Thus, we may regard R_{BC} on H as an endomorphism of H and we have $R_{BC} \in sp(n) \oplus sp(1)$.

As usual, we write $R(A, B, C, D) = g(R_{A,B}C, D)$.

Definition 4.3.1. The Ricci 2-forms ρ_i are defined by

$$\rho_i(B, C) = \frac{1}{4n} R(B, C, e_a, I_i e_a).$$

We decompose the curvature on H into $sp(n) \oplus sp(1)$-parts. Let $R^0_{BC} \in sp(n)$ denote the $sp(n)$-component.

Lemma 4.3.2.

a) The curvature of the Biquard connection decomposes on H as follows

$$R_{BC}X = R^0_{BC}X + \rho_1(B, C)I_1X + \rho_2(B, C)I_2X + \rho_3(B, C)I_3X,$$

$$R_{BC}I_iX - I_iR_{BC}X = 2(-\rho_j(B, C)I_kX + \rho_k(B, C)I_jX), \tag{4.60}$$

$$\rho_i(B, C) = \frac{1}{2}(d\alpha_i + \alpha_j \wedge \alpha_k)(B, C), \tag{4.61}$$

where the connection 1-forms α_s are determined in (4.16), (4.36) *and* (4.42).

b) The curvature of the Biquard connection on V is determined by

$$R(B, C, \xi_i, \xi_j) = 2\rho_k(B, C). \tag{4.62}$$

Proof. The first two identities follow directly from the definitions. Using (4.10), we calculate that on H we have

$$\begin{aligned} R_{BC}I_i - I_iR_{BC} &= \nabla_B\nabla_C I_i - \nabla_c\nabla_B I_i - \nabla_{[B,C]}I_i \\ &= \nabla_B(\alpha_k(C)I_j - \alpha_j(C)I_k) - \nabla_C(\alpha_k(B)I_j - \alpha_j(B)I_k) \\ &\qquad - (\alpha_k([B, C])I_j - \alpha_j([B, C])I_k) \\ &= -(d\alpha_j + \alpha_k \wedge \alpha_i)(B, C)I_k + (d\alpha_k + \alpha_i \wedge \alpha_j)(B, C)I_j. \end{aligned}$$

Now (4.61) follows from (4.60).

Similarly, using (4.10) and (4.61), we obtain

$$\begin{aligned} R(B, C)\xi_i &= -(d\alpha_j + \alpha_k \wedge \alpha_i)(B, C)\xi_k + (d\alpha_k + \alpha_i \wedge \alpha_j)(B, C)\xi_j \\ &= -2\rho_j(B, C)\xi_k + \rho_k(B, C)\xi_j. \end{aligned}$$

□

Definition 4.3.3. The quaternionic contact Ricci tensor (*qc-Ricci tensor* for short) and the qc-scalar curvature *Scal* of the Biquard connection are defined by

$$Ric(B, C) = R(e_a, B, C, e_a), \qquad Scal = Ric(e_a, e_a). \tag{4.63}$$

Proof. Since ∇ preserves the splitting $H \oplus V$, the first Bianchi identity (4.66) and (4.62) together with (4.11) and (4.34) imply

$$\begin{aligned}
2\rho_i(X,Y) &= R(X,Y,\xi_j,\xi_k) = b(X,Y,\xi_j,\xi_k) = (\nabla_{\xi_j}T)(X,Y,\xi_k) \\
&\quad + T(T(X,Y),\xi_j),\xi_k) + T(T(Y,\xi_j),X),\xi_k) + T(T(\xi_j,X),Y),\xi_k) \\
&= 2(\nabla_{\xi_j}\omega_k)(X,Y) - 2T(X,Y,\nabla_{\xi_j}\xi_k) \\
&\quad + 2\omega_i(X,Y)T(\xi_i,\xi_j,\xi_k) + 2\omega_k(T(\xi_j,X),Y) - 2\omega_k(T(\xi_j,Y),X), \qquad (4.77)
\end{aligned}$$

where we used the fact that $T(\xi_s,X)$ is a horizontal vector field, $T(\xi,X) \in H$ to conclude the vanishing of terms of the type $(\nabla_A T)(X,\xi_j,\xi_k)$.

Applying (4.10) and (4.11) we deduce that the third line in (4.77) vanishes because

$$T(X,Y,\nabla_{\xi_j}\xi_k) = -\alpha_i(\xi_j)\omega_j(X,Y) + \alpha_j(\xi_j)\omega_i(X,Y) = (\nabla_{\xi_j}\omega_k)(X,Y).$$

With the help of (4.41), (4.20), Lemma 4.2.6 and Lemma 4.2.7 we obtain from (4.77) that

$$\begin{aligned}
\rho_i(X,Y) &= -\lambda\omega_i(X,Y) - T(\xi_j,X,I_kY) + T(\xi_j,Y,I_kX) \\
&= -\lambda\omega_i(X,Y) - 2U(I_iX,Y) - T^0(\xi_j,X,I_kY) + T^0(\xi_j,Y,I_kX) \\
&= -\lambda\omega_i(X,Y) - 2U(I_iX,Y) + \frac{1}{2}(T^0(X,I_iY) - T^0(I_iX,Y)). \qquad (4.78)
\end{aligned}$$

In view of (4.60), we have

$$\begin{aligned}
Ric(B,I_{\cancel{s}}Y) \ + \ 4n\zeta_{\cancel{s}}(B,Y) \\
&= R(e_a,B,I_{\cancel{s}}Y,e_a) \ + \ R(e_a,B,Y,I_{\cancel{s}}e_a) \\
&= -2\rho_j(e_a,B)\omega_k(Y,e_a) \ + \ 2\rho_k(e_a,B)\omega_j(Y,e_a)) \\
&= 2\rho_j(B,I_kY) \ - \ 2\rho_k(B,I_jY). \qquad (4.79)
\end{aligned}$$

Consequently, applying (4.78), we derive from (4.79) and the properties of the torsion listed in Lemmas 4.2.6 and 4.2.7 that

$$\begin{aligned}
Ric(X,I_sY) \ + \ 4n\zeta_s(X,Y) &= -4\lambda\omega_s(X,Y) \\
&\quad + 4U(X,I_sY) + \frac{1}{2}\Big(T^0(X,I_sY) + T^0(I_sX,Y)\Big). \qquad (4.80)
\end{aligned}$$

The first Bianchi identity (4.66) implies

$$\begin{aligned}
&4n(\tau_s(X,Y) + 2\zeta_s(X,Y)) \\
&\quad = R(e_a,I_se_a,X,Y) + R(X,e_a,I_se_a,Y) + R(I_se_a,X,e_a,Y) \\
&\quad = b(e_a,I_se_a,X,Y). \qquad (4.81)
\end{aligned}$$

Theorem 4.3.11. *On a QC manifold the curvature of the Biquard connection satisfies the equalities:*

$$
\begin{aligned}
&R(X,Y,Z,V) - R(Z,V,X,Y) \\
&\qquad = 2\sum_{s=1}^{3}\Big[\omega_s(X,Y)U(I_sZ,V) - \omega_s(Z,V)U(I_sX,Y)\Big] \\
&\qquad - 2\sum_{s=1}^{3}\Big[\omega_s(X,Z)T^0(\xi_s,Y,V) + \omega_s(Y,V)T^0(\xi_s,Z,X)\Big] \\
&\qquad + 2\sum_{s=1}^{3}\Big[\omega_s(Y,Z)T^0(\xi_sX,V) + \omega_s(X,V)T^0(\xi_s,Z,Y)\Big]. \qquad (4.103)
\end{aligned}
$$

The [3]-componenet of the horizontal curvature with respect to the first two arguments is given by

$$
\begin{aligned}
&3R(X,Y,Z,V) - \sum_{s=1}^{3} R(I_sX,I_sY,Z,V)) \\
&\qquad = 2\Big[g(Y,Z)T^0(X,V) + g(X,V)T^0(Z,Y)\Big] \\
&\qquad - 2\Big[g(Z,X)T^0(Y,V) + g(V,Y)T^0(Z,X)\Big] \\
&\qquad - 2\sum_{s=1}^{3}\Big[\omega_s(Y,Z)T^0(X,I_sV) + \omega_s(X,V)T^0(\cancel{Z},I_s\cancel{Y})\Big] \\
&\qquad + 2\sum_{s=1}^{3}\Big[\omega_s(\cancel{Z},\cancel{X})T^0(Y,I_sV) + \omega_s(\cancel{V},\cancel{Y})T^0(\cancel{Z},I_s\cancel{X})\Big] \\
&+ \sum_{s=1}^{3}\Big[2\omega_s(X,Y)\Big(T^0(Z,I_sV) - T^0(I_sZ,V)\Big) - 8\omega_s(Z,V)U(I_sX,Y)\Big] \\
&\qquad - \frac{Scal}{2n(n+2)}\sum_{s=1}^{3}\omega_s(X,Y)\omega_s(Z,V). \qquad (4.104)
\end{aligned}
$$

In coordinates, with the obvious notation, the multiplication formula is

$$\begin{aligned}
t'^a &= t^a + t^a_o, \quad x'^a = x^a + x^a_o, \\
y'^a &= y^a + x^a_o, \quad z'^a = z^a + z^a_o, \\
x' &= x + x_o + 2(x^a_o t^a - t^a_o x^a + z^a_o y^a - y^a_o z^a) \\
y' &= y + y_o + 2(y^a_o t^a - z^a_o x^a - t^a_o y^a + x^a_o z^a) \\
z' &= z + z_o + 2(z^a_o t^a + y^a_o x^a - x^a_o y^a - t^a_o z^a).
\end{aligned} \tag{4.123}$$

A basis of left invariant horizontal vector fields

$$T_a, \quad X_a = I_1 T_a, \quad Y_a = I_2 T_a, \quad Z_a = I_3 T_a$$

is given by

$$\begin{aligned}
T_a &= \frac{\partial}{\partial t_\alpha} + 2x^a \frac{\partial}{\partial x} + 2y^a \frac{\partial}{\partial y} + 2z^a \frac{\partial}{\partial z} \\
X_a &= \frac{\partial}{\partial x_\alpha} - 2t^a \frac{\partial}{\partial x} - 2z^a \frac{\partial}{\partial y} + 2y^a \frac{\partial}{\partial z} \\
Y_a &= \frac{\partial}{\partial y_\alpha} + 2z^a \frac{\partial}{\partial x} - 2t^a \frac{\partial}{\partial y} - 2x^a \frac{\partial}{\partial z} \\
Z_a &= \frac{\partial}{\partial z_\alpha} - 2y^a \frac{\partial}{\partial x} + 2x^a \frac{\partial}{\partial y} - 2t^a \frac{\partial}{\partial z}.
\end{aligned} \tag{4.124}$$

The central (vertical) vector fields ξ_1, ξ_2, ξ_3 are described as follows

$$\xi_1 = 2\frac{\partial}{\partial x} \quad \xi_2 = 2\frac{\partial}{\partial y} \quad \xi_3 = 2\frac{\partial}{\partial z}. \tag{4.125}$$

A small calculation shows the following commutator relations

$$[I_j T_a, T_a] = 2\xi_j \qquad [I_j T_a, I_i T_a] = 2\xi_k. \tag{4.126}$$

The standard quaternionic contact form $\tilde{\Theta} = (\tilde{\Theta}_1, \tilde{\Theta}_2, \tilde{\Theta}_3)$ is

$$2\tilde{\Theta} = d\omega - \vec{q}' \cdot d\vec{q}' + d\vec{q}' \cdot \vec{q}', \tag{4.127}$$

which, in coordinates reads

$$\begin{aligned}
\tilde{\Theta}_1 &= \frac{1}{2}\, dx - x^a dt^a + t^a dx^a - z^a dy^a + y^a dz^a \\
\tilde{\Theta}_2 &= \frac{1}{2}\, dy - y^a dt^a + z^a dx^a + t^a dy^a - x^a dz^a \\
\tilde{\Theta}_3 &= \frac{1}{2}\, dz - z^a dt^a - y^a dx^a + x^a dy^a + t^a dz^a.
\end{aligned}$$

The Biquard connection coincides with the flat left-invariant connection on $\boldsymbol{G}(\mathbb{H})$ and the described horizontal and vertical vector fields are parallel