ & (f_{14}+f_{23}-2f_2f_3-2f_1f_4)e^{-2f}(\overline{e}^{12}-\overline{e}^{34})+(f_{11}+2f_2^2+f_{33}+2f_4^2+t^2/s^2e^{-2f}+4t^2e^{-2f})e^{-2f}\overline{e}^{13}\\
& -(f_{12}-2f_1f_2-f_{34}+2f_3f_4)e^{-2f}(\overline{e}^{14}-\overline{e}^{23})-(2f_1^2+f_{22}+2f_3^2+f_{44}+t^2/s^2e^{-2f})e^{-2f}\,\overline{e}^{24},\\[4pt]
(\Omega^-)_{\bar 5}^{\bar 2} =\ & 2e^{-3f}t/s\left[-f_3(\overline{e}^{12}-\overline{e}^{34})+f_2(\overline{e}^{13}+\overline{e}^{24})+f_1(\overline{e}^{14}-\overline{e}^{23})\right],\\[4pt]
(\Omega^-)_{\bar 6}^{\bar 2} =\ & 2e^{-3f}t/s\left[(f_4-2f_2s)(\overline{e}^{12}-\overline{e}^{34})+(f_1-2f_3s)(\overline{e}^{13}+\overline{e}^{24})-(f_2+2f_4s)(\overline{e}^{14}-\overline{e}^{23})\right],\\[4pt]
(\Omega^-)_{\bar 4}^{\bar 3} =\ & -(f_{11}+f_{22}+2f_3^2+2f_4^2+2t^2/s^2e^{-2f})e^{-2f}\overline{e}^{12}+(f_{14}-f_{23}+2f_2f_3-2f_1f_4)e^{-2f}(\overline{e}^{13}+\overline{e}^{24})\\
& -(f_{13}+f_{24}-2f_1f_3-2f_2f_4-2t^2/se^{-2f})e^{-2f}(\overline{e}^{14}-\overline{e}^{23})-(2f_1^2+2f_2^2+f_{33}+f_{44}+4t^2e^{-2f})\,e^{-2f}\overline{e}^{34},\\[4pt]
(\Omega^-)_{\bar 5}^{\bar 3} =\ & 2e^{-3f}t/s\left[f_2(\overline{e}^{12}-\overline{e}^{34})+f_3(\overline{e}^{13}+\overline{e}^{24})+f_4(\overline{e}^{14}-\overline{e}^{23})\right],\\[4pt]
(\Omega^-)_{\bar 6}^{\bar 3} =\ & 2e^{-3f}t/s\left[-(f_1+2f_3s)(\overline{e}^{12}-\overline{e}^{34})+(f_4+2f_2s)(\overline{e}^{13}+\overline{e}^{24})-(f_3-2f_1s)(\overline{e}^{14}-\overline{e}^{23})\right],\\[4pt]
(\Omega^-)_{\bar 5}^{\bar 4} =\ & 2e^{-3f}t/s\left[-f_1(\overline{e}^{12}-\overline{e}^{34})+f_4(\overline{e}^{13}+\overline{e}^{24})-f_3(\overline{e}^{14}-\overline{e}^{23})\right],\\[4pt]
(\Omega^-)_{\bar 6}^{\bar 4} =\ & 2e^{-3f}t/s\left[-(f_2+2f_4s)(\overline{e}^{12}-\overline{e}^{34})-(f_3+2f_1s)(\overline{e}^{13}+\overline{e}^{24})-(f_4-2f_2s)(\overline{e}^{14}-\overline{e}^{23})\right],\\[4pt]
(\Omega^-)_{\bar 6}^{\bar 5} =\ & 2e^{-4f}t^2/s^2\left[-(\overline{e}^{12}-\overline{e}^{34})+2s(\overline{e}^{14}-\overline{e}^{23})\right].
\end{aligned}
$$

Proposition 3.3. *The first Pontrjagin form of ∇^- is a scalar multiple of e^{1234} given by*

$$
(3.14) \qquad \pi^2 p_1(\nabla^-) = \left[\sum_{1\le i<j\le 4}\left(det(f_{ij})+(f_i^2f_j)_j+(f_if_j^2)_i\right)+\sum_{i=1}^4(f_i^3)_i-\frac{3}{2}t^2\kappa^2\triangle e^{-2f}\right]e^{1234}.
$$

Proof. The proof of (3.14) is a long straightforward calculations using the formulas for the curvature 2-form of ∇^-. $\qquad\square$

Note that even though the curvature 2-forms of ∇^- are quadratic in the gradient of the dilaton, remarkably, the Pontrjagin form of ∇^- is also quadratic in these terms.

3.4. **A conformally compact solution with negative α'. Proof of Theorem 1.1.** Here we give the proof of Theorem 1.1.

By [31] there is no compact solution of Strominger's system for positive α' in the case of the Chern connection on torus bundle over $\mathbb{T}^4$. On the other hand, the existence of a solution on a 2-torus bundle over K3-surfaces given in [31] seems to depend on the assumption $\alpha' > 0$, whereas the existence of a solution with negative α' is not clear.

The proof of Theorem 1.1 occupies the remaining part of Section 3.4. We begin with a Proposition defining the instanton bundle.

Proposition 3.4. *Let A_λ, $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ be the linear connection on H_5 whose non-zero 1-forms are given as follows*

$$(\omega^{A_\lambda})_{\bar{2}}^{\bar{1}} = -(\omega^{A_\lambda})_{\bar{1}}^{\bar{2}} = -(\omega^{A_\lambda})_{\bar{4}}^{\bar{3}} = (\omega^{A_\lambda})_{\bar{3}}^{\bar{4}} = -\lambda_1 \, \bar{e}^6,$$

$$(\omega^{A_\lambda})_{\bar{3}}^{\bar{1}} = -(\omega^{A_\lambda})_{\bar{1}}^{\bar{3}} = (\omega^{A_\lambda})_{\bar{4}}^{\bar{2}} = -(\omega^{A_\lambda})_{\bar{2}}^{\bar{4}} = -\lambda_2 \, \bar{e}^6,$$

$$(\omega^{A_\lambda})_{\bar{4}}^{\bar{1}} = -(\omega^{A_\lambda})_{\bar{1}}^{\bar{4}} = -(\omega^{A_\lambda})_{\bar{3}}^{\bar{2}} = (\omega^{A_\lambda})_{\bar{2}}^{\bar{3}} = -\lambda_3 \, \bar{e}^6.$$

Then, A_λ is an $SU(3)$-instanton which preserves the metric. Furthermore, the first Pontrjagin form of A_λ is

$$(3.15) \qquad 8\pi^2 p_1(A_\lambda) = -8t^2(1 + \kappa^2)|\lambda|^2 \, e^{1234}, \qquad |\lambda|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2.$$

Proof. A direct calculation shows that the non-zero curvature forms $(\Omega^{A_\lambda})_{\bar{j}}^{\bar{i}}$ of the connection A_λ are:

$$(\Omega^{A_\lambda})_{\bar{2}}^{\bar{1}} = -(\Omega^{A_\lambda})_{\bar{1}}^{\bar{2}} = -(\Omega^{A_\lambda})_{\bar{4}}^{\bar{3}} = (\Omega^{A_\lambda})_{\bar{3}}^{\bar{4}} = -\lambda_1 \, d\bar{e}^6 = 2t\lambda_1 e^{-2f}(\bar{e}^{12} - \bar{e}^{34}) + \tfrac{t}{s}\lambda_1 e^{-2f}(\bar{e}^{14} - \bar{e}^{23}),$$

$$(\Omega^{A_\lambda})_{\bar{3}}^{\bar{1}} = -(\Omega^{A_\lambda})_{\bar{1}}^{\bar{3}} = (\Omega^{A_\lambda})_{\bar{4}}^{\bar{2}} = -(\Omega^{A_\lambda})_{\bar{2}}^{\bar{4}} = -\lambda_2 \, d\bar{e}^6 = 2t\lambda_2 e^{-2f}(\bar{e}^{12} - \bar{e}^{34}) + \tfrac{t}{s}\lambda_2 e^{-2f}(\bar{e}^{14} - \bar{e}^{23}),$$

$$(\Omega^{A_\lambda})_{\bar{4}}^{\bar{1}} = -(\Omega^{A_\lambda})_{\bar{1}}^{\bar{4}} = -(\Omega^{A_\lambda})_{\bar{3}}^{\bar{2}} = (\Omega^{A_\lambda})_{\bar{2}}^{\bar{3}} = -\lambda_3 \, d\bar{e}^6 = 2t\lambda_3 e^{-2f}(\bar{e}^{12} - \bar{e}^{34}) + \tfrac{t}{s}\lambda_3 e^{-2f}(\bar{e}^{14} - \bar{e}^{23}).$$

It is straightforward to see that A_λ satisfies (2.5) and therefore it is an $SU(3)$-instanton. After another lengthy calculation we see $8\pi^2 p_1(A_\lambda) = -4t^2(4 + 1/s^2)|\lambda|^2 e^{-4f} \bar{e}^{1234}$, which in view of (3.8) and (3.6) implies formula (3.15). $\qquad\square$

Now, we suppose that the function f depends on one variable, say $f = f(x^1)$. Using $(f'' - 2f'^2)e^{-2f} = -\tfrac{1}{2}(e^{-2f})''$ we have from (3.14)

$$(3.16) \qquad 8\pi^2 p_1(\nabla^-) = 4\left(2f'^3 - 3t^2\kappa^2\left(e^{-2f}\right)'\right)' e^{1234}.$$

Furthermore, from (3.9)

$$(3.17) \qquad d\bar{T} = -\left(\left(e^{2f}\right)'' + 8t^2\kappa^2\right)e^{1234}.$$

In view of (3.15), (3.16) and (3.17) the anomaly cancellation condition (2.3), i.e., $d\bar{T} = \tfrac{\alpha'}{4}8\pi^2\left(p_1(\nabla^-) - p_1(A_\lambda)\right)$, takes the form of a single ODE for the function f

$$(3.18) \qquad \left(\left(e^{2f}\right)' - 3\alpha't^2\kappa^2\left(e^{-2f}\right)' + 2\alpha'f'^3\right)' + 8t^2\kappa^2 + 2\alpha't^2(1 + \kappa^2)|\lambda|^2 = 0.$$

For a negative α' we choose κ^2 or $|\lambda|^2$ so that $8t^2\kappa^2 + 2\alpha't^2(1 + \kappa^2)|\lambda|^2 = 0$, i.e., we let

$$\alpha' = -\alpha^2, \qquad 4\kappa^2 = \alpha^2(1 + \kappa^2)|\lambda|^2,$$

which simplifies (3.18) to the ordinary differential equation

$$(3.19) \qquad \left(e^{2f}\right)' + 3\alpha^2 t^2\kappa^2\left(e^{-2f}\right)' - 2\alpha^2 f'^3 = A = const.$$

At this point we let $u = \alpha^{-2}e^{2f}$. With this substitution the left-hand side of (3.19) becomes

$$\left(e^{2f}\right)' - 3\alpha't^2\left(e^{-2f}\right)' + 2\alpha'f'^3 = \frac{\alpha^2 u'}{4u^3}\left(4u^3 - 12\frac{t^2\kappa^2}{\alpha^2}u - u'^2\right).$$

For $A = 0$ consider the following ordinary differential equation for the function $u = u(x^1) > 0$

$$(3.20) \qquad u'^2 = 4u^3 - 12\frac{t^2\kappa^2}{\alpha^2}u = 4u\,(u-a)\,(u+a)\,, \qquad a = \kappa|t|\sqrt{3}/\alpha.$$

Equation (3.20) can also be considered in the complex plane by replacing the real derivative with the complex derivative which turns it into the Weierstrass' equation

$$\left(\frac{d\mathcal{P}}{dz}\right)^2 = 4\mathcal{P}\,(\mathcal{P}-a)\,(\mathcal{P}+a)$$

for the doubly periodic Weierstrass $\mathcal{P}$ function with a pole at the origin where it has the expansion

$$\mathcal{P}(z) = \frac{1}{z^2} + \frac{a^2}{5}z^2 + bz^6 + \cdots,$$

(no z^4 term and only even powers). In addition, as well known [24] and [1], letting $\tau_\pm$ be the basic half-periods such that tau_+ is real and τ_- is purely imaginary we have that $\mathcal{P}$ is real valued on the lines $\mathfrak{Re}\,z = m\tau_+$ or $\mathfrak{Im}\,z = im\tau_-$, $m \in \mathbb{Z}$. Furthermore, in the fundamental region centered at the origin, where $\mathcal{P}$ has a pole of order two, we have that $\mathcal{P}(z)$ decreases from $+\infty$ to a to 0 to $-a$ to $-\infty$ as z varies along the sides of the half-period rectangle from 0 to τ_+ to $\tau_+ + \tau_-$ to τ_- to 0.

Thus, $u(x^1) = \mathcal{P}(x^1)$ defines a non-negative $2\tau_+$-periodic function with singularities at the points $2n\tau_+$, $n \in \mathbb{Z}$, which solves the real equation (3.20). From the Laurent expansion of the Weierstrass' function it follows

$$u(x_1) = \frac{1}{(x^1)^2}\left(1 + \frac{a^2}{5}(x^1)^4 + \cdots\right).$$

By construction, $f = \frac{1}{2}\ln(\alpha^2 u)$ is a periodic function with singularities on the real line which is a solution to equation (3.18) sufficient for the anomaly cancellation condition. Therefore the $SU(3)$ structure defined by $\bar{F}$ and the non-degenerate (3,0) form $\bar{\Psi} = (\bar{e}^1 + i\bar{e}^2) \wedge (\bar{e}^3 + i\bar{e}^4) \wedge (\bar{e}^5 + i\bar{e}^6)$ descends to the 6-dimensional nilmanifold $M^6 = \Gamma\backslash H_5$ with singularity, determined by the singularity of u, where H_5 is the 2-step nilpotent Lie group with Lie algebra $\mathfrak{h}_5$, defined by (3.5), and Γ is a lattice with the same period as f, i.e., $2\tau_+$ in all variables. In fact, as seen from the asymptotic behavior of u, M^6 is the total space of a $\mathbb{T}^2$ bundle over the asymptotically hyperbolic manifold M^4 with metric

$$\bar{g}_H = u(x^1)\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2\right),$$

which is a conformally compact 4-torus with conformal boundary at infinity a flat 3-torus. Thus, we conclude that there is a complete solution with non-constant dilaton, non-trivial instanton and flux and with a negative α' parameter. This completes the proof of Theorem 1.1.

A few remarks are in order. First, since the function u has a $\mathbb{Z}_2$-symmetry determined by the symmetry with respect to the line $x^1 = \tau_+$ we also obtain a solution on the quotient $M^6/\mathbb{Z}_2$.

Second, the function $v(x_1) = \mathcal{P}(\tau_+ + ix_1)$, which is the restriction of $\mathcal{P}$ to the line $\mathfrak{Re}\,z = \tau_+$ leads to a solution "equivalent" to the one described above, taking into account the invariance under translation in x_1. Indeed, clearly $(v')^2 = -\left(\frac{d\mathcal{P}}{dz}\right)^2$ hence v satisfies

$$(v')^2 = -4v\,(v-a)\,(v+a)\,, \qquad a = \kappa|t|\sqrt{3}/\alpha.$$

The above equation is (3.20) with the substitution $u = \frac{a^2}{v}$ which shows that we obtain the same six dimensional $SU(3)$ structure. We note that v is an even non-negative periodic function with period $2i\tau_-$ (without a loss of generality $i\tau_- > 0$) such that $v(-i\tau_-) = v(i\tau_-) = 0$, $v(0) = a$ and v increases on the interval $(-i\tau_-, 0)$.

3.5. Complete solution with positive α'. Proof of Theorem 1.2.

Let us consider a connection $A_{a,d}$ depending on parameters $a, d \in \mathbb{R}$, $d \neq 0$, whose non-zero connection 1-forms $(\omega^{A_{a,d}})^{\bar{i}}_{\bar{j}}$ in the basis $\{\bar{e}^1, \ldots, \bar{e}^6\}$ are as follows

$$(\omega^{A_{a,d}})^{\bar{1}}_{\bar{2}} = e^{-f} \left[f_2\, \bar{e}^1 - f_1\, \bar{e}^2 + f_4\, \bar{e}^3 - f_3\, \bar{e}^4 \right], \qquad (\omega^{A_{a,d}})^{\bar{1}}_{\bar{3}} = e^{-f} \left[f_3\, \bar{e}^1 - f_4\, \bar{e}^2 - f_1\, \bar{e}^3 + f_2\, \bar{e}^4 \right],$$

$$(\omega^{A_{a,d}})^{\bar{1}}_{\bar{4}} = e^{-f} \left[f_4\, \bar{e}^1 + f_3\, \bar{e}^2 - f_2\, \bar{e}^3 - f_1\, \bar{e}^4 \right], \qquad (\omega^{A_{a,d}})^{\bar{1}}_{\bar{5}} = -\frac{a}{d}\, e^{-2f} \bar{e}^3,$$

$$(\omega^{A_{a,d}})^{\bar{1}}_{\bar{6}} = ae^{-2f} \left[2\, \bar{e}^2 + 1/d\, \bar{e}^4 \right], \qquad (\omega^{A_{a,d}})^{\bar{2}}_{\bar{3}} = e^{-f} \left[f_4\, \bar{e}^1 + f_3\, \bar{e}^2 - f_2\, \bar{e}^3 - f_1\, \bar{e}^4 \right],$$

$$(\omega^{A_{a,d}})^{\bar{2}}_{\bar{4}} = e^{-f} \left[-f_3\, \bar{e}^1 + f_4\, \bar{e}^2 + f_1\, \bar{e}^3 - f_2\, \bar{e}^4 \right], \qquad (\omega^{A_{a,d}})^{\bar{2}}_{\bar{5}} = -\frac{a}{d}\, e^{-2f} \bar{e}^4,$$

$$(\omega^{A_{a,d}})^{\bar{2}}_{\bar{6}} = -a\, e^{-2f} \left[2\, \bar{e}^1 + \frac{1}{d} \bar{e}^3 \right], \qquad (\omega^{A_{a,d}})^{\bar{3}}_{\bar{4}} = e^{-f} \left[f_2\, \bar{e}^1 - f_1\, \bar{e}^2 + f_4\, \bar{e}^3 - f_3\, \bar{e}^4 \right],$$

$$(\omega^{A_{a,d}})^{\bar{3}}_{\bar{5}} = \frac{a}{d}\, e^{-2f} \bar{e}^1, \qquad (\omega^{A_{a,d}})^{\bar{3}}_{\bar{6}} = ae^{-2f} \left[\frac{1}{d}\, \bar{e}^2 - 2\, \bar{e}^4 \right],$$

$$(\omega^{A_{a,d}})^{\bar{4}}_{\bar{5}} = \frac{a}{d}\, e^{-2f}\, \bar{e}^2, \qquad (\omega^{A_{a,d}})^{\bar{4}}_{\bar{6}} = -ae^{-2f} \left[\frac{1}{d}\, \bar{e}^1 - 2\, \bar{e}^3 \right].$$

Lemma 3.5. *$A_{a,d}$ is an instanton with respect to the $SU(3)$ structure defined with the help of the basis* (3.6) *if and only if the dilaton function satisfies*

$$(3.21) \qquad \triangle e^{2f} = -8\tau^2 a^2, \qquad \tau^2 = \frac{1}{2}\left(2 + \frac{1}{d^2}\right).$$

Proof. Observe that the connection 1-forms of $A_{a,d}$ are given by (3.13) replacing t with a and s with d. Then the assertion follows from Corollary 3.2. $\qquad \square$

Lemma 3.5 shows, in particular, that $A_{a,d}$ is an instanton with respect to the $SU(3)$ structure determined by the basis (3.6) if

$$(3.22) \qquad e^{2f} = h(x) - a^2\tau^2|x|^2,$$

where h is a harmonic function on $\mathbb{R}^4$.

The expression (3.14) yield that the difference between the first Pontrjagin forms of ∇^- and $A_{a,d}$ is given by the formula

$$(3.23) \quad 8\pi^2\Big(p_1(\nabla^-) - p_1(A_{a,d})\Big) = \frac{12}{d^2 s^2}\left[a^2(1+2d^2)s^2 - d^2(1+2s^2)t^2 \right] e^{-6f}\Big(2|f|^2 - \triangle f\Big)\bar{e}^{1234}$$

$$= -24\left(t^2\kappa^2 - a^2\tau^2\right)e^{-4f}\Big(\triangle e^{-2f}\Big)\bar{e}^{1234} = -24\left(t^2\kappa^2 - a^2\tau^2\right)\Big(\triangle e^{-2f}\Big)e^{1234}.$$

On the other hand recalling (3.9) and taking into account (3.23), we have that the anomaly cancellation condition

$$d\bar{T} - \frac{\alpha'}{4}8\pi^2\Big(p_1(\nabla^-) - p_1(A_{a,d})\Big) = -\left[\triangle e^{2f} + 8t^2\kappa^2 - 3\alpha'(t^2\kappa^2 - a^2\tau^2)\triangle e^{-2f}\right]e^{1234} = 0$$

simplifies to the single equation for the dilaton $\triangle e^{2f} + 8t^2\kappa^2 - 3\alpha'(t^2\kappa^2 - a^2\tau^2)\triangle e^{-2f} = 0$.

Thus a non-trivial dilaton is given by (3.22) which satisfies the equation

$$(3.24) \qquad (t^2\kappa^2 - a^2\tau^2)\Big(8 - 3\alpha'\triangle e^{-2f}\Big) = 0.$$

We analyze the solution in the next two cases.

Case 1. Here $t^2\kappa^2 - a^2\tau^2 = 0$, hence by (3.23) the anomaly condition is trivially satisfied for any α', provided the torsion is closed. In this case the solution is given by the solutions of (3.21).

Furthermore, taking into account Corollary 3.2 both ∇^- and $A_{a,d}$ are instantons. For example, a particular case of (3.22) is the solution

$$e^{2f} = a^2\tau^2(1 - |x|^2)$$

defined in the unit ball.

Notice that in this case we also obtain a solution of the type II theory, see [17] for the case of the Iwasawa manifold and [34, Section VII] for the general case of two flat directions fibered over a four dimensional base M_0.

Case 2. Here $t^2\kappa^2 - a^2\tau^2 \neq 0$, hence the anomaly condition is non-trivial.

We need to solve the system of the two equations (3.21) and (3.24). To get a solution we take $a = 0$ in (3.21) and arrive to the next two equations for the dilaton f:

$$\triangle e^{2f} = 0, \qquad \triangle e^{-2f} = 8/(3\alpha').$$

Hence the solution with a singularity is given by

(3.25)
$$e^{2f} = \frac{3\alpha'}{|x - b|^2}, \quad b \in \mathbb{R}^4,$$

As a result of the above arguments we obtain a non-compact solution with non-constant dilaton, non-trivial instanton and flux with positive α'. This solution is similar to the multi-instanton solution considered in [13]. Taking into account that H_5 is a $\mathbb{R}^2$-bundle over $\mathbb{R}^4$, and using logarithmic radial coordinates near the singularity as in [13] it follows that the $4 - D$ metric induced on $\mathbb{R}^4$ is actually complete. In fact, taking the singularity at the origin, in the coordinate $q = \sqrt{3\alpha'}/2 \ln\left(|x|^2/3\alpha'\right) = -\sqrt{3\alpha'}\,f$, we have that the dilaton and the $4 - D$ metric can be expressed as follows

$$\bar{g}_H = \sum_{i=1}^{4} e^{2f}(e^i)^2 = dq^2 + 3\alpha' ds_3^2, \qquad f = -q\sqrt{3\alpha'},$$

where ds_3^2 is the metric on the unit three-dimensional sphere in the four dimensional Euclidean space. The completeness of the horizontal metric implies that the metric

$$\bar{g} = \bar{g}_H + (e^5)^2 + (e^6)^2$$

is also complete. This finishes the proof of Theorem 1.2.

4. Contraction and the Lie algebra $\mathfrak{h}_3$

Taking into account that the Lie algebra $\mathfrak{h}_3$ is a contraction of the Lie algebra $\mathfrak{h}_5$ we can obtain solutions for the Lie algebra $\mathfrak{h}_3$ from the solution on $\mathfrak{h}_5$. Indeed, letting $\frac{t}{s} \to 0$ in (3.5) we obtain

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = 0, \quad de^6 = -2t\,(e^{12} - e^{34})$$

which are the structure equations of $\mathfrak{h}_3$.

Correspondingly, using equations (3.6) we obtain the structure equations

$$d\bar{e}^1 = -e^{-f}\left(f_2\,\bar{e}^{12} + f_3\,\bar{e}^{13} + f_4\,\bar{e}^{14}\right), \qquad d\bar{e}^2 = e^{-f}\left(f_1\,\bar{e}^{12} - f_3\,\bar{e}^{23} - f_4\,\bar{e}^{24}\right),$$

$$d\bar{e}^3 = e^{-f}\left(f_1\,\bar{e}^{13} + f_2\,\bar{e}^{23} - f_4\,\bar{e}^{34}\right), \qquad d\bar{e}^4 = e^{-f}\left(f_1\,\bar{e}^{14} + f_2\,\bar{e}^{24} + f_3\,\bar{e}^{34}\right),$$

$$d\bar{e}^5 = 0, \qquad\qquad\qquad\qquad\qquad d\bar{e}^6 = -2te^{-2f}(\bar{e}^{12} - \bar{e}^{34}).$$

As before, we consider the ∇^- connection described in (3.13). We define the connection A_a by letting the parameter $d \to \infty$ in (3.5), or equivalently $\frac{a}{d} \to 0$. All remaining calculation in Section

3.2 are valid by taking the above described limits. As a result, Theorem 1.1 and Theorem 1.2 give solutions with non-constant dilaton on $\mathfrak{h}_3$.

Remark 4.1. *It can be checked from the expression for the curvature 2-forms of ∇^- using* (2.6) *and* (3.9) *that the connection ∇^+ is an $SU(3)$-instanton if and only if f is a constant function and $\frac{t}{s} \to 0$, i.e. the connection ∇^+ is an $SU(3)$-instanton if and only if f is constant and the group is H_3.*

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