QUATERNIONIC CONTACT MANIFOLDS WITH A CLOSED FUNDAMENTAL 4-FORM

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Abstract. We show that the fundamental 4-form on a quaternionic contact manifold of dimension at least eleven is closed if and only if the torsion endomorphism of the Biquard connection vanishes. This condition characterizes quaternionic contact structures which are locally qc homothetic to 3-Sasakian structures.

1. Introduction

A quaternionic contact (qc) structure, introduced in [4, 5], appears naturally as the conformal boundary at infinity of the quaternionic hyperbolic space. Such structures have been considered in connection with the quaternionic contact Yamabe problem, [25, 15, 16]. A particular case of this problem amounts to finding the extremals and the best constant in the L^2 Folland-Stein Sobolev-type embedding, [10] and [11], on the quaternionic Heisenberg group, see [12] and[16].

A qc structure on a real (4n+3)-dimensional manifold M is a codimension three distribution H, called the horizontal space, locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$, such that, the three 2-forms $d\eta_i|_H$ are the fundamental 2-forms of a quaternionic structure on H. The 1-form η is determined up to a conformal factor and the action of $SO(3)$ on $\mathbb{R}^3$. Therefore H is equipped with a conformal class $[g]$ of Riemannian metrics and a 2-sphere bundle of almost complex structures, the quaternionic bundle $\mathbf{Q}$. The 2-sphere bundle of one forms determines uniquely the associated metric and a conformal change of the metric is equivalent to a conformal change of the one forms. To every metric in the fixed conformal class one can associate a complementary to H distribution V spanned by the Reeb vector fields ξ_1, ξ_2, ξ_3 and a linear connection ∇ preserving the qc structure and the splitting $TM = H \oplus V$ provided $n > 1$ [4]. This connection is known as the Biquard connection. The qc Ricci tensor, the qc scalar curvature $Scal$ of the Biquard connection are obtained from the curvature tensor by taking horizontal traces.

The transformations preserving a given qc structure η, i.e. $\bar\eta = \mu\Psi \cdot \eta$ for a positive smooth function μ and a $SO(3)$ matrix Ψ with smooth functions as entries, are called *quaternionic contact conformal (qc conformal) transformations.* If the function μ is constant we have *quaternionic contact homothetic (qc homothetic) transformations.* The Biquard connection is invariant under qc homothetic transformations.

Examples of qc manifolds can be found in [4, 5, 15, 9]. In particular, any totally umbilic hypersurface of a quaternionic Kähler or hyperkähler manifold carries such a structure. An extensiveley studied class of examples of quaternionic contact structures are provided by the 3-Sasakian manifolds. The latter can be defined as $(4n+3)$-dimensional (pseudo) Riemannian manifold of signature either (4n+3,0) or (4n,3) whose Riemannian cone is a hyperkähler manifold of signature (4n+4,0) or (4n,4), respectively. It was shown in [15] that the torsion endomorphism of the Biquard connection is the obstruction for a given qc-structure to be locally qc homothethic to a 3-Sasakian one provided

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the qc scalar curvature $Scal$ is not identically zero. Explicit examples of qc manifolds with zero or non-zero torsion endomorphism were recently given in [2]. The quaternionic Heisenberg group, the quaternionic sphere of dimension $4n+3$ with its standard 3-Sasakian structure and the qc structures locally qc conformal to them are characterized in [17] by the vanishing of a tensor invariant, the qc-conformal curvature defined in terms of the curvature and torsion of the Biquard connection. Explicit examples of non-qc conformally flat qc manifolds are constructed in [2].

In this article we consider the 4-form Ω defining the $Sp(n)Sp(1)$ structure on the horizontal distribution and call it *the fundamental four-form.*

The purpose of the paper is to show that when the dimension of the manifold is greater than seven, the fundamental 4-form form is closed if and only if the qc structure is locally qc homothetic to a 3-Sasakian one provided the qc scalar curvature does not vanish. We prove the following main result.

Theorem 1.1. *Let $(M^{4n+3},\eta,\mathbf{Q})$ be a $4n+3$-dimensional qc manifold. For $n>1$ the following conditions are equivalent*

i) The fundamental four form is closed, $d\Omega=0$;
ii) The torsion endomorphism of the Biquard connection vanishes;
iii) Each Reeb vector field ξ_l, defined in (2.3), preserves the fundamental four form, $\mathbb{L}_{\xi_l}\Omega=0$.

Any of the above conditions imply that the qc scalar curvature is constant and the vertical distribution is integrable.

Combining the last Theorem with Theorem 1.3 and Theorem 7.11 in [15] we obtain

Theorem 1.2. *Let $(M^{4n+3},\eta,\mathbf{Q})$ be a $4n+3$-dimensional qc manifold. For $n>1$ the following conditions are equivalent*

a) $(M^{4n+3},\eta,\mathbf{Q})$ has closed fundamental four form, $d\Omega=0$;
b) The torsion endomorphism of the Biquard connection vanishes;
c) $(M^{4n+3},g,\mathbf{Q})$ is a qc-Einstein manifold (the trace-free part of the qc Ricci tensor is zero);
d) Each Reeb vector ξ_l field preserves the horizontal metric and the quaternionic structure simultaneously, $\mathbb{L}_{\xi_l}g=0, \quad \mathbb{L}_{\xi_l}\mathbf{Q}\subset\mathbf{Q}$;
e) Each Reeb vector field ξ_l preserves the fundamental four form, $\mathbb{L}_{\xi_l}\Omega=0$.

If in addition the qc scalar curvature is non-zero, $Scal\neq 0$, then each of a), b), c), d) and e) is equivalent to the following condition f).

f) M^{4n+3} is locally qc homothetic to a 3-Sasakian manifold, i.e., locally, there exists a $SO(3)$-matrix Ψ with smooth entries depending on an auxiliary parameter, such that, the local qc structure $(\epsilon\frac{Scal}{16n(n+2)}\Psi\cdot\eta,\mathbf{Q})$, $\epsilon=sign(Scal)$ is 3-Sasakian.

As an application of Theorem 1.1 we give in the last section a proof of the equivalence of a) and f) in Theorem 1.2. Thus, when the dimension of the qc manifold is greater than seven, we establish in a slightly different manner Theorem 3.1 in [15].

Remark 1.3. *On a seven dimensional qc manifold, if the torsion endomorphism of the Biquard connection vanishes then the fundamental four form is closed. We do not know whether the converse holds or if there exists an example of a seven dimensional qc manifold with a closed fundamental four form and a non-vanishing torsion endomorphism. This might be related to the well known fact that in dimension eight an almost quaternion hermitian structure with a closed fundamental four form is not necessarily quaternionic Kähler since Salamon* [23] *gave a compact counter-example (see* [2] *for non-compact counter-examples).*

Organization of the paper. The paper relies heavily on the notion of Biquard connection introduced in [4] and the properties of its torsion and curvature discovered in [15]. In order to make the present paper self-contained, in Section 2 we give a review of the notion of a quaternionic contact structure and collect formulas and results from [4] and [15] that will be used.

Convention 1.4. *a) We shall use X, Y, Z, U to denote horizontal vector fields, i.e. $X, Y, Z, U \in H$;*
b) $\{e_1, \dots, e_{4n}\}$ denotes an orthonormal basis of the horizontal space H;
c) The triple (i, j, k) denotes any cyclic permutation of $(1, 2, 3)$.
d) l and m will be any numbers from the set $\{1, 2, 3\}$, $\quad l, m \in \{1, 2, 3\}$.

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2. Quaternionic contact manifolds

In this section we will briefly review the basic notions of quaternionic contact geometry and recall some results from [4] and [15]. For the purposes of this paper, a quaternionic contact (qc) manifold $(M, g, \mathbf{Q})$ is a $4n + 3$ dimensional manifold M with a codimension three distribution H equipped with a metric g and an Sp(n)Sp(1) structure, i.e., we have

i) a 2-sphere bundle $\mathbf{Q}$ over M of almost complex structures $I_l : H \to H$, $\quad I_l^2 = -1$, satisfying the commutation relations of the imaginary quaternions $I_1 I_2 = -I_2 I_1 = I_3$ and $\mathbf{Q} = \{aI_1 + bI_2 + cI_3 : \ a^2 + b^2 + c^2 = 1\}$;

ii) H is locally the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ satisfying the compatibility condition $\quad 2g(I_l X, Y) = d\eta_l(X, Y)$.

The fundamental 2-forms ω_l of the quaternionic structure $\mathbf{Q}$ are determined by

$$2\omega_{l|H} = d\eta_{l|H}, \qquad \xi \lrcorner \omega_l = 0, \quad \xi \in V. \tag{2.1}$$

The 4-form Ω defining the $Sp(n)Sp(1)$ structure on the horizontal distribution, called here the fundamental four-form, is defined (globally) on the horizontal distribution H by

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3. \tag{2.2}$$

On a quaternionic contact manifold there exists a canonical connection defined in [4] when the dimension $(4n + 3) > 7$, and in [8] in the 7-dimensional case.

Theorem 2.1. [4] *Let $(M, g, \mathbf{Q})$ be a quaternionic contact manifold of dimension $4n + 3 > 7$ and a fixed metric g on H in the conformal class $[g]$. Then there exists a unique connection ∇ with torsion T on M^{4n+3} and a unique supplementary subspace V to H in TM, such that:*

i) ∇ preserves the decomposition $H \oplus V$ and the $Sp(n)Sp(1)$-structure on H;
ii) for $X, Y \in H$, one has $T(X, Y) = -[X, Y]_{|V}$;
iii) for $\xi \in V$, the endomorphism $T(\xi, .)_{|H}$ of H lies in $(sp(n) \oplus sp(1))^{\perp} \subset gl(4n)$;

We shall call the above connection *the Biquard connection*. Biquard [4] also described the supplementary subspace V. Locally, V is generated by vector fields $\{\xi_1, \xi_2, \xi_3\}$, such that

$$\eta_l(\xi_k) = \delta_{lk}, \qquad (\xi_l \lrcorner d\eta_l)_{|H} = 0, \quad (\xi_l \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_l)_{|H}. \tag{2.3}$$

The vector fields ξ_1, ξ_2, ξ_3 are called Reeb vector fields or fundamental vector fields. If the dimension of M is seven, the conditions (2.3) do not always hold. Duchemin shows in [8] that if we assume, in addition, the existence of Reeb vector fields as in (2.3), then Theorem 2.1 holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2.3).

The torsion endomorphism $T_\xi = T(\xi, .) : H \to H, \quad \xi \in V$, plays an important role in the qc geometry. Decomposing the endomorphism $T_\xi \in (sp(n) + sp(1))^\perp$ into symmetric part T^0_ξ and skew-symmetric part b_ξ, $T_\xi = T^0_\xi + b_\xi$ Biquard shows in [4] that T_ξ is completely trace-free, $tr\, T_\xi = tr\, T_\xi \circ I = 0, I \in \mathbf{Q}$ and describes the properties of the two components. Using the two $Sp(n)Sp(1)$-invariant trace-free symmetric 2-tensors T^0 and U on H defined in [15] by

$$T^0(X,Y) \overset{def}{=} g((T^0_{\xi_1}I_1 + T^0_{\xi_2}I_2 + T^0_{\xi_3}I_3)X,Y), \quad U(X,Y) \overset{def}{=} -g(I_l b_{\xi_l}X,Y),$$

the properties of T_ξ outlined in [4] give the following identities, cf. [15],

$$T^0(X,Y) + T^0(I_1X,I_1Y) + T^0(I_2X,I_2Y) + T^0(I_3X,I_3Y) = 0, \tag{2.4}$$

$$3U(X,Y) - U(I_1X,I_1Y) - U(I_2X,I_2Y) - U(I_3X,I_3Y) = 0. \tag{2.5}$$

If $n = 1$ then U vanishes identically, $U = 0$, and the torsion is a symmetric tensor, $T_\xi = T^0_\xi$.

The covariant derivatives with respect to the Biquard connection of the almost complex structures and the vertical vectors are given by

$$\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j, \qquad \nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j. \tag{2.6}$$

It turns out that the vanishing of the $sp(1)$-connection 1-forms on H is equivalent to the vanishing of the torsion endomorphism of the Biquard connection, $T^0 = U = 0$ [15].

The first equation in (2.6) together with (2.2) imply that the fundamental four form is parallel with respect to ∇, $\nabla\Omega = 0$ but it may not be closed because of the torsion of the Biquard connection.

Let $R = [\nabla,\nabla] - \nabla_{[\ ,\ ]}$ be the curvature tensor of ∇. We denote the curvature tensor of type (0,4) by the same letter, $R(A,B,C,D) := g(R(A,B)C,D)$, $A,B,C,D \in \Gamma(TM)$. The *qc-Ricci forms* and *qc-scalar curvature* are defined by $4n\rho_l(A,B) = \sum_{a=1}^{4n} R(A,B,e_a,I_le_a)$, $\quad Scal = \sum_{a,b=1}^{4n} R(e_b,e_a,e_a,e_b)$, respectively. It was shown in [15] that the $sp(1)$-part of R is determined by the Ricci 2-forms and the connection 1-forms by

$$R(A,B,\xi_i,\xi_j) = 2\rho_k(A,B) = (d\alpha_k + \alpha_i \wedge \alpha_j)(A,B), \qquad A,B \in \Gamma(TM). \tag{2.7}$$

It is important to note that the horizontal part of the Ricci 2-forms can be expressed in terms of the torsion of the Biquard connection [15]. For ease of reading, we collect the necessary facts from [15, Theorem 1.3, Theorem 3.12, Corollary 3.14, Proposition 4.3 and Proposition 4.4] with slight modifications, using the equality $4T^0(\xi_l,I_lX,Y) = T^0(X,Y) - T^0(I_lX,I_lY)$, and present them in the form described in [17].

Theorem 2.2. [15] *On a $(4n+3)$-dimensional qc manifold, $n>1$ the next formulas hold*

$$\begin{aligned}
\rho_l(X,I_lY) &= -\frac{1}{2}\Big[T^0(X,Y) + T^0(I_lX,I_lY)\Big] - 2U(X,Y) - \frac{Scal}{8n(n+2)}g(X,Y),\\
T(\xi_i,\xi_j) &= -\frac{Scal}{8n(n+2)}\xi_k - [\xi_i,\xi_j]_H, \qquad Scal = -8n(n+2)g(T(\xi_1,\xi_2),\xi_3)\\
T(\xi_i,\xi_j,X) &= -\rho_k(I_iX,\xi_i) = -\rho_k(I_jX,\xi_j), \qquad \rho_i(\xi_i,\xi_j) + \rho_k(\xi_k,\xi_j) = \frac{1}{16n(n+2)}\xi_j(Scal);\\
\rho_i(X,\xi_i) &= -\frac{X(Scal)}{32n(n+2))} + \frac{1}{2}\left(-\rho_i(\xi_j,I_kX) + \rho_j(\xi_k,I_iX) + \rho_k(\xi_i,I_jX)\right).
\end{aligned}$$

In particular, the vanishing of the horizontal trace-free part of the Ricci forms is equivalent to the vanishing of the torsion endomorphism of the Biquard connection. In this case the vertical distribution is integrable, the qc scalar curvature is constant and if $Scal \neq 0$ then the qc-structure is

3-Sasakian up to a multiplication by a constant and an $SO(3)$-matrix with smooth entries depending on an auxiliary parameter.

For the last part of the above Theorem we have adopted the definition that a $4n+3$-dimensional (pseudo) Riemannian manifold (M, g_M) of signature either $(4n+3,0)$ or $(4n,3)$ has a *3-Sasakian structure* if the cone metric $t^2 g_M + \epsilon dt^2$ on $M \times \mathbb{R}$ is a hyperkähler metric of signature $(4n+4,0)$ (*positive 3-Sasakian structure,* $\epsilon = 1$) or (4n,4) (*negative 3-Sasakian structure,* $\epsilon = -1$), respectively, see [21, 24, 18] for the negative case. In other words, the cone metric has holonomy contained in $Sp(n+1)$ (see [7]) or in $Sp(n,1)$ (see [1]), respectively.

We remind that, usually, a $4n+3$-dimensional Riemannian manifold (M,g) is called 3-Sasakian only in the positive case, while the term pseudo 3-Sasakain is used in the negative case. However, we find it convenient to use the more general definition.

A 3-Sasakian manifold of dimension $(4n+3)$ is Einstein [19, 24, 1]. If the metric is positive definite then it is with positive Riemannian scalar curvature $(4n+2)(4n+3)$ [19] and if complete it is compact with finite fundamental group due to Myer's theorem. There are known many examples of positive 3-Sasakian manifold of dimension $(4n+3)$, see [6] and references therein for a nice overview of positive 3-Sasakian spaces. Certain $SO(3)$-bundles over quaternionic Kähler manifolds with negative scalar curvature constructed in [21, 24, 18, 1] supply examples of negative 3-Sasakian manifolds. A natural definite metric on the negative 3-Sasakian manifolds is considered in [24, 18] by changing the sign of the metric on the vertical $SO(3)$-factor. With respect to this metric the negative 3-Sasakian manifold becomes an A-manifold in the terminology of [13], its Riemannian Ricci tensor has precisely two constant eigenvalues, $-4n-14$ (of multiplicity $4n$) and $4n+2$ (of multiplicity 3) see [18], and the Riemannian scalar curvature is the negative constant $-16n^2 - 44n + 6$ [24, 18]. Explicit examples of negative 3-Sasakian manifolds are constructed in [2].

3. Local structure equations of qc manifolds

We derive the local structure equations of a qc structure in terms of the $sp(1)$-connection forms of the Biquard connection and the qc scalar curvature.

Proposition 3.1. *Let $(M^{4n+3}, \eta, \mathbf{Q})$ be a (4n+3)- dimensional qc manifold with qc scalar curvature $Scal$. Let $s = \frac{Scal}{8n(n+2)}$ be the normalized qc scalar curvature. The following equations hold*

$$2\omega_i = d\eta_i + \eta_j \wedge \alpha_k - \eta_k \wedge \alpha_j + s\eta_j \wedge \eta_k, \tag{3.1}$$

$$d\omega_i = \omega_j \wedge (\alpha_k + s\eta_k) - \omega_k \wedge (\alpha_j + s\eta_j) - \rho_k \wedge \eta_j + \rho_j \wedge \eta_k + \frac{1}{2} ds \wedge \eta_j \wedge \eta_k, \tag{3.2}$$

$$d\Omega = \sum_{(ijk)} \Big[2\eta_i \wedge (\rho_k \wedge \omega_j - \rho_j \wedge \omega_k) + ds \wedge \omega_i \wedge \eta_j \wedge \eta_k \Big], \tag{3.3}$$

where α_l are the $sp(1)$-connection 1-forms of the Biquard connection, ρ_l are the Ricci 2-forms and $\sum_{(ijk)}$ is the cyclic sum of even permutations of $\{1,2,3\}$.

In particular, for a 3-Sasakian manifold the structure equations have the form

$$d\eta_i = 2\omega_i + 2\epsilon \eta_j \wedge \eta_k \tag{3.4}$$

and the normalized qc scalar curvature is $s = 2\epsilon$, where $\epsilon = 1$ if the 3-Sasakian structure is positive and $\epsilon = -1$ in the negative 3-Sasakian case.

Proof. From the definition (2.1) of the fundamental 2-forms ω_l we have (see also [15])

$$2\omega_m = (d\eta_m)_{|H} = d\eta_m - \sum_{l=1}^{3} \eta_l \wedge (\xi_l \lrcorner d\eta_m) + \sum_{1 \le l < p \le 3} d\eta_m(\xi_l, \xi_p) \eta_l \wedge \eta_p. \tag{3.5}$$

It is shown in [4] that the $sp(1)$-connection 1-forms α_l on H are given by

$$\alpha_i(X) = d\eta_k(\xi_j, X) = -d\eta_j(\xi_k, X), \quad X \in H, \quad \xi_i \in V. \tag{3.6}$$

The $sp(1)$-connection 1-forms α_l on the vertical space V were determined in [15]:

$$\alpha_i(\xi_l) \;=\; d\eta_l(\xi_j,\xi_k) - \;\delta_{il}\left(\frac{Scal}{16n(n+2)} \;+\; \frac{1}{2}\,\left(d\eta_1(\xi_2,\xi_3) \;+\; d\eta_2(\xi_3,\xi_1) \;+\; d\eta_3(\xi_1,\xi_2)\right)\right). \tag{3.7}$$

A straightforward calculation using (3.6) and (3.7) gives the equivalence of (3.5) and (3.1). Taking the exterior derivative of (3.1), followed by an application of (3.1) and (2.7) implies (3.2). The last formula, (3.3), follows from (3.2) and definition (2.2).

For the last part of the theorem consider the cone $N = M^{4n+3} \times \mathbb{R}^+$ equipped with a (pseudo) almost hyperhermitian structure (G_N, ϕ_l) where G_N is a (pseudo) Riemannian metric of signature (4n+4,0) (resp. (4n,4)) for $\epsilon = 1$ (resp. $\epsilon = -1$) and ϕ_l, $l = 1, 2, 3$, are three anti-commuting almost complex structures. The 1-form dt on $\mathbb{R}^+$ and the three almost complex structures are related to three 1-forms η_l on M^{4n+3} defined by $\eta_l = \epsilon\,\phi_l\,(\frac{1}{t}dt)$, where we used the same notation for both a tensor and its lift to a tensor on the tangent bundle of N identifying M with the slice $t = 1$ of N. We may write the metric G_N and the three Kähler 2-forms on N as follows:

$$G_N = t^2 g + \epsilon t^2(\eta_1^2 + \eta_2^2 + \eta_3^2) + \epsilon dt^2; \qquad F_i^\epsilon = t^2\omega_i + \epsilon t^2 \eta_j \wedge \eta_k - t\eta_i \wedge dt, \tag{3.8}$$

where $g = G_{N|_H}$ and $H = \cap_{l=1}^3 Ker\,\eta_l$. A qc structure on M^{4n+3} is defined by the three 1-forms η_l [4].

It is straightforward to check from the second equation in (3.8) that the 2-forms F_l^ϵ are closed precisely when (3.4) holds. Therefore the cone metric G_N is hyperkähler, i.e. M^{4n+3} is 3-Sasakian, if and only if (3.4) is fulfilled due to Hitchin's theorem [14], which is valid with the same proof in the case of non-positive definite metrics.

To compute the qc scalar curvature of a 3-Sasakian manifold, we use equations (3.4) to find $\xi_i \lrcorner d\eta_{j|H} = 0$, $d\eta_i(\xi_j,\xi_k) = 2\epsilon$, $d\eta_i(\xi_i,\xi_k) = d\eta_i(\xi_i,\xi_j) = 0$. We calculate from (3.6) and (3.7) the $sp(1)$-connection 1-forms $\alpha_l = -\Big(\frac{Scal}{16n(n+2)} + \epsilon\Big)\eta_l$. The last identity and (2.7) yield $\rho_l(X,Y) = \frac{1}{2}d\alpha_l(X,Y) = -\Big(\frac{Scal}{16n(n+2)} + \epsilon\Big)\omega_l(X,Y)$, which compared with the first equation in Theorem 2.2 gives $T = U = 0$, $Scal = 16n(n+2)\epsilon$, see [15].

□

4. Proof of Theorem 1.1

First we show that if $T^0 = U = 0$, then $d\Omega = 0$. Indeed, in this case, Theorem 2.2 implies

$$\rho_l(X,Y) = -s\omega_l(X,Y), \quad \rho_l(\xi_m, X) = 0, \quad \rho_i(\xi_i,\xi_j) + \rho_k(\xi_k,\xi_j) = 0, \tag{4.1}$$

since $Scal$ is constant and the horizontal distribution is integrable. Using the just obtained identities in (3.3) gives $d\Omega = 0$ which proves the implication $ii) \to i)$.

To finish the proof of the theorem we shall apply the next Lemma.

Lemma 4.1. *On a qc manifold of dimension $(4n+3) > 7$ we have the identities*

$$U(X,Y) = -\frac{1}{16n}\sum_{a=1}^{4n}\Big[d\Omega(\xi_i, X, I_kY, e_a, I_je_a) + d\Omega(\xi_i, I_iX, I_jY, e_a, I_je_a)\Big] \tag{4.2}$$

$$T^0(X,Y) = \frac{1}{8(1-n)}\sum_{(ijk)}\sum_{a=1}^{4n}\Big[d\Omega(\xi_i, X, I_kY, e_a, I_je_a) - d\Omega(\xi_i, I_iX, I_jY, e_a, I_je_a)\Big]. \tag{4.3}$$

Proof. Equation (3.3) together with the first equality in Theorem 2.2 yield

$$d\Omega(\xi_i, X, I_kY, e_a, I_je_a) = 4(n-1)\rho^0_k(X, I_kY) + 2\rho^0_j(X, I_jY) - 2\rho^0_j(I_iX, I_kY), \tag{4.4}$$

where ρ^0 is the horizontal trace-free part of ρ given by

$$\rho^0_l(X, I_lY) \;=\; -\frac{1}{2}\Big[T^0(X,Y) + T^0(I_lX, I_lY)\Big] - 2U(X,Y). \tag{4.5}$$

A substitution of (4.5) in (4.4), combined with the properties of the torsion, (2.4) and (2.5) give

$$2(n-1)\Big[T^0(X,Y) + T^0(I_kX, I_kY)\Big] + 8nU(X,Y) = -\sum_{a=1}^{4n} d\Omega(\xi_i, X, I_kY, e_a, I_je_a). \tag{4.6}$$

Applying again (2.4) and (2.5) in (4.6) we see that U and T^0 satisfy (4.2) and (4.3), respectively which proves the lemma. □

The well known Cartan formula yields $\mathbb{L}_{\xi_l}\Omega = \xi_l \lrcorner d\Omega + d(\xi_l \lrcorner \Omega) = \xi_l \lrcorner d\Omega$, since Ω is horizontal. The latter formula together with the already proved implication $ii) \to i)$ and Lemma 4.1 complete the proof of Theorem 1.1.

From Lemma 4.1 we easily derive

Corollary 4.2. *If one of the Reeb vector fields preserves the fundamental four form on a qc manifold of dimension $(4n+3) > 7$ then $U = 0$ and the torsion endomorphism of the Biquard connection is symmetric, $T_{\xi_l} = T^0_{\xi_l}$.*

5. Proof of Theorem 1.2

In this section we give the proof of the equivalence of parts a) and f) of Theorem 1.2. The remaining claims follow from Theorem 1.1 and [15, Theorem 1.3 and Theorem 7.11]. The idea is the same as in the proof of Theorem 3.1 in [15], namely we show that both a) and f) are equivalent to the fact that the cone over M is locally hyperkähler. However, here, the proof is based on the fundamental 4-form. In one direction, let $d\Omega = 0$. Theorem 1.1 implies that the torsion of the Biquard connection vanishes, while Theorem 2.2 shows that the qc scalar curvature is constant and the vertical distribution is integrable. Let $Scal \neq 0$ and $\epsilon = sign(Scal)$. The qc structure $\eta' = \epsilon\frac{Scal}{16n(n+2)}\eta$ has normalized qc scalar curvature $s' = 2\epsilon$ and $d\Omega' = 0$. For simplicity, we shall denote η' with η and, in fact, drop the $'$ everywhere.

In the first step of the proof we show that the cone $N = M \times \mathbb{R}^+$ with the structure (G_N, F^ϵ_l) defined by (3.8) has holonomy contained in $Sp(n+1)$ for $\epsilon > 0$ and in $Sp(n,1)$ for $\epsilon < 0$. To this end we consider the following four form on N

$$F = F^\epsilon_1 \wedge F^\epsilon_1 + F^\epsilon_2 \wedge F^\epsilon_2 + F^\epsilon_3 \wedge F^\epsilon_3. \tag{5.1}$$

Applying (3.1), (3.2) and (3.3), we calculate

$$\begin{aligned} (5.2)\quad dF_i &= tdt \wedge (2\omega_i + 2\epsilon\eta_j \wedge \eta_k - d\eta_i) + t^2 d(\omega_i + \epsilon\eta_j \wedge \eta_k) \\ &= t\,dt \wedge (2\omega_i + 2\epsilon\eta_j \wedge \eta_k + \eta_j \wedge \alpha_k - \eta_k \wedge \alpha_k + s\eta_j \wedge \eta_k) + \epsilon t^2(2\omega_j + \eta_i \wedge \alpha_k) \wedge \eta_k - \epsilon t^2(2\omega_k - \eta_i \wedge \alpha_j) \wedge \eta_j) \\ &\quad + t^2\left(\omega_j \wedge (\alpha_k + s\eta_k) - \omega_k \wedge (\alpha_j + s\eta_j) - \rho_k \wedge \eta_j + \rho_j \wedge \eta_k + \frac{1}{2}ds \wedge \eta_j \wedge \eta_k\right). \end{aligned}$$

A short computation, using (3.1), (3.2), (3.3) and (5.2), gives

$$
\begin{aligned}
(5.3)\quad dF = 2\sum_{i=1}^{3} dF_i\wedge F_i &= t^4\sum_{(ijk)}\left[2\eta_i\wedge(\omega_j\wedge\rho_k-\omega_k\wedge\rho_j)+ds\wedge\omega_i\wedge\eta_j\wedge\eta_k\right]\\
&+t^3dt\wedge\sum_{(ijk)}\left[4\epsilon\omega_i\wedge\eta_k\wedge\eta_j-2s\omega_i\wedge\eta_j\wedge\eta_k-4\rho_k\wedge\eta_i\wedge\eta_j-ds\wedge\eta_i\wedge\eta_j\wedge\eta_k\right]\\
&=2t^4d\Omega-4t^3\sum_{(ijk)}dt\wedge(\rho_i+2\epsilon\omega_i)\wedge\eta_j\wedge\eta_k=0,
\end{aligned}
$$

taking into account the first equality in Theorem 2.2, (4.1) and $s=2\epsilon$, which hold when $d\Omega=0$ by Theorem 1.1.

Hence, $dF=0$ and the holonomy of the cone metric is contained either in $Sp(n+1)Sp(1)$ or in $Sp(n,1)Sp(1)$ provided $n>1$ [22], i.e. the cone is quaternionic Kähler manifold provided $n>1$. Note that when $n=1$ this conclusion can not be reached in the positive definite case due to the 8-dimensional compact counter-example constructed by S. Salamon [23] (for non compact counter-examples see [2]).

It is a classical result (see e.g [3] and references therein) that a quaternionic Kähler manifolds of dimension bigger than four (of arbitrary signature) are Einstein. This fact implies that the cone $N=M\times\mathbb{R}^+$ with the metric g_N must be Ricci flat (see e.g. [3, p.267]) and therefore it is locally hyperkähler since the $sp(1)$-part of the Riemannian curvature vanishes and therefore it can be trivialized locally by a parallel sections (see e.g. [3, p.397]). This means that locally there exists a $SO(3)$-matrix Ψ with smooth entries, possibly depending on t, such that the triple of two forms $(\tilde F_1,\tilde F_2,\tilde F_3)=\Psi\cdot(F_1,F_2,F_3)^T$ consists of closed 2-forms constitute the fundanental 2-forms of the local hyperkähler structure. Consequently $(M,\Psi\cdot\eta)$ is locally a 3-Sasakian manifold.

The fact that f) implies a) is trivial since the 4-form Ω is invariant under rotations and rescales by a constant when the metric on the horizontal space H is replaced by a homothetic to it metric.

Remark 5.1. *It follows from the above discussion that the cone over a (4n+3)-dimensional qc manifold carries either a $Sp(n+1)Sp(1)$ or a $Sp(n,1)Sp(1)$ structure which is closed exactly when $d\Omega=0$ provided $n>1$.*

Remark 5.2. *An example of a qc structure satisfying $T^0=U=Scal=0$ can be obtained as follows. Let M^{4n} be a hyperkähler manifold with closed and locally exact Kähler forms $\omega_l=d\eta_l$. The total space of an $\mathbb{R}^3$-bundle over the hyperkähler manifold M^{4n} with connection 1-forms η_l is an example of a qc structure with $T^0=U=Scal=0$. The qc structure is determined by the three 1-forms η_l satisfying $d\eta_l=\omega_l$ which yield $T^0=U=Scal=0$. In particular, the quaternionic Heisenberg group which locally is the unique qc structure with flat Biquard connection on H, see [17], can be considered as an $\mathbb{R}^3$ bundle over a 4n-dimensional flat hyperkähler $\mathbb{R}^{4n}$. A compact example is provided by a T^3-bundle over a compact hyperkähler manifold M^{4n} such that each closed Kähler form ω_l represents integral cohomology classes. Indeed, since $[\omega_l]$, $1\le l\le 3$ define integral cohomology classes on M^{4n}, the well-known result of Kobayashi [20] implies that there exists a circle bundle $S^1\hookrightarrow M^{4n+1}\to M^{4n}$, with connection 1-form η_1 on M^{4n+1} whose curvature form is $d\eta_1=\omega_1$. Because ω_l $(l=2,3)$ defines an integral cohomology class on M^{4n+1}, there exists a principal circle bundle $S^1\hookrightarrow M^{4n+2}\to M^{4n+1}$ corresponding to $[\omega_2]$ and a connection 1-form η_2 on M^{4n+2} such that $\omega_2=d\eta_2$ is the curvature form of η_2. Using again the result of Kobayashi, one gets a T^3-bundle over M^{4n} whose total space has a qc structure satisfying $d\eta_l=\omega_l$ which yield $T^0=U=Scal=0$.*

We do not know whether there are other examples satisfying the conditions $T^0=U=Scal=0$.

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