THE QUATERNIONIC HEISENBERG GROUP AND HETEROTIC STRING SOLUTIONS WITH NON-CONSTANT DILATON IN DIMENSIONS 7 AND 5

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Abstract. New smooth solutions of the Strominger system with non vanishing flux, non-trivial instanton and non-constant dilaton based on the quaternionic Heisenberg group are constructed. We show that through appropriate contractions the solutions found in the G_2-heterotic case converge to the heterotic solutions on 6-dimensional inner non-Kähler spaces previously found by the authors and, moreover, to new heterotic solutions with non-constant dilaton in dimension 5. All the solutions satisfy the heterotic equations of motion up to the first order of α'.

Contents

1. Introduction

We investigate smooth solutions with non-trivial fluxes to the heterotic equations of motion preserving at least one supersymmetry up to the first order of the string tension α' in dimensions seven and five. Using the quaternionic Heisenberg group we propose an explicit construction leading to new smooth solutions of the Killing spinor equations and the Green-Schwarz anomaly cancellation, the system of equations known as the Strominger system, with a non-constant dilaton. The found solutions satisfy the heterotic equations of motion up to the first order of α'.

Another goal of the paper is to point that through contractions of the quaternion Heisenberg algebra, the geometric structures, the partial differential equations and their solutions found in the G_2-heterotic case converge to the heterotic solutions on 6-dimensional inner non-Kähler spaces found in [26] and to the new 5-dimensional heterotic solutions with non-constant dilaton.

The bosonic fields of the ten-dimensional supergravity which arises as low energy effective theory of the heterotic string are the spacetime metric g, the NS three-form field strength (flux) H, the dilaton ϕ and the gauge connection A with curvature 2-form F^A. The bosonic geometry is of

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the form $\mathbb{R}^{1,9-d} \times M^d$, where the bosonic fields are non-trivial only on M^d, $d \leq 8$. We consider the two connections $\nabla^{\pm} = \nabla^g \pm \frac{1}{2}H$, where ∇^g is the Levi-Civita connection of the Riemannian metric g. Both connections preserve the metric, $\nabla^{\pm}g = 0$ and have totally skew-symmetric torsion $\pm H$, respectively. We denote by $R^g, R^{\pm}$ the corresponding curvature.

We consider the heterotic supergravity theory with an α' expansion where $1/2\pi\alpha'$ is the heterotic string tension. The bosonic part of the ten-dimensional supergravity action in the string frame is ([46], [13], $R = R^-$)

$$S = \frac{1}{2k^2}\int d^{10}x\sqrt{-g}e^{-2\phi}\Big[Scal^g + 4(\nabla^g\phi)^2 - \frac{1}{2}|H|^2 - \frac{\alpha'}{4}\Big(Tr|F^A|^2) - Tr|R|^2\Big)\Big]. \tag{1.1}$$

The string frame field equations (the equations of motion induced from the action (1.1)) of the heterotic string up to the first order of α' in sigma model perturbation theory in the notations in [36] are [44, 46]

$$\begin{aligned} &Ric^g_{ij} - \frac{1}{4}H_{imn}H_j^{mn} + 2\nabla^g_i\nabla^g_j\phi - \frac{\alpha'}{4}\Big[(F^A)_{imab}(F^A)_j^{mab} - R_{imnq}R_j^{mnq}\Big] = 0,\\ &\nabla^g_i(e^{-2\phi}H^i_{jk}) = 0, \qquad \nabla^+_i(e^{-2\phi}(F^A)^i_j) = 0. \end{aligned} \tag{1.2}$$

The field equation of the dilaton ϕ is implied from the first two equations above.

The Green-Schwarz anomaly cancellation mechanism requires that the three-form Bianchi identity receives an α' correction of the form

$$dH = \frac{\alpha'}{4}8\pi^2(p_1(M^d) - p_1(E)) = \frac{\alpha'}{4}\Big(Tr(R\wedge R) - Tr(F^A\wedge F^A)\Big), \tag{1.3}$$

where $p_1(M^d)$ and $p_1(E)$ are the first Pontrjagin forms of M^d with respect to a connection ∇ with curvature R and the vector bundle E with connection A, respectively.

A class of heterotic-string backgrounds for which the Bianchi identity of the three-form H receives a correction of type (1.3) are those with (2,0) world-volume supersymmetry. Such models were considered in [47]. The target-space geometry of (2,0)-supersymmetric sigma models has been extensively investigated in [47, 67, 43]. Recently, there is revived interest in these models [20, 32, 17, 51, 52, 33, 34, 36] as string backgrounds and in connection with heterotic-string compactification with fluxes mainly in dimension six [16, 6, 7, 8, 55, 29, 30, 9, 41, 40, 10, 39, 38, 63, 11, 3, 4, 5, 35, 14, 59, 61, 2, 58, 22, 62].

Equations (1.3), (1.1) and (1.2) involve a subtlety due to the choice of the connection ∇ on TM^d since anomalies can be canceled independently of the choice [45]. Different connections correspond to different regularization schemes in the two-dimensional worldsheet non-linear sigma model. Hence the background fields given for the particular choice of ∇ must be related to those for a different choice by a field redefinition [64]. Connections on M^d proposed to investigate the anomaly cancellation (1.3) are ∇^g [67, 34], ∇^+ [17, 19, 24], ∇^- [45, 13, 16, 36, 49, 53, 54, 57, 60, 48], Chern connection ∇^c when $d = 6$ [67, 55, 29, 30, 9].

A heterotic geometry preserves supersymmetry iff in ten dimensions there exists at least one Majorana-Weyl spinor ϵ such that the following Killing-spinor equations hold [67, 13]

$$\begin{aligned} &\delta_\lambda = \nabla_m\epsilon = \left(\nabla^g_m + \frac{1}{4}H_{mnp}\Gamma^{np}\right)\epsilon = \nabla^+\epsilon = 0,\\ &\delta_\Psi = \left(\Gamma^m\partial_m\phi - \frac{1}{12}H_{mnp}\Gamma^{mnp}\right)\epsilon = (d\phi - \frac{1}{2}H)\cdot\epsilon = 0,\\ &\delta_\xi = F^A_{mn}\Gamma^{mn}\epsilon = F^A\cdot\epsilon = 0, \end{aligned} \tag{1.4}$$

where λ, Ψ, ξ are the gravitino, the dilatino and the gaugino fields, Γ_i generate the Clifford algebra $\{\Gamma_i, \Gamma_j\} = 2g_{ij}$ and $\cdot$ means Clifford action of forms on spinors.

The system of Killing spinor equations (1.4) together with the anomaly cancellation condition (1.3) is known as the *Strominger system* [67]. The last equation in (1.4) is the instanton condition which means that the curvature F^A is contained in a Lie algebra of a Lie group which is a stabilizer of a non-trivial spinor. In dimension 7 this group is G_2. Denoting the G_2 three-form by Θ, the G_2-instanton condition has the form

$$\sum_{k,l=1}^{7} (F^A)^i_j(E_k, E_l)\Theta(E_k, E_l, E_m) = 0. \tag{1.5}$$

In the presence of a curvature term $Tr(R \wedge R)$ the solutions of the Strominger system (1.4), (1.3) obey the second and the third equations of motion (the second and the third equations in (1.2)) but do not always satisfy the Einstein equations of motion (see [24, 23, 25] where a sufficient quadratic condition on R is found). It was proved in [50] that the solutions of the Strominger system ((1.4) and (1.3)) also solve the heterotic supersymmetric equations of motion (1.2) if and only if R is an instanton in dimensions 5,6,7,8 (see [57, 61] for higher dimensions and different proofs). In particular, in dimension 7, R is required to be an G_2-instanton.

The physically relevant connection on the tangent bundle to be considered in (1.3), (1.1), (1.2) is the $(-)$-connection [13, 45]. One reason is that the curvature R^- of the $(-)$-connection is an instanton up to the first order of α' which is a consequence of the first equation in (1.4), (1.3) and the well known identity

$$R^+(X, Y, Z, U) - R^-(Z, U, X, Y) = \frac{1}{2}dH(X, Y, Z, U). \tag{1.6}$$

Indeed, (1.3) together with (1.6) imply $R^+(X, Y, Z, U) - R^-(Z, U, X, Y) = O(\alpha')$ and the first equation in (1.4) yields that the holonomy group of ∇^+ is contained in G_2, i.e. the curvature 2-form $R^+(X, Y) \subset \mathfrak{g}_2$ and therefore R^- satisfies the instanton condition (1.5) up to the first order of α'. Hence, a solution to the Strominger system with first Pontrjagin form of the $(-)$-connection always satisfies the heterotic equations of motion (1.2) up to the first order of α' (see e.g.[57] and references therein).

In dimension 7 the only known heterotic/type I solutions with non-zero fluxes to the equations of motion preserving at least one supersymmetry (satisfying (1.4) and (1.3) without the curvature term, $R = 0$) are those constructed [42]. All these solutions are noncompact and conformal to a flat space. Noncompact solutions to (1.4) and (1.3) in dimension 7 are presented also in [49]. The first compact heterotic/type I solutions with non-zero fluxes and constant dilaton to the equations of motion preserving at least one supersymmetry (satisfying (1.4) and (1.3)) in dimension seven are constructed in [25]

In dimension $d = 5$, if the field strength vanishes, $H = 0$, then the 5-dimensional case reduces to dimension four since any five dimensional Riemannian spin manifold admitting ∇^g-parallel spinor is reducible. Non compact solutions on circle bundle over 4-dimensional base endowed with a hyper Kähler metric (when the 4-dimensional metric is Eguchi-Hanson, Taub-NUT, Atiyah-Hitchin) have appeared in [56, 31, 65, 12, 63], the compact cases are discussed in [34] where a cohomological obstruction is presented. The first compact heterotic/type I solutions with non-zero fluxes and constant dilaton to the equations of motion preserving at least one supersymmetry (satisfying (1.4) and (1.3)) in dimension five are constructed in [23].

In this paper we construct smooth solutions with non vanishing flux and non-constant dilaton to the Strominger system using the first Pontrjagin form of the $(-)$-*connection* on 7-dimensional complete non-compact manifold equipped with conformally cocalibrated G_2 structures of pure type coupled with carefully chosen instanton bundle. The source of the construction is the already constructed smooth compact solutions to the Strominger system with constant dilaton on nilmanifods presented in [25] and the ideas outlined there to consider special three-torus bundles over either conformally $\mathbb{T}^4$ or K3 manifold.

Our first family of solutions are complete G_2 manifolds which are $\mathbb{T}^3$ bundles over conformally compact asymptotically hyperbolic metric on $\mathbb{T}^4$ with conformal boundary at infinity a flat torus $\mathbb{T}^3$. Using the first Pontrjagin form of the $(-)$-connection together with the first Pontrjagin form of a carefully chosen instanton we satisfied the anomaly cancellation condition with a *negative* α' and a non-constant dilaton which a real slice of an elliptic function of order two.

In Section 5 we present another smooth non-compact complete solution to the Strominger system using the first Pontrjagin form of the $(-)$-connection with positive string tension on certain $\mathbb{T}^3$ bundles over $\mathbb{R}^4$ with non-vanishing torsion, non-trivial instanton and non-constant dilaton. The non-constant dilaton function here is determined by the fundamental solution of the Laplacian on $\mathbb{R}^4$.

Conventions. The connection 1-forms ω_{ji} of a metric connection $\nabla, \nabla g = 0$ with respect to a local orthonormal basis $\{E_1, \dots, E_d\}$ are given by $\omega_{ji}(E_k) = g(\nabla_{E_k} E_j, E_i)$, since we write $\nabla_X E_j = \omega_j^s(X)\, E_s$.

The curvature 2-forms Ω^i_j of ∇ are given in terms of the connection 1-forms ω^i_j by $\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j, \quad \Omega_{ji} = d\omega_{ji} + \omega_{ki} \wedge \omega_{jk}, \quad R^l_{ijk} = \Omega^l_k(E_i, E_j), \quad R_{ijkl} = R^s_{ijk} g_{ls}$.

The first Pontrjagin class is represented by the 4-form $8\pi^2 p_1(\nabla) = \sum_{1\le i<j\le d} \Omega^i_j \wedge \Omega^i_j$.

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2. The supersymmetry equations and the geometric model

Geometrically, the vanishing of the gravitino variation is equivalent to the existence of a non-trivial real spinor parallel with respect to the metric connection ∇^+ with totally skew-symmetric torsion $T = H$. The presence of ∇^+-parallel spinor leads to restriction of the holonomy group $Hol(\nabla^+)$ of the torsion connection ∇^+.

2.1. Dimension 7.

In dimension seven $Hol(\nabla^+)$ has to be contained in the exceptional group G_2 [27, 32, 34, 28]. The precise conditions to have a solution to the gravitino Killing spinor equation in dimension 7 were found in [27]. Namely, there exists a non-trivial parallel spinor with respect to a G_2-connection with torsion 3-form T if and only if there exists an integrable G_2-structure Θ, i.e. $d*\Theta = \theta^7 \wedge *\Theta$, where $\theta^7 = -\frac{1}{3} * (*d\Theta \wedge \Theta) = \frac{1}{3} * (*d*\Theta \wedge *\Theta)$ is the Lee form. In this case, the connection ∇^+ is unique and the torsion 3-form T is given by the formula [27]

$$H = T = \frac{1}{6}(d\Theta, *\Theta)\,\Theta - *d\Theta + *(\theta^7 \wedge \Theta).$$

The necessary conditions to have a solution to the system of dilatino and gravitino Killing spinor equations (the first two equations in (1.4)) in dimension seven were derived in [32, 27, 28], and the sufficiency was proved in [27, 28]. The general existence result [27, 28] states that there exists a non-trivial solution to both dilatino and gravitino Killing spinor equations (the first two equations in (1.4)) in dimension 7 if and only if there exists a globally conformal co-calibrated G_2-structure (Θ, g) of pure type and with exact Lee form θ^7, i.e. a G_2-structure (Θ, g) satisfying the equations

$$d * \Theta = \theta^7 \wedge *\Theta, \quad d\Theta \wedge \Theta = 0, \quad \theta^7 = -2d\phi. \tag{2.1}$$

Consequently, the torsion 3-form (the flux H) is given by $H = T = - * d\Theta - 2 * (d\phi \wedge \Theta)$ and the Riemannian scalar curvature satisfies $s^g = 8||d\phi||^2 - \frac{1}{12}||T||^2 - 6\,\delta d\phi$. The equations (2.1) hold exactly when the G_2-structure $(\bar\Theta = e^{-\frac{3}{2}\phi}\Theta, \bar g = e^{-\phi}g)$ obeys the equations $d\bar{*}\bar\Theta = d\bar\Theta \wedge \bar\Theta = 0$, i.e., it is co-calibrated of pure type.

A geometric model which fits the above structures was proposed in [25] as a certain $\mathbb{T}^3$-bundle over a Calabi-Yau surface. For this, let Γ_i, $1 \le i \le 3$, be three closed anti-self-dual 2-forms on a Calabi-Yau surface M^4, which represent integral cohomology classes. Denote by ω_1 and by $\omega_2 + \sqrt{-1}\omega_3$ the (closed) Kähler form and the holomorphic volume form on M^4, respectively. Then, there is a compact 7-dimensional manifold $M^{1,1,1}$ which is the total space of a $\mathbb{T}^3$-bundle over M^4 and has a G_2-structure

$$\Theta = \omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2 - \omega_3 \wedge \eta_3 + \eta_1 \wedge \eta_2 \wedge \eta_3,$$

solving the first two Killing spinor equations in (1.4) with constant dilaton in dimension 7, where η_i, $1 \le i \le 3$, is a 1-form on

$M^{1,1,1}$ such that $d\eta_i = \Gamma_i$, $1 \le i \le 3$.

For any smooth function f on M^4, the G_2-structure on $M^{1,1,1}$ given by

$$\Theta_f = e^{2f}\Big[\omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2 - \omega_3 \wedge \eta_3\Big] + \eta_1 \wedge \eta_2 \wedge \eta_3$$

solves the first two Killing spinor equations in (1.4) with non-constant dilaton $\phi = -2f$. The metric has the form

$$g_f = e^{2f} g_{cy} + \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3.$$

To achieve a smooth solution to the Strominger system we still have to determine an auxiliary vector bundle with an instanton and a linear connection on $M^{1,1,1}$ in order to satisfy the anomaly cancellation condition (1.3).

2.2. **Dimension 5.** The existence of ∇^+-parallel spinor in dimension 5 determines an almost contact metric structure and, equivalently, a reduction of the structure group $SO(5)$ to $SU(2)$. The properties of the almost contact metric structure as well as solutions to gravitino and dilatino Killing-spinor equations are investigated in [27, 28] and presented in terms of reduction to $SU(2)$ in [23].

2.2.1. *Almost contact structure point of view.* We recall that an almost contact metric structure consists of an odd dimensional manifold M^{2k+1} equipped with a Riemannian metric g, vector field ξ of length one, its dual 1-form η as well as an endomorphism ψ of the tangent bundle such that

$$\psi(\xi) = 0, \quad \psi^2 = -id + \eta \otimes \xi, \quad g(\psi., \psi.) = g(.,.) - \eta \otimes \eta.$$

The Reeb vector field ξ is determined by the equations $\eta(\xi) = 1, \quad \xi \lrcorner d\eta = 0$, where $\lrcorner$ denotes the interior multiplication. The Nijenhuis tensor N, the fundamental form F and the Lee form θ of an almost contact metric structure are defined by

$$N = [\psi., \psi.] + \psi^2[.,.] - \psi[\psi.,.] - \psi[.,\psi.] + d\eta \otimes \xi, \quad F(.,.) = g(.,\psi.), \quad \theta = \frac{1}{2}F \lrcorner dF.$$

It was shown in [28] that the gravitino and the dilatino equation admit a solution in dimension five if and only if the Nijenhuis tensor is totally skew-symmetric, the Reeb vector field ξ is a Killing vector field and the next equalities hold $2d\phi = \theta, \quad *_{\mathbb{H}} d\eta = -d\eta$, where $*_{\mathbb{H}}$ denotes the Hodge operator acting on the 4-dimensional orthogonal complement $\mathbb{H}$ of the vector ξ, $\mathbb{H} = \mathrm{Ker}\,\eta$.

2.2.2. *The SU(2)-structure point of view.* The reduction of the structure group $SO(5)$ to $SU(2)$ is described in terms of forms by Conti and Salamon in [18] (see also [31]) as follows: an $SU(2)$-structure on a 5-dimensional manifold M is $(\eta, F = \omega_1, \omega_2, \omega_3)$, where η is a 1-form dual to ξ via the metric and ω_s, $s = 1, 2, 3$, are 2-forms on M satisfying $\omega_s \wedge \omega_t = \delta_{st}\, v, \quad v \wedge \eta \neq 0$ for some 4-form v, and $X \lrcorner \omega_1 = Y \lrcorner \omega_2 \Rightarrow \omega_3(X, Y) \geq 0$. The 2-forms ω_s, $s = 1, 2, 3$, can be chosen to form a basis of the $\mathbb{H}$-self-dual 2-forms [18].

It was shown in [23] that the first two equations in (1.4) admit a solution in dimension five exactly when there exists a five dimensional manifold M endowed with an $SU(2)$-structure $(\eta, F = \omega_1, \omega_2, \omega_3)$ satisfying the structure equations:

$$d\omega_s = 2df \wedge \omega_s, \qquad *_{\mathbb{H}} d\eta = -d\eta, \qquad df(\xi) = 0. \tag{2.2}$$

The flux H is given by [27, 28]

$$H = T = \eta \wedge d\eta + 2d^{\psi} f \wedge F, \quad \text{where} \quad d^{\psi} f(X) = -df(\psi X). \tag{2.3}$$

The dilaton ϕ is equal to $\phi = 2f$.

In other words, the gravitino and dilatino equations in dimension five are satisfied if and only if the manifold is special conformal to a quasi-Sasaki manifold with $\mathbb{H}$-anti-self-dual exterior derivative of the almost contact form and the metric has the form

$$g_f = e^{2f} g_{|_{\mathbb{H}}} + \eta \otimes \eta.$$

It was proposed in [25] to investigate S^1 bundles over a conformally hyper-Kähler manifold. This ansatz guaranties solution to the first two equations in (1.4). To achieve a smooth solution to the Strominger system we still have to determine a linear connection on the tangent bundle and an auxiliary vector bundle with an $SU(2)$-instanton, i.e., a connection A with curvature 2-form F^A satisfying

$$(F^A)^i_j(\psi E_k, \psi E_l) = (F^A)^i_j(E_k, E_l), \qquad \sum_{k=1}^{5} (F^A)^i_j(E_k, \psi E_k) = 0 \tag{2.4}$$

so that the anomaly cancellation condition (1.3) is satisfied.

3. The quaternionic Heisenberg group

The seven dimensional quaternionic Heisenberg group $G(\mathbb{H})$ is the connected simply connected Lie group with a group multiplication $[.,.]$ determined by the Lie algebra $\mathfrak{g}(\mathbb{H})$ with structure equations

$$d\gamma^1 = d\gamma^2 = d\gamma^3 = d\gamma^4 = 0, \quad d\gamma^5 = \gamma^{12} - \gamma^{34}, \quad d\gamma^6 = \gamma^{13} + \gamma^{24}, \quad d\gamma^7 = \gamma^{14} - \gamma^{23}. \tag{3.1}$$

In order to obtain results in dimensions less than seven through contractions of $\mathfrak{g}(\mathbb{H})$ it will be convenient to consider the orbit of $G(\mathbb{H})$ under the natural action of $GL(3, \mathbb{R})$ on the *span* $\{\gamma^5, \gamma^6, \gamma^7\}$. Accordingly let K_A be a seven-dimensional real Lie group with Lie bracket $[x, x']_A = A[A^{-1}x, A^{-1}x']$ for $A \in GL(3, \mathbb{R})$ defined by a basis of left-invariant 1-forms $\{e^1, \ldots, e^7\}$ such that $e^i = \gamma^i$ for $1 \leq i \leq 4$ and $(e^5\ e^6\ e^7) = A\,(\gamma^5\ \gamma^6\ \gamma^7)^T$. Hence, the structure equations of the Lie algebra $\mathfrak{K}_A$ of the group K_A are

$$de^1 = de^2 = de^3 = de^4 = 0, \qquad de^{4+i} = \sum_{j=1}^{3} a_{ij}\, \sigma_j, \quad i = 1, 2, 3, \tag{3.2}$$

where $\sigma_1 = e^{12} - e^{34}$, $\sigma_2 = e^{13} + e^{24}$, $\sigma_3 = e^{14} - e^{23}$ are the three anti-self-dual forms on $\mathbb{R}^4$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{3.3}$$

We will denote the norm of A by $|A|$, $|A|^2 = \sum_{i,j=1}^{3} a_{ij}^2$.

Since $\mathfrak{K}_A$ is isomorphic to $\mathfrak{g}(\mathbb{H})$, if K_A is connected and simply connected it is isomorphic to $G(\mathbb{H})$. Furthermore, any lattice Γ_A gives rise to a (compact) nilmanifold $M_A = K_A/\Gamma_A$, which is a $\mathbb{T}^3$-bundle over a $\mathbb{T}^4$ with connection 1-forms of anti-self-dual curvature on the four torus.

Following [25] we consider the G_2 structure on the Lie group K_A defined by the 3-form

$$\Theta = \omega_1 \wedge e^7 + \omega_2 \wedge e^5 - \omega_3 \wedge e^6 + e^{567}, \tag{3.4}$$

where

$$\omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} - e^{24}, \quad \omega_3 = e^{14} + e^{23}$$

are the three closed self-dual 2-forms on $\mathbb{R}^4$. The corresponding Hodge dual 4-form $*\Theta$ is given by

$$* \Theta = \omega_1 \wedge e^{56} + \omega_2 \wedge e^{67} + \omega_3 \wedge e^{57} + \frac{1}{2}\omega_1 \wedge \omega_1. \tag{3.5}$$

It is easy to check using (3.2) and the property $\sigma_i \wedge \omega_j = 0$ for $1 \le i, j \le 3$ that

$$d * \Theta = 0, \quad d\Theta \wedge \Theta = 0, \tag{3.6}$$

i.e. Θ is co-calibrated of pure type. According to [27, 28] this G_2 structure solves the gravitino and dilatino equations with constant dilaton.

Let f be a smooth function on $\mathbb{R}^4$. Following [25] we consider the G_2 form given by

$$\bar{\Theta} = e^{2f}\Big[\omega_1 \wedge e^7 + \omega_2 \wedge e^5 - \omega_3 \wedge e^6\Big] + e^{567}. \tag{3.7}$$

The corresponding metric $\bar{g}$ on K_A has an orthonormal basis of 1-forms given by

$$\bar{e}^1 = e^f e^1, \quad \bar{e}^2 = e^f e^2, \quad \bar{e}^3 = e^f e^3, \quad \bar{e}^4 = e^f e^4, \quad \bar{e}^5 = e^5, \quad \bar{e}^6 = e^6, \quad \bar{e}^7 = e^7 \tag{3.8}$$

and self-dual form $\bar{\omega}_i$ and anti-self-dual forms $\bar{\sigma}_i$ given by

$$\bar{\omega}_i = e^{2f}\omega_i, \qquad \bar{\sigma}_i = e^{2f}\sigma_i, \quad i = 1, 2, 3. \tag{3.9}$$

The corresponding Hodge dual 4-form $\bar{*}\bar{\Theta}$ is

$$\bar{*}\bar{\Theta} = e^{2f}\Big[\omega_1 \wedge e^{56} + \omega_2 \wedge e^{67} + \omega_3 \wedge e^{57} + \frac{e^{2f}}{2}\omega_1 \wedge \omega_1\Big]. \tag{3.10}$$

It was shown in [25, Theorem 6.1] using (3.6) that

$$d\bar{*}\bar{\Theta} = 2df \wedge \bar{*}\bar{\Theta}, \quad d\bar{\Theta} \wedge \bar{\Theta} = 0. \tag{3.11}$$

Then the Lie form $\bar{\theta}$ is given by

$$\bar{\theta} = 2df \tag{3.12}$$

and the G_2 structure $\bar{\Theta}$ solves the gravitino and dilatino equations with non-constant dilaton $\phi = -2f$ [27, 28].

According to [27, 28], the torsion of the $(+)$-connection ∇^+ is the 3-form

$$T = - * d\Theta + *(\theta \wedge \Theta). \tag{3.13}$$

We calculate from (3.2) and (3.7) that

$$d\bar{\Theta} = 2df \wedge \bar{\Theta} - 2df \wedge e^{567} + de^{567}. \tag{3.14}$$

A substitution of (3.14) in (3.13), and using (3.12), gives

$$
\begin{aligned}
(3.15)\quad \bar{T} = \bar{*}(2df\wedge e^{567} - de^{567}) = e^{-f}\Big[-2f_1\,\bar{e}^{234} + 2f_2\,\bar{e}^{134} - 2f_3\,\bar{e}^{124} + 2f_4\,\bar{e}^{123}\Big] \\
+e^{-2f}\Big[(a_{11}\,\bar{\sigma}_1 + a_{12}\,\bar{\sigma}_2 + a_{13}\,\bar{\sigma}_3)\wedge\bar{e}^5 + (a_{21}\,\bar{\sigma}_1 + a_{22}\,\bar{\sigma}_2 + a_{23}\,\bar{\sigma}_3)\wedge\bar{e}^6 + (a_{31}\,\bar{\sigma}_1 + a_{32}\,\bar{\sigma}_2 + a_{33}\,\bar{\sigma}_3)\wedge\bar{e}^7\Big],
\end{aligned}
$$

where $f_i = \frac{\partial f}{\partial x_i}$, $1\leq i\leq 4$, and $\bar{\sigma}_1 = \bar{e}^{12} - \bar{e}^{34}$, $\bar{\sigma}_2 = \bar{e}^{13} + \bar{e}^{24}$ and $\bar{\sigma}_3 = \bar{e}^{14} - \bar{e}^{23}$. Letting $f_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}$, $1\leq i,j\leq 4$, a short calculation gives

$$
d\bar{T} = -e^{-4f}\left[\triangle e^{2f} + 2|A|^2\right]\bar{e}^{1234} = -\left[\triangle e^{2f} + 2|A|^2\right]e^{1234}, \tag{3.16}
$$

where $\triangle e^{2f} = (e^{2f})_{11} + (e^{2f})_{22} + (e^{2f})_{33} + (e^{2f})_{44}$ is the standard Laplacian on $\mathbb{R}^4$.

3.1. **The first Pontrjagin form of the $(-)$-connection.** From Koszul's formula, we have that the Levi-Civita connection 1-forms $(\omega^{\bar{g}})^{\bar{\imath}}_{\bar{\jmath}}$ of the metric $\bar{g}$ are given by

$$
\begin{aligned}
(\omega^{\bar{g}})^{\bar{\imath}}_{\bar{\jmath}}(\bar{e}_k) &= -\tfrac{1}{2}\Big(\bar{g}(\bar{e}_i,[\bar{e}_j,\bar{e}_k]) - \bar{g}(\bar{e}_k,[\bar{e}_i,\bar{e}_j]) + \bar{g}(\bar{e}_j,[\bar{e}_k,\bar{e}_i])\Big) \\
&= \tfrac{1}{2}\Big(d\bar{e}^i(\bar{e}_j,\bar{e}_k) - d\bar{e}^k(\bar{e}_i,\bar{e}_j) + d\bar{e}^j(\bar{e}_k,\bar{e}_i)\Big)
\end{aligned} \tag{3.17}
$$

taking into account $\bar{g}(\bar{e}_i,[\bar{e}_j,\bar{e}_k]) = -d\bar{e}^i(\bar{e}_j,\bar{e}_k)$. With the help of (3.17) we compute the expressions for the connection 1-forms $(\omega^-)^{\bar{\imath}}_{\bar{\jmath}}$ of the connection ∇^-,

$$
(\omega^-)^{\bar{\imath}}_{\bar{\jmath}} = (\omega^{\bar{g}})^{\bar{\imath}}_{\bar{\jmath}} - \frac{1}{2}(\bar{T})^{\bar{\imath}}_{\bar{\jmath}}, \qquad \text{where} \qquad (\bar{T})^{\bar{\imath}}_{\bar{\jmath}}(\bar{e}_k) = \bar{T}(\bar{e}_i,\bar{e}_j,\bar{e}_k). \tag{3.18}
$$

Now, (3.18), (3.17) and (3.15) show that the possibly non-zero connection 1-forms $(\omega^-)^{\bar{\imath}}_{\bar{\jmath}}$ are given in terms of the basis $\{\bar{e}^1,\dots,\bar{e}^7\}$ by:

$$
\begin{aligned}
&(\omega^-)^{\bar{1}}_{\bar{2}} = (\omega^-)^{\bar{3}}_{\bar{4}} = e^{-f}\left(f_2\,\bar{e}^1 - f_1\,\bar{e}^2 + f_4\,\bar{e}^3 - f_3\,\bar{e}^4\right), \\
&(\omega^-)^{\bar{1}}_{\bar{3}} = -(\omega^-)^{\bar{2}}_{\bar{4}} = e^{-f}\left(f_3\,\bar{e}^1 - f_4\,\bar{e}^2 - f_1\,\bar{e}^3 + f_2\,\bar{e}^4\right), \\
&(\omega^-)^{\bar{1}}_{\bar{4}} = (\omega^-)^{\bar{2}}_{\bar{3}} = e^{-f}\left(f_4\,\bar{e}^1 + f_3\,\bar{e}^2 - f_2\,\bar{e}^3 - f_1\,\bar{e}^4\right), \\
&(\omega^-)^{\bar{1}}_{\bar{5}} = e^{-2f}\left(-a_{11}\,\bar{e}^2 - a_{12}\,\bar{e}^3 - a_{13}\,\bar{e}^4\right), \quad (\omega^-)^{\bar{1}}_{\bar{6}} = e^{-2f}\left(-a_{21}\,\bar{e}^2 - a_{22}\,\bar{e}^3 - a_{23}\,\bar{e}^4\right), \\
&(\omega^-)^{\bar{1}}_{\bar{7}} = e^{-2f}\left(-a_{31}\,\bar{e}^2 - a_{32}\,\bar{e}^3 - a_{33}\,\bar{e}^4\right), \quad (\omega^-)^{\bar{2}}_{\bar{5}} = e^{-2f}\left(a_{11}\,\bar{e}^1 + a_{13}\,\bar{e}^3 - a_{12}\,\bar{e}^4\right), \\
&(\omega^-)^{\bar{2}}_{\bar{6}} = e^{-2f}\left(a_{21}\,\bar{e}^1 + a_{23}\,\bar{e}^3 - a_{22}\,\bar{e}^4\right), \quad (\omega^-)^{\bar{2}}_{\bar{7}} = e^{-2f}\left(a_{31}\,\bar{e}^1 + a_{33}\,\bar{e}^3 - a_{32}\,\bar{e}^4\right), \\
&(\omega^-)^{\bar{3}}_{\bar{5}} = e^{-2f}\left(a_{12}\,\bar{e}^1 - a_{13}\,\bar{e}^2 + a_{11}\,\bar{e}^4\right), \quad (\omega^-)^{\bar{3}}_{\bar{6}} = e^{-2f}\left(a_{22}\,\bar{e}^1 - a_{23}\,\bar{e}^2 + a_{21}\,\bar{e}^4\right), \\
&(\omega^-)^{\bar{3}}_{\bar{7}} = e^{-2f}\left(a_{32}\,\bar{e}^1 - a_{33}\,\bar{e}^2 + a_{31}\,\bar{e}^4\right), \quad (\omega^-)^{\bar{4}}_{\bar{5}} = e^{-2f}\left(a_{13}\,\bar{e}^1 + a_{12}\,\bar{e}^2 - a_{11}\,\bar{e}^3\right), \\
&(\omega^-)^{\bar{4}}_{\bar{6}} = e^{-2f}\left(a_{23}\,\bar{e}^1 + a_{22}\,\bar{e}^2 - a_{21}\,\bar{e}^3\right), \quad (\omega^-)^{\bar{4}}_{\bar{7}} = e^{-2f}\left(a_{33}\,\bar{e}^1 + a_{32}\,\bar{e}^2 - a_{31}\,\bar{e}^3\right).
\end{aligned} \tag{3.19}
$$

A long straightforward calculation using (3.19) gives in terms of the basis $\{\bar e^1, \dots, \bar e^7\}$ the following formulas for the curvature 2-forms $(\Omega^-)^{\bar i}_{\bar j}$ of the connection ∇^-:

$$\begin{aligned}
(\Omega^-)^{\bar 1}_{\bar 2} = &-e^{-2f}[f_{11}+f_{22}+2f_3^2+2f_4^2+(a_{11}^2+a_{21}^2+a_{31}^2)e^{-2f}]\,\bar e^{12}\\
&+e^{-2f}[f_{14}-f_{23}-2f_1f_4+2f_2f_3-(a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32})e^{-2f}]\,\bar\sigma_2\\
&-e^{-2f}[f_{13}+f_{24}-2f_1f_3-2f_2f_4+(a_{11}a_{13}+a_{21}a_{23}+a_{31}a_{33})e^{-2f}]\,\bar\sigma_3\\
&-e^{-2f}[f_{33}+f_{44}+2f_1^2+2f_2^2+(a_{12}^2+a_{22}^2+a_{32}^2+a_{13}^2+a_{23}^2+a_{33}^2)e^{-2f}]\,\bar e^{34},\\
(\Omega^-)^{\bar 1}_{\bar 3} = &-e^{-2f}[f_{14}+f_{23}-2f_1f_4-2f_2f_3+(a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32})e^{-2f}]\,\bar\sigma_1\\
&-e^{-2f}[f_{11}+f_{33}+2f_2^2+2f_4^2+(a_{12}^2+a_{22}^2+a_{32}^2)e^{-2f}]\,\bar e^{13}\\
&+e^{-2f}[f_{12}-f_{34}-2f_1f_2+2f_3f_4+(a_{12}a_{13}+a_{22}a_{23}+a_{32}a_{33})e^{-2f}]\,\bar\sigma_3\\
&+e^{-2f}[f_{22}+f_{44}+2f_1^2+2f_3^2+(a_{11}^2+a_{21}^2+a_{31}^2+a_{13}^2+a_{23}^2+a_{33}^2)e^{-2f}]\,\bar e^{24},\\
(\Omega^-)^{\bar 1}_{\bar 4} = &\,e^{-2f}[f_{13}-f_{24}-2f_1f_3+2f_2f_4-(a_{11}a_{13}+a_{21}a_{23}+a_{31}a_{33})e^{-2f}]\,\bar\sigma_1\\
&-e^{-2f}[f_{12}+f_{34}-2f_1f_2-2f_3f_4+(a_{12}a_{13}+a_{22}a_{23}+a_{32}a_{33})e^{-2f}]\,\bar\sigma_2\\
&-e^{-2f}[f_{11}+f_{44}+2f_2^2+2f_3^2+(a_{13}^2+a_{23}^2+a_{33}^2)e^{-2f}]\,\bar e^{14}\\
&-e^{-2f}[f_{22}+f_{33}+2f_1^2+2f_4^2+(a_{11}^2+a_{21}^2+a_{31}^2+a_{12}^2+a_{22}^2+a_{32}^2)e^{-2f}]\,\bar e^{23},
\end{aligned}$$

$$(\Omega^-)^{\bar 1}_{\bar 5} = 2e^{-3f}\left[(a_{11}f_1-a_{13}f_3+a_{12}f_4)\,\bar\sigma_1+(a_{12}f_1+a_{13}f_2-a_{11}f_4)\,\bar\sigma_2+(a_{13}f_1-a_{12}f_2+a_{11}f_3)\,\bar\sigma_3\right],$$
$$(\Omega^-)^{\bar 1}_{\bar 6} = 2e^{-3f}\left[(a_{21}f_1-a_{23}f_3+a_{22}f_4)\,\bar\sigma_1+(a_{22}f_1+a_{23}f_2-a_{21}f_4)\,\bar\sigma_2+(a_{23}f_1-a_{22}f_2+a_{21}f_3)\,\bar\sigma_3\right],$$
$$(\Omega^-)^{\bar 1}_{\bar 7} = 2e^{-3f}\left[(a_{31}f_1-a_{33}f_3+a_{32}f_4)\,\bar\sigma_1+(a_{32}f_1+a_{33}f_2-a_{31}f_4)\,\bar\sigma_2+(a_{33}f_1-a_{32}f_2+a_{31}f_3)\,\bar\sigma_3\right],$$

$$\begin{aligned}
(\Omega^-)^{\bar 2}_{\bar 3} = &\,e^{-2f}[f_{13}-f_{24}-2f_1f_3+2f_2f_4+(a_{11}a_{13}+a_{21}a_{23}+a_{31}a_{33})e^{-2f}]\,\bar\sigma_1\\
&-e^{-2f}[f_{12}+f_{34}-2f_1f_2-2f_3f_4-(a_{12}a_{13}+a_{22}a_{23}+a_{32}a_{33})e^{-2f}]\,\bar\sigma_2\\
&-e^{-2f}[f_{11}+f_{44}+2f_2^2+2f_3^2+(a_{11}^2+a_{21}^2+a_{31}^2+a_{12}^2+a_{22}^2+a_{32}^2)e^{-2f}]\,\bar e^{14}\\
&-e^{-2f}[f_{22}+f_{33}+2f_1^2+2f_4^2+(a_{13}^2+a_{23}^2+a_{33}^2)e^{-2f}]\,\bar e^{23},\\
(\Omega^-)^{\bar 2}_{\bar 4} = &\,e^{-2f}[f_{14}+f_{23}-2f_1f_4-2f_2f_3-(a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32})e^{-2f}]\,\bar\sigma_1\\
&+e^{-2f}[f_{11}+f_{33}+2f_2^2+2f_4^2+(a_{11}^2+a_{21}^2+a_{31}^2+a_{13}^2+a_{23}^2+a_{33}^2)e^{-2f}]\,\bar e^{13}\\
&-e^{-2f}[f_{12}-f_{34}-2f_1f_2+2f_3f_4+(a_{12}a_{13}+a_{22}a_{23}+a_{32}a_{33})e^{-2f}]\,\bar\sigma_3\\
&-e^{-2f}[f_{22}+f_{44}+2f_1^2+2f_3^2+(a_{12}^2+a_{22}^2+a_{32}^2)e^{-2f}]\,\bar e^{24},
\end{aligned}$$

$$(\Omega^-)^{\bar 2}_{\bar 5} = 2e^{-3f}\left[(a_{11}f_2-a_{12}f_3-a_{13}f_4)\,\bar\sigma_1-(a_{13}f_1-a_{12}f_2-a_{11}f_3)\,\bar\sigma_2+(a_{12}f_1+a_{13}f_2+a_{11}f_4)\,\bar\sigma_3\right],$$
$$(\Omega^-)^{\bar 2}_{\bar 6} = 2e^{-3f}\left[(a_{21}f_2-a_{22}f_3-a_{23}f_4)\,\bar\sigma_1-(a_{23}f_1-a_{22}f_2-a_{21}f_3)\,\bar\sigma_2+(a_{22}f_1+a_{23}f_2+a_{21}f_4)\,\bar\sigma_3\right],$$
$$(\Omega^-)^{\bar 2}_{\bar 7} = 2e^{-3f}\left[(a_{31}f_2-a_{32}f_3-a_{33}f_4)\,\bar\sigma_1-(a_{33}f_1-a_{32}f_2-a_{31}f_3)\,\bar\sigma_2+(a_{32}f_1+a_{33}f_2+a_{31}f_4)\,\bar\sigma_3\right],$$

$$\begin{aligned}
(\Omega^-)^{\bar 3}_{\bar 4} = &-e^{-2f}[f_{11}+f_{22}+2f_3^2+2f_4^2+(a_{12}^2+a_{22}^2+a_{32}^2+a_{13}^2+a_{23}^2+a_{33}^2)e^{-2f})\bar e^{12}\\
&+e^{-2f}[f_{14}-f_{23}-2f_1f_4+2f_2f_3+(a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32})e^{-2f}]\,\bar\sigma_2\\
&-e^{-2f}[f_{13}+f_{24}-2f_1f_3-2f_2f_4-(a_{11}a_{13}+a_{21}a_{23}+a_{31}a_{33})e^{-2f}]\,\bar\sigma_3\\
&-e^{-2f}[f_{33}+f_{44}+2f_1^2+2f_2^2+(a_{11}^2+a_{21}^2+a_{31}^2)e^{-2f})\bar e^{34},
\end{aligned}$$

$$(\Omega^-)^{\bar 3}_{\bar 5} = 2e^{-3f}\left[(a_{13}f_1+a_{12}f_2+a_{11}f_3)\,\bar\sigma_1-(a_{11}f_2-a_{12}f_3+a_{13}f_4)\,\bar\sigma_2-(a_{11}f_1-a_{13}f_3-a_{12}f_4)\,\bar\sigma_3\right],$$
$$(\Omega^-)^{\bar 3}_{\bar 6} = 2e^{-3f}\left[(a_{23}f_1+a_{22}f_2+a_{21}f_3)\,\bar\sigma_1-(a_{21}f_2-a_{22}f_3+a_{23}f_4)\,\bar\sigma_2-(a_{21}f_1-a_{23}f_3-a_{22}f_4)\,\bar\sigma_3\right],$$
$$(\Omega^-)^{\bar 3}_{\bar 7} = 2e^{-3f}\left[(a_{33}f_1+a_{32}f_2+a_{31}f_3)\,\bar\sigma_1-(a_{31}f_2-a_{32}f_3+a_{33}f_4)\,\bar\sigma_2-(a_{31}f_1-a_{33}f_3-a_{32}f_4)\,\bar\sigma_3\right],$$

$$(\Omega^-)^{\bar 4}_{\bar 5} = 2e^{-3f}\left[-(a_{12}f_1 - a_{13}f_2 - a_{11}f_4)\,\bar\sigma_1 + (a_{11}f_1 + a_{13}f_3 + a_{12}f_4)\,\bar\sigma_2 - (a_{11}f_2 + a_{12}f_3 - a_{13}f_4)\,\bar\sigma_3\right],$$
$$(\Omega^-)^{\bar 4}_{\bar 6} = 2e^{-3f}\left[-(a_{22}f_1 - a_{23}f_2 - a_{21}f_4)\,\bar\sigma_1 + (a_{21}f_1 + a_{23}f_3 + a_{22}f_4)\,\bar\sigma_2 - (a_{21}f_2 + a_{22}f_3 - a_{23}f_4)\,\bar\sigma_3\right],$$
$$(\Omega^-)^{\bar 4}_{\bar 7} = 2e^{-3f}\left[-(a_{32}f_1 - a_{33}f_2 - a_{31}f_4)\,\bar\sigma_1 + (a_{31}f_1 + a_{33}f_3 + a_{32}f_4)\,\bar\sigma_2 - (a_{31}f_2 + a_{32}f_3 - a_{33}f_4)\,\bar\sigma_3\right],$$

$$(\Omega^-)^{\bar 5}_{\bar 6} = 2e^{-4f}\left[(a_{12}a_{23} - a_{13}a_{22})\,\bar\sigma_1 - (a_{11}a_{23} - a_{13}a_{21})\,\bar\sigma_2 + (a_{11}a_{22} - a_{12}a_{21})\,\bar\sigma_3\right],$$
$$(\Omega^-)^{\bar 5}_{\bar 7} = 2e^{-4f}\left[(a_{12}a_{33} - a_{13}a_{32})\,\bar\sigma_1 - (a_{11}a_{33} - a_{13}a_{31})\,\bar\sigma_2 + (a_{11}a_{32} - a_{12}a_{31})\,\bar\sigma_3\right],$$
$$(\Omega^-)^{\bar 6}_{\bar 7} = 2e^{-4f}\left[(a_{22}a_{33} - a_{23}a_{32})\,\bar\sigma_1 - (a_{21}a_{33} - a_{23}a_{31})\,\bar\sigma_2 + (a_{21}a_{32} - a_{22}a_{31})\,\bar\sigma_3\right].$$

A long calculation based on the formulas for the curvature 2-form $(\Omega^-)^{\bar i}_{\bar j}$ of ∇^- gives

Proposition 3.1. *The first Pontrjagin form of ∇^- is a scalar multiple of e^{1234} given by*

$$\pi^2 p_1(\nabla^-) = \left[\mathfrak{F}_2[f] + \triangle_4 f - \frac{3}{8}|A|^2\triangle e^{-2f}\right] e^{1234}, \tag{3.20}$$

where $\mathfrak{F}_2[f]$ is the 2-Hessian of f, i.e., the sum of all principle 2×2-minors of the Hessian, and $\triangle_4 f = div(|\nabla f|^2\nabla f)$ is the 4-Laplacian of f.

The above Proposition shows, in particular, that even though the curvature 2-forms of ∇^- are quadratic in the gradient of the dilaton, the Pontrjagin form of ∇^- is also quadratic in these terms. Furthermore, if f depends on two of the variables then $\mathfrak{F}_2[f] = det(Hess f)$ while if f is a function of one variable $\mathfrak{F}_2[f]$ vanishes.

4. A conformally compact solution with negative α'

In this section we give our first main result. Recall that K_A is the connected simply connected Lie group with Lie algebra $\mathfrak{K}_A$ determined by (3.2). Due to the results recalled in Section 2.1 the remaining part is to solve the anomaly cancellation condition. This we will achieve for the G_2 structure (3.7) with the torsion term (3.16), the Pontrjagin form (3.20) of the ∇^- connection, and the G_2-instanton defined below.

Proposition 4.1. *Let D_Λ, $\Lambda = (\lambda_{ij}) \in \mathfrak{gl}_3(\mathbb{R})$, be the linear connection on the Lie group K_A whose possibly non-zero 1-forms are given as follows*

$$(\omega^{\mathrm{D}_\Lambda})^{\bar 1}_{\bar 2} = -(\omega^{\mathrm{D}_\Lambda})^{\bar 2}_{\bar 1} = -(\omega^{\mathrm{D}_\Lambda})^{\bar 3}_{\bar 4} = (\omega^{\mathrm{D}_\Lambda})^{\bar 4}_{\bar 3} = \lambda_{11}\,\bar e^5 + \lambda_{12}\,\bar e^6 + \lambda_{13}\,\bar e^7,$$
$$(\omega^{\mathrm{D}_\Lambda})^{\bar 1}_{\bar 3} = -(\omega^{\mathrm{D}_\Lambda})^{\bar 3}_{\bar 1} = (\omega^{\mathrm{D}_\Lambda})^{\bar 2}_{\bar 4} = -(\omega^{\mathrm{D}_\Lambda})^{\bar 4}_{\bar 2} = \lambda_{21}\,\bar e^5 + \lambda_{22}\,\bar e^6 + \lambda_{23}\,\bar e^7,$$
$$(\omega^{\mathrm{D}_\Lambda})^{\bar 1}_{\bar 4} = -(\omega^{\mathrm{D}_\Lambda})^{\bar 4}_{\bar 1} = -(\omega^{\mathrm{D}_\Lambda})^{\bar 2}_{\bar 3} = (\omega^{\mathrm{D}_\Lambda})^{\bar 3}_{\bar 2} = \lambda_{31}\,\bar e^5 + \lambda_{32}\,\bar e^6 + \lambda_{33}\,\bar e^7.$$

Then, D_Λ is a G_2-instanton with respect to the G_2 structure defined by (3.7) which preserves the metric if and only if $\mathrm{rank}(\Lambda) \leq 1$.

Proof. Let us use the notation $\Lambda_{ijkl} = \lambda_{ik}\lambda_{jl} - \lambda_{jk}\lambda_{il} = \det\begin{pmatrix}\lambda_{ik} & \lambda_{il}\\ \lambda_{jk} & \lambda_{jl}\end{pmatrix}$ for the 2×2 minors of Λ. A direct calculation using (3.2) shows that the possibly non-zero curvature forms $(\Omega^{\mathrm{D}_\Lambda})^{\bar i}_{\bar j}$ of the connection D_Λ are:

$$\begin{aligned}
(\Omega^{\mathrm{D}_\Lambda})^{\bar 1}_{\bar 2} &= -(\Omega^{\mathrm{D}_\Lambda})^{\bar 2}_{\bar 1} = -(\Omega^{\mathrm{D}_\Lambda})^{\bar 3}_{\bar 4} = (\Omega^{\mathrm{D}_\Lambda})^{\bar 4}_{\bar 3} = \quad e^{-2f}(a_{11}\lambda_{11}+a_{21}\lambda_{12}+a_{31}\lambda_{13})\,\bar\sigma_1\\
&\quad +e^{-2f}(a_{12}\lambda_{11}+a_{22}\lambda_{12}+a_{32}\lambda_{13})\,\bar\sigma_2 + e^{-2f}(a_{13}\lambda_{11}+a_{23}\lambda_{12}+a_{33}\lambda_{13})\,\bar\sigma_3\\
&\quad + 2\Lambda_{2312}\,\bar e^{56} + 2\Lambda_{2313}\,\bar e^{57} + 2\Lambda_{2323}\,\bar e^{67},\\
(\Omega^{\mathrm{D}_\Lambda})^{\bar 1}_{\bar 3} &= -(\Omega^{\mathrm{D}_\Lambda})^{\bar 3}_{\bar 1} = (\Omega^{\mathrm{D}_\Lambda})^{\bar 2}_{\bar 4} = -(\Omega^{\mathrm{D}_\Lambda})^{\bar 4}_{\bar 2} = \quad e^{-2f}(a_{11}\lambda_{21}+a_{21}\lambda_{22}+a_{31}\lambda_{23})\,\bar\sigma_1\\
&\quad +e^{-2f}(a_{12}\lambda_{21}+a_{22}\lambda_{22}+a_{32}\lambda_{23})\,\bar\sigma_2 + e^{-2f}(a_{13}\lambda_{21}+a_{23}\lambda_{22}+a_{33}\lambda_{23})\,\bar\sigma_3\\
&\quad - 2\Lambda_{1312}\,\bar e^{56} - 2\Lambda_{1313}\,\bar e^{57} - 2\Lambda_{1323}\,\bar e^{67},\\
(\Omega^{\mathrm{D}_\Lambda})^{\bar 1}_{\bar 4} &= -(\Omega^{\mathrm{D}_\Lambda})^{\bar 4}_{\bar 1} = -(\Omega^{\mathrm{D}_\Lambda})^{\bar 2}_{\bar 3} = (\Omega^{\mathrm{D}_\Lambda})^{\bar 3}_{\bar 2} = \quad e^{-2f}(a_{11}\lambda_{31}+a_{21}\lambda_{32}+a_{31}\lambda_{33})\,\bar\sigma_1\\
&\quad +e^{-2f}(a_{12}\lambda_{31}+a_{22}\lambda_{32}+a_{32}\lambda_{33})\,\bar\sigma_2 + e^{-2f}(a_{13}\lambda_{31}+a_{23}\lambda_{32}+a_{33}\lambda_{33})\,\bar\sigma_3\\
&\quad + 2\Lambda_{1212}\,\bar e^{56} + 2\Lambda_{1213}\,\bar e^{57} + 2\Lambda_{1223}\,\bar e^{67}.
\end{aligned}$$

Now, it is straightforward to see that D_Λ satisfies (1.5) if and only if all the 2×2 minors Λ_{ijkl} of the matrix Λ vanish. Therefore, D_Λ is a G_2-instanton if and only rank$(\Lambda)\le 1$. □

Corollary 4.2. *For $\Lambda=(\lambda_{ij})\in\mathfrak{gl}_3(\mathbb{R})$ a matrix of rank one, let D_Λ be the G_2-instanton defined in Proposition 4.1. Then, the first Pontrjagin form $p_1(\mathrm{D}_\Lambda)$ of the G_2-instanton D_Λ is given by*

$$8\pi^2 p_1(\mathrm{D}_\Lambda) = -4\lambda^2\,e^{1234}, \tag{4.1}$$

where $\lambda=|\Lambda\,A|$ is the norm of the product matrix $\Lambda\,A$.

Proof. Since the 2×2 minors Λ_{ijkl} are all zero, the formulas for the curvature forms $(\Omega^{\mathrm{D}_\Lambda})^{\bar i}_{\bar j}$ given in the proof of Proposition 4.1 imply the claimed identity.

□

We turn to the proof of our first main result.

Theorem 4.3. *The conformally compact manifold $M^7=(\Gamma\backslash K_A,\bar\Theta,\nabla^-,D_\Lambda,f)$ is a G_2-manifold which solves the Strominger system with non-constant dilaton f, non-trivial flux $H=\bar T$, non-flat instanton D_Λ using the first Pontrjagin form of ∇^- and negative α'. The dilaton f depends on one variable and is determined as a real slice of the Weierstrass' elliptic function.*

The conformally compact manifold $M^7=(\Gamma\backslash K_A,\bar\Theta,\nabla^-,D_\Lambda,f)$ satisfies the heterotic equations of motion (1.2) up to first order of α'.

Proof. By the construction in Section 2.1 we are left with solving the anomaly cancellation condition $d\bar T=\frac{\alpha'}{4}8\pi^2\Big(p_1(\nabla^-)-p_1(D_\Lambda)\Big)$, which in our case taking into account (3.16), (3.20) and (4.1) becomes *the single* non-linear equation

$$\triangle e^{2f}+2|A|^2+\frac{\alpha'}{4}\Big[8\mathcal{F}_2[f]+8\triangle_4 f-3|A|^2\triangle e^{-2f}+4\lambda^2\Big]=0. \tag{4.2}$$

Up to relabelling the constants, this is the same equation as the one obtained through the anomaly cancellation that appeared in [26, Section 4.2]. Accordingly, we assume that the function f depends on one variable, $f=f(x^1)$, and for a *negative* α' we choose $2|A|^2+\alpha'\lambda^2=0$, i.e., we let $\alpha'=-\alpha^2$ so that $2|A|^2=\alpha^2\lambda^2$. This simplifies (4.2) to the ordinary differential equation

$$\Big(e^{2f}\Big)'+\frac{3}{4}\alpha^2|A|^2\Big(e^{-2f}\Big)'-2\alpha^2 f'^3 = C_0 = const. \tag{4.3}$$

A solution of the last equation for $C_0=0$ was found in [26, Section 4.2]. For ease of reading we repeat the key steps of the derivation in order to obtain a seven dimensional solution of the

Strominger system. The substitution $u = \alpha^{-2}e^{2f}$ allows us to write (4.3) in the form

$$\left(e^{2f}\right)' + \frac{3}{4}\alpha^2|A|^2\left(e^{-2f}\right)' - 2\alpha^2 f'^3 = \frac{\alpha^2 u'}{4u^3}\left(4u^3 - 3\frac{|A|^2}{\alpha^2}u - u'^2\right).$$

For $C_0 = 0$ we shall solve the following ordinary differential equation for the function $u = u(x^1) > 0$

$$u'^2 = 4u^3 - 3\frac{|A|^2}{\alpha^2}u = 4u\,(u-d)\,(u+d)\,, \qquad d = \sqrt{3|A|^2}/\alpha. \tag{4.4}$$

Replacing the real derivative with the complex derivative leads to the Weierstrass' equation

$$\left(\frac{d\,\mathcal{P}}{dz}\right)^2 = 4\mathcal{P}\,(\mathcal{P}-d)\,(\mathcal{P}+d) \tag{4.5}$$

for the doubly periodic Weierstrass $\mathcal{P}$ function with a pole at the origin. As well known, [21] and [1], near the origin $\mathcal{P}$ has the expansion

$$\mathcal{P}(z) = \frac{1}{z^2} + \frac{d^2}{5}z^2 + d_1 z^6 + \cdots\,,$$

which has no z^4 term and only even powers of z. Furthermore, see [21] and [1], letting $\tau_\pm$ be the basic half-periods such that τ_+ is real and τ_- is purely imaginary we have that $\mathcal{P}$ is real valued on the lines $\mathfrak{Re}\, z = m\tau_+$ or $\mathfrak{Im}\, z = im\tau_-$, $m \in \mathbb{Z}$. In the fundamental region centered at the origin, where $\mathcal{P}$ has a pole of order two, we have that $\mathcal{P}(z)$ decreases from $+\infty$ to a to 0 to $-a$ to $-\infty$ as z varies along the sides of the half-period rectangle from 0 to τ_+ to $\tau_+ + \tau_-$ to τ_- to 0.

Thus, $u(x^1) = \mathcal{P}(x^1)$ defines a non-negative $2\tau_+$-periodic function with singularities at the points $2n\tau_+$, $n \in \mathbb{Z}$, which solves the real equation (4.4). From the Laurent expansion of the Weierstrass' function it follows

$$u(x_1) = \frac{1}{(x^1)^2}\left(1 + \frac{d^2}{5}(x^1)^4 + \cdots\right).$$

By construction, $f = \frac{1}{2}\ln(\alpha^2 u)$ is a periodic function with singularities on the real line which is a solution to equation (4.2). Therefore the G_2 structure defined by $\bar\Theta$ descends to the 7-dimensional nilmanifold $M^7 = \Gamma\backslash K_A$ with singularity, determined by the singularity of u, where K_A is the 2-step nilpotent Lie group with Lie algebra $\mathfrak{K}_A$, defined by (3.2), and Γ is a lattice with the same period as f, i.e., $2\tau_+$ in all variables. In fact, as seen from the asymptotic behavior of u, M^7 is the total space of a $\mathbb{T}^3$ bundle over the asymptotically hyperbolic manifold M^4 with metric

$$\bar g_H = u(x^1)\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2\right),$$

which is a conformally compact 4-torus with conformal boundary at infinity a flat 3-torus. Thus, we conclude that there is a complete solution with non-constant dilaton, non-trivial instanton and flux and with a negative α' parameter.

The last statement follows from the fact that the $(-)$-connection is an instanton up to the first order of α'. This completes the proof of Theorem 4.3. □

From the apparent $\mathbb{Z}_2$-symmetry of u determined by the symmetry with respect to the line $x^1 = \tau_+$ we also obtain a solution on the quotient $M^7/\mathbb{Z}_2$.

5. A complete solution with positive α'

In this section we exhibit a solution of the Strominger system using again the G_2 structure (3.7) by solving the anomaly cancellation condition with torsion term (3.16), the Pontrjagin form (3.20) of the ∇^- connection, and the G_2-instanton defined with the help of Lemma 5.1.

As usual, the $(\pm)$-connections of the G_2 structure $\bar\Theta$ are defined by the formula $\nabla^\pm = \nabla^{\bar g} \pm \frac12 \bar T$, where $\nabla^{\bar g}$ is the Levi-Civita connection of the metric $\bar g$ and the torsion is determined in (3.15). The curvature of the connections $\nabla^\pm$ are denoted by $R^\pm$.

Lemma 5.1. *The $(-)$-connection of the G_2 structure $\bar\Theta$ is a G_2 instanton with respect to $\bar\Theta$ if and only if the torsion 3-form is closed, $d\bar T = 0$, i.e. the dilaton function f satisfies the equality*

$$\triangle e^{2f} + 2|A|^2 = 0. \tag{5.1}$$

Proof. Let $\{\bar e_1, \dots, \bar e_7\}$ be the orthonormal basis dual to $\{\bar e^1, \dots, \bar e^7\}$. Using (1.6) we investigate the G_2 instanton condition (1.5) for R^- as follows

$$\begin{aligned} 0 &= \textstyle\sum_{i,j=1}^7 R^-(\bar e_i, \bar e_j, \bar e_l, \bar e_m)\bar\Theta(\bar e_i, \bar e_j, \bar e_k) = \sum_{i,j=1}^7 \left[R^+ - d\bar T\right](\bar e_i, \bar e_j, \bar e_l, \bar e_m)\bar\Theta(\bar e_i, \bar e_j, \bar e_k) \\ &= -\textstyle\sum_{i,j=1}^7 d\bar T(\bar e_i, \bar e_j, \bar e_l, \bar e_m)\bar\Theta(\bar e_i, \bar e_j, \bar e_k), \end{aligned} \tag{5.2}$$

where we used the fact that the holonomy of ∇^+ is contained in G_2, i.e. $\sum_{i,j=1}^7 R^+(\bar e_i, \bar e_j, \bar e_l, \bar e_m)\bar\Theta(\bar e_i, \bar e_j, \bar e_k) = 0$. Now, applying (3.16) and (3.7) we conclude that (5.2) is satisfied if and only if (5.1) holds.

□

Let D_B be the ∇^- connection obtained by replacing A with the matrix B in Lemma 5.1, but allowing B to be singular, $B \in \mathfrak{gl}_3(\mathbb{R})$. Hence, the connection D_B is a G_2-instanton with respect to the G_2 structure defined by (3.7) iff the dilaton function satisfies

$$\triangle e^{2f} = -2|B|^2. \tag{5.3}$$

Equation (3.20) shows that the difference between the first Pontrjagin forms of ∇^- and D_B is given by the formula

$$8\pi^2\Big(p_1(\nabla^-) - p_1(\mathrm{D}_B)\Big) = -3\Big(|A|^2 - |B|^2\Big)\left(\triangle e^{-2f}\right) e^{1234}. \tag{5.4}$$

Therefore, recalling (3.16) and taking into account (5.4), the anomaly cancellation condition is

$$d\bar T - \frac{\alpha'}{4} 8\pi^2\Big(p_1(\nabla^-) - p_1(\mathrm{D}_B)\Big) = -\left[\triangle e^{2f} + 2|A|^2 - \frac34\alpha'\Big(|A|^2 - |B|^2\Big)\left(\triangle e^{-2f}\right)\right] e^{1234} = 0$$

coupled with (5.3). Notice that at this point the analysis can proceed exactly as in [26, Section 5.2]. As a result we obtain the following results depending on the $|A|^2 - |B|^2$ being zero or non-zero.

For $B = O$, where O is the zero matrix in $\mathfrak{gl}_3(\mathbb{R})$, and a fixed $e \in \mathbb{R}^4$ we let

$$e^{2f} = \frac{3\alpha'}{4|x-e|^2}, \quad x \in \mathbb{R}^4. \tag{5.5}$$

Using logarithmic radial coordinates near the singularity (as e.g. in [15]) it follows that the $4-D$ metric induced on $\mathbb{R}^4$ is actually complete. In fact, taking the singularity at the origin, in the coordinate $t = \sqrt{3\alpha'}/2 \, \ln\left(4|x|^2/3\alpha'\right) = -\sqrt{3\alpha'}\, f$, we have that the dilaton and the $4-D$ metric can be expressed as follows

$$f = -t\sqrt{3\alpha'}, \qquad \bar g_H = \sum_{i=1}^4 e^{2f}(e^i)^2 = dt^2 + 3\alpha' ds_3^2,$$

where ds_3^2 is the metric on the unit three-dimensional sphere in the four dimensional Euclidean space. The completeness of the horizontal metric implies that the metric $\bar g = \bar g_H + (e^5)^2 + (e^6)^2 + (e^7)^2$ is also complete. Thus, we proved

Theorem 5.2. *The non-compact complete simply connected manifold* $(K_A, \bar{\Theta}, \nabla^-, \mathrm{D}_O, f)$ *described above is a complete* G_2 *manifold which solves the Strominger system with non-constant dilaton* f *determined by* (5.5)*, non-zero flux* $H = \bar{T}$ *and non-flat instanton* D_O *using the first Pontrjagin form of* ∇^- *and positive* α'*. Furthermore,* $(K_A, \bar{\Theta}, \nabla^-, \mathrm{D}_O, f)$ *also solves the heterotic equations of motion* (1.2) *up to the first order of* α'*.*

On the other hand, in the case $|A|^2 = |B|^2 \neq 0$ the anomaly condition is trivially satisfied for any α', provided the torsion is closed, see Lemma 5.1. In this case the solution is given by the solutions of (5.1). Furthermore, both ∇^- and D_B are G_2-instantons. For example, a particular solution is obtained by taking

$$e^{2f} = \frac{|A|^2}{4}(1 - |x|^2)$$

defined in the unit ball.

6. Solutions through contractions

In this section we consider appropriate contractions of the quaternion Heisenberg algebra, the geometric structures, the partial differential equations and their solutions found in sections 4 and 5 in the G_2-heterotic case, and we show that they converge to the heterotic solutions on 6-dimensional inner non-Kähler spaces constructed in [26]. Furthermore, this method allows us to find new heterotic solutions with non-constant dilaton in dimension 5.

6.1. **Six dimensional solutions.** Using the classification results of [68] it was shown in [26], that the 2-step nilmanifolds which are $\mathbb{T}^2$ bundles over $\mathbb{T}^4$ with connection 1-forms of anti-self-dual curvature are precisely the invariant balanced Hermitian metrics with Abelian complex structure J, i.e., $[JX, JY] = [X, Y]$. Moreover, in such case the Lie algebra underlying M is isomorphic to $\mathfrak{h}_3$ or $\mathfrak{h}_5$. Here, $\mathfrak{h}_3$ is the Lie algebra underlying the nilmanifold given by the product of the 5-dimensional generalized Heisenberg nilmanifold by S^1, while $\mathfrak{h}_5$ is the Lie algebra underlying the Iwasawa manifold. The structure equations of the Lie algebra $\mathfrak{h}_5$ are

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = b\,\sigma_2, \quad de^6 = a\,\sigma_1 - b\,\sigma_3, \tag{6.1}$$

while $\mathfrak{h}_3$ is given by

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = 0, \quad de^6 = a\,\sigma_1,$$

where $a, b \in \mathbb{R}^*$ and σ_i are the anti-self-dual forms on $\mathbb{R}^4$, see after (3.2). Clearly $\mathfrak{h}_3$ is a contraction of $\mathfrak{h}_5$ and both are contractions of $\mathfrak{g}(\mathbb{H})$, see (3.2).

It is a remarkable fact that the geometric structures, the partial differential equations and their solutions found in sections 4 and 5 converge to the heterotic solutions on 6-dimensional inner non-Kähler spaces found in [26] as we explain next in details for $\mathfrak{h}_5$. The $SU(3)$ structure and corresponding solution based on $\mathfrak{h}_3$ is handled analogously.

Clearly $\mathfrak{h}_5$ is a contraction of $\mathfrak{K}_A$ when $\varepsilon \to 0$ using, for example,

$$A_\varepsilon \overset{def}{=} \begin{pmatrix} 0 & b & 0 \\ a & 0 & -b \\ 0 & 0 & \varepsilon \end{pmatrix}.$$

Notice that by (3.8) we have $\bar{e}^7 = e^7_\varepsilon = \varepsilon\gamma^7 \to 0$ as $\varepsilon \to 0$. With the above choice of A_ε we write the G_2-form (3.7) in the usual way as

$$\bar{\Theta}_\varepsilon = \bar{F} \wedge e^7_\varepsilon + \bar{\Psi}^+, \qquad \bar{F} = e^{2f}\omega_1 + e^{56}, \qquad \bar{\Psi}^+ = e^{2f}(\omega_2 \wedge e^5 - \omega_3 \wedge e^6)$$

using (3.8) and indicating with subscript ε the dependence on ε through the matrix A_ε. In addition, we let $\bar\Psi^- = e^{2f}(\omega_2\wedge e^6 + \omega^3\wedge e^5)$. In the limit $\varepsilon\to 0$, the forms $\bar F$, $\bar\Psi^\pm$ define an $SU(3)$ structure $(\bar F,\bar\Psi^\pm)$ on a six dimensional space, obtained through the ansatz proposed in [37] from a $\mathbb{T}^2$ bundle over $\mathbb{T}^4$ (corresponding to $f=0$), see [26, Section 3.2] for details in the case of $\mathfrak{h}_5$. Therefore, this $SU(3)$ structure solves the first two Killing spinor equations. Furthermore, the Pontrjagin form of the ∇^- connection is given again by (3.20) as shown in [26, Section 3]. In fact, the connection forms (3.8) and the corresponding curvature 2-forms (notice that $(\Omega^-_\varepsilon)^{\bar i}_{\bar 7}\to 0$ for all i) converge to those of the ∇^- connection of the $SU(3)$ case. Similarly, the seven dimensional anomaly cancellation conditions of Sections 4 and 5 turn into the anomaly cancellation conditions for the corresponding six dimensional structures. As a consequence we obtain the six-dimensional solutions with non-constant dilaton found in [26].

6.2. **Five dimensional solutions.** We begin with recalling the five dimensional Lie algebra $\mathfrak{h}(2,1)$ [23] with structure equations

$$de^j=0,\ j=1,2,3,4,\qquad de^5=\sum_{i=1}^3 a_i\,\sigma_i,\quad a_i\in\mathbb{R},\quad (a_1,a_2,a_3)\neq(0,0,0). \tag{6.2}$$

Without loss of generality we will suppose next that $a_1\neq 0$. Clearly $\mathfrak{h}(2,1)$ is a contraction of $\mathfrak{K}_A$, see (3.2), using, for example,

$$A_\varepsilon \overset{def}{=} \begin{pmatrix} a_1 & a_2 & a_3\\ 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon\end{pmatrix}$$

and letting $\varepsilon\to 0$. Notice that by (3.8) we have

$$\bar e^i = e^i_\varepsilon = \varepsilon\gamma^i \to 0,\quad i=6,7 \tag{6.3}$$

when $\varepsilon\to 0$.

It was shown in [23, Section 4] that the $SU(2)$-structure $(e^5,\omega_1,\omega_2,\omega_3)$ is the unique family of left invariant solutions (with constant dilaton) to the first two Killing spinor equations on a five dimensional Lie group. Furthermore, [23] continued on showing that for $\nabla=\nabla^+$ or $\nabla=\nabla^g$ and suitably defined instantons one can obtain compact (nilmanifolds) heterotic solutions with constant dilaton. However, since the first Pontrjagin form of the connection ∇^- vanishes there is no compact solution with constant dilaton to the heterotic supersymmetry equations satisfying the anomaly cancellation condition with $\nabla=\nabla^-$.

From the current point of view, we consider the case $\nabla=\nabla^-$ as a contraction limit of the G_2-solutions in Sections 4 and 5. As a result we will obtain five dimensional solutions with non-constant dilaton. Indeed, applying (6.3) and (6.2) in (3.15) we obtain the expression for the torsion in dimension five described in (2.3). In other words, the torsion in dimension five is obtained as a dimensional reduction of the torsion in dimension seven. Furthermore, the Pontrjagin form of the ∇^- connection is given again by (3.20) taking into account (6.2). In fact, the connection forms (3.8) and the corresponding curvature 2-forms (notice that $(\Omega^-_\varepsilon)^{\bar i}_{\bar 6}\to 0$ and $(\Omega^-_\varepsilon)^{\bar i}_{\bar 7}\to 0$ for all i) converge to those of the ∇^- connection of the $SU(2)$ structure in dimension five. Similarly, the seven dimensional anomaly cancellation conditions of Sections 4 and 5 turn into the anomaly cancellation conditions for the corresponding five dimensional structures. At this point we turn to the construction of the five dimensional solutions with non-constant dilaton.

The five dimensional version of Theorem 4.3 is Theorem 6.2 below. In the statement of Theorem 6.2 we use (3.9), i.e., $\bar\omega_i = e^{2f}\omega_i$, $i=1,2,3,4$. Let $H(2,1)$ be the five dimensional connected simply connected Lie group $H(2,1)$ with Lie algebra $\mathfrak{h}(2,1)$, We consider a lattice Γ in the Lie group $H(2,1)$ with period $2\tau_+$ in all variables, where $2\tau_+$ is the period of the Weirstrass' $\mathcal{P}$ function (4.5).

The $SU(2)$ instanton D_Λ below corresponds to the instanton obtained from the one in Proposition 4.1 by setting the last two columns equal to zero or letting $\varepsilon \to 0$, see (6.3).

Lemma 6.1. *Let* D_Λ, $\Lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$, *be the linear connection on the Lie group* $H(2,1)$ *whose possibly non-zero 1-forms are given as follows*

$$(\omega^{\mathrm{D}_\Lambda})^{\bar 1}_{\bar 2} = -(\omega^{\mathrm{D}_\Lambda})^{\bar 2}_{\bar 1} = -(\omega^{\mathrm{D}_\Lambda})^{\bar 3}_{\bar 4} = (\omega^{\mathrm{D}_\Lambda})^{\bar 4}_{\bar 3} = \lambda_1\,\bar e^5,$$
$$(\omega^{\mathrm{D}_\Lambda})^{\bar 1}_{\bar 3} = -(\omega^{\mathrm{D}_\Lambda})^{\bar 3}_{\bar 1} = (\omega^{\mathrm{D}_\Lambda})^{\bar 2}_{\bar 4} = -(\omega^{\mathrm{D}_\Lambda})^{\bar 4}_{\bar 2} = \lambda_2\,\bar e^5,$$
$$(\omega^{\mathrm{D}_\Lambda})^{\bar 1}_{\bar 4} = -(\omega^{\mathrm{D}_\Lambda})^{\bar 4}_{\bar 1} = -(\omega^{\mathrm{D}_\Lambda})^{\bar 2}_{\bar 3} = (\omega^{\mathrm{D}_\Lambda})^{\bar 3}_{\bar 2} = \lambda_3\,\bar e^5.$$

Then, D_Λ *is an* $SU(2)$*-instanton with respect to the* $SU(2)$ *structure defined by* $(e^5, \bar\omega_1, \bar\omega_2, \bar\omega_3)$.

We skip the proof which is similar to the proof of Proposition 4.1. The five dimensional version of Theorem 4.3 follows.

Theorem 6.2. *Let* $(e^5, \bar\omega_1, \bar\omega_2, \bar\omega_3)$ *be the* $SU(2)$ *structure on the Lie group* $H(2,1)$. *The conformally compact five manifold* $M^5 = (\Gamma\backslash H(2,1), \eta^5, \bar\omega_1, \bar\omega_2, \bar\omega_3, \nabla^-, D_\Lambda, f)$ *is a conformally quasi-Sasakian five manifold which solves the Strominger system with non-constant dilaton* f, *non-trivial flux* $H = \bar T$ *and non-flat instanton* D_Λ *using the first Pontrjagin form of* ∇^- *and negative* α'. *The dilaton* f *depends on one variable and is determined as a real slice of the Weierstrass' elliptic function. In addition,* M^5 *satisfies the heterotic equations of motion* (1.2) *up to first order of* α'.

In order to obtain the five dimensional version of Theorem 5.2 we use the following property of the ∇^- connection whose 1-forms are

$$(\omega^-)^{\bar 1}_{\bar 2} = (\omega^-)^{\bar 3}_{\bar 4} = e^{-f}\left(f_2\,\bar e^1 - f_1\,\bar e^2 + f_4\,\bar e^3 - f_3\,\bar e^4\right),$$
$$(\omega^-)^{\bar 1}_{\bar 3} = -(\omega^-)^{\bar 2}_{\bar 4} = e^{-f}\left(f_3\,\bar e^1 - f_4\,\bar e^2 - f_1\,\bar e^3 + f_2\,\bar e^4\right),$$
$$(\omega^-)^{\bar 1}_{\bar 4} = (\omega^-)^{\bar 2}_{\bar 3} = e^{-f}\left(f_4\,\bar e^1 + f_3\,\bar e^2 - f_2\,\bar e^3 - f_1\,\bar e^4\right),$$
$$(\omega^-)^{\bar 1}_{\bar 5} = e^{-2f}\left(-a_{11}\,\bar e^2 - a_{12}\,\bar e^3 - a_{13}\,\bar e^4\right),\quad (\omega^-)^{\bar 2}_{\bar 5} = e^{-2f}\left(a_{11}\,\bar e^1 + a_{13}\,\bar e^3 - a_{12}\,\bar e^4\right),$$
$$(\omega^-)^{\bar 3}_{\bar 5} = e^{-2f}\left(a_{12}\,\bar e^1 - a_{13}\,\bar e^2 + a_{11}\,\bar e^4\right),\quad (\omega^-)^{\bar 4}_{\bar 5} = e^{-2f}\left(a_{13}\,\bar e^1 + a_{12}\,\bar e^2 - a_{11}\,\bar e^3\right),$$

which are obtained from (3.19) taking into account (6.3).

Lemma 6.3. *The* $(-)$*-connection of the* $SU(2)$ *structure* $(e^5, \bar\omega_1, \bar\omega_2, \bar\omega_3)$ *is an* $SU(2)$ *instanton iff the torsion 3-form is closed,* $d\bar T = 0$, *i.e., the dilaton function* f *satisfies equation* (5.1).

The proof of Lemma 6.3 is very similar to the proof of Lemma 5.1 and involves a direct calculation. Let D_O be the $SU(2)$ instanton constructed by Lemma 6.3 in the case $A = O$-the zero:

Theorem 6.4. *The non-compact simply connected five manifold* $(H(2,1), e^5, \bar\omega_1, \bar\omega_2, \bar\omega_3, \nabla^-, D_O, f)$ *is a complete conformally quasi-Sasakian five manifold which solves the Strominger system with non-constant dilaton* f *determined by* (5.5), *non-trivial flux* $H = \bar T$, *non-flat instanton* D_O *using the first Pontrjagin form of* ∇^- *and positive* α'.

The complete five manifold $(H(2,1), e^5, \bar\omega_1, \bar\omega_2, \bar\omega_3, \nabla^-, D_O, f)$ *satisfies the heterotic equations of motion* (1.2) *up to first order of* α'.

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