Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem

Stefan Ivanov

Ivan Minchev

Dimiter Vassilev

Author address:

University of Sofia, Faculty of Mathematics and Informatics, blvd. James Bourchier 5, 1164, Sofia, Bulgaria and Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany

Current address: University of Sofia, Faculty of Mathematics and Informatics, blvd. James Bourchier 5, 1164, Sofia, Bulgaria

E-mail address: `ivanovsp@fmi.uni-sofia.bg`

University of Sofia, Sofia, Bulgaria, and Institut für Mathematik, Humboldt Universität zu Berlin, Unter den Linden 6, Berlin D-10099, Germany

Current address: University of Sofia, Sofia, Bulgaria

E-mail address: `minchev@fmi.uni-sofia.bg`

Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87131-0001, and, University of California, Riverside, Riverside, CA 92521

Current address: University of New Mexico, Albuquerque, New Mexico, 87131-0001

E-mail address: `vassilev@math.unm.edu`

Contents

Abstract

A partial solution of the quaternionic contact Yamabe problem on the quaternionic sphere is given. It is shown that the torsion of the Biquard connection vanishes exactly when the trace-free part of the horizontal Ricci tensor of the Biquard connection is zero and this occurs precisely on 3-Sasakian manifolds. All conformal transformations sending the standard flat torsion-free quaternionic contact structure on the quaternionic Heisenberg group to a quaternionic contact structure with vanishing torsion of the Biquard connection are explicitly described. A '3-Hamiltonian form' of infinitesimal conformal automorphisms of quaternionic contact structures is presented.

Received by the editor January 25, 2008.

2010 *Mathematics Subject Classification.* 58G30, 53C17.

Key words and phrases. Yamabe equation, quaternionic contact structures, Einstein structures.

This project has been funded in part by the National Academy of Sciences under the [Collaboration in Basic Science and Engineering Program 1 Twinning Program] supported by Contract No. INT-0002341 from the National Science Foundation. The contents of this publication do not necessarily reflect the views or policies of the National Academy of Sciences or the National Science Foundation, nor does mention of trade names, commercial products or organizations imply endorsement by the National Academy of Sciences or the National Science Foundation.

CHAPTER 1

Introduction

The Riemannian [LP] and CR Yamabe problems [JL1, JL2, JL3, JL4] have been a fruitful subject in geometry and analysis. Major steps in the solutions is the understanding of the conformally flat cases. A model for this setting is given by the corresponding spheres, or equivalently, the Heisenberg groups with, respectively, 0-dimensional and 1-dimensional centers. The equivalence is established through the Cayley transform [K], [CDKR1] and [CDKR2], which in the Riemannian case is the usual stereographic projection.

In the present paper we consider the Yamabe problem on the quaternionic Heisenberg group (three dimensional center). This problem turns out to be equivalent to the quaternionic contact Yamabe problem on the unit (4n+3)-dimensional sphere in the quaternionic space due to the quaternionic Cayley transform, which is a conformal quaternionic contact transformation (see the proof of Theorem 1.2).

The central notion is the quaternionic contact structure (QC structure for short), [Biq1, Biq2], which appears naturally as the conformal boundary at infinity of the quaternionic hyperbolic space, see also [P, GL, FG]. Namely, a QC structure $(\eta, \mathbb{Q})$ on a (4n+3)-dimensional smooth manifold M is a codimension 3 distribution H, such that, at each point $p \in M$ the nilpotent Lie algebra $H_p \oplus (T_pM/H_p)$ is isomorphic to the quaternionic Heisenberg algebra $\mathbb{H}^m \oplus Im\ \mathbb{H}$. This is equivalent to the existence of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$, such that, $H = Ker\ \eta$ and the three 2-forms $d\eta_{i|H}$ are the fundamental 2-forms of a quaternionic structure $\mathbb{Q}$ on H. A special phenomena here, noted by Biquard [Biq1], is that the 3-contact form η determines the quaternionic structure as well as the metric on the horizontal bundle in a unique way. Of crucial importance is the existence of a distinguished linear connection, see [Biq1], preserving the QC structure and its Ricci tensor and scalar curvature $Scal$, defined in (3.34), and called correspondingly qc-Ricci tensor and qc-scalar curvature. The Biquard connection will play a role similar to the Tanaka-Webster connection [We] and [T] in the CR case.

The quaternionic contact Yamabe problem, in the considered setting, is about the possibility of finding in the conformal class of a given QC structure one with constant qc-scalar curvature.

The question reduces to the solvability of the Yamabe equation (5.8). As usual if we take the conformal factor in a suitable form the gradient terms in (5.8) can be removed and one obtains the more familiar form of the Yamabe equation. In fact, taking the conformal factor of the form $\bar{\eta} = u^{1/(n+1)}\eta$ reduces (5.8) to the equation

$$\mathcal{L}u \ \equiv\ 4\frac{n+2}{n+1}\, \triangle u - \ u\, Scal \ =\ -u^{2^*-1}\, \overline{Scal},$$

where $\triangle$ is the horizontal sub-Laplacian, cf. (5.2), and $Scal$ and $\overline{Scal}$ are the qc-scalar curvatures correspondingly of $(M,\, \eta)$ and $(M,\, \bar{\eta})$, $2^* = \frac{2Q}{Q-2}$, and $Q = 4n+6$

is the so called homogeneous dimension. In the case of the quaternionic Heisenberg group, cf. Section 4.1, the equation is

$$\mathcal{L}u \equiv \sum_{\alpha=1}^{n} \left(T_\alpha^2 u + X_\alpha^2 u + Y_\alpha^2 u + Z_\alpha^2 u\right) = -\frac{n+1}{4(n+2)}\, u^{2^*-1}\, \overline{Scal}.$$

This is also, up to a scaling, the Euler-Lagrange equation of the non-negative extremals in the L^2 Folland-Stein embedding theorem [Fo] and [FSt], see [GV1] and [Va2]. On the other hand, on a compact quaternionic contact manifold M with a fixed conformal class $[\eta]$ the Yamabe equation characterizes the non-negative extremals of the Yamabe functional defined by

$$\Upsilon(u) = \int_M 4\frac{n+2}{n+1}\, |\nabla u|^2 + Scal\cdot u^2\, dv_g, \qquad \int_M u^{2^*}\, dv_g = 1,\ u>0,$$

where dv_g is the volume form associated to te natural Riemannian metric g on M, cf. (2.11). When the Yamabe constant

$$\lambda(M) \overset{def}{=} \lambda(M,[\eta]) = \inf\{\Upsilon(u) : \int_M u^{2^*}\, dv_g = 1,\ u>0\}$$

is less than the corresponding constant for the 3-Sasakian sphere the existence of solutions can be constructed with the use of suitable coordinates see [W] and [JL2].

The present paper can be considered as a contribution towards the soluiton of the Yamabe problem in the case when the Yamabe constant of the considered quaternionic contact manifold is equal to the Yamabe constant of the unit sphere with its standard quaternionic contact structure, which is induced from the embedding in the quaternion $(n+1)$-dimensional space. It is also natural to conjecture that if the quaternionic contact structure is not locally equivalent to the standard sphere then the Yamabe constant is less than that of the sphere, see [JL4] for a proof in the CR case. The results of the present paper will be instrumental for the analysis of these and some other questions concerning the geometric analysis on quaternionic contact structures.

In this article we provide a partial solution of the Yamabe problem on the quaternionic sphere with its standard contact quaternionic structure or, equivalently, the quaternionic Heisenberg group. Note that according to [GV2] the extremals of the above variational problem are $\mathcal{C}^\infty$ functions, so we will not consider regularity questions in this paper. Furthermore, according to [Va1] or [Va2] the infimum is achieved, and the extremals are solutions of the Yamabe equation. Let us observe that [GV2] solves the same problem in a more general setting, but under the assumption that the solution is invariant under a certain group of rotation. If one is on the flat models, i.e., the groups of Iwasawa type [CDKR1] the assumption in [GV2] is equivalent to the a-priori assumption that, up to a translation, the solution is radial with respect to the variables in the first layer. The proof goes on by using the moving plane method and showing that the solution is radial also in the variables from the center, after which a very non-trivial identity is used to determine all cylindrical solutions. In this paper the a-priori assumption is of a different nature, see further below, and the method has the potential of solving the general problem.

The strategy, following the steps of [LP] and [JL3], is to solve the Yamabe problem on the quaternionic sphere by replacing the non-linear Yamabe equation by an appropriate geometrical system of equations which could be solved.

Our first observation is that if if $n > 1$ and the qc-Ricci tensor is trace-free (qc-Einstein condition) then the qc-scalar curvature is constant (Theorem 4.9). Studying conformal transformations of QC structures preserving the qc-Einstein condition, we describe explicitly all global functions on the quaternionic Heisenberg group sending conformally the standard flat QC structure to another qc-Einstein structure. Our result here is the following Theorem.

THEOREM 1.1. *Let $\Theta = \frac{1}{2h}\tilde{\Theta}$ be a conformal transformation of the standard qc-structure $\tilde{\Theta}$ on the quaternionic Heisenberg group $G(\mathbb{H})$. If Θ is also qc-Einstein, then up to a left translation the function h is given by*

$$h = c\left[\left(1 + \nu\,|q|^2\right)^2 + \nu^2\,(x^2 + y^2 + z^2)\right],$$

where c and ν are positive constants. All functions h of this form have this property.

The crucial fact which allows the reduction of the Yamabe equation to the system preserving the qc-Einstein condition is Proposition 8.2 which asserts that, under some "extra" conditions, QC structure with constant qc-scalar curvature obtained by a conformal transformation of a qc-Einstein structure on compact manifold must be again qc-Einstein. The prove of this relies on detailed analysis of the Bianchi identities for the Biquard connection. Using the quaternionic Cayley transform combined with Theorem 1.1 lead to our second main result.

THEOREM 1.2. *Let $\eta = f\,\tilde{\eta}$ be a conformal transformation of the standard qc-structure $\tilde{\eta}$ on the quaternionic sphere S^{4n+3}. Suppose η has constant qc-scalar curvature. If the vertical space of η is integrable then up to a multiplicative constant η is obtained from $\tilde{\eta}$ by a conformal quaternionic contact automorphism. In the case $n > 1$ the same conclusion holds when the function f is the real part of an anti-CRF function.*

The definition of conformal quaternionic contact automorpism can be found in Definition 7.6. The solutions (conformal factors) we find agree with those conjectured in [GV1]. It might be possible to dispense of the "extra" assumption in Theorem 1.2. In a subsequent paper [IMV] we give such a proof for the seven dimensional sphere.

Studying the geometry of the Biquard connection, our main geometrical tool towards understanding the geometry of the Yamabe equation, we show that the qc-Einstein condition is equivalent to the vanishing of the torsion of Biquard connection. In our third main result we give a local characterization of such spaces as 3-Sasakian manifolds.

THEOREM 1.3. *Let $(M^{4n+3}, g, \mathbb{Q})$ be a QC manifold with positive qc scalar curvature $Scal > 0$, assumed to be constant if $n = 1$. The next conditions are equivalent:*

a) $(M^{4n+3}, g, \mathbb{Q})$ is a qc-Einstein manifold.

b) M is locally 3-Sasakian, i.e., locally there exists an $SO(3)$-matrix Ψ with smooth entries, such that, the local QC structure $(\frac{16n(n+2)}{Scal}\Psi\cdot\eta, \mathbb{Q})$ is 3-Sasakian.

c) The torsion of the Biquard connection is identically zero.

In particular, a qc-Einstein manifold of positive qc scalar curvature, assumed in addition to be constant if $n = 1$, is an Einstein manifold of positive Riemannian scalar curvature.

In addition to the above results, in Theorem 7.10 we show that the above conditions are equivalent to the property that every Reeb vector field, defined in (2.10), is an infinitesimal generator of a conformal quaternionic contact automorphism, cf. Definition 7.7.

Finally, we also develop useful tools necessary for the geometry and analysis on QC manifolds. We define and study some special functions, which will be relevant in the geometric analysis on quaternionic contact and hypercomplex manifolds as well as properties of infinitesimal automorphisms of QC structures. In particular, the considered anti-regular functions will be relevant in the study of qc-pseudo-Einstein structures, cf. Definition 6.1.

Organization of the paper: In the following two chapters we describe in details the notion of a quaternionic contact manifold, abbreviate sometimes to QC-manifold, and the Biquard connection, which is central to the paper.

In Chapter 4 we write explicitly the Bianchi identities and derive a system of equations satisfied by the divergences of some important tensors. As a result we are able to show that qc-Einstein manifolds, i.e., manifolds for which the restriction to the horizontal space of the qc-Ricci tensor is proportional to the metric, have constant scalar curvature, see Theorem 4.9. The proof uses Theorem 4.8 in which we derive a relation between the horizontal divergences of certain $Sp(n)Sp(1)$-invariant tensors. By introducing an integrability condition on the horizontal bundle we define hyperhermitian contact structures, see Definition 4.14, and with the help of Theorem 4.8 we prove Theorem 1.3.

Chapter 5 describes the conformal transformations preserving the qc-Einstein condition. Note that here a conformal quaternionic contact transformation between two quaternionic contact manifold is a diffeomorphism Φ which satisfies $\Phi^*\eta = \mu\ \Psi\cdot\eta$, for some positive smooth function μ and some matrix $\Psi \in SO(3)$ with smooth functions as entries and $\eta = (\eta_1, \eta_2, \eta_3)^t$ is considered as an element of $\mathbb{R}^3$. One defines in an obvious manner a point-wise conformal transformation. Let us note that the Biquard connection does not change under rotations as above, i.e., the Biquard connection of $\Psi\cdot\eta$ and η coincides. In particular, when studying conformal transformations we can consider only transformations with $\Phi^*\eta = \mu\,\eta$. We find all conformal transformations preserving the qc-Einstein condition on the quaternionis Heisenberg group or, equivalently, on the quaternionic sphere with their standard contact quaternionic structures proving Theorem 1.1.

Chapter 6 concerns a special class of functions, which we call anti-regular, defined respectively on the quaternionic space, real hyper-surface in it, or on a quaternionic contact manifold, cf. Definitions 6.6 and 6.15 as functions preserving the quaternionic structure. The anti-regular functions play a role somewhat similar to those played by the CR functions, but the analogy is not complete. The real parts of such functions will be also of interest in connection with conformal transformation preserving the qc-Einstein tensor and should be thought of as generalization of pluriharmonic functions. Let us stress explicitly that regular quaternionic functions have been studied extensively, see [**S**] and many subsequent papers, but they are not as relevant for the considered geometrical structures. Anti-regular functions on hyperkähler and quaternionic Kähler manifolds are studied in [**CL1, CL2, LZ**] in a different context, namely in connection with minimal surfaces and quaternionic maps between quaternionic Kähler manifolds. The notion of hypercomplex contact structures will appear in this section again since on such manifolds the real part of anti-CRF functions, see (6.18) for the definition, have some interesting properties, cf. Theorem 6.20

In Chapter 7 we study infinitesimal conformal automorphisms of QC structures (QC-vector fields) and show that they depend on three functions satisfying some differential conditions thus establishing a '3-hamiltonian' form of the QC-vector fields (Proposition 7.8). The formula becomes very simple expression on a 3-Sasakian manifolds (Corollary 7.9). We characterize the vanishing of the torsion of Biquard connection in terms of the existence of three vertical vector fields whose flow preserves the metric and the quaternionic structure. Among them, 3-Sasakian manifolds are exactly those admitting three transversal QC-vector fields (Theorem 7.10, Corollary 7.13).

In the last section we complete the proof of our main result Theorem 1.2.

Notation 1.4. *a) Let us note explicitly, that in this paper for a one form θ we use*

$$d\theta(X,Y) \;=\; X\theta(Y) \;-\; Y\theta(X) \;-\; \theta([X,Y]).$$

b) We shall denote with ∇h the horizontal gradient of the function h, see (5.1)*, while dh means as usual the differential of the function h.*

c) The triple $\{i,j,k\}$ will denote a cyclic permutation of $\{1,2,3\}$, unless it is explicitly stated otherwise.

Acknowledgements S.Ivanov is a Senior Associate to the Abdus Salam ICTP. The paper was completed during the visit of S.I. in Max-Plank-Institut für Mathematik, Bonn. S.I. thanks ICTP and MPIM, Bonn for providing the support and an excellent research environment. I.Minchev is a member of the Junior Research Group "Special Geometries in Mathematical Physics" founded by the Volkswagen Foundation" The authors would like to thank The National Academies for the support. It is a pleasure to acknowledge the role of Centre de Recherches Mathématiques and CIRGET, Montréal where the project initiated. The authors also would like to thank University of California, Riverside and University of Sofia for hosting the respective visits of the authors.

CHAPTER 2

Quaternionic contact structures and the Biquard connection

The notion of *Quaternionic Contact Structure* has been introduced by O. Biquard in [Biq1] and [Biq2]. Namely, a quaternionic contact structure (QC structure for short) on a (4n+3)-dimensional smooth manifold M is a codimension 3 distribution H, such that, at each point $p \in M$ the nilpotent step two Lie algebra $H_p \oplus (T_pM/H_p)$ is isomorphic to the quaternionic Heisenberg algebra $\mathbb{H}^n \oplus Im\ \mathbb{H}$. The quaternionic Heisenberg algebra structure on $\mathbb{H}^n \oplus Im\ \mathbb{H}$ is obtained by the identification of $\mathbb{H}^n \oplus Im\ \mathbb{H}$ with the algebra of the left invariant vector fields on the quaternionic Heisenberg group, see Section 5.2. In particular, the Lie bracket is given by the formula $[(q_o, \omega_o), (q, \omega)] = 2\ Im\ q_o \cdot \bar{q}$, where $q = (q^1, q^2, \dots, q^n)$, $q_o = (q_o^1, q_o^2, \dots, q_o^n) \in \mathbb{H}^n$ and $\omega,\ \omega_o \in Im\ \mathbb{H}$ with $q_o \cdot \bar{q} = \sum_{\alpha=1}^{n} q_o^{\alpha} \cdot \overline{q^{\alpha}}$, see Section 6.1.0.1 for notations concerning $\mathbb{H}$. It is important to observe that if M has a quaternionic contact structure as above then the definition implies that the distribution H and its commutators generate the tangent space at every point.

The following is another, more explicit, definition of a quaternionic contact structure.

Definition 2.1. [Biq1] *A quaternionic contact structure (*QC-structure*) on a $4n+3$ dimensional manifold M, $n > 1$, is the data of a codimension three distribution H on M equipped with a CSp(n)Sp(1) structure, i.e., we have:*

i) a fixed conformal class $[g]$ of metrics on H;

ii) a 2-sphere bundle $\mathbb{Q}$ over M of almost complex structures, such that, locally we have $\mathbb{Q} = \{aI_1 + bI_2 + cI_3 :\ a^2 + b^2 + c^2 = 1\}$, where the almost complex structures $I_s : H \to H$, $I_s^2 = -1$, $s = 1,\ 2,\ 3$, satisfy the commutation relations of the imaginary quaternions $I_1 I_2 = -I_2 I_1 = I_3$;

iii) H is locally the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ and the following compatibility condition holds

$$2g(I_s X, Y) = d\eta_s(X, Y), \quad s = 1, 2, 3, \quad X, Y \in H. \tag{2.1}$$

A manifold M with a structure as above will be called also quaternionic contact manifold (QC manifold) and denoted by $(M, [g], \mathbb{Q})$. The distribution H is frequently called the *horizontal space* of the QC-structure. With a slight abuse of notation we shall use the letter $\mathbb{Q}$ to also denote the rank-three bundle consisting of endomorphisms of H locally generated by three almost complex structures I_1, I_2, I_3 on H. The meaning should be clear from the context. We note that if in some local chart $\bar{\eta}$ is another form, with corresponding $\bar{g} \in [g]$ and almost complex structures

$\bar{I}_s$, $s=1,2,3$, then $\bar{\eta} = \mu\,\Psi\,\eta$ for some $\Psi \in SO(3)$ and a positive function μ Typical examples of manifolds with QC-structures are totally umbilical hypersurfaces in quaternionic Kähler or hyperkähler manifold, see Proposition 6.12 for the latter.

It is instructive to consider the case when there is a globally defined one-form η. The obstruction to the global existence of η is encoded in the first Pontrjagin class [**AK**]. Besides clarifying the notion of a QC-manifold, most of the time, for example when considering the Yamabe equation, we shall work with a QC-structure for which we have a fixed globally defined contact form. In this case, if we rotate the $\mathbb{R}^3$-valued contact form and the almost complex structures by the same rotation we obtain again a contact form, almost complex structures and a metric (the latter is unchanged) satisfying the above conditions. On the other hand, it is important to observe that given a contact form the almost complex structures and the horizontal metric are unique if they exist. Finally, if we are given the horizontal bundle and a metric on it, there exists at most one sphere of associated contact forms with a corresponding sphere $\mathbb{Q}$ of almost complex structures [**Biq1**]. These observations are summed-up in the next Lemma.

LEMMA 2.2. [**Biq1**]

a) If (η, I_s, g) and (η, I'_s, g') are two QC structures on M, then $I_s = I'_s$, $s=1,2,3$ and $g = g'$.

b) If (η, g) and (η', g) are two QC structures on M with $Ker(\eta) = Ker(\eta') = H$ then $\mathbb{Q} = \mathbb{Q}'$ and $\eta' = \Psi\,\eta$ for some matrix $\Psi \in SO(3)$ with smooth functions as entries.

PROOF. a) Let us fix a basis $\{e_1, ..., e_{4n}\}$ of H . Suppose the tensors $g, d\eta_{1|H}$, $d\eta_{2|H}, d\eta_{3|H}$, I_1, I_2, I_3 (tensors on H) are given in local coordinates, respectively, by the matrices $G, N_1, N_2, N_3, J_1, J_2, J_3 \in GL(4n)$. From (2.1) it follows

$$GJ_s = \frac{1}{2}N_s, \quad s = 1,2,3. \tag{2.2}$$

Let (i,j,k) be any cyclic permutation of $(1,2,3)$. Using (2.2) we compute that

$$J_k = J_iJ_j = -J_i^{-1}G^{-1}GJ_j = -(GJ_i)^{-1}(GJ_j) = -N_i^{-1}N_j, \tag{2.3}$$

which accomplishes the proof.

b) The condition $Ker(\eta) = Ker(\eta') = H$ implies that

$$\eta'_k = \sum_{l=1}^{3} \Psi_{kl}\,\eta_l, \quad k = 1,2,3 \tag{2.4}$$

for some matrix $\Psi \in GL(3)$ with smooth functions Ψ_{ij} as entries. Applying the exterior derivative in (2.4) we find

$$d\eta'_k = d\Psi_{kl} \wedge \eta_l + \Psi_{kl}\,d\eta_l, \qquad k = 1,2,3. \tag{2.5}$$

Let the H tensors I_k and I'_k be defined as usual with (2.1) using respectively η and η'. Restricting the equation (2.5) to H and using the metric tensor g on H we have

$$g(I_kX, Y) = \Psi_{kl}g(I_lX, Y), \quad X, Y \in H \tag{2.6}$$

or the equivalent equations $I_k = \Psi_{kl}I_l$ on H. It is easy to see that this is possible if and only if $\Psi \in SO(3)$. $\square$

Besides the non-uniqueness due to the action of $SO(3)$, the 1-form η can be changed by a conformal factor, in the sense that if η is a form for which we can find associated almost complex structures and metric g as above, then for any $\Psi \in SO(3)$ and a positive function μ, the form $\mu\,\Psi\,\eta$ also has an associated complex structures and metric. In particular, when $\mu = 1$ we obtain a whole unit sphere of contact forms, and we shall denote, as already mentioned, by $\mathbb{Q}$ the corresponding sphere bundle of associated triples of almost complex structures. With the above consideration in mind we introduce the following notation.

NOTATION 2.3. *We shall denote with (M, η) a QC-manifold with a fixed globally defined contact form. $(M, g, \mathbb{Q})$ will denote a QC-manifold with a fixed metric g and a sphere bundle of almost complex structures $\mathbb{Q}$. In this case we have in fact a $Sp(n)Sp(1)$ structure, i.e., we are working with a fixed metric on the horizontal space. Correspondingly, we shall denote with η any (locally defined) associated contact form.*

We recall the definition of the Lie groups $Sp(n)$, $Sp(1)$ and $Sp(n)Sp(1)$. Let us identify $\mathbb{H}^n = \mathbb{R}^{4n}$ and let $\mathbb{H}$ acts on $\mathbb{H}^n$ by right multiplications, $\lambda(q)(W) = W \cdot q^{-1}$. This defines a homomorphism $\lambda : \{\text{unit quaternions}\} \longrightarrow SO(4n)$ with the convention that $SO(4n)$ acts on $\mathbb{R}^{4n}$ on the left. The image is the Lie group $Sp(1)$. Let $\lambda(i) = I_0, \lambda(j) = J_0, \lambda(k) = K_0$. The Lie algebra of $Sp(1)$ is

$$sp(1) = span\{I_0, J_0, K_0\}.$$

The group $Sp(n)$ is

$$Sp(n) = \{O \in SO(4n) : OB = BO,\ B \in Sp(1)\}$$

or $Sp(n) = \{O \in M_n(\mathbb{H}) : O\,\bar{O}^t = I\}$, and $O \in Sp(n)$ acts by

$$(q^1, q^2, \dots, q^n)^t \mapsto O\,(q^1, q^2, \dots, q^n)^t.$$

Denote by $Sp(n)Sp(1)$ the product of the two groups in $SO(4n)$. Abstractly, $Sp(n)Sp(1) = (Sp(n) \times Sp(1))/\mathbb{Z}_2$. The Lie algebra of the group $Sp(n)Sp(1)$ is $sp(n) \oplus sp(1)$.

Any endomorphism Ψ of H can be decomposed with respect to the quaternionic structure $(\mathbb{Q}, g)$ uniquely into four $Sp(n)$-invariant parts

$$\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+},$$

where Ψ^{+++} commutes with all three I_i, Ψ^{+--} commutes with I_1 and anti-commutes with the others two and etc. Explicitly,

$$4\Psi^{+++} = \Psi - I_1\Psi I_1 - I_2\Psi I_2 - I_3\Psi I_3,\ \ 4\Psi^{+--} = \Psi - I_1\Psi I_1 + I_2\Psi I_2 + I_3\Psi I_3,$$
$$4\Psi^{-+-} = \Psi + I_1\Psi I_1 - I_2\Psi I_2 + I_3\Psi I_3,\ \ 4\Psi^{--+} = \Psi + I_1\Psi I_1 + I_2\Psi I_2 - I_3\Psi I_3.$$

The two $Sp(n)Sp(1)$-invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \qquad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}. \tag{2.7}$$

Denoting the corresponding (0,2) tensor via g by the same letter one sees that the $Sp(n)Sp(1)$-invariant components are the projections on the eigenspaces of the Casimir operator

$$\dagger\ =\ I_1 \otimes I_1\ +\ I_2 \otimes I_2\ +\ I_3 \otimes I_3 \tag{2.8}$$

corresponding, respectively, to the eigenvalues 3 and -1, see [CSal]. If $n = 1$ then the space of symmetric endomorphisms commuting with all $I_i, i = 1, 2, 3$ is

1-dimensional, i.e. the [3]-component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_3 = \frac{tr(\Psi)}{4} Id_{|H}$.

There exists a canonical connection compatible with a given quaternionic contact structure. This connection was discovered by O. Biquard [Biq1] when the dimension $(4n+3) > 7$ and by D. Duchemin [D] in the 7-dimensional case. The next result due to O. Biquard is crucial in the quaternionic contact geometry.

THEOREM 2.4. [Biq1] *Let $(M, g, \mathbb{Q})$ be a quaternionic contact manifold of dimension $4n+3 > 7$ and a fixed metric g on H in the conformal class $[g]$. Then there exists a unique connection ∇ with torsion T on M^{4n+3} and a unique supplementary subspace V to H in TM, such that:*

i) ∇ preserves the decomposition $H \oplus V$ and the metric g;
ii) for $X, Y \in H$, one has $T(X,Y) = -[X,Y]_{|V}$;
iii) ∇ preserves the $Sp(n)Sp(1)$-structure on H, i.e., $\nabla g = 0$ and $\nabla \mathbb{Q} \subset \mathbb{Q}$;
iv) for $\xi \in V$, the endomorphism $T(\xi, .)_{|H}$ of H lies in $(sp(n) \oplus sp(1))^{\perp} \subset gl(4n)$;
v) the connection on V is induced by the natural identification φ of V with the subspace $sp(1)$ of the endomorphisms of H, i.e. $\nabla \varphi = 0$.

In (iv) the inner product on $End(H)$ is given by

$$g(A,B) = tr(B^* A) = \sum_{a=1}^{4n} g(A(e_a), B(e_a)), \tag{2.9}$$

where $A, B \in End(H)$, $\{e_1, ..., e_{4n}\}$ is some g-orthonormal basis of H.

We shall call the above connection *the Biquard connection.* Biquard [Biq1] also described the supplementary "vertical" (sub-)space V explicitly, namely, locally V is generated by vector fields $\{\xi_1, \xi_2, \xi_3\}$, such that

$$\begin{aligned} \eta_s(\xi_k) = \delta_{sk}, \qquad & (\xi_s \lrcorner d\eta_s)_{|H} = 0, \\ (\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}, \qquad & s, k \in \{1,2,3\}. \end{aligned} \tag{2.10}$$

The vector fields ξ_1, ξ_2, ξ_3 are called Reeb vector fields or fundamental vector fields.

If the dimension of M is seven, the conditions (2.10) do not always hold. Duchemin shows in [D] that if we assume, in addition, the existence of Reeb vector fields as in (2.10), then Theorem 2.4 holds. Henceforth, by a QC structure in dimension 7 we shall always mean a QC structure satisfying (2.10).

Notice that equations (2.10) are invariant under the natural $SO(3)$ action. Using the Reeb vector fields we extend g to a metric on M by requiring

$$span\{\xi_1, \xi_2, \xi_3\} = V \perp H \text{ and } g(\xi_s, \xi_k) = \delta_{sk}. \tag{2.11}$$

The extended metric does not depend on the action of $SO(3)$ on V, but it changes in an obvious manner if η is multiplied by a conformal factor. Clearly, the Biquard connection preserves the extended metric on $TM, \nabla g = 0$. We shall also extend the quternionic structure by setting $I_{s|V} = 0$.

Suppose the Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ have been fixed. The restriction of the torsion of the Biquard connection to the vertical space V satisfies [Biq1]

$$T(\xi_i, \xi_j) = \lambda \xi_k - [\xi_i, \xi_j]_{|H}, \tag{2.12}$$

where λ is a smooth function on M, which will be determined in Corollary 3.14. We recall that in the paper the indices follow the convention 1.4. Further properties of the Biquard connection are encoded in the properties of the torsion endomorphism

$$T_\xi = T(\xi, .) : H \to H, \quad \xi \in V.$$

Decomposing the endomorphism $T_\xi \in (sp(n) + sp(1))^\perp$ into its symmetric part T^0_ξ and skew-symmetric part b_ξ,

$$T_\xi = T^0_\xi + b_\xi,$$

we summarize the description of the torsion due to O. Biquard in the following Proposition.

PROPOSITION 2.5. [**Biq1**] *The torsion T_ξ is completely trace-free,*

$$trT_\xi = \sum_{a=1}^{4n} g(T_\xi(e_a), e_a) = 0, \quad trT_\xi \circ I = \sum_{a=1}^{4n} g(T_\xi(e_a), Ie_a) = 0, \quad I \in Q, \tag{2.13}$$

where $e_1 \dots e_{4n}$ is an orthonormal basis of H. The symmetric part of the torsion has the properties:

$$T^0_{\xi_i} I_i = -I_i T^0_{\xi_i}, \qquad i = 1, 2, 3. \tag{2.14}$$

In addition, we have

$$\begin{aligned} I_2(T^0_{\xi_2})^{+--} &= I_1(T^0_{\xi_1})^{-+-}, \qquad I_3(T^0_{\xi_3})^{-+-} = I_2(T^0_{\xi_2})^{--+}, \\ I_1(T^0_{\xi_1})^{--+} &= I_3(T^0_{\xi_3})^{+--}. \end{aligned} \tag{2.15}$$

The skew-symmetric part can be represented in the following way

$$b_{\xi_i} = I_i u, \quad i = 1, 2, 3, \tag{2.16}$$

where u is a traceless symmetric (1,1)-tensor on H which commutes with I_1, I_2, I_3.

If $n = 1$ then the tensor u vanishes identically, $u = 0$ and the torsion is a symmetric tensor, $T_\xi = T^0_\xi$.

CHAPTER 3

The torsion and curvature of the Biquard connection

Let $(M^{4n+3}, g, \mathbb{Q})$ be a quaternionic contact structure on a $4n+3$-dimensional smooth manifold. Working in a local chart, we fix the vertical space

$$V = span\{\xi_1, \xi_2, \xi_3\}$$

by requiring the conditions (2.10). The fundamental 2-forms $\omega_i, i = 1,2,3$ of the quaternionic structure $\mathbb{Q}$ are defined by

$$2\omega_{i|H} \;=\; d\eta_{i|H}, \qquad \xi\lrcorner\omega_i = 0, \quad \xi \in V. \tag{3.1}$$

Define three 2-forms $\theta_i, i = 1,2,3$ by the formulas

$$\begin{aligned} \theta_i &= \frac{1}{2}\{d((\xi_j\lrcorner d\eta_k)_{|H}) + (\xi_i\lrcorner d\eta_j)\wedge(\xi_i\lrcorner d\eta_k)\}_{|H} \\ &= \frac{1}{2}\{d(\xi_j\lrcorner d\eta_k) + (\xi_i\lrcorner d\eta_j)\wedge(\xi_i\lrcorner d\eta_k)\}_{|H} - d\eta_k(\xi_j,\xi_k)\omega_k + d\eta_k(\xi_i,\xi_j)\omega_i. \end{aligned} \tag{3.2}$$

Define, in addition, the corresponding $(1,1)$ tensors A_i by

$$g(A_i(X), Y) = \theta_i(X,Y), X, Y \in H.$$

We recall (2.11), which we shall use to define the orthogonal projections to the horizontal and vertical spaces H and V, respectively.

3.1. The torsion tensor

Due to (3.1), the torsion restricted to H has the form

$$T(X,Y) = -[X,Y]_{|V} = 2\sum_{s=1}^{3}\omega_s(X,Y)\xi_s, \qquad X, Y \in H. \tag{3.3}$$

The next two Lemmas provide some useful technical facts.

LEMMA 3.1. *Let D be any differentiation of the tensor algebra of H. Then we have the identities*

$$\begin{aligned} D(I_i)\,I_i \;&=\; -I_i\,D(I_i),\; i = 1,2,3, \\ I_1\,D(I_1)^{-+-} \;=\; I_2\,D(I_2)^{+--}, &\quad I_1\,D(I_1)^{-+-} \;=\; I_2\,D(I_2)^{+--}, \\ I_1\,D(I_1)^{-+-} \;&=\; I_2\,D(I_2)^{+--}. \end{aligned}$$

PROOF. The proof is a straightforward consequence of the next identities

$$0 = I_2(D(I_1) - I_2D(I_1)I_2) + I_1(D(I_2) - I_1D(I_2)I_1) = I_2D(I_1)^{-+-} + I_1D(I_2)^{+--},$$
$$0 = D(-\mathrm{Id}_V) = D(I_iI_i) = D(I_i)I_i + I_iD(I_i).$$

$\square$

With $\mathcal{L}$ denoting the Lie derivative, we shall denote by $\mathcal{L}^{'}$ its projection on the horizontal space, i.e., $\mathcal{L}^{'}_A(X)=[\mathcal{L}_A(X)]_H,\ A\in TM,\ X\in H.$

LEMMA 3.2. *The following identities hold true.*

$$\mathcal{L}^{'}_{\xi_1}I_1 = -2T^0_{\xi_1}I_1+d\eta_1(\xi_1,\xi_2)I_2+d\eta_1(\xi_1,\xi_3)I_3, \tag{3.4}$$

$$\begin{aligned}\mathcal{L}^{'}_{\xi_1}I_2 &= -2{T^0_{\xi_1}}^{--+}I_2-2I_3\tilde{u}+d\eta_1(\xi_2,\xi_1)I_1\\ &\quad+\frac{1}{2}(d\eta_1(\xi_2,\xi_3)-d\eta_2(\xi_3,\xi_1)-d\eta_3(\xi_1,\xi_2))I_3,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mathcal{L}^{'}_{\xi_2}I_1 &= -2{T^0_{\xi_2}}^{--+}I_1+2I_3\tilde{u}+d\eta_2(\xi_1,\xi_2)I_2\\ &\quad-\frac{1}{2}(-d\eta_1(\xi_2,\xi_3)+d\eta_2(\xi_3,\xi_1)-d\eta_3(\xi_1,\xi_2))I_3,\end{aligned} \tag{3.6}$$

where the symmetric endomorphism $\tilde{u}$ on H, commuting with I_1,I_2,I_3, is defined by

$$2\tilde{u}=I_3((\mathcal{L}^{'}_{\xi_1}I_2)^{--+})+\frac{1}{2}(d\eta_1(\xi_2,\xi_3)-d\eta_2(\xi_3,\xi_1)-d\eta_3(\xi_1,\xi_2))Id_H. \tag{3.7}$$

In addition, we have six more identities, which can be obtained with a cyclic permutation of (1,2,3).

PROOF. For all $k,l=1,2,3$ we have

$$\mathcal{L}_{\xi_k}\omega_l(X,Y)=\mathcal{L}_{\xi_k}g(I_lX,Y)+g((\mathcal{L}_{\xi_k}I_l)X,Y). \tag{3.8}$$

Cartan's formula yields

$$\mathcal{L}_{\xi_k}\omega_l=\xi_k\lrcorner(d\omega_l)+d(\xi_k\lrcorner\omega_l). \tag{3.9}$$

A direct calculation using (3.1) gives

$$2\omega_l=(d\eta_l)_{|H}=d\eta_l-\sum_{s=1}^{3}\eta_s\wedge(\xi_s\lrcorner d\eta_l)+\sum_{1\le s<t\le 3}d\eta_l(\xi_s,\xi_t)\eta_s\wedge\eta_t. \tag{3.10}$$

Combining (3.10) and (3.9) we obtain, after a short calculation, the following identities

$$(\mathcal{L}_{\xi_1}\omega_1)_{|H}=(d\eta_1(\xi_1,\xi_2)\omega_2+d\eta_1(\xi_1,\xi_3)\omega_3)_{|H} \tag{3.11}$$

$$2(\mathcal{L}_{\xi_i}\omega_j)_{|H}=(d(\xi_i\lrcorner d\eta_j)-(\xi_i\lrcorner d\eta_k)\wedge(\xi_k\lrcorner d\eta_j))_{|H}, \tag{3.12}$$

where $i\neq j\neq k\neq i,\quad i,j,k\in\{1,2,3\}$. Clearly, (3.11) and (3.8) imply (3.4).

Furthermore, using (2.10) and (3.12) twice for $i=1,j=2$ and $i=2,j=1$, we find

$$\begin{aligned}(\mathcal{L}_{\xi_1}\omega_2+\mathcal{L}_{\xi_2}\omega_1)_{|H} &= \frac{1}{2}(d(\xi_1\lrcorner d\eta_2)+d(\xi_2\lrcorner d\eta_1))_{|H}\\ &= d\eta_1(\xi_2,\xi_1)\omega_1+d\eta_2(\xi_1,\xi_2)\omega_2+(d\eta_1(\xi_2,\xi_3)+d\eta_2(\xi_1,\xi_3))\omega_3.\end{aligned} \tag{3.13}$$

On the other hand, (3.8) implies

$$\begin{aligned}&2T^0_{\xi_1}I_2+\mathcal{L}^{'}_{\xi_1}I_2+2T^0_{\xi_2}I_1+\mathcal{L}^{'}_{\xi_2}I_1\\ &\quad= d\eta_1(\xi_2,\xi_1)I_1+d\eta_2(\xi_1,\xi_2)I_2+(d\eta_1(\xi_2,\xi_3)+d\eta_2(\xi_1,\xi_3))I_3.\end{aligned} \tag{3.14}$$

Let us decompose (3.14) into $Sp(n)$-invariant components:

$$(\mathcal{L}'_{\xi_1} I_2)^{+--} = -2T^{0\ --+}_{\xi_1} I_2 + d\eta_1(\xi_2,\xi_1) I_1,$$
$$(\mathcal{L}'_{\xi_2} I_1)^{-+-} = -2T^{0\ --+}_{\xi_2} I_1 + d\eta_2(\xi_1,\xi_2) I_2, \tag{3.15}$$

$$\left(\mathcal{L}'_{\xi_1} I_2 + \mathcal{L}'_{\xi_2} I_1\right)^{--+} = (d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_1,\xi_3)) I_3. \tag{3.16}$$

Using (3.16) and (3.7), we obtain

$$2\tilde{u} = -I_3((\mathcal{L}'_{\xi_2} I_1)^{--+}) + \frac{1}{2}(-d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2)) Id_H.$$

The latter, together with (3.7), tells us that $\tilde{u}$ commutes with all $I \in Q$. Now, Lemma 3.1 with $D = \mathcal{L}'$ implies (3.5) and (3.6). The vanishing of the symmetric part of the left hand side in (3.8) for $k = 1, \quad l = 2$, combined with (3.18) and (3.5) yields $0 = -2g(I_3\tilde{u}X, Y) - 2g(I_3\tilde{u}Y, X)$. As $\tilde{u}$ commutes with all $I \in \mathbb{Q}$ we conclude that $\tilde{u}$ is symmetric.

The rest of the identities can be obtained through a cyclic permutation of (1,2,3). ∎

We describe the properties of the quaternionic contact torsion more precisely in the next Proposition.

Proposition 3.3. *The torsion of the Biquard connection satisfies the identities:*

$$T_{\xi_i} = T^0_{\xi_i} + I_i u, \quad i = 1,2,3, \tag{3.17}$$

$$T^0_{\xi_i} = \frac{1}{2} L_{\xi_i} g, \quad i = 1,2.3, \tag{3.18}$$

$$u = \tilde{u} - \frac{tr(\tilde{u})}{4n} Id_H, \tag{3.19}$$

where the symmetric endomorphism $\tilde{u}$ on H commuting with I_1, I_2, I_3 satisfies

$$\begin{aligned}\tilde{u} &= \frac{1}{2} I_1 A_1^{+--} + \frac{1}{4}\big(-d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2)\big) Id_H \\ &= \frac{1}{2} I_2 A_2^{-+-} + \frac{1}{4}\big(d\eta_1(\xi_2,\xi_3) - d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2)\big) Id_H \\ &= \frac{1}{2} I_3 A_3^{--+} + \frac{1}{4}\big(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2)\big) Id_H.\end{aligned} \tag{3.20}$$

For $n = 1$ the tensor $u = 0$ and $\tilde{u} = \frac{tr(\tilde{u})}{4} Id_H$.

Proof. Expressing the Lie derivative in terms of the Biquard connection, using that ∇ preserves the splitting $H \oplus V$, we see that for $X, Y \in H$ we have

$$\mathcal{L}_{\xi_i} g(X,Y) = g(\nabla_X \xi_i, Y) + g(\nabla_Y \xi_i, X) + g(T_{\xi_i} X, Y) + g(T_{\xi_i} Y, X) = 2g(T^0_{\xi_i} X, Y).$$

To show that $\tilde{u}$ satisfies (3.20), insert (3.12) into (3.2) to get

$$\theta_3 = (\mathcal{L}_{\xi_1}\omega_2)_{|H} - d\eta_2(\xi_1,\xi_2)\omega_3 + d\eta_2(\xi_3,\xi_1)\omega_3. \tag{3.21}$$

A substitution of (3.8) and (3.5) in (3.21) gives

$$\begin{aligned}A_3 &= 2T^{0\ -+-}_{\xi_1} I_2 - 2I_3\tilde{u} \\ &+ d\eta_1(\xi_2,\xi_1) I_1 - d\eta_2(\xi_1,\xi_2) I_2 + \frac{1}{2}(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2)) I_3.\end{aligned} \tag{3.22}$$

Now, by comparing the (+++) component on both sides of (3.22) we see the last equality of (3.20). The rest of the identities can be obtained with a cyclic permutation of (1,2,3).

Turning to the rest of the identities, let Σ^2 and Λ^2 denote, respectively, the subspaces of symmetric and skew-symmetric endomorphisms of H. Let $skew : End(H) \to \Lambda^2$ be the natural projection with kernel Σ^2. We have

$$\begin{aligned} 4[T_{\xi_i}]_{(\Sigma^2\oplus sp(n))^\perp} &= 3skew(T_{\xi_i}) + I_1 skew(T_{\xi_i})I_1 + I_2 skew(T_{\xi_i})I_2 + I_3 skew(T_{\xi_i})I_3 \\ &= \sum_{s=1}^{3}(skew(T_{\xi_i}) \ + \ I_s skew(T_{\xi_i})I_s). \end{aligned}$$

According to Theorem 2.4, $T_\xi X \in H$ for $X \in H, \xi \in V$. Hence,

$$T(\xi, X) = \nabla_\xi X - [\xi, X]_H = \nabla_\xi X - \mathcal{L}'_\xi(X). \tag{3.23}$$

An application of (3.23) gives

$$\begin{aligned} g(4[T_{\xi_i}]_{(\Sigma^2\oplus sp(n))^\perp}X, Y) = &-\sum_{s=1}^{3} g\big((\nabla_{\xi_i} I_s)X, I_s Y\big) \\ &+ \frac{1}{2}\sum_{s=1}^{3}\{g\big((\mathcal{L}_{\xi_i} I_s)X, I_s Y\big) - g((\mathcal{L}_{\xi_i} I_s)Y, I_s X)\}. \end{aligned} \tag{3.24}$$

Let $B(H)$ be the orthogonal complement of $\Sigma^2\oplus sp(n)\oplus sp(1)$ in $\mathrm{End}(H)$. Obviously, $B(H)\subset\Lambda^2$ and we have the following splitting of $\mathrm{End}(H)$ into mutually orthogonal components

$$\mathrm{End}(H) = \Sigma^2 \oplus sp(n) \oplus sp(1) \oplus B(H). \tag{3.25}$$

If Ψ is an arbitrary section of the bundle Λ^2 of M, the orthogonal projection of Ψ into $B(H)$ is given by

$$[\Psi]_{B(H)} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+} - [\Psi]_{sp(1)},$$

where $[\Psi]_{sp(1)}$ is the orthogonal projection of Ψ onto $sp(1)$. We also have

$$[\Psi]_{sp(1)} = \frac{1}{4n}\sum_{s=1}^{3}\sum_{a=1}^{4n} g(\Psi e_a, I_s e_a)I_s.$$

Theorem 2.4 - (iv) and the decomposition (3.25) yield

$$\begin{aligned} T_{\xi_i} = [T_{\xi_i}]_{(sp(n)\oplus sp(1))^\perp} = [T_{\xi_i}]_{\Sigma^2} + [T_{\xi_i}]_{B(H)} \\ = T^0_{\xi_i} + [T_{\xi_i}]_{(\Sigma^2\oplus sp(n))^\perp} - [T_{\xi_i}]_{sp(1)}. \end{aligned} \tag{3.26}$$

Using (3.24), Lemma 3.2 and the fact that $I_s(\nabla_{\xi_i} I_s) \in sp(1)$, we compute

$$\begin{aligned} &4[T_{\xi_i}]_{(\Sigma^2\oplus sp(n))^\perp} \ - \ [T_{\xi_i}]_{sp(1)} \\ &= \ -\sum_{s=1}^{3}\big\{\, skew(I_s(\mathcal{L}'_{\xi_i} I_s)) - [I_s(\mathcal{L}'_{\xi_i} I_s)]_{sp(1)}\big\} \ = \ \sum_{s=1}^{3} skew(2I_s T^0_{\xi_i} I_s) + 4u \ = \ 4u. \end{aligned} \tag{3.27}$$

A substitution of (3.27) in (3.26) completes the proof. □

The $Sp(n)$-invariant splitting of (3.22) leads to the following Corollary.

Corollary 3.4. *The (1,1)-tensors A_i satisfy the equalities*

$$A_3^{+++} = 2T^{0\ -+-}_{\xi_1} I_2, \quad A_3^{+--} = d\eta_1(\xi_2,\xi_1)I_1, \quad A_3^{-+-} = -d\eta_2(\xi_1,\xi_2)I_2,$$
$$A_3^{--+} = -2I_3\tilde{u} + \frac{1}{2}(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2))I_3.$$

Analogous formulas for A_1 and A_2 can be obtained by a cyclic permutation of $(1,2,3)$.

Proposition 3.5. *The covariant derivative of the quaternionic contact structure with respect to the Biquard connection is given by*

$$\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j, \tag{3.28}$$

where the $sp(1)$-connection 1-forms α_s are determined by

$$\alpha_i(X) = d\eta_k(\xi_j, X) = -d\eta_j(\xi_k, X), \quad X \in H, \quad \xi_i \in V, \tag{3.29}$$
$$\alpha_i(\xi_s) = d\eta_s(\xi_j,\xi_k) - \delta_{is}\left(\frac{tr(\tilde{u})}{2n} + \frac{1}{2}\left(d\eta_1(\xi_2,\xi_3) + d\eta_2(\xi_3,\xi_1) + d\eta_3(\xi_1,\xi_2)\right)\right), \tag{3.30}$$

for $s = 1,2,3$ and (i,j,k) any cyclic permutation of $(1,2,3)$.

Proof. The equality (3.29) is proved by Biquard in [**Biq1**]. Using (3.23), we obtain

$$\nabla_{\xi_s} I_i = [T_{\xi_s}, I_i] + \mathcal{L}'_{\xi_s} I_i = [T^0_{\xi_s}, I_i] + u[I_s, I_i] + \mathcal{L}'_{\xi_s} I_i.$$

An application of Lemma 3.2 completes the proof. □

Corollary 3.6. *The covariant derivative of the distribution V is given by*

$$\nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j.$$

We finish this section by expressing the Biquard connection in terms of the Levi-Civita connection D^g of the metric g, namely, we have

$$\nabla_B Y = D^g_B Y + \sum_{s=1}^3 \{((D^g_B \eta_s)Y)\xi_s + \eta_s(B)(I_s u - I_s)Y\}, \quad B \in TM, \quad Y \in H. \tag{3.31}$$

Indeed, for $B = X \in H$ formula (3.31) follows from the equation $\nabla_X Y = [D^g_X Y]_H$. If $B \in V$ we may assume $B = \xi_1$ and for $Z \in H$ we compute

$$\begin{aligned} 2g(D^g_{\xi_1} Y, Z) &= \xi_1 g(Y,Z) + g([\xi_1, Y], Z) - g([\xi_1, Z], Y) - g([Y,Z],\xi_1) = \\ &= (\mathcal{L}_{\xi_1} g)(Y,Z) + 2g([\xi_1,Y],Z) + d\eta_1(Y,Z) = 2g(T_{\xi_1} Y + [\xi_1, Y], Z) \\ &- 2g(I_1 u Y, Z) + 2g(I_1 Y, Z) = 2g(\nabla_{\xi_1} Y, Z) - 2g((I_1 u - I_1)Y, Z). \end{aligned}$$

In the above calculation we used (3.23) and Proposition 3.3.

Note that the covariant derivatives $\nabla_B \xi_s$ are also determined by (3.31) in view of the relation $g(\nabla_B \xi_s, \xi_k) = \frac{1}{4n} g(\nabla_B I_s, I_k)$, $s,k = 1,2,3$

3.2. The Curvature Tensor

Let $R = [\nabla, \nabla] - \nabla_{[\ ,\ ]}$ be the curvature tensor of ∇. For any $B, C \in \Gamma(TM)$ the curvature operator R_{BC} preserves the QC structure on M since ∇ preserves it. In particular R_{BC} preserves the distributions H and V, the quaternionic structure $\mathbb{Q}$ on H and the $(2,1)$ tensor φ. Moreover, the action of R_{BC} on V is completely determined by its action on H, $R_{BC}\xi_i = \varphi^{-1}([R_{BC}, I_i])$, $\ i = 1, 2, 3$. Thus, we may regard R_{BC} as an endomorphism of H and we have $R_{BC} \in sp(n) \oplus sp(1)$.

Definition 3.7. *The Ricci 2-forms ρ_i are defined by*

$$\rho_i(B, C) = \frac{1}{4n}\sum_{a=1}^{4n} g(R(B, C)e_a, I_i e_a),\ \ B, C \in \Gamma(TM).$$

Hereafter $\{e_1, \dots, e_{4n}\}$ will denote an orthonormal basis of H. We decompose the curvature into $sp(n) \oplus sp(1)$-parts. Let $R^0_{BC} \in sp(n)$ denote the $sp(n)$-component.

Lemma 3.8. *The curvature of the Biquard connection decomposes as follows*

$$R_{BC} = R^0_{BC} + \rho_1(B, C)I_1 + \rho_2(B, C)I_2 + \rho_3(B, C)I_3.$$

$$[R_{BC}, I_i] = 2(-\rho_j(B, C)I_k + \rho_k(B, C)I_j),\ \ B, C \in \Gamma(TM), \tag{3.32}$$

$$\rho_i = \frac{1}{2}(d\alpha_i + \alpha_j \wedge \alpha_k), \tag{3.33}$$

where the connection 1-forms α_s are determined in (3.29), (3.30).

Proof. The first two identities follow directly from the definitions. Using (3.28), we calculate

$$[R_{BC}, I_1] = \nabla_B(\alpha_3(C)I_2 - \alpha_2(C)I_3) - \nabla_C(\alpha_3(B)I_2 - \alpha_2(B)I_3) - (\alpha_3([B, C])I_2 - \alpha_2([B, C])I_3) = -(d\alpha_2 + \alpha_3 \wedge \alpha_1)(B, C)I_3 + (d\alpha_3 + \alpha_1 \wedge \alpha_2)(B, C)I_2.$$

Now (3.32) completes the proof. $\square$

Definition 3.9. *The quaternionic contact Ricci tensor (*qc-Ricci tensor *for short) and the qc-scalar curvature Scal of the Biquard connection are defined by*

$$Ric(B, C) = \sum_{a=1}^{4n} g(R(e_a, B)C, e_a), \qquad Scal = \sum_{a=1}^{4n} Ric(e_a, e_a). \tag{3.34}$$

It is known, cf. [Biq1], that the qc-Ricci tensor restricted to H is a symmetric tensor. In addition, we define six Ricci-type tensors ζ_i, τ_i, $i = 1, 2, 3$ as follows

$$\begin{aligned} \zeta_i(B, C) &= \frac{1}{4n}\sum_{a=1}^{4n} g(R(e_a, B)C, I_i e_a), \\ \tau_i(B, C) &= \frac{1}{4n}\sum_{a=1}^{4n} g(R(e_a, I_i e_a)B, C). \end{aligned} \tag{3.35}$$

We shall show that all Ricci-type contractions evaluated on the horizontal space H are determined by the components of the torsion. First, define the following 2-tensors on H using the tensors from Proposition 3.3

$$\begin{aligned} T^0(X, Y) &\overset{def}{=} g((T^0_{\xi_1}I_1 + T^0_{\xi_2}I_2 + T^0_{\xi_3}I_3)X, Y), \\ U(X, Y) &\overset{def}{=} g(uX, Y), \quad X, Y \in H. \end{aligned} \tag{3.36}$$

LEMMA 3.10. *The tensors T^0 and U are $Sp(n)Sp(1)$-invariant trace-free symmetric tensors with the properties:*

$$T^0(X,Y)+T^0(I_1X,I_1Y)+T^0(I_2X,I_2Y)+T^0(I_3X,I_3Y)=0, \tag{3.37}$$
$$3U(X,Y)-U(I_1X,I_1Y)-U(I_2X,I_2Y)-U(I_3X,I_3Y)=0. \tag{3.38}$$

PROOF. The lemma follows directly from (2.14), (2.16) of Proposition 2.5. □

We turn to a Lemma, which shall be used later.

LEMMA 3.11. *For any $X,Y\in H$, $\quad B\in H\oplus V$, we have*

$$Ric(B,I_iY)+4n\zeta_i(B,Y) \;=\; 2\rho_j(B,I_kY)-2\rho_k(B,I_jY), \tag{3.39}$$
$$\begin{aligned}\zeta_i(X,Y) \;=\; & -\frac{1}{2}\rho_i(X,Y)+\frac{1}{2n}g(I_iuX,Y)+\frac{2n-1}{2n}g(T^0_{\xi_i}X,Y)\\ & + \frac{1}{2n}g(I_jT^0_{\xi_k}X,Y)-\frac{1}{2n}g(I_kT^0_{\xi_j}X,Y).\end{aligned} \tag{3.40}$$

The Ricci 2-forms evaluated on H satisfy

$$\begin{aligned}\rho_1(X,Y) &= 2g(T^{0--+}_{\xi_2}I_3X,Y)-2g(I_1uX,Y)-\frac{tr(\tilde u)}{n}\omega_1(X,Y),\\ \rho_2(X,Y) &= 2g(T^{0+--}_{\xi_3}I_1X,Y)-2g(I_2uX,Y)-\frac{tr(\tilde u)}{n}\omega_2(X,Y),\\ \rho_3(X,Y) &= 2g(T^{0-+-}_{\xi_1}I_2X,Y)-2g(I_3uX,Y)-\frac{tr(\tilde u)}{n}\omega_3(X,Y).\end{aligned} \tag{3.41}$$

The 2-forms τ_s evaluated on H satisfy

$$\begin{aligned}\tau_1(X,Y) &= \rho_1(X,Y)+2g(I_1uX,Y)+\frac{4}{n}g(T^{0--+}_{\xi_2}I_3X,Y),\\ \tau_2(X,Y) &= \rho_2(X,Y)+2g(I_2X,Y)+\frac{4}{n}g(T^{0+--}_{\xi_3}I_1X,Y),\\ \tau_3(X,Y) &= \rho_3(X,Y)+2g(I_3X,Y)+\frac{4}{n}g(T^{0-+-}_{\xi_1}I_2X,Y).\end{aligned} \tag{3.42}$$

For $n=1$ the above formulas hold with $U=0$.

PROOF. From (3.32) we have

$$\begin{aligned}Ric(B,I_1Y)\;+\;4n\zeta_1(B,Y) \;&=\; \sum_{a=1}^{4n}\{R(e_a,B,I_1Y,e_a)\;+\;R(e_a,B,Y,I_1e_a)\}\\ &= \sum_{a=1}^{4n}\{-2\rho_2(e_a,B)\omega_3(Y,e_a)\;+\;2\rho_3(e_a,B)\omega_2(Y,e_a))\}\\ &= 2\rho_2(B,I_3Y)\;-\;2\rho_3(B,I_2Y),\end{aligned}$$

Using (3.2) and (3.33) we obtain $\rho_1(X,Y)=A_1(X,Y)-\frac{1}{2}\alpha_1([X,Y]_V)=A_1(X,Y)+\sum_{s=1}^3\omega_s(X,Y)\alpha_1(\xi_s)$. Now, Corollary 3.4 and Corollary 3.5 imply the first equality in (3.41). The other two equalities in (3.41) can be obtained in the same manner.

Letting $b(X,Y,Z,W) = 2\sigma_{X,Y,Z}\{\sum_{l=1}^{3}\omega_l(X,Y)g(T_{\xi_l}Z,W)\}$, where $\sigma_{X,Y,Z}$ is the cyclic sum over X,Y,Z, we have

$$\sum_{a=1}^{4n} b(X,Y,e_a,I_1e_a) = 4g(I_1uX,Y) + 8g(I_2T^{0-+-}_{\xi_3}X,Y), \tag{3.43}$$

$$\begin{aligned}\sum_{a=1}^{4n} b(e_a,I_1e_a,X,Y) &= (8n-4)g(T^0_{\xi_1}X,Y) + (8n+4)g(I_1uX,Y)\\ &\quad + 4g(T^0_{\xi_2}I_3X,Y) - 4g(T^0_{\xi_3}I_2X,Y).\end{aligned} \tag{3.44}$$

The first Bianchi identity gives

$$\begin{aligned}&4n\,(\tau_1(X,Y) + 2\zeta_1(X,Y))\\ &= \sum_{a=1}^{4n}\{R(e_a,I_1e_a,X,Y) + R(X,e_a,I_1e_a,Y) + R(I_1e_a,X,e_a,Y)\}\\ &\qquad = \sum_{a=1}^{4n} b(e_a,I_1e_a,X,Y)\end{aligned} \tag{3.45}$$

and also

$$4n(\tau_1(X,Y) - \rho_1(X,Y)) = \sum_{a=1}^{4n}\{R(e_a,I_1e_a,X,Y) - R(X,Y,e_a,I_1e_a)\} \tag{3.46}$$

$$= \frac{1}{2}\sum_{a=1}^{4n}\{b(e_a,I_1e_a,X,Y) - b(e_a,I_1e_a,Y,X) - b(e_a,X,Y,I_1e_a) + b(I_1e_a,X,Y,e_a)\}.$$

Taking into account (3.43), (3.44), (3.45) and (3.46) yield the first set of equalities in (3.42) and (3.40). The other equalities in (3.42) and (3.40) can be shown similarly. This completes the proof of Lemma 3.11. $\square$

Theorem 3.12. *Let $(M^{4n+3},g,\mathbb{Q})$ be a quaternionic contact $(4n+3)$-dimensional manifold, $n>1$. For any $X,Y\in H$ the qc-Ricci tensor and the qc-scalar curvature satisfy*

$$\begin{aligned} Ric(X,Y) &= (2n+2)T^0(X,Y) + (4n+10)U(X,Y)\\ &\qquad + (2n+4)\frac{tr(\tilde u)}{n}g(X,Y),\\ Scal &= (8n+16)tr(\tilde u).\end{aligned} \tag{3.47}$$

For $n=1$ *we have* $Ric(X,Y) = 4T^0(X,Y) + 6\frac{tr(\tilde u)}{n}g(X,Y)$.

Proof. The proof follows from Lemma 3.11, (3.40), (3.41) and (3.39). If $n=1$, recall $U=0$ to obtain the last equality. $\square$

Corollary 3.13. *The qc-scalar curvature satisfies the equalities*

$$\frac{Scal}{2(n+2)} = \sum_{a=1}^{4n}\rho_i(I_ie_a,e_a) = \sum_{a=1}^{4n}\tau_i(I_ie_a,e_a) = -2\sum_{a=1}^{4n}\zeta_i(I_ie_a,e_a), \quad i=1,2,3.$$

We determine the function λ in (2.12) in the next Corollary.

Corollary 3.14. *The torsion of the Biquard connection restricted to V satisfies the equality*

$$T(\xi_i,\xi_j) = -\frac{Scal}{8n(n+2)}\xi_k - [\xi_i,\xi_j]_H. \tag{3.48}$$

Proof. A small calculation using Corollary 3.6 and Proposition 3.5, gives

$$T(\xi_i,\xi_j) = \nabla_{\xi_i}\xi_j - \nabla_{\xi_j}\xi_i - [\xi_i,\xi_j] = -\frac{tr(\tilde{u})}{n}\xi_k - [\xi_i,\xi_j]_H.$$

Now, the assertion follows from the second equality in (3.47). $\square$

Corollary 3.15. *The tensors $T^0, U, \tilde{u}$ do not depend on the choice of the local basis.*

CHAPTER 4

QC-Einstein quaternionic contact structures

The goal of this section is to show that the vanishing of the torsion of the quaternionic contact structure implies that the qc-scalar curvature is constant and to prove our classification Theorem 1.3. The Bianchi identities will have an important role in the analysis.

Definition 4.1. *A quaternionic contact structure is* qc-Einstein *if the qc-Ricci tensor is trace-free,*

$$Ric(X,Y) = \frac{Scal}{4n} g(X,Y), \quad X,Y \in H.$$

Proposition 4.2. *A quaternionic contact manifold $(M, g, \mathbb{Q})$ is a qc-Einstein if and only if the quaternionic contact torsion vanishes identically, $T_\xi = 0, \xi \in V$.*

Proof. If $(\eta, \mathbb{Q})$ is qc-Einstein structure then $T^0 = U = 0$ because of (3.47). We will use the same symbol T^0 for the corresponding endomorphism of the 2-tensor T^0 on H. According to (3.36), we have $T^0 = T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3$. Using first (2.14) and then (2.15), we compute

$$(T^0)^{+--} = (T^0_{\xi_2})^{--+} I_2 + (T^0_{\xi_3})^{-+-} I_3 = 2(T^0_{\xi_2})^{--+} I_2. \tag{4.1}$$

Hence, $T_{\xi_2} = T^0_{\xi_2} + I_2 u$ vanishes. Similarly $T_{\xi_1} = T_{\xi_3} = 0$. The converse follows from (3.47). □

Proposition 4.3. *For $X \in V$ and any cyclic permutation (i,j,k) of $(1,2,3)$ we have*

$$\rho_i(X,\xi_i) = -\frac{X(Scal)}{32n(n+2)} + \frac{1}{2}(\omega_i([\xi_j,\xi_k],X) - \omega_j([\xi_k,\xi_i],X) - \omega_k([\xi_i,\xi_j],X)), \tag{4.2}$$

$$\rho_i(X,\xi_j) = \omega_j([\xi_j,\xi_k],X), \quad \rho_i(X,\xi_k) = \omega_k([\xi_j,\xi_k],X), \tag{4.3}$$

$$\rho_i(I_kX,\xi_j) = -\rho_i(I_jX,\xi_k) = g(T(\xi_j,\xi_k),I_iX) = \omega_i([\xi_j,\xi_k],X). \tag{4.4}$$

Proof. Since ∇ preserves the splitting $H \oplus V$, the first Bianchi identity, (3.48) and (3.32) imply

$$\begin{aligned} 2\rho_i(X,\xi_i) + 2\rho_j(X,\xi_j) &= g(R(X,\xi_i)\xi_j,\xi_k) + g(R(\xi_j,X)\xi_i,\xi_k) \\ &= \sigma_{\xi_i,\xi_j,X}\{g((\nabla_{\xi_i}T)(\xi_j,X),\xi_k) + g(T(T(\xi_i,\xi_j),X),\xi_k)\} \\ &= g((\nabla_X T)(\xi_i,\xi_j),\xi_k) + g(T(T(\xi_i,\xi_j),X),\xi_k) = -\frac{X(Scal)}{8n(n+2)} - 2\omega_k([\xi_i,\xi_j],X). \end{aligned} \tag{4.5}$$

Summing the first two equalities in (4.5) and subtracting the third one, we obtain (4.2). Similarly,

$$
\begin{aligned}
2\rho_k(\xi_j, X) = g(R(\xi_j, X)\xi_i, \xi_j) &= \sigma_{\xi_i,\xi_j,X}\{g((\nabla_{\xi_i} T)(\xi_j, X), \xi_j) + g(T(T(\xi_i, \xi_j), X), \xi_j)\} \\
&= g(T(T(\xi_i, \xi_j), X), \xi_j) = g(T(-[\xi_i, \xi_j]_H, X), \xi_j) = g([[\xi_i, \xi_j]_H, X], \xi_j) \\
&= -d\eta_j([\xi_i, \xi_j]_H, X) = -2\omega_j([\xi_i, \xi_j], X).
\end{aligned}
$$

Hence, the second equality in (4.3) follows. Analogous calculations show the validity of the first equality in (4.3). Then, (4.4) is a consequence of (4.3) and (3.48). □

The vertical derivative of the qc-scalar curvature is determined in the next Proposition.

PROPOSITION 4.4. *On a QC manifold we have*

$$
\rho_i(\xi_i, \xi_j) + \rho_k(\xi_k, \xi_j) = \frac{1}{16n(n+2)}\xi_j(Scal). \tag{4.6}
$$

PROOF. Since ∇ preserves the splitting $H \oplus V$, the first Bianchi identity and (3.48) imply

$$
\begin{aligned}
&-2(\rho_i(\xi_i, \xi_j) + \rho_k(\xi_k, \xi_j)) = g(\sigma_{\xi_i,\xi_j,\xi_k}\{R(\xi_i, \xi_j)\xi_k\}, \xi_j) \\
&\quad = g(\sigma_{\xi_i,\xi_j,\xi_k}\{(\nabla_{\xi_i} T)(\xi_j, \xi_k) + T(T(\xi_i, \xi_j), \xi_k)\}, \xi_j) = -\frac{1}{8n(n+2)}\xi_j(Scal).
\end{aligned}
$$

□

4.1. The Bianchi identities

In order to derive the essential information contained in the Bianchi identities we need the next Lemma, which is an application of a standard result in differential geometry.

LEMMA 4.5. *In a neighborhood of any point $p \in M^{4n+3}$ and a $\mathbb{Q}$-orthonormal basis*

$$
\{X_1(p), X_2(p) = I_1X_1(p) \dots, X_{4n}(p) = I_3X_{4n-3}(p), \xi_1(p), \xi_2(p), \xi_3(p)\}
$$

of the tangential space at p, there exists a $\mathbb{Q}$ - orthonormal frame field

$$
\begin{aligned}
\{X_1, X_2 = I_1X_1, \dots, X_{4n} = I_3X_{4n-3}, \xi_1, \xi_2, \xi_3\}, X_{a_{|p}} = X_a(p), \xi_{s_{|p}} = \xi_s(p), \\
a = 1, \dots, 4n, i = 1, 2, 3,
\end{aligned}
$$

such that the connection 1-forms of the Biquard connection are all zero at the point p, i.e., we have

$$
(\nabla_{X_a} X_b)_{|}p = (\nabla_{\xi_i} X_b)_{|}p = (\nabla_{X_a} \xi_t)_{|}p = (\nabla_{\xi_t} \xi_s)_{|}p = 0, \tag{4.7}
$$

for $a, b = 1, \dots, 4n, s, t, r = 1, 2, 3$. In particular,

$$
((\nabla_{X_a} I_s)X_b)_{|}p = ((\nabla_{X_a} I_s)\xi_t)_{|}p = ((\nabla_{\xi_t} I_s)X_b)_{|}p = ((\nabla_{\xi_t} I_s)\xi_r)_{|}p = 0.
$$

PROOF. Since ∇ preserves the splitting $H \oplus V$ we can apply the standard arguments for the existence of a normal frame with respect to a metric connection (see e.g. [Wu]). We sketch the proof for completeness.

Let $\{\tilde{X}_1, \dots, \tilde{X}_{4n}, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3\}$ be a $\mathbb{Q}$-orthonormal basis around p such that $\tilde{X}_{a_{|p}} = X_a(p)$, $\tilde{\xi}_{i_{|p}} = \xi_i(p)$. We want to find a modified frame $X_a = o_a^b \tilde{X}_b$, $\xi_i = o_i^j \tilde{\xi}_j$, which satisfies the normality conditions of the lemma.

Let ϖ be the $sp(n)\oplus sp(1)$-valued connection 1-forms with respect to the frame $\{\tilde X_1,\dots,\tilde X_{4n},\tilde\xi_1,\tilde\xi_2,\tilde\xi_3\}$, i.e.,

$$\nabla\tilde X_b=\varpi_b^c\tilde X_c,\quad \nabla\tilde\xi_s=\varpi_s^t\tilde\xi_t,\quad B\in\{\tilde X_1,\dots,\tilde X_{4n},\tilde\xi_1,\tilde\xi_2,\tilde\xi_3\}.$$

Let $\{x^1,\dots,x^{4n+3}\}$ be a coordinate system around p such that

$$\frac{\partial}{\partial x^a}(p)=X_a(p),\quad \frac{\partial}{\partial x^{4n+t}}(p)=\xi_t(p),\quad a=1,\dots,4n,\quad t=1,2,3.$$

One can easily check that the matrices with entries

$$o_a^b=exp\left(-\sum_{c=1}^{4n+3}\varpi_a^b(\frac{\partial}{\partial x^c})_{|p}x^c\right)\in Sp(n),\ o_t^s=exp\left(-\sum_{c=1}^{4n+3}\varpi_t^s(\frac{\partial}{\partial x^c})_{|p}x^c\right)\in Sp(1)$$

are the desired matrices making the identities (4.7) true.

Now, the last identity in the lemma is a consequence of the fact that the choice of the orthonormal basis of V does not depend on the action of $SO(3)$ on V combined with Corollary 3.6 and Proposition 3.5. □

Definition 4.6. *We refer to the orthonormal frame constructed in Lemma 4.5 as a* qc-normal frame.

Let us fix a qc-normal frame $\{e_1,\dots,e_{4n},\xi_1,\xi_2,\xi_3\}$. We shall denote with X,Y,Z horizontal vector fields $X,Y,Z\in H$ and keep the notation for the torsion of type (0,3) $T(B,C,D)=g(T(B,C),D)$, $B,C,D\in H\oplus V$.

Proposition 4.7. *On a quaternionic contact manifold $(M^{4n+3},g,\mathbb{Q})$ the following identities hold true*

$$2\sum_{a=1}^{4n}(\nabla_{e_a}Ric)(e_a,X)-X(Scal)=4\sum_{r=1}^{3}Ric(\xi_r,I_rX)-8n\sum_{r=1}^{3}\rho_r(\xi_r,X),\tag{4.8}$$

$$Ric(\xi_s,I_sX)=2\rho_q(I_tX,\xi_s)+2\rho_t(I_sX,\xi_q)+\sum_{a=1}^{4n}(\nabla_{e_a}T)(\xi_s,I_sX,e_a),\tag{4.9}$$

$$\begin{aligned}&4n(\rho_s(X,\xi_s)-\zeta_s(\xi_s,X))\\&\qquad=2\rho_q(I_tX,\xi_s)+2\rho_t(I_sX,\xi_q)-\sum_{a=1}^{4n}(\nabla_{e_a}T)(\xi_s,X,I_se_a),\end{aligned}\tag{4.10}$$

$$\zeta_s(\xi_s,X)=-\frac{1}{4n}\sum_{a=1}^{4n}(\nabla_{e_a}T)(\xi_s,I_sX,e_a),\tag{4.11}$$

where $s\in\{1,2,3\}$ is fixed and (s,t,q) is an even permutation of $(1,2,3)$.

Proof. The second Bianchi identity implies

$$\begin{aligned}2\sum_{a=1}^{4n}(\nabla_{e_a}Ric)(e_a,X)-X(Scal)+2\sum_{a=1}^{4n}Ric(T(e_a,X),e_a)\\+\sum_{a,b=1}^{4n}R(T(e_b,e_a),X,e_b,e_a)=0.\end{aligned}$$

An application of (3.3) to the last equality gives (4.8).

The first Bianchi identity combined with (2.13), (3.3) and the fact that ∇ preserves the orthogonal splitting $H \oplus V$ yield

$$Ric(\xi_s, I_sX) = \sum_{a=1}^{4n}\left((\nabla_{e_a}T)(\xi_s, I_sX, e_a) + 2\sum_{r=1}^{3}\omega_r(I_sX, e_a)T(\xi_r, \xi_s, e_a)\right) =$$
$$= \sum_{a=1}^{4n}(\nabla_{e_a}T)(\xi_s, I_sX, e_a) + 2T(\xi_s, \xi_t, I_qX) + 2T(\xi_q, \xi_s, I_tX),$$

which, together with (4.4), completes the proof of (4.9).

In a similar fashion, from the first Bianchi identity, (2.13), (3.3) and the fact that ∇ preserves the orthogonal splitting $H \oplus V$ we can obtain the proof of (4.10). Finally, take (3.39) with $B = \xi_i$ and combine the result with (4.9) to get (4.11). $\square$

The following Theorem gives relations between $Sp(n)Sp(1)$-invariant tensors and is crucial for the solution of the Yamabe problem, which we shall undertake in the last Chapter. We define the horizontal divergence ∇^*P of a (0,2)-tensor field P with respect to Biquard connection to be the (0,1)-tensor defined by

$$\nabla^*P(.) = \sum_{a=1}^{4n}(\nabla_{e_a}P)(e_a, .),$$

where $e_a, a = 1, \dots, 4n$ is an orthonormal basis on H.

THEOREM 4.8. *The horizontal divergences of the curvature and torsion tensors satisfy the system* $B\,b = 0$*, where*

$$\mathbf{B} = \begin{pmatrix} -1 & 6 & 4n-1 & \frac{3}{16n(n+2)} & 0 \\ -1 & 0 & n+2 & \frac{3}{16(n+2)} & 0 \\ 1 & -3 & 4 & 0 & -1 \end{pmatrix},$$

$$\mathbf{b} = \Big(\ \nabla^* T^0, \quad \nabla^* U, \quad \mathbb{A}, \quad d\,Scal\,|_H, \quad \textstyle\sum_{j=1}^3 Ric\ (\xi_j, I_j\ .\ )\ \Big)^t,$$

with T^0 *and* U *defined in* (3.36) *and*

$$\mathbb{A}(X) = g(I_1[\xi_2, \xi_3] + I_2[\xi_3, \xi_1] + I_3[\xi_1, \xi_2], X).$$

PROOF. Throughout the proof of Theorem 4.8 (s, t, q) will denote an even permutation of $(1, 2, 3)$. Equations (4.2) and (4.4) yield

$$\sum_{r=1}^{3}\rho_r(X, \xi_r) = -\frac{3}{32n(n+2)}X(Scal) - \frac{1}{2}\mathbb{A}(X), \tag{4.12}$$

$$\sum_{s=1}^{3}\rho_q(I_tX, \xi_s) = \mathbb{A}(X). \tag{4.13}$$

Using the properties of the torsion described in Proposition 3.3 and (2.14), we obtain

$$\sum_{s=1}^{3}\sum_{a=1}^{4n}(\nabla_{e_a}T)(\xi_s, I_sX, e_a) \;=\; \nabla^*T^0(X) - 3\nabla^*U(X), \tag{4.14}$$

$$\sum_{s=1}^{3}\sum_{a=1}^{4n}(\nabla_{e_a}T)(\xi_s, X, I_se_a) \;=\; \nabla^*T^0(X) + 3\nabla^*U(X). \tag{4.15}$$

Substituting (4.13) and (4.14) in the sum of (4.9) written for $s = 1, 2, 3$, we obtain the third row of the system. The second row can be obtained by inserting (4.11) into (4.10), taking the sum over $s = 1, 2, 3$ and applying (4.12), (4.13), (4.14), (4.15).

The second Bianchi identity and applications of (3.3) give

$$\begin{aligned}(4.16)\quad \sum_{s=1}^{3}\Big(\sum_{a=1}^{4n}\Big[\ (\nabla_{e_a}Ric)(I_sX,I_se_a)+4n(\nabla_{e_a}\zeta_s)(I_sX,e_a)\ \Big]\Big)\\ -2\sum_{s=1}^{3}(Ric(\xi_s,I_sX)+8n\zeta_s(\xi_s,X))\\ +8n\sum_{s=1}^{3}\Big[\zeta_s(\xi_t,I_qX)-\zeta_s(\xi_q,I_tX)-\rho_s(\xi_t,I_qX)+\rho_s(\xi_q,I_tX)\Big]=0.\end{aligned}$$

Using (3.39), (3.41) as well as (2.14),(2.15) and (4.1) we obtain the next three identities

$$\begin{aligned}&\sum_{s=1}^{3}\Big[Ric(I_sX,I_se_a)+4n\zeta_s(I_sX,e_a)\Big]\\ &\qquad=2\sum_{s=1}^{3}\Big[\rho_s(I_qX,I_te_a)-\rho_s(I_tX,I_qe_a)\Big]\\ &\qquad\qquad= -4T^0(X,e_a)+24U(X,e_a)+\frac{3\,Scal}{2n(n+2)}g(X,e_a),\\ (4.17)\quad &8n\sum_{s=1}^{3}\Big[\zeta_s(\xi_t,I_qX)-\zeta_s(\xi_q,I_tX)\Big]\\ &\qquad=\sum_{s=1}^{3}\Big[4Ric(\xi_s,I_sX)-8\rho_s(\xi_s,X)+4\rho_s(\xi_t,I_qX)-4\rho_s(\xi_q,I_tX\Big],\\ &\sum_{s=1}^{3}\Big[-2Ric(\xi_s,I_sX)+8n\zeta_s(\xi_s,X)\Big]\\ &\qquad=\sum_{s=1}^{3}\Big[-4Ric(\xi_s,I_sX)-4\rho_s(\xi_t,I_qX)+4\rho_s(\xi_q,I_tX)\Big],\end{aligned}$$

A substitution of (4.17) in (4.16), and then a use (4.12) and (4.13) give the first row of the system. $\square$

We are ready to prove one of our main observations.

Theorem 4.9. *The qc-scalar curvature of a qc-Einstein quaternionic contact manifold of dimension bigger than seven is a global constant. In addition, the vertical distribution V of a qc-Einstein structure is integrable. On a seven dimensional qc-Einstein manifold the constancy of the qc scalar curvature is equivalent to the* *integrability of the vertical distribution. In both cases the Ricci tensors are given by*

$$\rho_{t|H} = \tau_{t|H} = -2\zeta_{t|H} = -\frac{Scal}{8n(n+2)}\omega_t \quad s,t = 1,2,3.,$$
$$Ric(\xi_s, X) = \rho_s(X,\xi_t) = \zeta_s(X,\xi_t) = 0, \quad s,t = 1,2,3.$$

PROOF. The statement for $n = 1$ follows directly from Theorem 4.8 and the fact that in dimension seven $U = 0$. Suppose the quaternionic contact manifold is qc-Einstein. According to Proposition 4.2, the quaternionic contact torsion vanishes, $T_\xi = 0, \xi \in V$. Since $n > 1$, Theorem 4.8 gives immediately that the horizontal gradient of the scalar curvature vanishes, i.e., $X(Scal) = 0, \quad X \in H$. Notice that this fact implies also $\xi(Scal) = 0, \xi \in V$, taking into account that for any $p \in M$ one has $[e_a, I_s e_a]_{|p} = T(e_a, I_s e_a)_{|p} = 2\xi_{s|p}$. Now, (4.9), (4.3), (3.40), (3.41) and (3.42) complete the proof. $\square$

4.2. Examples of qc-Einstein structures

EXAMPLE 4.10. *The flat model.*

The quaternionic Heisenberg group $G(\mathbb{H})$ with its standard left invariant quaternionic contact structure (see Section 5.2) is the simplest example. The Biquard connection coincides with the flat left-invariant connection on $G(\mathbb{H})$. More precisely, we have the following Proposition.

PROPOSITION 4.11. *Any quaternionic contact manifold $(M, g, \mathbb{Q})$ with flat Biquard connection is locally isomorphic to $G(\mathbb{H})$.*

PROOF. Since the Biquard connection ∇ is flat, there exists a local $\mathbb{Q}$-orthonormal frame

$$\{T_a,\, I_1T_a,\, I_2T_a,\, I_3T_a,\, \xi_1,\, \xi_2,\, \xi_3 : \ a = 1,\dots,n\},$$

which is ∇-parallel. Theorem 4.9 shows that the quaternionic contact torsion vanishes and the vertical distribution is integrable. In addition, (3.48) and (3.3) yield $[\xi_i, \xi_j] = 0$ with the only non-zero commutators $[I_iT_a, T_a] = 2\xi_i,\ i,j = 1,2,3$ (cf. (5.12)). Hence, the manifold has a local Lie group structure which is locally isomorphic to $G(\mathbb{H})$ by the Lie theorems. In other words, there is a local diffeomorphism $\Phi : M \to G(\mathbb{H})$ such that $\eta = \Phi^*\Theta$, where Θ is the standard contact form on $G(\mathbb{H})$, see (5.13). $\square$

EXAMPLE 4.12. *The 3-Sasakian Case.*

Suppose (M, g) is a (4n+3)-dimensional Riemannian manifold with a given 3-Sasakian structure, i.e., the cone metric on $M \times \mathbb{R}$ is a hyperkähler metric, namely, it has holonomy contained in $Sp(n+1)$ [BGN]. Equivalently, there are three Killing vector fields $\{\xi_1, \xi_2, \xi_3\}$, which satisfy

(i) $g(\xi_i, \xi_j) = \delta_{ij},\ i,j = 1,2,3$

(ii) $[\xi_i, \xi_j] = -2\xi_k$, for any cyclic permutation (i,j,k) of $(1,2,3)$

(iii) $(D_B\tilde{I}_i)C = g(\xi_i, C)B - g(B,C)\xi_i,\ i = 1,2,3,\ B, C \in \Gamma(TM)$, where $\tilde{I}_i(B) = D_B\xi_i$ and D denotes the Levi-Civita connection.

A 3-Sasakian manifold of dimension $(4n+3)$ is Einstein with positive Riemannian scalar curvature $(4n+2)(4n+3)$ [Kas] and if complete it is compact with finite fundamental group due to Mayer's theorem (see [BG] for a nice overview of 3-Sasakian spaces).

Let $H=\{\xi_1,\xi_2,\xi_3\}^{\perp}$. Then

$$\tilde I_i(\xi_j)=\xi_k,\quad \tilde I_i\circ\tilde I_j(X)=\tilde I_kX,\quad \tilde I_i\circ\tilde I_i(X)=-X,\ X\in H,$$
$$d\eta_i(X,Y)=2g(\tilde I_iX,Y),\ X,Y\in H.$$

Defining $V=span\{\xi_1,\xi_2,\xi_3\},\quad I_{i|H}=\tilde I_{i|H},\ I_{i|V}=0$ we obtain a quaternionic contact structure on M [Biq1]. It is easy to calculate that

$$\xi_i\lrcorner d\eta_{j\,|H}=0,\ d\eta_i(\xi_j,\xi_k)=2,\ d\eta_i(\xi_i,\xi_k)=d\eta_i(\xi_i,\xi_j)=0, \tag{4.18}$$
$$A_1=A_2=A_3=0 \text{ cf. (3.2)},\qquad \tilde u=\frac12 Id_H\ \text{ cf. (3.7)}.$$

This quaternionic contact structure satisfies the conditions (2.10) and therefore it admits the Biquard connection ∇. More precisely, we have

(i) $\nabla_XI_i=0, X\in H,\qquad \nabla_{\xi_i}I_i=0,\qquad \nabla_{\xi_i}I_j=-2I_k,\ \nabla_{\xi_j}I_i=2I_k,$
(ii) $T(\xi_i,\xi_j)=-2\xi_k,$
(iii) $T(\xi_i,X)=0,\ X\in H.$

From Proposition 4.2, Theorem 4.9, (3.30) and (3.33), we obtain the following Corollary.

Corollary 4.13. *Any 3-Sasakian manifold is a qc-Einstein with positive qc-scalar curvature*

$$Scal=16n(n+2).$$

For any $s,t,r=1,2,3$, the Ricci-type tensors are given by

$$\begin{aligned}\rho_{t|H}&=\tau_{t|H}=-2\zeta_{t|H}=-2\omega_t\\ Ric(\xi_s,X)&=\rho_s(X,\xi_t)=\zeta_s(X,\xi_t)=0\\ \rho_s(\xi_t,\xi_r)&=0.\end{aligned}\tag{4.19}$$

The nonzero parts of the curvature R of the Biquard connection is expressed in terms of the curvature of the Levi-Civita connection R^g as follows

i) $R(X,Y,Z,W)=R^g(X,Y,Z,W)$
$+\sum_{s=1}^3\{\omega_s(Y,Z)\omega_s(X,W)-\omega_s(X,Z)\omega_s(Y,W)-2\omega_s(X,Y)\omega_s(Z,W)\},$

ii) $R(\xi,Y,Z,W)=-R(Y,\xi,Z,W)=R^g(\xi,Y,Z,W),$

iii) $R(\xi,\bar\xi,Z,W)=R^g(\xi,\bar\xi,Z,W),$

iv) $R(X,Y,\xi,\bar\xi)=-4\{\eta_1\wedge\eta_2(\xi,\bar\xi)\omega_3(X,Y)+\eta_2\wedge\eta_3(\xi,\bar\xi)\omega_1(X,Y)$
newline $+\eta_3\wedge\eta_1(\xi,\bar\xi)\omega_2(X,Y)\},$

where $X,Y,Z,W\in H$ and $\xi,\bar\xi\in V$.

In fact, 3-Sasakian spaces are locally the only qc-Einstein manifolds (cf. Theorem 1.3). Before we turn to the proof of this fact we shall consider some special cases of QC-structures suggested by the above example. These structures will be relevant in Chapter 6, see for ex. Theorem 6.20. We recall that the Nijenhuis tensor N_{I_i} corresponding to I_i on H is defined as usual by $N_{I_i}(X,Y)=[I_iX,I_iY]-[X,Y]-I_i[I_iX,Y]-I_i[X,I_iY],\quad X,Y\in H.$

Definition 4.14. *A quaternionic contact structure $(M,g,\mathbb{Q})$ is said to be* hypERhermitian contact *(abbr.* HC structure*) if the horizontal bundle H is formally integrable with respect to I_1,I_2,I_3 simultaneously, i.e. for $i=1,2,3$ and any*

$X,\, Y \in H$, *we have*

$$N_{I_i}(X,Y) = 0 \quad mod \quad V. \tag{4.20}$$

In fact a QC structure is locally a HC structure exactly when two of the almost complex structures on H are formally integrable due to the next identity essentially established in [AM, (3.4.4)]

$$2N_{I_3}(X,Y) - N_{I_1}(X,Y) + I_2N_{I_1}(I_2X,Y) + I_2N_{I_1}(X,I_2Y) - N_{I_1}(I_2X,I_2Y) - \\ N_{I_2}(X,Y) + I_1N_{I_2}(I_1X,Y) + I_1N_{I_2}(X,I_1Y) - N_{I_2}(I_1X,I_1Y) = 0 \quad \mathrm{m}od \quad V.$$

On the other hand, the Nijenhuis tensor has the following expression in terms of a connection ∇ with torsion T satisfying (3.28)(see e.g. [Iv])

$$N_{I_i}(X,Y) = T^{0,2}_{Ii}(X,Y) + \beta_i(Y)I_jX - \beta_i(X)I_jY - I_i\beta_i(Y)I_kX + I_i\beta_i(X)I_kY, \tag{4.21}$$

where the 1-forms β_i and the (0,2)-part of the torsion $T^{0,2}_{I_i}$ with respect to the almost complex structure I_i are defined on H, correspondingly, by

$$\beta_i = \alpha_j + I_i\alpha_k, \tag{4.22}$$

$$T^{0,2}_{I_i}(X,Y) = T(X,Y) - T(I_iX,I_iY) + I_iT(I_iX,Y) + I_iT(X,I_iY). \tag{4.23}$$

Applying the above formulas to the Biquard connection and taking into account (3.3) one sees that (4.20) is equivalent to $(\beta_i)_{|_H} = 0$. Hence we have the following proposition.

Proposition 4.15. *A quaternionic contact structure $(M,g,\mathbb{Q})$ is a hyperhermitian contact structure if and only if the connection 1-forms satisfy the relations*

$$\alpha_j(X) = \alpha_k(I_iX), \quad X \in H \tag{4.24}$$

The Nijenhuis tensors of a HC structure satisfy $N_{I_i}(X,Y) = T^{0,2}_{I_i}(X,Y), \quad X,Y \in H$.

Given a QC structure $(M,g,\mathbb{Q})$ let us consider the three almost complex structures $(\eta_i,\tilde{I}_i)$

$$\tilde{I}_iX = I_iX, \quad X \in H, \quad \tilde{I}_i(\xi_j) = \xi_k, \quad \tilde{I}_i(\xi_i) = 0. \tag{4.25}$$

With these definitions $(\eta_i,\tilde{I}_i)$ are almost CR structures (i.e. possibly non-integrable) exactly when the QC structure is HC since the condition $d\eta_i(\tilde{I}_iX,\tilde{I}_i\xi_j) = d\eta_i(X,\xi_j)$ is equivalent to $\alpha_k(X) = -\alpha_j(I_iX)$ in view of (3.29). Hence, $d\eta_i$ is a (1,1)-form with respect to $\tilde{I}_i$ on $\xi_i^{\perp} = H \oplus \{\xi_j,\xi_k\}$ and a HC structure supports a non integrable hyper CR-structure $(\eta_i,\tilde{I}_i)$.

A natural question is to examine when $\tilde{I}_i$ is formally integrable, i.e $N_{\tilde{I}_i} = 0 \quad \mathrm{m}od \quad \xi_i$.

Proposition 4.16. *Let $(M,g,\mathbb{Q})$ be a hyperhermitian contact structure. Then the CR structures $(\eta_i,\tilde{I}_i)$ are integrable if and only if the next two equalities hold*

$$d\eta_j(\xi_k,\xi_i) = d\eta_k(\xi_i,\xi_j), \qquad d\eta_j(\xi_j,\xi_i) - d\eta_k(\xi_k,\xi_i) = 0. \tag{4.26}$$

Proof. From (3.3) it follows $T^{0,2}_{\tilde{I}_1}(X,Y) = 0$ using also (4.23). Substituting the latter into (4.21) taken with respect to $\tilde{I}_i$ shows $N_{\tilde{I}_i}|_H = 0 \quad mod \quad \xi_i$ is equivalent to (4.24). Corollary 3.6 implies

$$N_{\tilde I_i}(X,\xi_j)=(\alpha_j(I_iX)+\alpha_k(X))\xi_i+(\alpha_j(\xi_k)+\alpha_k(\xi_j))I_kX+(\alpha_j(\xi_j)-\alpha_k(\xi_k))I_jX+ \\ +T(\xi_k,I_iX)-I_iT(\xi_k,X)-T(\xi_j,X)-I_iT(\xi_j,I_iX).$$

Taking the trace part and the trace-free part in the right-hand side allows us to conclude that $N_{\tilde I_i}(X,\xi_j)=0 \mod \xi_i$ is equivalent to the system

$$T(\xi_k,I_1X)-I_1T(\xi_k,X)-T(\xi_j,X)-I_1T(\xi_j,I_1X)=0, \\ \alpha_j(\xi_k)+\alpha_k(\xi_j)=0 \quad \alpha_j(\xi_j)-\alpha_k(\xi_k)=0.$$

An application of Proposition 3.3, (2.14) and (2.15) shows the first equality is trivially satisfied, while (3.30) tells us that the other equalities are equivalent to (4.26). □

4.3. Proof of Theorem 1.3

Proof of Theorem 1.3. The equivalence of a) and c) was proved in Proposition 4.2. We are left with proving the implication a) implies b). Let $(M,\tilde g,\mathbb{Q})$ be a qc-Einstein manifold with qc-scalar curvature $\overline{Scal}$. According to Theorem 4.9, $\overline{Scal}$ is a global constant on M. We define $\eta=\frac{\overline{Scal}}{16n(n+2)}\,\tilde\eta$. Then $(M,g,\mathbb{Q})$ is a qc-Einstein manifold with qc-scalar curvature $Scal=16n(n+2)$, horizontal distribution $H=Ker(\eta)$ and involutive vertical distribution $V=span\{\xi_1,\xi_2,\xi_3\}$ (see (5.1),(5.7) and (5.8)).

We shall show that the Riemannian cone is a hyperkähler manifold. Consider the structures defined by (4.25). We have the relations

$$\eta_i(\xi_j)=\delta_{ij},\quad \eta_i\tilde I_j=-\eta_j\tilde I_i=\eta_k,\qquad \tilde I_i\xi_j=-\tilde I_j\xi_i=\xi_k$$
$$\tilde I_i\tilde I_j-\eta_j\otimes\xi_i=-\tilde I_j\tilde I_i+\eta_i\otimes\xi_j=\tilde I_k \tag{4.27}$$
$$\tilde I_i^2=-Id+\eta_i\otimes\xi_i,\qquad \eta_i\tilde I_i=0,\qquad \tilde I_i\xi_i=0,\qquad g(\tilde I_i.,\tilde I_i.)=g(.,.)-\eta_i(.)\,\eta_i(.).$$

Let D be the Levi-Civita connection of the metric g on M determined by the structure (η,Q). The next step is to show

$$D\tilde I_i=Id\otimes\eta_i-g\otimes\xi_i-\sigma_j\otimes\tilde I_k+\sigma_k\otimes\tilde I_j, \tag{4.28}$$

for some appropriate 1-forms σ_s on M. We consider all possible cases.
Case 1$[X,Y,Z\in H]$ The well known formula

$$2\,g(D_AB,C)\ =\ Ag(B,C)\ +\ Bg(A,C)\ -\ Cg(A,B) \\ +g([A,B],C)-g([B,C],A)+g([C,A],B),\qquad A,B,C\in\Gamma(TM) \tag{4.29}$$

yields

$$2\,g(\,(D_X\tilde I_i)\,Y,Z)=d\omega_i(X,Y,Z)-d\omega_i(X,I_iY,I_iZ)+g(N_i(Y,Z),I_iX). \tag{4.30}$$

We compute $d\omega_i$ in terms of the Biquard connection. Using (3.3), (3.28) and (4.22), we calculate

(4.31)
$$d\omega_i(X,Y,Z)-d\omega_i(X,I_iY,I_iZ)\ ==\ -2\,\alpha_j(X)\,\omega_k(Y,Z)+2\alpha_k(X)\,\omega_j(Y,Z) \\ -\beta_i(Y)\,\omega_k(Z,X)\ -\ I_i\beta_i(Y)\,\omega_j(Z,X)\ -\ \beta_i(Z)\,\omega_k(X,Y)\ -\ I_i\beta_i(Z)\,\omega_j(X,Y).$$

A substitution of (4.21) and (4.31) in (4.30) gives

$$g((D_X\tilde I_i)Y,Z)=-\alpha_j(X)\omega_k(Y,Z)+\alpha_k(X)\omega_j(Y,Z). \tag{4.32}$$

Letting $\sigma_i(X)=\alpha_i(X)$, we obtain equation (4.28).
Case 2 [$\xi_s,\xi_t\in V$ and $Z\in H$] Using the integrability of the vertical distribution V and (4.29), we compute

$$2g((D_{\xi_s}\tilde{I}_i)\xi_t,Z)=2g(D_{\xi_s}\tilde{I}_i\xi_t,Z)+2g(D_{\xi_s}\xi_t,I_iZ)=$$
$$-g([\tilde{I}_i\xi_t,Z],\xi_s)-g([\xi_s,Z],\tilde{I}_i\xi_t)-g([\xi_s,I_iZ],\xi_t)-g([\xi_t,I_iZ],\xi_s).$$

An application of (2.10) allows to conclude $g((D_{\xi_s}\tilde{I}_i)\xi_t,Z)=0$ for any $i,s,t\in\{1,2,3\}$.
Case 3 [$X,Y\in H$ and $C\in V$] First, let $C=\xi_1$. We have

$$\begin{aligned}2g((D_X\tilde{I}_1)Y,\xi_1)&=2g(D_X\tilde{I}_1Y,\xi_1)\\ &=\ -\xi_1g(X,I_1Y)+g([X,I_1Y],\xi_1)-g([X,\xi_1],I_1Y)-g([I_1Y,\xi_1],X)\\ &=-(\mathcal{L}_{\xi_1}g)(X,I_1Y)+\eta_1([X,I_1Y])=-d\eta_1(X,I_1Y)=-2g(X,Y).\end{aligned}$$

after using (3.18), $T_{\xi_s}=0,\ s=1,2,3$, and (2.1).
For $C=\xi_2$, we calculate applying (4.27) and (4.29) that

$$\begin{aligned}2g((D_X\tilde{I}_1)Y,\xi_2)&=2g(D_X\tilde{I}_1Y,\xi_2)+2g(D_XY,\xi_3)\\ &=\ -\xi_1g(X,I_iY)-\xi_3g(X,Y)+g([X,I_iY],\xi_2)-g([X,\xi_2],I_iY)\\ &\quad-g([I_iY,\xi_2],X)+g([X,Y],\xi_3)-g([X,\xi_3],Y)-g([Y,\xi_3],X)\\ &=-(\mathcal{L}_{\xi_2}g)(X,I_1Y)-(\mathcal{L}_{\xi_3}g)(X,Y)+\eta_2([X,I_1Y])+\eta_3(]X,Y])=0.\end{aligned}$$

The other possibilities in this case can be checked in a similar way.
Case 4 [$X\in H$ and $A,B\in V$]. We verify (4.28) for $\tilde{I}_1,A=\xi_1,B=\xi_2$ and $\tilde{I}_1,A=\xi_2,B=\xi_3$ since the other verifications are similar. Using the integrability of V, (4.29), (4.27) and (3.29), we find

$$2g((D_X\tilde{I}_1)\xi_1,\xi_2)=2g(D_X\xi_1,\xi_3)=g([X,\xi_1],\xi_3)-g([X,\xi_3],\xi_1)=-2\alpha_2(X),$$
$$2g((D_X\tilde{I}_1)\xi_2,\xi_3)=2g(D_X\xi_3,\xi_3)-2g(D_X\xi_2,\xi_2)=0.$$

Case 5 [$A,B,C\in V$] Let us extend the definition of the three 1-forms σ_s on V as follows

$$\begin{aligned}\sigma_i(\xi_i)&=1+\frac{1}{2}(d\eta_i(\xi_j,\xi_k)-d\eta_j(\xi_k,\xi_i)-d\eta_k(\xi_i,\xi_j))\\ \sigma_i(\xi_j)&=d\eta_j(\xi_j,\xi_k),\qquad \sigma_i(\xi_k)=d\eta_k(\xi_j,\xi_k).\end{aligned}\tag{4.33}$$

A small calculation leads to the formula

$$g(\tilde{I}_iA,B)=(\eta_j\wedge\eta_k)(A,B).\tag{4.34}$$

On the other hand, we have

$$\begin{aligned}&2(D_A\eta_i)(B)=2g((D_A\xi_i,B)\\ &\quad=A\eta_i(B)+\xi_ig(A,B)-B\eta_i(A)+g([A,\xi_i],B)-\eta_i([A,B])-g([\xi_i,B],A)\\ &=\sum_{s,t=1}^{3}\Big[\eta_s(A)\,\eta_t(B)\,d\eta_i(\xi_s,\xi_t)\ -\ \eta_s(A)\,\eta_t(B)\,d\eta_t(\xi_s,\xi_i)\ -\ \eta_s(B)\,\eta_t(A)\,d\eta_t(\xi_s,\xi_i)\Big]\\ &\qquad=\ 2\,\eta_j\wedge\eta_k(A,B)\ -\ 2\,\sigma_j(A)\,\eta_k(B)\ +\ 2\,\sigma_k(A)\,\eta_j(B).\end{aligned}\tag{4.35}$$

With the help of (4.34) and (4.35) we see

$$\begin{aligned}(4.36)\quad g((D_A\tilde{I}_i)B,C) &= D_A(\eta_j\wedge\eta_k)(B,C) = [D_A(\eta_j)\wedge\eta_k+\eta_j\wedge D_A(\eta_k)](B,C)\\ &= ((A\lrcorner(\eta_k\wedge\eta_i)-\sigma_k(A)\eta_i+\sigma_i(A)\eta_k)\wedge\eta_k)(B,C)\\ &\quad+(\eta_j\wedge(A\lrcorner(\eta_i\wedge\eta_j)-\sigma_i(A)\eta_j+\sigma_j(A)\eta_i))(B,C)\\ &= (\eta_k(A)\eta_i\wedge\eta_k(B,C)+\eta_j(A)\eta_i\wedge\eta_j(B,C))-\sigma_j(A)g(\tilde{I}_kB,C)+\sigma_k(A)g(\tilde{I}_jB,C)\\ &= \eta_i(B)g(A,C)-g(A,B)\eta_i(C)-\sigma_j(A)g(\tilde{I}_kB,C)+\sigma_k(A)g(\tilde{I}_jB,C).\end{aligned}$$

Case 6 $[A\in V$ and $Y,Z\in H]$. Let $A=\xi_s,\ s\in\{1,2,3\}$. The right hand side of (4.28) is equal to $-\sigma_j(\xi_s)\omega_k(Y,Z)+\sigma_k(\xi_s)\omega_j(Y,Z)$. On the left hand side of (4.28), we have

$$\begin{aligned}(4.37)\quad 2g((D_{\xi_s}\tilde{I}_i)Y,Z) &= 2g(D_{\xi_s}(I_iY),Z)+2g(D_{\xi_s}Y,I_iZ)\\ &= \{\xi_sg(I_iY,Z)+g([\xi_s,I_iY],Z)-g([\xi_s,Z],I_iY)-g([I_iY,Z],\xi_s)\}\\ &\quad+\{\xi_sg(Y,I_iZ)+g([\xi_s,Y],I_iZ)-g([\xi_s,I_iZ],Y)-g([Y,I_iZ],\xi_s)\}\\ &= g((\mathcal{L}_{\xi_s}I_i)Y,Z)-g((\mathcal{L}_{\xi_s}I_i)Z,Y)+\omega_i(I_sY,Z)+\omega_i(Y,I_sZ)\end{aligned}$$

Now, recall Lemma 3.2 to compute the skew symmetric part of $g((\mathcal{L}_{\xi_s}I_i)Y,Z)$, and also use formulas (4.33), to prove the case $i=s$

$$\begin{aligned}g((D_{\xi_i}\tilde{I}_i)Y,Z) &= d\eta_i(\xi_i,\xi_j)\,\omega_j(Y,Z)+d\eta_i(\xi_i,\xi_k)\,\omega_k(Y,Z)\\ &= -\sigma_j(\xi_i)\,\omega_k(Y,Z)+\sigma_k(\xi_i)\,\omega_j(Y,Z).\end{aligned}$$

Similarly, for $s=j$ we have

$$\begin{aligned}g((D_{\xi_j}\tilde{I}_i)Y,Z) &= d\eta_j(\xi_i,\xi_j)\,\omega_j(Y,Z)\\ &\quad-\frac{1}{2}\Big(-d\eta_i(\xi_j,\xi_k)+d\eta_j(\xi_k,\xi_i)-d\eta_k(\xi_i,\xi_j)\Big)\,\omega_k(Y,Z)\\ &\quad-\omega_k(Y,Z) = -\sigma_j(\xi_j)\,\omega_k(Y,Z)+\sigma_k(\xi_j)\,\omega_j(Y,Z),\end{aligned}$$

which completes the proof of (4.28).

At this point, consider the Riemannian cone $N=M\times\mathbb{R}^+$ with the cone metric $g_N=t^2g+dt^2$ and the almost complex structures

$$\phi_i(E,f\frac{d}{dt})=(\tilde{I}_iE+\frac{f}{t}\xi_i,-t\eta_i(E)\frac{d}{dt}),\quad i=1,2,3,\quad E\in\Gamma(TM).$$

Using the O'Neill formulas for warped product [**On**, p.206], (4.27) and the just proved (4.28) we conclude (see also [**MO**]) that the Riemannian cone $(N,g_N,\phi_i,i=1,2,3)$ is a quatrnionic Kähler manifold with connection 1-forms defined by (4.32) and (4.33). It is classical result (see e.g [**Bes**]) that a quaternionic Kähler manifolds are Einstein. This fact implies that the cone $N=M\times\mathbb{R}^+$ with the warped product metric g_N must be Ricci flat (see e.g. [**Bes**, p.267]) and therefore it is locally hyperkähler (see e.g. [**Bes**, p.397]). This means that locally there exists a $SO(3)$-matrix Ψ with smooth entries such that the triple $(\tilde{\phi}_1,\tilde{\phi}_2,\tilde{\phi}_3)=\Psi\cdot(\phi_1,\phi_2,\phi_3)^t$ is D-parallel. Consequently $(M,\Psi\cdot\eta)$ is locally 3-Sasakian. Example 4.12 and Proposition 4.2 complete the proof. □

Corollary 4.17. *Let $n>1$ and $(M,g,\mathbb{Q})$ be a QC structure on a (4n+3)-dimensional manifold with positive qc-scalar curvature, Scal >0. The following two conditions are equivalent.*

i) The structure $(M, \frac{16n(n+2)}{Scal}g, \mathbb{Q})$ is locally 3-Sasakian.
ii) There exists a (local) 1-form η such that the connection 1-forms of the Biquard connection vanish on H, $\alpha_i(X) = -d\eta_j(\xi_k, X) = 0, X \in H$, $i, j, k = 1, 2, 3$ and $Scal$ is constant if $n = 1$.

Proof. In view of Theorem 1.3 and Example 4.12 it is sufficient to prove the following Lemma. □

Lemma 4.18. *If a QC structure has zero connection one forms restricted to the horizontal space H then it is qc-Einstein, or equivalently, it has zero torsion.*

Proof. If $\alpha_i(X) = 0$ for $i = 1, 2, 3$ and $X \in H$ then (3.33) together with (3.3) yield

$$2\rho_i(X, Y) = -\alpha_i([X, Y]) = \alpha_i(T(X, Y)) = 2\sum_{s=1}^{3}\alpha_i(\xi_s)\omega_s(X, Y).$$

Substitute the latter into (3.41) to conclude considering the $Sp(n)Sp(1)$-invariant parts of the obtained equalities that $T^0(X, Y) = U(X, Y) = \alpha_i(\xi_j) = 0, \quad \alpha_i(\xi_i) = -\frac{Scal}{8n(n+2)}$. □

Corollary 4.19. *Any totally umbilical hypersurface M in a quaternionic-Kähler manifold admits a canonical qc-Einstein structure with a non-zero scalar curvature.*

Proof. Let M be a hypersurface in the quaternionic-Kähler manifold $(\tilde{M}, \tilde{g})$. With N standing for the unit normal vector field on M the second fundamental form is given by $II(A, B) = -\tilde{g}(\tilde{D}_A N, B)$, with $A, B \in TM$ and $\tilde{D}$ being the Levi-Civita connection of $(\tilde{M}, \tilde{g})$. Since M is a totally umbilical hypersurface of an Einstein manifold $\tilde{M}$, taking a suitable trace in the Codazzi equation [KoNo, Proposition 4.3], we find $II(A, B) = -Const\,\tilde{g}(A, B)$. Thus, after a homothety of $\tilde{M}$ we can assume $II(A, B) = -\tilde{g}(A, B)$.

Consider a local basis $\tilde{J}_1, \tilde{J}_2, \tilde{J}_3$ of the quaternionic structure of $\tilde{M}$ satisfying the quaternionic identities. We define the horizontal distribution H of M to be the maximal subspace of TM invariant under the action of $\tilde{J}_1, \tilde{J}_2, \tilde{J}_3$, whose restriction to M will be denoted with I_1, I_2, I_3. We claim that (H, I_1, I_2, I_3) is a qc-structure on M. Defining $\eta_i(A) = \tilde{g}(\tilde{J}_i(N), A)$ and $\xi_i = \tilde{J}_i N$, a small calculation shows

$$d\eta_i(X, Y) = II(X, I_iY) - II(I_iX, Y) = 2\tilde{g}(I_iX, Y), \quad X, Y \in H,$$

Hence, the conditions in the definition of a qc-structure are satisfied.

Let D be the Levi-Civita connection of the restriction g of $\tilde{g}$ to M. Then we have

$$\tilde{D}_A B = D_A B + II(A, B)N, \qquad A, B \in TM \tag{4.38}$$

Define $\tilde{I}_i(A) = \tilde{J}_i(A)_{TM}$ the orthogonal projection on TM, $A \in TM$. Since by assumption $\tilde{M}$ is a quaternionic-Kähler manifold we have $\tilde{D}\tilde{J}_i = -\sigma_j \otimes \tilde{J}_k + \sigma_k \otimes \tilde{J}_j$. This together with (4.38), after some computation gives (compare with (4.28))

$$D\tilde{I}_i = Id \otimes \eta_i - g \otimes \xi_i - \sigma_j \otimes \tilde{I}_k + \sigma_k \otimes \tilde{I}_j. \tag{4.39}$$

Using (4.39) we will show that the torsion of the qc-structure is zero. The same computation as in the Case 3 in the proof of Theorem 1.3 gives

$$\begin{aligned}
-2g(X,Y) &= 2g((D_X\tilde{I}_i)Y,\xi_i) = 2g(D_X\tilde{I}_iY,\xi_i)\\
&= -\xi_i g(X,I_iY) + g([X,I_iY],\xi_i) - g([X,\xi_i],I_iY) - g([I_iY,\xi_i],X)\\
&\qquad\qquad = -(\mathcal{L}_{\xi_i}g)(X,I_iY) - 2g(X,Y).
\end{aligned}$$

Hence $0 = (\mathcal{L}_{\xi_i}g)(X,Y) = 2T^0_{\xi_i}(X,Y)$. A computation analogous to the the proof of Theorem 1.3, Case 6 gives

$$\begin{aligned}
-2\sigma_3(\xi_1)\omega_1(Y,Z) + 2\sigma_1(\xi_1)\omega_3(Y,Z) &= 2g((D_{\xi_1}\tilde{I}_2)Y,Z)\\
= g((\mathcal{L}_{\xi_1}I_2)Y,Z) - g((\mathcal{L}_{\xi_1}I_2)Z,Y) &+ \omega_2(I_1Y,Z) + \omega_2(Y,I_1Z).
\end{aligned} \tag{4.40}$$

From Lemma 3.2 it follows

$$\begin{aligned}
-2\sigma_3(\xi_1)\omega_1(Y,Z) &+ 2\sigma_1(\xi_1)\omega_3(Y,Z) =\\
&= -4g(I_3\tilde{u}Y,Z) + 2d\eta_1(\xi_2,\xi_1)\omega_1(Y,Z)\\
&\quad + \left(d\eta_1(\xi_2,\xi_3) - d\eta_2(\xi_3,\xi_1) - d\eta_3(\xi_1,\xi_2) - 2\right)\omega_3(Y,Z).
\end{aligned}$$

Working similarly in the remaining cases we conclude that the traceless part u of the tensor $\tilde{u}$ vanishes. $\square$

CHAPTER 5

Conformal transformations of a qc-structure

Let h be a positive smooth function on a QC manifold $(M, g, \mathbb{Q})$. Let $\bar\eta = \frac{1}{2h}\eta$ be a conformal transformation of the QC structure η (to be precise we should let $\bar g = \frac{1}{2h}g$ on H and consider $(M, \bar g, \mathbb{Q})$). We denote the objects related to $\bar\eta$ by overlining the same object corresponding to η. Thus, $d\bar\eta = -\frac{1}{2h^2}dh\wedge\eta + \frac{1}{2h}d\eta$ and $\bar g = \frac{1}{2h}g$. The new triple $\{\bar\xi_1, \bar\xi_2, \bar\xi_3\}$ is determined by (2.10). We have

$$\bar\xi_s = 2h\xi_s + I_s\nabla h, \quad s = 1,2,3, \tag{5.1}$$

where ∇h is the horizontal gradient defined by $g(\nabla h, X) = dh(X), X \in H$.

The (horizontal) sub-Laplacian and the norm of the horizontal gradient are defined respectively by

$$\triangle h \ = \ tr^g_H(\nabla dh) \ = \ \sum_{\alpha=1}^{4n}\nabla dh(e_\alpha, e_\alpha), \qquad |\nabla h|^2 \ = \ \sum_{\alpha=1}^{4n} dh(e_\alpha)^2. \tag{5.2}$$

The Biquard connections ∇ and $\bar\nabla$ are connected by a (1,2) tensor S, $\bar\nabla_A B = \nabla_A B + S_A B, A, B \in \Gamma(TM)$. The condition (3.3) yields

$$g(S_XY,Z) - g(S_YX,Z) = -h^{-1}\textstyle\sum_{s=1}^3\omega_s(X,Y)dh(I_sZ), \quad X,Y,Z \in H.$$

From $\bar\nabla\bar g = 0$ we get $g(S_XY,Z) + g(S_XZ,Y) = -h^{-1}dh(X)g(Y,Z), \quad X,Y,Z \in H.$

The last two equations determine $g(S_XY,Z)$ for $X,Y,Z \in H$ due to the equality

$$\begin{aligned} g(S_XY,Z) = -(2h)^{-1}\{&dh(X)g(Y,Z) - \sum_{s=1}^3 dh(I_sX)\omega_s(Y,Z) + dh(Y)g(Z,X)\\ &+\sum_{s=1}^3 dh(I_sY)\omega_s(Z,X) - dh(Z)g(X,Y)Z + \sum_{s=1}^3 dh(I_sZ)\omega_s(X,Y)\}.\end{aligned}$$

Using Biquard's Theorem 2.4, we obtain after some calculations that

$$\begin{aligned} g(\bar T_{\bar\xi_1}X,Y) - 2hg(T_{\xi_1}X,Y) - g(S_{\bar\xi_1}X,Y) = \\ -\nabla dh(X,I_1Y) + h^{-1}(dh(I_3X)dh(I_2Y) - dh(I_2X)dh(I_3Y)).\end{aligned} \tag{5.3}$$

The identity $d^2 = 0$ yields $\nabla dh(X,Y) - \nabla dh(Y,X) = -dh(T(X,Y))$. Applying (3.3), we can write

$$\nabla dh(X,Y) = [\nabla dh]_{[sym]}(X,Y) - \sum_{s=1}^3 dh(\xi_s)\omega_s(X,Y), \tag{5.4}$$

where $[.]_{[sym]}$ denotes the symmetric part of the correspondin (0,2)-tensor.

Decomposing (5.3) into [3] and [-1] parts according to (2.7), using the properties of the torsion tensor T_{ξ_i} and (3.36) we come to the next transformation formulas:

$$\overline{T}^0(X,Y) \ = \ T^0(X,Y) \ + \ h^{-1}[\nabla dh]_{[sym][-1]}, \tag{5.5}$$

$$\bar{U}(X,Y) \ = \ U(X,Y) \ + \ (2h)^{-1}[\nabla dh - 2h^{-1}dh\otimes dh]_{[3][0]}, \tag{5.6}$$

$$\begin{aligned} g(S_{\bar\xi_1}X,Y) \ = \ &-\frac{1}{4}\Big[-\nabla dh(X,I_1Y)+\nabla dh(I_1X,Y)-\nabla dh(I_2X,I_3Y)\\ &+\nabla dh(I_3X,I_2Y)\Big]- \ (2h)^{-1}\Big[dh(I_3X)dh(I_2Y)\\ &-dh(I_2X)dh(I_3Y)+dh(I_1X)dh(Y)-dh(X)dh(I_1Y)\Big]\\ &+\ \frac{1}{4n}\left(-\triangle h+2h^{-1}|\nabla h|^2\right)g(I_1X,Y)-dh(\xi_3)g(I_2X,Y)+dh(\xi_2)g(I_3X,Y), \end{aligned}$$

where $[\,.\,]_{[sym][-1]}$ and $[\,.\,]_{[3][0]}$ denote the symmetric $[-1]$-component and the traceless [3] part of the corresponding (0,2) tensors on H, respectively. Observe that for $n=1$ (5.6) is trivially satisfied.
Thus, using (3.47), we proved the following Proposition.

Proposition 5.1. *Let $\overline{\eta}=\frac{1}{2h}\eta$ be a conformal transformation of a given QC structure η. Then the trace-free parts of the corresponding qc-Ricci tensors are related by the equation*

$$\begin{aligned} &Ric_0(X,Y)-\overline{Ric}_0(X,Y)\\ &=-(2n{+}2)h^{-1}[\nabla dh]_{[sym][-1]}(X,Y)-(2n{+}5)h^{-1}\Big[\nabla dh-2h^{-1}dh\otimes dh\Big]_{[3][0]}(X,Y). \end{aligned} \tag{5.7}$$

For $n=1$, $\qquad Ric_0(X,Y)-\overline{Ric}_0(X,Y)=-4h^{-1}[\nabla dh]_{[sym][-1]}(X,Y).$

In addition, the qc-scalar curvature transforms by the formula [Biq1], which is the qc-Yamabe equation,

$$\overline{Scal}=2h(Scal)-8(n+2)^2h^{-1}|\nabla h|^2+8(n+2)\triangle h. \tag{5.8}$$

5.1. Conformal transformations preserving the qc-Einstein condition

In this section we investigate the question of conformal transformations, which preserve the qc-Einstein condition. A straightforward consequence of (5.7) is the following

Proposition 5.2. *Let $\bar\eta=\frac{1}{2h}\eta$ be a conformal transformation of a given qc-structure $(M,g,\mathbb{Q})$. Then the trace-free part of the qc-Ricci tensor does not change if and only if the function h satisfies the differential equations*

$$3(\nabla_X dh)Y-\sum_{s=1}^3(\nabla_{I_sX}dh)I_sY \ = \ -4\sum_{s=1}^3 dh(\xi_s)\omega_s(X,Y), \tag{5.9}$$

$$\begin{aligned} &(\nabla_X dh)Y-2h^{-1}dh(X)dh(Y)\\ &\qquad+\sum_{s=1}^3\left[(\nabla_{I_sX}dh)I_sY-2h^{-1}dh(I_sX)dh(I_sY)\right] = \ \lambda g(X,Y), \end{aligned} \tag{5.10}$$

for some smooth function λ and any $X,Y\in H$.

Note that for $n=1$ (5.10) is trivially satisfied. Let us fix a qc-normal frame, cf. definition 4.6, $\{T_\alpha, X_\alpha=I_1T_\alpha, Y_\alpha=I_2T_\alpha, Z_\alpha=I_3T_\alpha, \xi_1,\xi_2,\xi_3\}, \alpha=1\dots,n$ at a point $p\in M$.

LEMMA 5.3. *If h satisfies (5.9) then we have at $p\in M$ the relations*

$$\begin{aligned}(I_j\,T_\alpha)\,T_\alpha\,h \;&=\; -\,T_\alpha\,(I_j\,T_\alpha)\,h \;=\; \xi_j\,h\\ (I_j\,T_\alpha)\,(I_i\,T_\alpha)\,h \;&=\; -\,(I_i\,T_\alpha)\,(I_j\,T_\alpha)\,h \;=\; \xi_k\,h.\end{aligned} \tag{5.11}$$

PROOF. Working with the fixed qc-normal frame, equation (5.9) gives

$$4T_\alpha X_\alpha h\,(p)\;-\;[T_\alpha,X_\alpha]h\,(p)\;+\;[Y_\alpha,Z_\alpha]h\,(p)\;=\;-4\xi_1 h\,(p).$$

Lemma 4.5 and (3.3) yield $[T_\alpha,X_\alpha]h\,(p)\;-\;[Y_\alpha,Z_\alpha]h\,(p)\;=\;0$. Hence, (5.11) follow. ∎

5.2. Quaternionic Heisenberg group. Proof of Theorem 1.1

The proof of Theorem 1.1 will be presented as separate Propositions and Lemmas in the rest of the Section, see (5.29) for the final formula. We use the following model of the quaternionic Heisenberg group $G(\mathbb{H})$. Define $G(\mathbb{H})\;=\;\mathbb{H}^n\times\operatorname{Im}\mathbb{H}$ with the group law given by $(q',\omega')\;=\;(q_o,\omega_o)\circ(q,\omega)\;=\;(q_o\,+\,q,\omega\,+\,\omega_o\,+\,2\,\operatorname{Im}\,q_o\,\bar q)$, where $q,\;q_o\in\mathbb{H}^n$ and $\omega,\omega_o\in\operatorname{Im}\mathbb{H}$. In coordinates, with the obvious notation, a basis of left invariant horizontal vector fields $T_\alpha, X_\alpha=I_1T_\alpha, Y_\alpha=I_2T_\alpha, Z_\alpha=I_3T_\alpha, \alpha=1\dots,n$ is given by

$$\begin{aligned}T_\alpha\;&=\;\partial_{t_\alpha}\;+2x^\alpha\partial_x+2y^\alpha\partial_y+2z^\alpha\partial_z &\qquad X_\alpha\;&=\;\partial_{x_\alpha}\;-2t^\alpha\partial_x-2z^\alpha\partial_y+2y^\alpha\partial_z\\ Y_\alpha\;&=\;\partial_{y_\alpha}\;+2z^\alpha\partial_x-2t^\alpha\partial_y-2x^\alpha\partial_z &\qquad Z_\alpha\;&=\;\partial_{z_\alpha}\;-2y^\alpha\partial_x+2x^\alpha\partial_y-2t^\alpha\partial_z\,.\end{aligned}$$

The central (vertical) vector fields ξ_1,ξ_2,ξ_3 are described as follows

$$\xi_1=2\partial_x\quad\;\xi_2=2\partial_y\quad\;\xi_3=2\partial_z\,.$$

A small calculation shows the following commutator relations

$$[I_j\,T_\alpha,T_\alpha]\;=\;2\xi_j\qquad\quad[I_j\,T_\alpha,I_i\,T_\alpha]\;=\;2\xi_k. \tag{5.12}$$

The standard 3-contact form $\;\tilde\Theta=(\tilde\Theta_1,\;\tilde\Theta_2,\;\tilde\Theta_3)$ is

$$2\tilde\Theta\;=\;d\omega\;-\;q'\cdot d\bar q'\;+\;dq'\,\cdot\bar q'. \tag{5.13}$$

The described horizontal and vertical vector fields are parallel with respect to the Biquard connection and constitute an orthonormal basis of the tangent space.

We turn to the proof of Theorem 1.1. We start with a Proposition in which we shall determine the vertical Hessian of h.

PROPOSITION 5.4. *If h satisfies (5.9) on $G(\mathbb{H})$ then we have the relations*

$$\xi_1^2(h)=\xi_2^2(h)=\xi_3^2(h)=8\mu_o,\quad\;\xi_i\xi_j(h)=0,\quad\;i\neq j=1,2,3, \tag{5.14}$$

where $\mu_o>0$ is a constant. In particular,

$$h(q,\omega)\;=\;g(q)\;+\;\mu_o\left[\;(x\;+\;x_o(q)\,)^2\;+\;(y\;+\;y_o(q)\,)^2\;+\;(z\;+\;z_o(q)\,)^2\;\right] \tag{5.15}$$

for some real valued functions $g,\;x_o\;\,y_o$ and z_o on $\mathbb{H}^n$. Furthermore we have

$$T_\alpha Z_\alpha X_\alpha^2(h)=T_\alpha Z_\alpha Y_\alpha^2(h)=0,\quad\;T_\alpha^2\xi_j(h)=0.$$

PROOF. Equations (5.11) and (5.12) yield the next sequence of equalities

$$
\begin{aligned}
2\xi_i\xi_j\ h\ &=\ -2T_\alpha\ (I_i\,T_\alpha)\ \xi_j\ h\ =\ -2T_\alpha\ \xi_j\ (I_i\,T_\alpha)\ h = \ T_\alpha\ [T_\alpha, I_j\,T_\alpha]\ (I_iT_\alpha)\ h\\
&=\ T_\alpha^2\,(I_j\,T_\alpha)\ (I_i\,T_\alpha)\ h\ -\ 2T_\alpha\ (I_j\,T_\alpha)\ T_\alpha\ (I_iT_\alpha)\ h = \ T_\alpha^2\ \xi_k\ h\ -\ \xi_i\ \xi_j\ h.
\end{aligned}
$$

Hence, $3\xi_i\xi_j\ h = T_\alpha^2\ \xi_k\ h$. Similarly, interchanging the roles of i and j together with $\{I_iT_\alpha, I_jT_\alpha\} = 0$ we find $3\xi_j\xi_i(h) = -T_\alpha^2\ \xi_k(h)$. Consequently $\xi_i\xi_j\ h\ =\ T_\alpha^2\ \xi_k\ h\ =\ 0$. An analogous calculation shows that $\xi_i\xi_k\ h\ =\ 0$. Furthermore, we have

$$
\begin{aligned}
2\xi_1^2(h) &= 2X_\alpha T_\alpha\xi_1(h) = 2X_\alpha\xi_1T_\alpha(h) = -X_\alpha[Y_\alpha Z_\alpha]T_\alpha(h)\\
&= X_\alpha Z_\alpha Y_\alpha T_\alpha(h) - X_\alpha Y_\alpha Z_\alpha T_\alpha(h) = \xi_2^2(h) + \xi_3^2(h).
\end{aligned}
$$

We derive similarly $2\xi_2^2(h) = \xi_1^2(h) + \xi_3^2(h), \quad 2\xi_3^2(h) = \xi_2^2(h) + \xi_1^2(h)$. Therefore $\xi_1^2(h) = \xi_2^2(h) = \xi_3^2(h), \quad \xi_i^3(h) = \xi_i\xi_j^2(h) = 0, \quad i \neq j = 1,2,3$ which proves part of (5.14).

Next we prove that the common value of the second derivatives is a constant. For this we differentiate the equation $T_\alpha^2\,\xi_k h = 0$ with respect to I_kT_α from where, (5.11) and (5.12), we get

$$
\begin{aligned}
0\ &=\ \xi_k\ (I_k\,T_\alpha)\ T_\alpha^2\ h\ =\ \xi_k\ T_\alpha\ (I_k\,T_\alpha)\ T_\alpha\ h\ +\ \xi_k\ [I_k\,T_\alpha,\ T_\alpha]\ h\\
&=\ T_\alpha\ \xi_k^2\ h\ +\ 2\ T_\alpha\ \xi_k^2\ h\ =\ 3\ T_\alpha\ \xi_k^2\ h.
\end{aligned}
$$

In order to see the vanishing of $(I_i\,T_\alpha)\ \xi_k^2\ h$ we shall need

$$
(I_i\,T_\alpha)^2\,\xi_k h = 0. \tag{5.16}
$$

The latter can be seen by the following calculation.

$$
\begin{aligned}
2\xi_i\xi_j\ h\ &=\ 2\xi_i\ (I_i\,T_\alpha)\,(I_k\,T_\alpha)\ h\ =\ 2(I_i\,T_\alpha)\,\xi_i\ (I_k\,T_\alpha)\ h\ =\ T_\alpha\,[T_\alpha, I_j\,T_\alpha]\,(I_iT_\alpha)\,h\\
=\ &(I_i\,T_\alpha)^2\ T_\alpha(I_k\,T_\alpha)\ h - (I_i\,T_\alpha)\ T_\alpha\ (I_i\,T_\alpha)\ (I_k\,T_\alpha)\ h = -\,(I_i\,T_\alpha)^2\ \xi_k\ h - \ \xi_i\ \xi_j\ h,
\end{aligned}
$$

from where $0 = 3\xi_i\xi_j h = -(I_i\,T_\alpha)^2\xi_k h$. Differentiate $(I_i\,T_\alpha)^2\ \xi_k h = 0$ with respect to I_jT_α to get

$$
\begin{aligned}
0\ &=\ \xi_k\ (I_j\,T_\alpha)\ (I_i\,T_\alpha)^2\ \xi_k\ h = \ \xi_k\ (I_i\,T_\alpha)\ (I_k\,T_\alpha)\ T_\alpha\ h + \xi_k\ [I_j\,T_\alpha, I_i\,T_\alpha]\ (I_i\,T_\alpha)\ h\\
&=\ (I_i\,T_\alpha)\ \xi_k^2\ h\ +\ 2\ (I_i\,T_\alpha)\ \xi_k^2\ h\ =\ 3\ (I_i\,T_\alpha)\ \xi_k^2\ h.
\end{aligned}
$$

We proved the vanishing of all derivatives of the common value of $\xi_j^2 h$, i.e., this common value is a constant, which we denote by $8\mu_o$. Let us note that $\mu_o > 0$ follows easily from the fact that $h > 0$ since g is independent of x, y and z.

The rest equalities of the proposition follow easily from (5.11) and (5.14). $\square$

In view of Proposition 5.4, we define $h\ =\ g\ +\ \mu_o\,f$, where

$$
f\ =\ (x + x_o(q))^2\ +\ (y + y_o(q))^2\ +\ (z + z_o(q))^2. \tag{5.17}
$$

The following simple Lemma is one of the keys to integrating our system.

LEMMA 5.5. *Let X and Y be two parallel horizontal vectors*

a) If $\omega_s(X,Y)\ =\ 0$ for $s = 1,2,3$ then

$$
4XYh - 2h^{-1}\Big[dh(X)\ dh(Y) + \sum_{s=1}^{3} dh(I_sX)\ dh(I_sY)\Big]\ =\ \lambda\,g(X,Y). \tag{5.18}
$$

b) If $g(X,Y) \;=\; 0$ *then*

$$
\begin{aligned}
(5.19)\quad 2\,XYh \;-\; h^{-1}\{\,dh(X)\;dh(Y) \\
+\;\sum_{s=1}^{3} dh(I_sX)\;dh(I_sY)\,\} \;=\; 2\,\sum_{s=1}^{3}\{\;(\xi_s h)\;\omega_s(X,Y)\;\}.
\end{aligned}
$$

c) If $g(X,Y) \;=\; \omega_s(X,Y) \;=\; 0$ *for* $s=1,2,3$ *we have for any* $j\in\{1,2,3\}$

$$
XY\,(\xi_j h) \;=\; 0, \tag{5.20}
$$

$$
8\,XYh \;=\; \mu_o\,\big\{(X\xi_j f)\,(Y\xi_j f) \;+\; \sum_{s=1}^{3}(I_sX\,\xi_j f)\,(I_sY\,\xi_j f)\,\big\}.
$$

Proof. The equation of a) and b) are obtained by adding (5.9) and (5.10). Let us prove part c). From (5.9) and (5.10) taking any two horizontal vectors satisfying $g(X,Y)=\omega_s(X,Y)=0$, we obtain $2h\nabla dh(X,Y) \;=\; dh(X)\,dh(Y) + \sum_{s=1}^{3} dh(I_sX)\;dh(I_sY)$.
If $X,\;Y$ are also parallel, differentiate along ξ_j twice to get consequently

$$
\begin{aligned}
(5.21)\quad & 2\xi_j h\,XYh \;+\; 2hXY\xi_j h \\
&= \;(X\xi_j h)\,(Yh) \;+\; \sum_{s=1}^{3}[(I_sX\,\xi_j h)\,(I_sY\,h) \;+\; (I_sX\,\xi_j h)\,(I_sY\,h)],\\
& 2\xi_j^2 h\,XYh \;+\; 4hXY\xi_j h \;=\; 2\big\{\,(X\xi_j h)\,(Y\xi_j h) \;+\; \sum_{s=1}^{3}(I_sX\,\xi_j h)\,(I_sY\,\xi_j h)\,\big\}.
\end{aligned}
$$

Differentiate three times along ξ_j and use $\xi_j^2 h$ =const, cf. (5.14) to get

$$
2\;(\xi_j^2 h)\;XY\,(\xi_j h) \;=\; 0,
$$

from where the first equality in (5.20) follows. With this information the second line in (5.21) reduces to the second equality in (5.20). $\square$

In order to see that after a suitable translation the functions x_o y_o and z_o can be made equal to zero we prove the following proposition.

Proposition 5.6. *If* h *satisfies* (5.9) *and* (5.10) *on* $G(\mathbb{H})$ *then we have*
a) For $s\in\{1,2,3\}$ *and* i,j,k *a cyclic permutation of* $1,2,3$

$$
\begin{aligned}
T_\alpha\;T_\beta\;(\xi_s h) \;&=\; (I_iT_\alpha)\;(I_iT_\beta)\;(\xi_s h) \;=\; 0 \quad \forall\,\alpha,\;\beta\\
(I_iT_\alpha)\;T_\beta\;(\xi_s h) \;&=\; (I_iT_\alpha)\;(I_iT_\beta)\;(\xi_s h) \;=\; 0,\;\alpha\neq\beta\\
(I_j\,T_\alpha)\,T_\alpha\;(\xi_s\,h) \;&=\; -\;T_\alpha\,(I_j\,T_\alpha)\;(\xi_s h) \;=\; 8\;\delta_{sj}\;\mu_o\\
(I_j\,T_\alpha)\,(I_i\,T_\alpha)\;(\xi_s h) \;&=\; -\;(I_i\,T_\alpha)\,(I_j\,T_\alpha)\;(\xi_s h) \;=\; 8\;\delta_{sk}\;\mu_o\;,
\end{aligned}
\tag{5.22}
$$

i.e., the horizontal Hessian of a vertical derivative of h *is determined completely.*
b) There is a point $(q_o,\omega_o)\in G(\mathbb{H})$, $q_o=(q_o^1,\,q_o^2,\ldots,q_o^n)\in\mathbb{H}^n$ *and* $\omega=ix_o+jy_o+kz_o\in Im(\mathbb{H})$*, such that,*

$$
ix_o(q) \;+\; jy_o(q) \;+\; kz_o(q) \;=\; w_o \;+\; 2\;Im\;q_o\,\bar q.
$$

Proof. a) Taking $\alpha\neq\beta$ and using $X=T_\alpha$ and $Y=T_\beta$ in (5.20), we obtain $T_\alpha T_\beta\xi_s h=0,\quad \alpha\neq\beta$. When $\alpha=\beta$ the same equality holds by (5.16).

The vanishing of the other derivatives can be obtained similarly. Finally, the rest of the second derivatives can be determined from (5.11).

b) From the identities in (5.22) all second derivatives of x_o, y_o and z_o vanish. Thus x_o, y_o and z_o are linear function. The fact that the coefficients are related as required amounts to the following system

$$\begin{aligned} T_\alpha\, x_o &= Z_\alpha\, y_o = -Y_\alpha\, z_o, & X_\alpha\, x_o &= Y_\alpha\, y_o = Z_\alpha\, z_o\\ Y_\alpha\, x_o &= X_\alpha\, y_o = -T_\alpha\, z_o, & Z_\alpha\, x_o &= -T_\alpha\, y_o = -X_\alpha\, z_o. \end{aligned}$$

¿From (5.15) we have $\xi_1 h = 4\mu_o(x+x_o(q))$, $\xi_2 h = 4\mu_o(y+y_o(q))$, $\xi_3 h = 4\mu_o(z+z_o(q))$. Therefore, the above system is equivalent to

$$\begin{aligned} T_\alpha\,\xi_1\ h &= Z_\alpha\,\xi_2\ h = -Y_\alpha\,\xi_3\ h, & X_\alpha\,\xi_1\ h &= Y_\alpha\,\xi_2\ h = Z_\alpha\,\xi_3\ h\\ Y_\alpha\,\xi_1\ h &= X_\alpha\,\xi_2\ h = -T_\alpha\,\xi_3\ h & Z_\alpha\,\xi_1\ h &= -T_\alpha\,\xi_2\ h = -X_\alpha\,\xi_3\ h. \end{aligned}$$

Let us prove the first line. Denote $a = T_\alpha\,\xi_1\ h$, $b = Z_\alpha\,\xi_2\ h$, $c = -Y_\alpha\,\xi_3\ h$. From (5.11) and (5.12) it follows

$$\begin{aligned} a &= T_\alpha Z_\alpha Y_\alpha\ h = Z_\alpha T_\alpha Y_\alpha\ h + [T_\alpha, Z_\alpha]Y_\alpha\ h = -b + 2c\\ b &= Z_\alpha Y_\alpha T_\alpha\ h = Y_\alpha Z_\alpha T_\alpha\ h + [Z_\alpha, Y_\alpha]T_\alpha\ h = -c + 2a\\ c &= Y_\alpha T_\alpha Z_\alpha\ h = T_\alpha Y_\alpha Z_\alpha\ h + [Y_\alpha, T_\alpha]Z_\alpha\ h = -a + 2b, \end{aligned}$$

which implies $a = b = c$. The rest of the identities of the system can be obtained analogously. □

So far we have proved that if h satisfies the system (5.9) and (5.10) on $G(\mathbb{H})$ then, in view of the translation invariance of the system, after a suitable translation we have

$$h(q,\omega) = g(q) + \mu_o\,(x^2 + y^2 + z^2).$$

Proposition 5.7. *If h satisfies the system (5.9) and (5.10) on $G(\mathbb{H})$ then after a suitable translation we have*

$$g(q) = (b + 1 + \sqrt{\mu}_o\ |q|^2)^2, \qquad b + 1 > 0.$$

Proof. Notice that $\xi_1 h = 4\mu_o x, \xi_2 h = 4\mu_o y, \xi_3 h = 4\mu_o z$. With this equations (5.11) become

$$\begin{aligned} T_\alpha X_\alpha(h) = Y_\alpha Z_\alpha(h) = -X_\alpha T_\alpha(h) = -Z_\alpha Y_\alpha(h) = -4\mu_o x,\\ T_\alpha Y_\alpha(h) = Z_\alpha X_\alpha(h) = -Y_\alpha T_\alpha(h) = -X_\alpha Z_\alpha(h) = -4\mu_o y,\\ T_\alpha Z_\alpha(h) = X_\alpha Y_\alpha(h) = -Z_\alpha T_\alpha(h) = -Y_\alpha X_\alpha(h) = -4\mu_o z. \end{aligned} \tag{5.23}$$

Let us also write explicitly some of the derivatives of f, which shall be used to express the derivatives of g by the derivatives of h. For all α and β we have

$$T_\beta f = 4(x^\beta x + y^\beta y + z^\beta z), \qquad X_\beta f = 4(-t^\beta x - z^\beta y + y^\beta z),$$
$$Y_\beta f = 4(z^\beta x - t^\beta y - x^\beta z), \qquad Z_\beta f = 4(-y^\beta x + x^\beta y - t^\beta z),$$
$$T_\alpha T_\beta f = 8(x^\alpha x^\beta + y^\alpha y^\beta + z^\alpha z^\beta), \qquad X_\alpha X_\beta f = 8(t^\alpha t^\beta + z^\alpha z^\beta + y^\alpha y^\beta),$$
$$Y_\alpha Y_\beta f = 8(z^\alpha z^\beta + t^\alpha t^\beta + x^\alpha x^\beta), \qquad Z_\alpha Z_\beta f = 8(y^\alpha y^\beta + x^\alpha x^\beta + t^\alpha t^\beta),$$
$$T_\alpha X_\beta f = -4\delta_{\alpha\beta} x + 8(-x^\alpha t^\beta - y^\alpha z^\beta + z^\alpha y^\beta),$$
$$T_\alpha Y_\beta f = -4\delta_{\alpha\beta} y + 8(x^\alpha z^\beta - y^\alpha t^\beta - z^\alpha x^\beta),$$
$$T_\alpha Z_\beta f = -4\delta_{\alpha\beta} z + 8(-x^\alpha y^\beta + y^\alpha x^\beta - z^\alpha t^\beta),$$
$$X_\alpha T_\beta f = 4\delta_{\alpha\beta} x + 8(-t^\alpha x^\beta - z^\alpha y^\beta + y^\alpha z^\beta),$$
$$X_\alpha Y_\beta f = -4\delta_{\alpha\beta} z + 8(-t^\alpha z^\beta + z^\alpha t^\beta - y^\alpha x^\beta),$$
$$X_\alpha Z_\beta f = 4\delta_{\alpha\beta} y + 8(t^\alpha y^\beta - z^\alpha x^\beta + y^\alpha t^\beta).$$

From the above formulas we see that the fifth order horizontal derivatives of f vanish. In particular the fifth order derivatives of h and g coincide.

Taking $X = Y = T_\alpha$ in (5.18) we obtain

$$4T_\alpha^2 h - 2h^{-1}\{(T_\alpha h)^2 + (X_\alpha h)^2 + (Y_\alpha h)^2 + (Z_\alpha h)^2\} = \lambda. \tag{5.24}$$

Using in the same manner X_α, Y_α and Z_α we see the equality of the second derivatives

$$T_\alpha^2 h = X_\alpha^2 h = Y_\alpha^2 h = Z_\alpha^2 h. \tag{5.25}$$

Therefore, using (5.11) and (5.14), we have

$$T_\alpha^3 h = T_\alpha X_\alpha^2 h = X_\alpha T_\alpha X_\alpha h + [T_\alpha, X_\alpha] X_\alpha h = -3X_\alpha \xi_1 h = 24\mu_o t^\alpha$$

and thus $T_\alpha^4 h = 24\mu_o$. In the same fashion we conclude

$$T_\alpha^3 h = 24\mu_o t^\alpha \qquad X_\alpha^3 h = 24\mu_o x^\alpha, \quad Y_\alpha^3 h = 24\mu_o y^\alpha, \quad Z_\alpha^3 h = 24\mu_o z^\alpha. \tag{5.26}$$

Similarly, taking $X = T_\alpha$, $Y = X_\beta$ and $j = 1$ in (5.20) we find

$$T_\alpha X_\beta h = 8\mu_o(-x^\alpha t^\beta + t^\alpha x^\beta - y^\alpha z^\beta + z^\alpha y^\beta), \qquad \alpha \neq \beta.$$

Plugging $X = T_\alpha$, $Y = T_\beta$ with $\alpha \neq \beta$ in (5.20) we obtain

$$T_\alpha T_\beta h = X_\alpha h X_\beta h = Y_\alpha h Y_\beta h = Z_\alpha h Z_\beta h = 8\mu_o(t^\alpha t^\beta + x^\alpha x^\beta + y^\alpha y^\beta + z^\alpha z^\beta).$$

The other mixed second order derivatives when $\alpha \neq \beta$ can be obtained by taking suitable X and Y. In view of the formulas for the derivatives of f and (5.23), we conclude

$$\begin{aligned}
&T_\alpha X_\beta\, g = 8\mu_o t^\alpha x^\beta, \quad T_\alpha Y_\beta\, g = 8\mu_o t^\alpha y^\beta, \quad T_\alpha Z_\beta\, g = 8\mu_o t^\alpha z^\beta\\
&X_\alpha Y_\beta\, g = 8\mu_o x^\alpha y^\beta, \quad X_\alpha Z_\beta\, g = 8\mu_o x^\alpha z^\beta, \quad Y_\alpha Z_\beta\, g = 8\mu_o y^\alpha z^\beta,\\
&T_\alpha T_\beta g = 8\mu_o t^\alpha t^\beta, \quad X_\alpha X_\beta g = 8\mu_o x^\alpha x^\beta, \quad Y_\alpha Y_\beta g = 8\mu_o y^\alpha y^\beta,\\
&Z_\alpha Z_\beta g = 8\mu_o z^\alpha z^\beta, \quad T_\alpha X_\alpha\, g = 8\mu_o t^\alpha x^\alpha, \quad T_\alpha Y_\alpha\, g = 8\mu_o t^\alpha y^\alpha,\\
&T_\alpha Z_\alpha\, g = 8\mu_o t^\alpha z^\alpha \quad X_\alpha Y_\alpha\, g = 8\mu_o x^\alpha y^\alpha, \quad X_\alpha Z_\alpha\, g = 8\mu_o x^\alpha z^\alpha,\\
&Y_\alpha Z_\alpha\, g = 8\mu_o y^\alpha z^\alpha.
\end{aligned} \tag{5.27}$$

A consequences of the considerations so far is the fact that all second order derivative are quadratic functions of the variables from the first layer, except the pure (unmixed) second derivatives, in which case we know (5.25) and (5.26). It is easy to see then that the fifth order horizontal derivatives of h vanish. With the information so far after a small argument we can assert that g ~~is a polynomial of degree 4 without terms of degree 3, and of~~ the form

$$g = \mu_o \sum_{\cancel{\alpha=1}}^{\cancel{n}} \cancel{(t_\alpha^4 + x_\alpha^4 + y_\alpha^4 + z_\alpha^4)} + p_2,$$

where p_2 is a polynomial of degree two. Furthermore, the mixed second order derivatives of g are determined, while the pure second order derivatives are equal. The latter follows from (5.24) taking $q = 0$, $\omega = 0$. Let us see that there are no terms of degree one on p_2. Taking $X = T_\alpha$, $Y = T_\beta$, $\alpha \neq \beta$ and $j = 1$ in (5.21) we find

$$\begin{aligned}
&(4\mu_o x)\ \{\ 4T_\alpha T_\beta g \ +\ 32\mu_o(x^\alpha x^\beta \ +\ y^\alpha y^\beta \ +\ z^\alpha z^\beta)\ \}\\
&= 2\Big\{(8x^\alpha)(T_\beta g + 4\mu_o(x^\beta x + y^\beta y + z^\beta z)) + (-8t^\alpha)(X_\beta g + 4\mu_o(-t^\beta x - z^\beta y + y^\beta z))\\
&+ (8z^\alpha)(Y_\beta g + 4\mu_o(z^\beta x - t^\beta y - x^\beta z)) + (-8y^\alpha)(Z_\beta g + 4\mu_o(-y^\beta x + x^\beta y - t^\beta z))\\
&+ (8x^\beta)(T_\alpha g + 4\mu_o(x^\alpha x + y^\alpha y + z^\alpha z)) + (-8t^\beta)(X_\alpha g + 4\mu_o(-t^\alpha x - z^\alpha y + y^\alpha z))\\
&+ (8z^\beta)(Y_\alpha g + 4\mu_o(z^\alpha x - t^\alpha y - x^\alpha z)) + (-8y^\beta)(Z_\alpha g + 4\mu_o(-y^\alpha x + x^\alpha y - t^\alpha z))\Big\}\\
&= 16\left(x^\alpha T_\beta g + x^\beta T_\alpha g - t^\alpha X_\beta g - t^\beta X_\alpha g + z^\alpha Y_\beta g + z^\beta Y_\alpha g - y^\alpha Z_\beta g - y^\beta Z_\alpha g\right)\\
&\qquad\qquad\qquad + 128\mu_o\, x\ (t^\alpha t^\beta \ +\ x^\alpha x^\beta \ +\ t^\alpha t^\beta \ +\ x^\alpha x^\beta\ )
\end{aligned}$$

Taking into account (5.27) we proved

$$x^\alpha T_\beta g + x^\beta T_\alpha g - t^\alpha X_\beta g - t^\beta X_\alpha g + z^\alpha Y_\beta g + z^\beta Y_\alpha g - y^\alpha Z_\beta g - y^\beta Z_\alpha g = 0, \quad \alpha \neq \beta.$$

Comparing coefficients in front of the linear terms implies that g has no first order terms. Thus, we can assert that g can be written in the following form

$$g \ =\ \left(1 \ +\ \sqrt{\mu_o}\,|q|^2\right)^2 \ +\ 2a\,|q|^2 \ +\ b. \tag{5.28}$$

Hence, $h \ =\ \left(1 \ +\ \sqrt{\mu_o}\,|q|^2\right)^2 \ +\ a\,|q|^2 \ +\ b \ +\ \mu_o\,(x^2 \ +\ y^2 \ +\ z^2)$. Taking $X = T_\alpha$, $Y = T_\beta$ in (5.19) we obtain $16\mu_0\,(1 \ +\ b) \ =\ 4\,(a \ +\ 2\sqrt{\mu_o})^2$. Therefore,

$$\begin{aligned}
g \ &=\ \mu_o\,|q|^4 \ +\ (a \ +\ 2\sqrt{\mu_o})\ \sqrt{\mu_o}\,|q|^2 \ +\ b \ +\ 1\\
&\qquad\qquad =\ 2\sqrt{b \ +\ 1}\,|q|^2 \ +\ b \ +\ 1 = \ \left(b \ +\ 1 \ +\ \sqrt{\mu_o}\,|q|^2\right)^2.
\end{aligned}$$

In turn the formula for h becomes

$$h \ =\ \left(b \ +\ 1 \ +\ \sqrt{\mu_o}\,|q|^2\right)^2 \ +\ \mu_o\,(x^2 \ +\ y^2 \ +\ z^2). \tag{5.29}$$

Setting $c \ =\ (b \ +\ 1)^2$ and $\nu \ =\ \frac{\sqrt{\mu_o}}{1+b} \ >\ 0$ the solution takes the form $h \ =\ c\,\left[\left(1 \ +\ \nu\,|q|^2\right)^2 \ +\ \nu^2\,(x^2 \ +\ y^2 \ +\ z^2)\right]$, which completes the proof of Theorem 1.1. $\square$

Let us note that the final conclusion can be reached also using the fact that a qc-Einstein structure has necessarily constant scalar curvature by Theorem 4.9, together with the result of [GV1] identifying all partially symmetric solutions of the Yamabe equation on $G(\mathbb{H})$, i.e., of the equation

$$\sum_{\alpha=1}^{n} \left(T_\alpha^2 u \ + \ X_\alpha^2 u \ + \ Y_\alpha^2 u \ + \ Z_\alpha^2 u\right) \ = \ -u^{\frac{Q+2}{Q-2}}.$$

The fact that we are dealing with such a solution follows from (5.28). The current solution depends on one more parameter as the scalar curvature can be an arbitrary constant. This constant will appear in the argument of [GV1] by first using scalings to reduce to a fixed scalar curvature one for example.

CHAPTER 6

Special functions and pseudo-Einstein quaternionic contact structures

Considering only the [3]-component of the Einstein tensor of the Biquard connection due to Theorem 3.12 and by analogy with the CR-case [L1], it seems useful to give the following Definition.

DEFINITION 6.1. *Let $(M, g, \mathbb{Q})$ be a quaternionic contact manifold of dimension bigger than 7. We call M* qc-pseudo-Einstein *if the trace-free part of the [3]-component of the qc-Einstein tensor vanishes.*

Observe that for $n = 1$ any QC structure is qc-pseudo-Einstein. According to Theorem 3.12 $(M, g, \mathbb{Q})$ is quaternionic qc-pseudo-Einstein exactly when the trace-free part of the [3]-component of the torsion vanishes, $U = 0$. Proposition 5.1 yields the following claim.

PROPOSITION 6.2. *Let $\bar{\eta} = u\eta$ be a conformal transformation of a given qc-structure. Then the trace-free part of the* [3] *component of the qc-Ricci tensor (i.e. U) is preserved if and only if the function u satisfies the differential equations*

$$(\nabla_X du)Y + (\nabla_{I_1X} du)I_1Y + (\nabla_{I_2X} du)I_2Y + (\nabla_{I_3X} du)I_3Y = \frac{1}{n}\Delta u\, g(X, Y). \tag{6.1}$$

In particular, the qc-pseudo-Einstein condition persists under conformal transformation $\tilde{\eta} = u\eta$ exactly when the function u satisfies (6.1).

PROOF. Defining $h = u^{-1}$ a small calculation shows

$$\nabla dh - 2h^{-1} dh \otimes dh = u^{-1}\nabla du. \tag{6.2}$$

Inserting (6.2) into (5.10) shows (6.1). $\square$

Our next goal is to investigate solutions to (6.1). We shall find geometrically defined functions, which are solutions of (6.1).

6.1. Quaternionic pluriharmonic functions

We start with some analysis on the quaternion space $\mathbb{H}^n$.

6.1.0.1. *Pluriharmonic functions in $\mathbb{H}^n$.* Let $\mathbb{H}$ be the four-dimensional real associative algebra of the quaternions. The elements of $\mathbb{H}$ are of the form $q = t+ix+jy+kz$, where $t, x, y, z \in \mathbb{R}$ and $i,\ j,\ k$ are the basic quaternions satisfying the multiplication rules $i^2 = j^2 = k^2 = -1$ and $ijk = -1$. For a quaternion q we define its conjugate $\bar{q} = t - ix - jy - kz$, and real and imaginary parts, correspondingly, by $\Re q = t$ and $\Im q = xi + yj + zk$.The most important operator for us is the Dirac-Feuter operator $\overline{\mathcal{D}} = \partial_t + i\partial_x + j\partial_y + k\partial_z$, i.e.,

$$\overline{\mathcal{D}}\, F = \partial_t F + i\partial_x F + j\partial_y F + k\partial_z F$$

and in addition
$$\mathcal{D}F \;=\; \partial_t F \;-\; i\partial_x F \;-\; j\partial_y F \;-\; k\partial_z F.$$
Note that if F is a quaternionic valued function due to the non-commutativity of the multiplication the above expression is not the same as $F\overline{\mathcal{D}} \;\overset{def}{=}\; \partial_t F \,+\, \partial_x F i \,+\, \partial_y F j \;+\; \partial_z F k$. Also, when conjugating $\overline{\overline{\mathcal{D}}\, F} \neq \mathcal{D}\overline{F}$.

Definition 6.3. *A function $F \;:\; \mathbb{H} \to \; \mathbb{H}$, which is continuously differentiable when regarded as a function of $\mathbb{R}^4$ into $\mathbb{R}^4$ is called* quaternionic anti-regular *(*quaternionic regular*), or just* anti-regular *(*regular*) for short, if $\mathcal{D}F = 0$ $(\overline{\mathcal{D}}\; F \;=\; 0)$.*

These functions were introduced by Fueter [**F**]. The reader can consult the paper of A. Sudbery [**S**] for the basics of the quaternionic analysis on $\mathbb{H}$. Let us note explicitly one of the most striking differences between complex and quaternionic analysis. As it is well known the theory of functions of a complex variable z is equivalent to the theory of power series of z. In the quaternionic case, each of the coordinates t, x, y and z can be written as a polynomial in q, see eq. (3.1) of [**S**], and hence the theory of power series of q is just the theory of real analytic functions. Our goal here is to consider functions of several quaternionic variables in $\mathbb{H}^n$ and on manifolds with quaternionic structure and present some applications in geometry.

For a point $q \in \mathbb{H}^n$ we shall write $q = (q^1, \dots, q^n)$ with $q^\alpha \in \mathbb{H}$, $q^\alpha = t^\alpha + ix^\alpha + jy^\alpha + kz^\alpha$ for $\alpha = 1, \dots, n$. Furthermore, $q^{\overline{\alpha}} = \overline{q^\alpha}$, i.e., $q^{\overline{\alpha}} = t^\alpha - ix^\alpha - jy^\alpha - kz^\alpha$.

We recall that a function $F \;:\; \mathbb{H}^n \to \; \mathbb{H}$, which is continuously differentiable when regarded as a function of $\mathbb{R}^{4n}$ into $\mathbb{R}^4$ is called quaternionic regular, or just regular for short, if
$$\overline{\mathcal{D}}_\alpha\; F \;=\; \partial_{t_\alpha} F \;+\; i\partial_{x_\alpha} F \;+\; j\partial_{y_\alpha} F \;+\; k\partial_{z_\alpha} F \;=\; 0, \quad \alpha = 1, \dots, n.$$
In other words, a real-differentiable function of several quaternionic variables is regular if it is regular in each of the variables (see [**Per1, Per2, Joy**]. The condition that $F = f + iw + ju + kv$ is regular is equivalent to the following Cauchy-Riemann-Feuter equations
$$\begin{aligned} &\partial_{t_\alpha} f - \partial_{x_\alpha} w - \partial_{y_\alpha} u - \partial_{z_\alpha} v \;=\; 0, \quad && \partial_{t_\alpha} w + \partial_{x_\alpha} f + \partial_{y_\alpha} v - \partial_{z_\alpha} u \;=\; 0,\\ &\partial_{t_\alpha} u - \partial_{x_\alpha} v + \partial_{y_\alpha} f + \partial_{z_\alpha} w \;=\; 0, \quad && \partial_{t_\alpha} v + \partial_{x_\alpha} u - \partial_{y_\alpha} w + \partial_{z_\alpha} f \;=\; 0. \end{aligned} \tag{6.3}$$

Definition 6.4. *A real-differentiable function $f : \mathbb{H}^n \mapsto \mathbb{R}$ is called* $\bar{Q}$-pluriharmonic *if it is the real part of a regular function.*

Proposition 6.5. *Let f be a real-differentiable function $f : \mathbb{H}^n \mapsto \mathbb{R}$. The following conditions are equivalent*

i) f is $\bar{Q}$-pluriharmonic, i.e., it is the real part of a regular function;
ii) $\overline{\mathcal{D}}_\beta\; \mathcal{D}_\alpha\; f \;=\; 0$ for every $\alpha, \beta \in \{1, \dots, n\}$, where $\mathcal{D}_\alpha \;=\; \partial_{t_\alpha} \;-\; i\partial_{x_\alpha} \;-\; j\partial_{y_\alpha} \;-\; k\partial_{z_\alpha}$;
iii) f satisfies the following system of PDEs
$$\begin{aligned} &f_{t_\beta t_\alpha} + f_{x_\beta x_\alpha} + f_{y_\beta y_\alpha} + f_{z_\beta z_\alpha} = 0, \quad && f_{x_\beta t_\alpha} - f_{t_\beta x_\alpha} - f_{y_\beta z_\alpha} + f_{z_\beta y_\alpha} = 0\\ -&f_{t_\beta y_\alpha} + f_{x_\beta z_\alpha} + f_{y_\beta t_\alpha} - f_{z_\beta x_\alpha} = 0, \quad && -f_{t_\beta z_\alpha} - f_{x_\beta y_\alpha} + f_{y_\beta x_\alpha} + f_{z_\beta t_\alpha} = 0. \end{aligned} \tag{6.4}$$

Proof. It is easy to check that $\overline{\mathcal{D}}_\beta\; \mathcal{D}_\alpha\; f = 0$ is equivalent to (6.4).

We turn to the proof of ii) implies i). Let f be real valued function on $\mathbb{H}^n$, such that, $\overline{\mathcal{D}}_\beta\; \mathcal{D}_\alpha\; f \;=\; 0$. We shall construct a real-differentiable regular function

$F : \mathbb{H}^n \mapsto \mathbb{H}$. In fact, for $q \in \mathbb{H}^n$ we define

$$F(q) \; = \; f(q) \; + \; \Im \int_0^1 s^2 \, (\mathcal{D}_\alpha \; f)(sq) \, q^\alpha \, ds.$$

In order to rewrite the imaginary part in a different way we compute

$$\begin{aligned}
&\Re \int_o^1 s^2 \, (\mathcal{D}_\alpha \; f)(sq) \, q_\alpha \, ds \\
&\quad = \Re \int_0^1 s^2 \Big(\partial_{t_\alpha} f - i\partial_{x_\alpha} f - j\partial_{y_\alpha} f - k\partial_{z_\alpha} f \Big) \, (sq) \, (t_\alpha + i x_\alpha + j y_\alpha + k z_\alpha) \, ds \\
&\quad = \int_0^1 s^2 \, \Big(\partial_{t_\alpha} f(sq) t_\alpha + \partial_{x_\alpha} f(sq) x_\alpha + \partial_{y_\alpha} f(sq) y_\alpha + \partial_{z_\alpha} f(sq) z_\alpha \Big) \, ds \\
&= \int_0^1 s^2 \frac{d}{ds} \Big(f(sq) \Big) \, ds \; = \; s^2 f(sq)|_0^1 \; - \; 2 \int_0^1 s f(sq) \, ds = \; f(q) \; - \; 2 \int_0^1 s f(sq) \, ds.
\end{aligned}$$

Therefore we have

$$\Im \int_o^1 s^2 \, (\mathcal{D}_\alpha \; f)(sq) \, q^\alpha \, ds \; = \; \int_o^1 s^2 \, (\mathcal{D}_\alpha \; f)(sq) \, q^\alpha \, ds \; - \; f(q) \; + \; 2 \int_0^1 s f(sq) \, ds.$$

In turn, the formula for $F(q)$ becomes

$$F(q) \; = \; \int_o^1 s^2 \, (\mathcal{D}_\alpha \; f)(sq) \, q^\alpha \, ds \; + \; 2 \int_0^1 s f(sq) \, ds.$$

Hence $\overline{\mathcal{D}}_\beta \; F(q) \; = \; \int_o^1 s^2 \, \overline{\mathcal{D}}_\beta \, \big[(\mathcal{D}_\alpha \; f)(sq) \, q^\alpha \big] \, ds + 2 \int_0^1 s \, \overline{\mathcal{D}}_\beta \, \big[f(sq) \big] \, ds$. We compute the first term,

$$\begin{aligned}
&\overline{\mathcal{D}}_\beta \, \big[(\mathcal{D}_\alpha \; f)(sq) \, q^\alpha \big] = (\partial_{t_\beta} + i\partial_{x_\beta} + j\partial_{y_\beta} + k\partial_{z_\beta}) \big[(\mathcal{D}_\alpha \; f)(sq) \, q^\alpha \big] \\
&\qquad = \; \overline{\mathcal{D}}_\beta \, \big[(\mathcal{D}_\alpha \; f)(sq) \big] \, q^\alpha + \; \mathcal{D}_\alpha \; f(sq) \; \partial_{t_\beta} q_\alpha + \; i \mathcal{D}_\alpha \; f(sq) \; \partial_{x_\beta} q_\alpha \\
&\qquad\qquad + \; j \mathcal{D}_\alpha \; f(sq) \; \partial_{y_\beta} q_\alpha \; + \; k \mathcal{D}_\alpha \; f(sq) \; \partial_{z_\beta} q_\alpha \\
&= \; \overline{\mathcal{D}}_\beta \, \big[(\mathcal{D}_\alpha \; f)(sq) \big] \, q^\alpha + \delta_{\alpha\beta} \{ \mathcal{D}_\alpha \; f(sq) + i \mathcal{D}_\alpha \; f(sq) i + j \mathcal{D}_\alpha \; f(sq) j + k \mathcal{D}_\alpha \; f(sq) k \}.
\end{aligned}$$

The last term can be simplified, using the fundamental property that the coordinates of a quaternion can be expressed by the quaternion only, as follows

$$\begin{aligned}
&\mathcal{D}_\beta \; f(sq) \; + \; i \mathcal{D}_\beta \; f(sq) i \; + \; j \mathcal{D}_\beta \; f(sq) j \; + \; k \mathcal{D}_\beta \; f(sq) k \\
&= \; (\partial_{t_\beta} f \; - \; i\partial_{x_\beta} f \; - \; j\partial_{y_\beta} f \; - \; k\partial_{z_\beta} f) \; + \; i(\partial_{t_\beta} f \; - \; i\partial_{x_\beta} f \; - \; j\partial_{y_\beta} f \; - \; k\partial_{z_\beta} f) i \\
&+ \; j(\partial_{t_\beta} f \; - \; i\partial_{x_\beta} f \; - \; j\partial_{y_\beta} f \; - \; k\partial_{z_\beta} f) j \; + \; k(\partial_{t_\beta} f \; - \; i\partial_{x_\beta} f \; - \; j\partial_{y_\beta} f \; - \; k\partial_{z_\beta} f) k \\
&\qquad = \; -2\partial_{t_\beta} f \; - \; i\partial_{x_\beta} f \; - \; j\partial_{y_\beta} f \; - \; k\partial_{z_\beta} f \; + \; i\partial_{x_\beta} f \; - \; j\partial_{y_\beta} f \; - \; k\partial_{z_\beta} f \\
&\qquad\quad - \; i\partial_{x_\beta} f \; + \; j\partial_{y_\beta} f \; - \; k\partial_{z_\beta} f \; - \; i\partial_{x_\beta} f \; - \; j\partial_{y_\beta} f \; + \; k\partial_{z_\beta} f \\
&\qquad\qquad = \; -2\partial_{t_\beta} f \; - \; 2i\partial_{x_\beta} f \; - \; 2j\partial_{y_\beta} f \; - \; 2k\partial_{z_\beta} f \; = \; -2\overline{\mathcal{D}}_\beta \; f(sq).
\end{aligned}$$

Going back to the computation of $\overline{\mathcal{D}}_\beta \; F(q)$, we find

$$\begin{aligned}
\overline{\mathcal{D}}_\beta \; F(q) \; &= \; \int_0^1 \overline{\mathcal{D}}_\beta \, \big[(\mathcal{D}_\alpha \; f)(sq) \big] \, q^\alpha \, ds - 2 \int_0^1 s^2 \overline{\mathcal{D}}_\beta \; f(sq) \, ds + 2 \int_0^1 s^2 \overline{\mathcal{D}}_\beta \; f(sq) \, ds \\
&= \; \int_0^1 \overline{\mathcal{D}}_\beta \, \big[(\mathcal{D}_\alpha \; f)(sq) \big] \, q^\alpha \, ds.
\end{aligned}$$

Hence, if $\overline{\mathcal{D}}_\beta \; \mathcal{D}_\alpha \; f \; = \; 0$ for every α and β we have $\overline{\mathcal{D}}_\beta \; F(q) \; = \; 0$.

Next we show that i) implies ii). Using (6.3), we have

$$f_{x_\beta t_\alpha} - f_{t_\beta x_\alpha} + f_{z_\beta y_\alpha} - f_{y_\beta z_\alpha} = w_{x_\beta x_\alpha} + u_{x_\beta y_\alpha} + v_{x_\beta z_\alpha} + w_{t_\beta t_\alpha} + v_{t_\beta y_\alpha} - u_{t_\beta z_\alpha} - u_{z_\beta t_\alpha} + v_{z_\beta x_\alpha} - w_{z_\beta z_\alpha} + v_{y_\beta t_\alpha} + u_{y_\beta x_\alpha} - w_{y_\beta y_\alpha}.$$

Both sides must be equal to zero by noticing that the left hand side is antisymmetric while on the right we have an expression symmetric with respect to exchanging α with β. The other identities can be obtained similarly. □

According to [**Sti**] there are exactly two kinds of Cauchy-Riemann equations for functions of several quaternionic variables. The second one turns out to be most suitable for the geometric purposes considered in this paper.

Definition 6.6. *A function $F : \mathbb{H}^n \to \mathbb{H}$, which is continuously differentiable when regarded as a function of $\mathbb{R}^{4n}$ into $\mathbb{R}^4$ is called* quaternionic anti-regular *(also* anti-regular*), if*

$$\mathcal{D}F = \partial_{t_\alpha} F - i\partial_{x_\alpha} F - j\partial_{y_\alpha} F - k\partial_{z_\alpha} F = 0, \quad \alpha = 1, \dots, n.$$

The condition that $F = f + iw + ju + kv$ is anti-regular function is equivalent to the following Cauchy-Riemann-Feuter equations

$$\begin{aligned} &\partial_{t_\alpha} f + \partial_{x_\alpha} w + \partial_{y_\alpha} u + \partial_{z_\alpha} v = 0, \quad \partial_{t_\alpha} w - \partial_{x_\alpha} f - \partial_{y_\alpha} v + \partial_{z_\alpha} u = 0, \\ &\partial_{t_\alpha} u + \partial_{x_\alpha} v - \partial_{y_\alpha} f - \partial_{z_\alpha} w = 0, \quad \partial_{t_\alpha} v - \partial_{x_\alpha} u + \partial_{y_\alpha} w - \partial_{z_\alpha} f = 0. \end{aligned} \tag{6.5}$$

See also (6.8) for an equivalent form of the above system.

Anti-regular functions on hyperkähler and quaternionic Kähler manifolds are studied in [**CL1, CL2, LZ**], under the name quaternionic maps, in connection with minimal surfaces and maps between quaternionic Kähler manifolds preserving the sphere of almost complex structures. Thus, the anti-regular functions considered here are quaternionic maps between $\mathbb{H}^n$ and $\mathbb{H}$ with a suitable choice of the coordinates.

Definition 6.7. *A real-differentiable function $f : \mathbb{H}^n \mapsto \mathbb{R}$ is called* quaternionic pluriharmonic *(* Q-pluriharmonic *for short) if it is the real part of an anti-regular function.*

The anti-regular functions and their real part play a significant role in the theory of hypercomplex manifold as well as in the theory of quaternionic contact (hypercomplex contact) manifolds as we shall see further in the paper. We need a real expression of the second order differential operator $\mathcal{D}_\alpha \overline{\mathcal{D}}_\beta f$ acting on a real function f.

We use the standard hypercomplex structure on $\mathbb{H}^n$ determined by the action of the imaginary quaternions

$$I_1\, dt^\alpha = dx^\alpha, \quad I_1\, dy^\alpha = dz^\alpha, \qquad I_2\, dt^\alpha = dy^\alpha, \quad I_2\, dx^\alpha = -dz^\alpha$$
$$I_3\, dt^\alpha = dz^\alpha, \qquad I_3\, dx^\alpha = dy^\alpha.$$

We recall a convention. For any p-form ψ we consider the p-form $I_s\psi$ and three (p+1)-forms $d_s\psi$, $s = 1, 2, 3$ defined by

$$I_s\psi(X_1, \dots, X_p) \overset{def}{=} (-1)^p \psi(I_sX_1, \dots, I_sX_p), \qquad d_s\psi \overset{def}{=} (-1)^p I_s d I_s \psi.$$

Consider the second order differential operators DD_{I_s} acting on the exterior algebra defined by [**HP**]

$$DD_{I_i} := dd_i + d_jd_k = dd_i - I_jdd_i = dd_i - I_kdd_i. \tag{6.6}$$

PROPOSITION 6.8. *Let f be a real-differentiable function $f:\mathbb{H}^n\to\mathbb{R}$. The following conditions are equivalent*

i) *f is Q-pluriharmonic, i.e. it is the real part of an anti-regular function;*
ii) $DD_{I_s} f \;=\; 0, \quad s=1,2,3.$*;*
iii) $\mathcal{D}_\alpha\, \overline{\mathcal{D}}_\beta\, f \;=\; 0$ *for every* $\alpha,\beta\in\{1,\dots,n\}$*, where*

$$\mathcal{D}_\alpha \;=\; \partial_{t_\alpha} \;-\; i\partial_{x_\alpha} \;-\; j\partial_{y_\alpha} \;-\; k\partial_{z_\alpha},$$

iv) *f satisfies the following system of PDEs*

$$f_{t_\beta t_\alpha}+f_{x_\beta x_\alpha}+f_{y_\beta y_\alpha}+f_{z_\beta z_\alpha} \;=\; 0, \quad -f_{x_\beta t_\alpha}+f_{t_\beta x_\alpha}-f_{y_\beta z_\alpha}+f_{z_\beta y_\alpha} \;=\; 0,$$
$$f_{t_\beta y_\alpha}+f_{x_\beta z_\alpha}-f_{y_\beta t_\alpha}-f_{z_\beta x_\alpha} \;=\; 0, \quad f_{t_\beta z_\alpha}-f_{x_\beta y_\alpha}+f_{y_\beta x_\alpha}-f_{z_\beta t_\alpha} \;=\; 0.$$

PROOF. A simple calculation of $\mathcal{D}_\beta\, \overline{\mathcal{D}}_\alpha\, f$ gives the equivalence between iii) and iv).

Next, we shall show that ii) is equivalent to iii). As $df=\partial_{t_\alpha} f\, dt^\alpha+\partial_{x_\alpha} f\, dx^\alpha+\partial_{y_\alpha} f\, dy^\alpha+\partial_{z_\alpha} f\, dz^\alpha$ we have $I_1 df=\partial_{t_\alpha} f\, dx^\alpha-\partial_{x_\alpha} f\, dt^\alpha+\partial_{y_\alpha} f\, dz^\alpha-\partial_{z_\alpha} f\, dy^\alpha$. A routine calculation gives the following formula

$$D\,D_{I_1} f \;=\; \sum_{\alpha,\beta}\Re(\mathcal{D}_\beta\,\overline{\mathcal{D}}_\alpha\, f)\,[\,dt^\alpha\wedge dx^\beta+dy^\alpha\wedge dz^\beta] \tag{6.7}$$
$$+\sum_{\alpha<\beta}\Re(i\mathcal{D}_\beta\,\overline{\mathcal{D}}_\alpha\, f)\,[-\,dt^\alpha\wedge dt^\beta-\,dx^\alpha\wedge dx^\beta+\,dy^\alpha\wedge dy^\beta+\,dz^\alpha\wedge dz^\beta]$$
$$+\sum_{\alpha,\beta}\Re(j\mathcal{D}_\beta\,\overline{\mathcal{D}}_\alpha\, f)\,[\,dt^\alpha\wedge dz^\beta-dx^\alpha\wedge dy^\beta]-\sum_{\alpha,\beta}\Re(k\mathcal{D}_\beta\,\overline{\mathcal{D}}_\alpha\, f)\,[\,dt^\alpha\wedge dy^\beta+dx^\alpha\wedge dz^\beta].$$

Similar formulas hold for DD_{I_2} and DD_{I_3}. Hence, the equivalence of ii) and iii) follows.

The proof of the implication iii) implies i) is analogous to the proof of the corresponding implication in Proposition 6.5. Define

$$F(q) \;=\; f(q) \;+\; \Im\int_o^1 s^2\,(\overline{\mathcal{D}}_\beta\, f)(sq)\, q^{\overline{\beta}}\, ds,$$

and a small calculation shows that this defines an anti-regular function, i.e., $\mathcal{D}_\alpha\, F \;=\; 0$ for every α.

In order to see that iii) follows from i) we can proceed as in Proposition 6.5 and hence we skip the details. See also another proof in Proposition 6.11 □

REMARK 6.9. *We note that Proposition 6.5 and Proposition 6.8 imply that the real part of a regular function is not in the kernel of the operators DD_{I_s} which is one of the main difference between regular and anti-regular function.*

6.2. Quaternionic pluriharmonic functions on hypercomplex manifold

We recall that a hypercomplex manifold is a smooth $4n$-dimensional manifold M together with a triple (I_1,I_2,I_3) of integrable almost complex structures satisfying the quaternionic relations $I_1I_2=-I_2I_1=I_3$. The second order differential operators DD_{I_i} defined in [**HP**] by (6.21) having the origin in the papers [**Sal1**, **Sal2**, **CSal**] play an important róle in the theory of quaternionic plurisubharmonic functions (i.e. a real function for which $DD_{I_s}(.,I_s.)$ is positive definite) on hypercomplex manifold [**A1**, **A2**, **V**, **AV**, **A3**] as well as the potential theory of HKT-manifolds. We recall that Riemannian metric g on a hypercomplex manifold compatible with the three complex structures is said to be HKT-metric [HP] if the three corresponding Kähler forms $\Omega_s = g(I_s.,.)$ satisfy $d_1\Omega_1 = d_2\Omega_2 = d_3\Omega_3$. A smooth real function is a HKT-potential if locally it generates the three Kähler forms, $\Omega_s = DD_{I_s}f$ [MS, GP], in particular such a function is quaternionic plurisubharmonic. The existence of a HKT potential on any HKT metric on $\mathbb{H}^n$ is proved in [MS] and for any HKT metric in [BS].

Regular functions on hypercomplex manifold are studied from analytical point [Per1, Per2], from algebraic point [Joy, Q]. However, as we have already mentioned, regular functions are not the appropriate functions for our purposes mainly because they have no direct connection with the second order differential operator DD_{I_s}.

Here we consider anti-regular functions and their real parts on hypercomplex manifold.

Definition 6.10. *Let (M, I_1, I_2, I_3) be a hypercomplex manifold. A quaternionic valued function $F : M \longrightarrow f + iw + ju + kv \in \mathbb{H}$ is said to be* anti-regular *if any one of the following relations between the differentials of the coordinates hold*

$$
\begin{aligned}
df &= d_1w + d_2u + d_3v \qquad & d_1f &= -dw + d_3u - d_2v \\
d_2f &= -d_3w - du + d_1v \qquad & d_3f &= d_2w - d_1u - dv.
\end{aligned} \tag{6.8}
$$

A real valued function $f : M \longrightarrow \mathbb{R}$ is said to be quaternionic pluriharmonic *(or* Q-pluriharmonic*) if it is the real part of an anti-regular function.*

Observe that the system (6.5) is equivalent to (6.8). We have the hypercomplex manifold analogue of Proposition 6.8

Proposition 6.11. *Let (M, I_1, I_2, I_3) be a hypercomplex manifold and let f be a real-differentiable function on M, $f : M \longrightarrow \mathbb{R}$. The following conditions are equivalent*

i) f is Q-pluriharmonic, i.e. it is the real part of an anti-regular function;
ii) $DD_{I_s}f \;=\; 0, \quad s = 1, 2, 3.$

Proof. It is easy to verify that if each I_s is integrable almost complex structure then we have the identities [HP]

$$
dd_s + d_sd = 0, \qquad d_sd_r + d_rd_s = 0, \qquad s, r = 1, 2, 3. \tag{6.9}
$$

Using the commutation relations (6.9), we get readily that i) implies ii). For example, (6.8) yields

$$
\begin{aligned}
dd_1f + d_2d_3f + d^2w - d_2^2w - dd_3u + d_2d_1u + dd_2v + d_2dv &= 0 \\
d_1df + d_3d_2f - d_1^2w + d_3^2w - d_1d_2u + d_3du - d_1d_3v - d_3d_1v &= 0.
\end{aligned}
$$

Subtracting the two equations and using the commutation relations (6.9) we get $DD_{I_1}f = 0$.

For the converse, observe that $DD_{I_1}f = 0 \Leftrightarrow dd_1f = I_2dd_1f$. The $\partial\bar\partial$-lemma for I_2 gives the existense of a smooth function A_1 such that $dd_1f = dd_2A_1$. Similarly, using the Poincare lemma, we obtain $d_1f - d_2A_1 - dB_1 = 0, \quad d_2f - d_3A_2 - dB_2 = 0, \quad d_3f - d_1A_3 - dB_3 = 0$ for a smooth functions $A_1, A_2, A_3, B_1, B_2, B_3$. The latter implies $df + d_1(A_2 + B_1) + d_2(B_2 - A_3) + d_3(A_1 - B_3) = 0$. Set $w = -A_2 - B_1$, $u = A_3 - B_2$, $v = B_3 - A_1$ to get the equivalence between i) and ii). $\square$

6.3. The hypersurface case

In this section we shall denote with $\langle .,. \rangle$ the Euclidean scalar product in $\mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$ and with $\tilde{I}_j$, $j = 1, 2, 3$, the standard almost complex structures on $\mathbb{H}^{n+1}$. Let M be a smooth hyper-surface in $\mathbb{H}^{n+1}$ with a defining function ρ, $M = \{\rho = 0\}$, $d\rho \neq 0$, and $i : M \hookrightarrow \mathbb{H}^{n+1}$ be the embedding. It is not hard to see that at every point $p \in M$ the subspace $H_p = \bigcap_{j=1}^{3} \tilde{I}_j\,(T_pM)$ of the tangent space T_pM of M at p is the largest subspace invariant under the almost complex structures and $\dim H_p = 4n$. We shall call H_p the horizontal space at p. Thus on the horizontal space H the almost complex structures I_j, $j = 1, 2, 3$, are the restrictions of the standard almost complex structures on $\mathbb{H}^{n+1}$. In particular, for a horizontal vector X we have

$$\tilde{I}_j\, i_* X \;=\; i_*\,(I_j X). \tag{6.10}$$

Let $\tilde{\theta}^j = \tilde{I}_j\ \frac{d\rho}{|d\rho|}$. We drop the " " in the notation of the almost complex structures when there is no ambiguity.

We define three one-forms on M by setting $\theta^j = i^*\tilde{\theta}^j = i^*(\tilde{I}_j\ \frac{d\rho}{|d\rho|})$, i.e.,

$$\theta_j\ (\ .\ )\ =\ -\frac{d\rho\,(\tilde{I}_j\ .\ )}{|d\rho|}\ =\ \langle\ .\ , \tilde{I}_j N\rangle,$$

where $N = \frac{D\rho}{|D\rho|}$ is the unit normal vector to M. We describe the hypersurfaces which inherit a natural quaternionic contact structure from the standard structures on $\mathbb{H}^{n+1}$ (see also [**D1**]) in the next

Proposition 6.12. *If M is a smooth hypersurface of $\mathbb{H}^{n+1}$ then we have*

$$d\theta_1(I_1X, Y) = d\theta_2(I_2X, Y) = d\theta_3(I_3X, Y) \quad (X,\, Y \in H) \tag{6.11}$$

if and only if the restriction of the second fundamental form of M to the horizontal space is invariant with respect to the almost complex structures, i.e. if X and Y are two horizontal vectors we have $II(I_jX, I_jY) = II(X, Y)$. Furthermore, if the restriction of the second fundamental form of M to the horizontal space is positive definite, $II(X, X) > 0$ for any non-zero horizontal vector X, then (M, θ, I_1, I_2) is a quaternionic contact manifold.

Proof. Let D be the Levi-Civita connection on $\mathbb{R}^{4n+4}$ and X, Y be two horizontal vectors. As the horizontal space is the intersection of the kernels of the one forms θ_j we have

$$\begin{aligned} d\theta_1(I_1X, Y) \;&=\; -\theta_1([\tilde{I}_1X, Y]) \;=\; -\langle[\tilde{I}_1X, Y], \tilde{I}_1N\rangle \\ &=\; -\langle D_{\tilde{I}_1X}Y - D_Y(\tilde{I}_1X), \tilde{I}_1N\rangle \;=\; -\langle D_{\tilde{I}_1X}Y, \tilde{I}_1N\rangle + \langle D_Y(\tilde{I}_1X), \tilde{I}_1N\rangle \\ &=\; \langle D_{\tilde{I}_1X}(\tilde{I}_1Y), N\rangle + \langle D_YX, N\rangle \;=\; II(\tilde{I}_1X, \tilde{I}_1Y) + II(X, Y). \end{aligned} \tag{6.12}$$

Therefore $d\theta_1(I_1X, Y) = d\theta_2(I_2X, Y)$ iff $II(I_jX, I_jY) = II(X, Y)$.

The last claim of the proposition is clear from the above formula. In particular, $g_H(X, Y) = II(X, Y)$ is a metric on the horizontal space when the second fundamental form is positive definite on the horizontal space and we have $d\theta_1(I_1X, Y) = 2g_H(X, Y)$. Hence, (M, θ, I, J) becomes a quaternionic contact structure. We denote the corresponding horizontal forms with ω_j, i.e., $\omega_j(X, Y) = g_H(I_jX, Y)$. $\square$

Let us note also that in the situation as above $g = g_H + \Sigma_{j=1}^3 \theta_j \otimes \theta_j$ is a Riemannian metric on M. In view of the above observations we define a QC-hypersurface of $\mathbb{H}^{n+1}$ as follows.

Definition 6.13. *We say that a smooth embedded hypersurface of $\mathbb{H}^{n+1}$ is a QC-*hypersurface *if the restriction of the second fundamental form of M to the horizontal space is a definite symmetric form, which is invariant with respect to the almost complex structures.*

Clearly every sphere in $\mathbb{H}^{n+1}$ is a QC-hypersurface and this is true also for the ellipsoids $\sum_a \frac{|q^a|^2}{b_a} = 1$. In fact, a hypersurface of $\mathbb{H}^{n+1}$ is a QC-hypersurface if and only if the (Euclidean) Hessian of the defining function ρ, considered as a quadratic form on the horizontal space, is a symmetric definite matrix from $GL(n, \mathbb{H})$, the latter being the linear group of invertible matrices which commute with the standard complex structures on $\mathbb{H}^n$. The same statement holds for hypersurfaces in quaternionic Kähler and hyperkähler manifolds.

Proposition 6.14. *Let $i : M \to \mathbb{H}^n$ be a QC hypersurface in $\mathbb{H}^n$, f a real-valued function on M. If $f = i^*F$ is the restriction to M of a Q-pluriharmonic function F defined on $\mathbb{H}^n$, i.e. F is the real part of an anti-regular function $F + iW + jU + kV$, then:*

$$df = d(i^*F) = d_1(i^*W) + d_2(i^*U) + d(i^*V) \qquad mod\ \eta, \tag{6.13}$$

$$DD_{I_1} f(X, I_1 Y) \;=\; -4dF(D\rho)\, g_H(X,Y) \;-\; 4(\xi_2 f)\,\omega_2(X,Y) \tag{6.14}$$

for any horizontal vector fields $X, Y \in H$.

Proof. Let us prove first (6.13). Denote with small letters the restrictions of the functions defined on $\mathbb{H}^n$. For $X \in H$ from (6.10) we have

$$\begin{aligned}(i^* \tilde{I}_1\, dW)(X) = (\tilde{I}_1 dW)(i_*\, X) = -dW(\tilde{I}_1\, i_*\, X)&\\ = -dW(i_*\,(I_1\ X)) = -dw(I_1\, X) = d_1 w\,(X).&\end{aligned}$$

Applying the same argument to the functions U and V we see the validity of (6.13).

Our goal is to write the equation for f on M, using the fact that $f = i^*F$. Let us consider the function λ, $\lambda = \frac{dF(D\rho)}{|D\rho|^2}$, and the one-form $d_M F$, $d_M F = dF - \lambda\, d\rho$. Thus the one-form df satisfies the equation $df = i^*(d_M F + \lambda\, d\rho) = i^*(d_M F)$, taking into account that $(\lambda \circ i)\, d(\rho \circ i) = 0$ as ρ is constant on M. From Proposition 6.8, the assumption on F is equivalent to $DD_{\tilde{I}_j} F = 0$. Therefore, we have

$$\begin{aligned} 0 \;=\; DD_{\tilde{I}_1} F \;&=\; d\tilde{I}_1 dF \;-\; \tilde{I}_2 d\tilde{I}_1 dF \\ &=\; d\,(\tilde{I}_1 d_M F \;+\; \lambda \tilde{I}_1 d\rho) \;-\; \tilde{I}_2 d\,(\tilde{I}_1 d_M F \;+\; \lambda \tilde{I}_1 d\rho) \\ &=\; d\,\tilde{I}_1 d_M F \;+\; d\lambda \wedge \tilde{I}_1 d\rho \;+\; \lambda d\tilde{I}_1 d\rho - \;\tilde{I}_2 d\,\tilde{I}_1 d_M F \\ &\qquad -\; \tilde{I}_2 (d\lambda \wedge \tilde{I}_1 d\rho) \;-\; \lambda \tilde{I}_2 \tilde{I}_1 d\rho. \end{aligned}$$

Restricting to M, and in fact, to the horizontal space H we find

$$\begin{aligned} 0 \;=\; i^*\,(DD_{\tilde{I}_1} F)|_H = \;& i^* d\,(\tilde{I}_1 d_M F)|_H \;+\; d(\lambda \circ i) \wedge i^*(\tilde{I}_1 d\rho)|_H \\ &+\; (\lambda \circ i) d i^*(\tilde{I}_1 d\rho)|_H \;-\; i^*(\tilde{I}_2 d\,\tilde{I}_1 d_M F)|_H \\ &-\; i^*(\tilde{I}_2 (d\lambda \wedge \tilde{I}_1 d\rho))|_H \;-\; (\lambda \circ i)\; i^*(\tilde{I}_2 d\,\tilde{I}_1 d\rho)|_H. \end{aligned} \tag{6.15}$$

Since the horizontal space is in the kernel of the one-forms $\theta_j = \tilde{I}_j\, d\rho|_H$ it follows that $i^*\,(\tilde{I}_j\, d\rho)|_H = 0$. Hence, two of the terms in (6.15) are equal to zero, and we have

$$(6.16)\quad 0 = i^*\,(DD_{\tilde{I}_1}F)_{|H} = i^*(d\,\tilde{I}_1 d_M F - \tilde{I}_2 d\,\tilde{I}_1 d_M F)\,_{|H} + (\lambda\circ i)\; i^*\big(d\,\tilde{I}_1 d\rho - \tilde{I}_2 d\,\tilde{I}_1 d\rho\big)_{|H}.$$

In other words for horizontal X and Y we have

$$(6.17)\quad i^*(d\,\tilde{I}_1 d_M F - \tilde{I}_2 d\,\tilde{I}_1 d_M F)(X, IY) = -(\lambda\circ i)\; i^*\big(d\,\tilde{I}_1 d\rho - \tilde{I}_2 d\,\tilde{I}_1 d\rho\big)(X, IY)$$

The right-hand side is proportional to the metric. Indeed, recalling

$$i^*\;(\tilde{I}_j\, d\,\rho)(X) = |d\rho|\;\theta_j(X) d\,\theta_j(X,Y) = 2g(I_j X, Y)$$

we obtain the identity

$$i^*\big(d\,\tilde{I}_1 d\rho - \tilde{I}_2 d\,\tilde{I}_1 d\rho\big)(X,Y) = 2|d\rho|\; g(I_1X,Y) - 2|d\rho|\; g(I_1I_2X, I_2Y) = 2|d\rho|\; g(I_1X,Y) - 2g(I_3X,I_2Y) = 4|d\rho|\; g(I_1X,Y).$$

Let us consider now the term in the left-hand side of (6.17). Decomposing $d_M F$ into horizontal and vertical parts we write $d_M F = d_H f + F_j\tilde{\theta}^j$. ¿From the definitions of the forms $\tilde{\theta}^j$ we have $\tilde{I}_1\tilde{\theta}^1 = \frac{d\rho}{|d\rho|}$, $\quad \tilde{I}_1\tilde{\theta}^2 = \tilde{\theta}^3$, $\quad \tilde{I}_1\tilde{\theta}^1\tilde{\theta}^3 = -\tilde{\theta}^2$. Therefore

$$d\tilde{I}_1 d_M F = d\tilde{I}_1 d_H F + dF_j \wedge \tilde{I}_1\theta^j + F_1 d(\frac{d\rho}{|d\rho|}) + F_2 d\tilde{\theta}^3 - F_3 d\tilde{\theta}^2$$
$$= d\tilde{I}_1 d_H F + dF_j \wedge \tilde{I}_1\theta^j - |d\rho|^{-2} d|d\rho| \wedge d\rho + F_2 d\tilde{\theta}^3 - F_3 d\tilde{\theta}^2$$

$$\tilde{I}_2 d\tilde{I}_1 d_M F = \tilde{I}_2 d\tilde{I}_1 d_H F + \tilde{I}_2 dF_j \wedge \tilde{I}_2\tilde{I}_1\theta^j - |d\rho|^{-2}\tilde{I}_2 d|d\rho| \wedge \tilde{I}_2 d\rho + F_2\tilde{I}_2 d\tilde{\theta}^3 - F_3\tilde{I}_2 d\tilde{\theta}^2.$$

From $I_2 d\theta^3 = -d\theta^3$, $I_2 d\theta^2 = d\theta^2$ and the above it follows

$$i^*(d\,\tilde{I}_1 d_M F - \tilde{I}_2 d\,\tilde{I}_1 d_M F)_{|H} = DD_{I_1} f + F_2 d\theta^3 - F_3 d\theta^2 + F_2 d\theta^3 + F_3 d\theta^2 = DD_{I_1} f + 4F_2\;\omega_3.$$

In conclusion, we proved $DD_{I_1} f(X,Y) = -4(\lambda\circ i)\,|\nabla\rho|\; g(I_1X,Y) - 4F_2\,\omega_3(X,Y)$ from where the claim of the Proposition. □

6.4. Anti-CRF functions on Quaternionic contact manifold

Let $(M, \eta, \mathbb{Q})$ be a (4n+3)-dimensional quaternionic contact manifold and ∇ denote the Biquard connection on M. The equation (6.13) suggests the following

Definition 6.15. *A smooth $\mathbb{H}$-valued function $F : M \longrightarrow \mathbb{H}$, $\quad F = f + iw + ju + kv$, is said to be an* anti-CRF function *if the smooth real valued functions f, w, u, v satisfy*

$$(6.18)\qquad df = d_1 w + d_2 u + d_3 v \quad mod \;\; \eta,$$

where $d_i = I_i \circ d$.

Choosing a local frame $\{T_a, X_a = I_1T_a, Y_a = I_2T_a, Z_a = I_3T_a, \xi_1, \xi_2, \xi_3\}$, $a = 1, \dots, n$ it is easy to check that a $\mathbb{H}$-valued function $F = f + iw + ju + kv$ is an anti-CRF function if it belongs to the kernel of the operators

$$D_{T_\alpha} \;=\; T_\alpha - iX_\alpha - jY_\alpha - kZ_\alpha, \qquad D_{T_\alpha}F \;=\; 0, \qquad \alpha = 1, \dots n. \tag{6.19}$$

REMARK 6.16. *We note that anti-CRF functions have different properties than the CRF functions* [**Per1, Per2**]*which are defined to be in the kernel of the operator*

$$\overline{D}_{T_\alpha} \;=\; T_\alpha + iX_\alpha + jY_\alpha + kZ_\alpha, \qquad \overline{D}_{T_\alpha}F \;=\; 0, \qquad \alpha = 1, \dots n.$$

Equation (6.18) and a small calculation give the following Proposition.

PROPOSITION 6.17. *A $\mathbb{H}$-valued function $F = f + iw + ju + kv$ is an anti-CRF function if and only if the smooth functions f, w, u, v satisfy the horizontal Cauchy-Riemann-Fueter equations*

$$\begin{aligned} T_\alpha f &= -X_\alpha w - Y_\alpha u - Z_\alpha v, \quad & X_\alpha f &= T_\alpha w + Z_\alpha u - Y_\alpha v,\\ Y_\alpha f &= -Z_\alpha w + T_\alpha u + X_\alpha v, \quad & Z_\alpha f &= Y_\alpha w - X_\alpha u + T_\alpha v. \end{aligned} \tag{6.20}$$

Having the quaternionic contact form η fixed, we may extend the definitions (6.6) of DD_{I_i} to the second order differential operator DD_{I_i} acting on the real-differentiable functions $f : M \to \mathbb{R}$ by

$$DD_{I_i}f := dd_if + d_jd_kf = dd_if - I_jdd_if = d(I_idf) - I_j(d(I_idf)). \tag{6.21}$$

The following proposition provides some formulas, which shall be used later.

PROPOSITION 6.18. *On a QC-manifold, for $X, Y \in H$, we have the following commutation relations*

$$\begin{aligned} DD_{I_i}f(X, I_iY) - DD_{I_k}f(X, I_kY) &= -I_iN_{I_j}(X, I_iY)(f) - N_{I_k}(I_jX, I_jY)(f),\\ d_id_jf(X, Y) + d_jd_if(X, Y) &= -N_{I_k}(I_jX, I_iY)(f),\\ dd_if(X, Y) + d_idf(X, Y) &= N_{I_i}(I_iX, Y)f, \end{aligned} \tag{6.22}$$

$$d_i^2f(X, Y) = -2(\xi_if)\omega_i(X, Y) + 2(\xi_jf)\omega_j(X, Y) + 2(\xi_kf)\omega_k(X, Y), \tag{6.23}$$

In particular, on a hyperhermitian contact manifold we have

$$\begin{aligned} DD_{I_i}f(X, I_iY) - DD_{I_k}(X, I_kY) &= 4\xi_i(f)\omega_i(X, Y) - 4\xi_j(f)\omega_j(X, Y),\\ d_id_jf(X, Y) + d_jd_if(X, Y) &= -4\left[(\xi_if)\omega_j(X, Y) + (\xi_jf)\omega_i(X, Y)\right],\\ dd_if(X, Y) + d_idf(X, Y) &= 4\left[(\xi_kf)\omega_j(X, Y) - (\xi_jf)\omega_k(X, Y)\right]. \end{aligned} \tag{6.24}$$

PROOF. By the definition (6.21) we obtain the second and the third formulas in (6.22) as well as $DD_i(X, Y) + (dd_k - d_jd_i)f(X, I_jY) = -I_iN_{I_j}(X, Y)f$. The first equality in (6.22) is a consequence of the latter and the second equality in (6.22). We have

$$d_i^2f(X, Y) = -I_id(I_i^2df)(X, Y) = d(df - \sum_{s=1}^{3}(\xi_sf)\eta_s)(I_iX, I_iY)$$

which is exactly (6.23). If H is formally integrable then the formula (4.21) reduces to $N_i(X, Y) = T_i^{0,2}(X, Y)$. The equation (6.24) is an easy consequences of the latter equality, (6.22) and (3.3) □

Let us make the conformal change $\bar\eta = \frac{1}{2h}\eta$. The endomorphisms $\bar I_i$ will coincide with I_i on the horizontal distribution H but they will have a different kernel - the new vertical space $span\{\bar\xi_1, \bar\xi_2, \bar\xi_3\}$, where $\bar\xi_s = 2h\xi_s + I_s(\nabla h)$ (see (5.1)). Hence, for any $P \in \Gamma(TM)$ we have

$$\bar I_i(P) = \bar I_i(P - \sum_{s=1}^{3} \tilde\eta_s(P)\bar\xi_s) = I_i(P - \frac{1}{2h}\sum_{s=1}^{3}\eta_s(P)(2h\xi_s + I_s(\nabla h))) \\ = I_i(P) + \frac{1}{2h}\{\eta_i(P)\nabla h - \eta_j(P)I_k\nabla h + \eta_k(P)I_j\nabla h\}. \tag{6.25}$$

Proposition 6.19. *Suppose $\bar\eta = \frac{1}{2h}\eta$ are two conformal to each other structures.*

a) The second order differential operator DD_{I_i} (restricted on functions) transforms as follows:

$$DD_{\bar I_i} f - DD_{I_i} f = -2h^{-1}df(\nabla h)\omega_i - 2h^{-1}df(I_j\nabla h)\omega_k \qquad mod\ \eta.$$

b) If f is the real part of the anti-CRF function $f+iw+ju+kv$ then the two forms

$$\Omega_i \ = \ DD_{I_i}\, f \ - \ \lambda\,\omega_i \ + \ 4(\xi_j f)\,\omega_k \qquad mod\ \eta$$

are conformally invariant, where $\lambda \ = \ 4\big(\xi_1 w + \xi_2 u + \xi_3 v\big)$.

Proof. a) For any $X, Y \in H$, we compute

$$d(\bar I_i df)(X,Y) = X(I_i df(Y)) - Y(I_i df(X)) - \bar I_i df[X,Y] \\ = d(I_i df)(X,Y) + df(\bar I_i[X,Y] - I_i[X,Y])$$

Here, we apply (6.25) to get

$$d(\bar I_i df)(X,Y) = d(I_i df)(X,Y) \\ + \frac{1}{h}\{-df(\nabla h)\omega_i(X,Y) + df(I_k\nabla h)\omega_j(X,Y) - df(I_j\nabla h)\omega_k(X,Y)\}.$$

Now, we apply the defining equation (6.21) which acomplishes the proof of part a).

b) Assuming that f is the real part of an anti-CRF function, from part a) we have

$$\begin{aligned}\bar\Omega_i \ - \ \Omega_i &= DD_{\bar I_i}\, f - DD_{I_i}\, f - \bar\lambda\,\bar\omega_i + \lambda\,\omega_i + 4(\bar\xi_j f)\,\bar\omega_k \ - \ 4(\xi_j f)\,\omega_k \quad \mathrm{mod}\ \eta \\ &= -4(\xi_1 w + \xi_2 u + \xi_3 v)\omega_i - 4\big((I_1 dh)w + (I_2 dh)u + (I_3 dh)v\big)\frac{\omega_i}{2h} \\ &- \frac{2}{h}g(df, dh)\omega_i - \frac{2}{h}g(df, d_j h)\omega_k + 4(\xi_1 w + \xi_2 u + \xi_3 v)\omega_i + 4g(df, d_j h)\frac{\omega_k}{2h} \\ &= \ 0 \quad \mathrm{mod}\ \eta,\end{aligned}$$

taking into account (6.18). □

We restrict our considerations to hyperhermitian contact manifolds.

Theorem 6.20. *If $f : M \to \mathbb{R}$ is the real part of an anti-CRF function $f + iw + ju + kv$ on a (4n+3)-dimensional ($n > 1$) hyperhermitian contact manifold (M, η, Q), then the following equivalent conditions hold true.*

i) The next equalities hold

$$DD_{I_i} f = \lambda\omega_i - 4(\xi_j f)\omega_k \qquad mod\ \eta. \tag{6.26}$$

ii) For any $X, Y \in H$ we have the equality

$$(\nabla_X df)(Y) + (\nabla_{I_1X} df)(I_1Y) + (\nabla_{I_2X} df)(I_2Y) + (\nabla_{I_3X} df)(I_3Y) \tag{6.27}$$
$$= \lambda g(X,Y) + df(X)\alpha_3(I_3Y) + df(I_1X)\alpha_3(I_2Y)$$
$$- df(I_2X)\alpha_3(I_1Y) - df(I_3X)\alpha_3(Y) + df(Y)\alpha_3(I_3X)$$
$$+ df(I_1Y)\alpha_3(I_2X) - df(I_2Y)\alpha_3(I_1X) - df(I_3Y)\alpha_3(X).$$

iii) The function f satisfies the second order system of partial differential equations

$$\begin{aligned}\Re(D_{T_\beta}\overline{D}_{T_\alpha} f) &= \lambda g(T_\beta, T_\alpha)\\ &+ df(\nabla_{T_\beta}T_\alpha) + df(\nabla_{I_1T_\beta}I_1T_\alpha) + df(\nabla_{I_2T_\beta}I_2T_\alpha) + df(\nabla_{I_3T_\beta}I_3T_\alpha)\\ &+ df(T_\beta)\alpha_3(I_3T_\alpha) + df(I_1T_\beta)\alpha_3(I_2T_\alpha) - df(I_2T_\beta)\alpha_3(I_1T_\alpha) - df(I_3T_\beta)\alpha_3(T_\alpha)\\ &+ df(T_\alpha)\alpha_3(I_3T_\beta) + df(I_1T_\alpha)\alpha_3(I_2T_\beta) - df(I_2T_\alpha)\alpha_3 I_1(T_\beta) - df(I_3T_\alpha)\alpha_3(T_\beta)\end{aligned} \tag{6.28}$$

$$\begin{aligned}\Re(iD_{T_\beta}\overline{D}_{T_\alpha} f) = \Re(D_{I_1T_\beta}\overline{D}_{T_\alpha} f), \quad \Re(jD_{T_\beta}\overline{D}_{T_\alpha} f) = \Re(D_{I_2T_\beta}\overline{D}_{T_\alpha} f),\\ \Re(jD_{T_\beta}\overline{D}_{T_\alpha} f) = \Re(D_{I_3T_\beta}\overline{D}_{T_\alpha} f).\end{aligned} \tag{6.29}$$

The function λ is determined by

$$\lambda \;=\; 4\left[(\xi_1 w) + (\xi_2 u) + (\xi_3 v)\right]. \tag{6.30}$$

Proof. The proof includes a number of steps and occupies the rest of the section. Suppose that there exists a smooth functions w, u, v such that $F = f + iw + ju + kv$ is an anti-CRF function.

i) The defining equation (6.18) yields

$$df = d_1w + d_2u + d_3v + \sum_{s=1}^{3}\xi_s(f)\eta_s, \tag{6.31}$$

Since $d_s\eta_t(X,Y) = 0$, for $s,t \in \{1,2,3\}, X,Y \in H$, applying (6.23) and (2.1), we obtain from (6.31)

$$(dd_1f - dd_3u + dd_2v - 2\xi_1(w)\omega_1 - 2\xi_2(w)\omega_2 - 2\xi_3(w)\omega_3)(X,Y) = 0,$$
$$(d_1df - d_1d_2u - d_1d_3v + 2\xi_1(w)\omega_1 - 2\xi_2(w)\omega_2 - 2\xi_3(w)\omega_3)(X,Y) = 0,$$
$$(d_2d_3f + d_2d_1u + d_2dv - 2\xi_1(w)\omega_1 + 2\xi_2(w)\omega_2 - 2\xi_3(w)\omega_3)(X,Y) = 0,$$
$$(d_3d_2f + d_3du - d_3d_1v + 2\xi_1(w)\omega_1 + 2\xi_2(w)\omega_2 - 2\xi_3(w)\omega_3)(X,Y) = 0.$$

Summing the first and the third equations, subtracting the second and the fourth and using the commutation relations (6.24) we obtain (6.26) with the condition (6.30) which proves *i)*.

iii) Equations (6.19) and (6.20) yield

$$\begin{aligned}2\Re(D_{T_\beta}\overline{D}_{T_\alpha} f) \;&=\; 2(T_\beta T_\alpha f + X_\beta X_\alpha f + Y_\beta Y_\alpha f + Z_\beta Z_\alpha f)\\ &= (\Re(D_{T_\beta}\overline{D}_{T_\alpha} f) + \Re(D_{T_\alpha}\overline{D}_{T_\beta} f)) + (\Re(D_{T_\beta}\overline{D}_{T_\alpha} f) - \Re(D_{T_\alpha}\overline{D}_{T_\beta} f)) =\\ &([T_\beta, T_\alpha] + [X_\beta, X_\alpha] + [Y_\beta, Y_\alpha] + [Z_\beta, Z_\alpha])f\\ &- ([T_\beta, X_\alpha] - [X_\beta, T_\alpha] + [Y_\beta, Z_\alpha] - [Z_\beta, Y_\alpha])w\\ &- ([T_\beta, Y_\alpha] - [X_\beta, Z_\alpha] - [Y_\beta, T_\alpha] + [Z_\beta, X_\alpha])u\\ &- ([T_\beta, Z_\alpha] + [X_\beta, Y_\alpha] - [Y_\beta, X_\alpha] - [Z_\beta, T_\alpha])v.\end{aligned}$$

Expanding the commutators and applying (6.18), (3.3), (3.28) and (4.24) gives (6.28). Similarly, one can check the validity of (6.29)

i) $\Leftrightarrow$ *ii)* $\Leftrightarrow$ *iii)* The next lemma establishes the equivalence between i), ii) and iii).

LEMMA 6.21. *For any $X, Y \in H$ on a quaternionic contact manifold we have the identity*

$$\begin{aligned} DD_{I_1}f(X, I_1Y) &= (\nabla_X df)Y + (\nabla_{I_1X}df)I_1Y + (\nabla_{I_2X}df)I_2Y \\ &+ (\nabla_{I_3X}df)I_3X - 4\xi_2(f)\omega_2(X,Y) - df(X)\alpha_3(I_3Y) + df(I_1X)\alpha_2(I_3Y) \\ &+ df(I_2X)\alpha_3(I_1Y) - df(I_3X)\alpha_2(I_1Y) - df(Y)\alpha_2(I_2X) \\ &- df(I_1Y)\alpha_3(I_2X) + df(I_2Y)\alpha_2(X) + df(I_3Y)\alpha_3(X). \end{aligned}$$

PROOF OF LEMMA 6.21. Using the definition and also (3.28), (3.3) and (5.4) we derive the next sequence of equalities

$$\begin{aligned} (dd_{I_1}f)(X,Y) &= -(\nabla_X df)(I_1Y) + (\nabla_Y df)(I_1X) - df(\nabla_X(I_1Y) \\ &- \nabla_Y(I_1X) - I_1[X,Y]) = -(\nabla_X df)(I_1Y) + (\nabla_Y df)(I_1X) \\ &+ \alpha_2(X)df(I_3Y) - \alpha_3(X)df(I_2Y) - \alpha_2(Y)df(I_3X) + \alpha_3(Y)df(I_2X) \\ &= -(\nabla_X df)I_1Y + (\nabla_{I_1X}df)Y - df(T(Y, I_1X)) \\ &+ \alpha_2(X)df(I_3Y) - \alpha_3(X)df(I_2Y) - \alpha_2(Y)df(I_3X) + \alpha_3(Y)df(I_2X). \end{aligned}$$

$$\begin{aligned} DD_{I_1}f(X, I_1Y) &= (dd_{I_1} - I_2 dd_{I_1})f(X, I_1Y) \\ &= (\nabla_X df)Y + (\nabla_{I_1X}df)I_1Y + (\nabla_{I_2X}df)I_2Y + (\nabla_{I_3X}df)I_3X \\ &- df(T(I_1Y, I_1X)) - df(T(I_3Y, I_3X)) - df(X)\alpha_3(I_3Y) + df(I_1X)\alpha_2(I_3Y) \\ &+ df(I_2X)\alpha_3(I_1Y) - df(I_3X)\alpha_2(I_1Y) - df(Y)\alpha_2(I_2X) \\ &- df(I_1Y)\alpha_3(I_2X) + df(I_2Y)\alpha_2(X) + df(I_3Y)\alpha_3(X). \end{aligned} \tag{6.32}$$

A short calculation using (3.3) gives

$$df(T(I_1Y, I_1X)) + df(T(I_3Y, I_3X)) = 4(\xi_2 f)\,\omega_2(X,Y).$$

Inserting the last equality in (6.32) completes the proof of Lemma 6.21. $\square$

Taking into account that the structure is hyperhermitian contact, with the help of (4.24) of Lemma 6.21 the proof of Theorem 6.20 follows. $\square$

We conjecture that the converse of the claim of Theorem 6.20 is true. At this point we can prove Lemma 6.23, which supports the conjecture. First we prove a useful technical result.

LEMMA 6.22. *Suppose M is a quaternionic contact manifold of dimension $(4n+3) > 7$. If ψ is a smooth closed two-form whose restriction to H vanishes, then ψ vanishes identically.*

PROOF OF LEMMA 6.22. The hypothesis on ψ show that ψ is of the form $\psi = \sum_{s=1}^{3} \sigma_s \wedge \eta_s$, where σ_s are 1-forms. Taking the exterior differential and using (3.10), we obtain for $X \in H$

$$0=\sum_{a=1}^{4n}d\psi(e_a,I_ie_a,X)=-2\sum_{a=1}^{4n}\sum_{s=1}^{3}\sigma_s\wedge\omega_s(e_a,I_ie_a,X)$$
$$=(4n-2)\sigma_i(X)+2\sigma_j(I_kX)-2\sigma_k(I_jX),$$

where $e_1,\dots,e_{4n}$ is an orthonormal basis on H. For $n>1$ the latter implies $\sigma_{s_{|_H}}=0, s=1,2,3$. Hence, we have $\psi=\sum_{1\leq s<t\leq}A_{st}\,\eta_s\wedge\eta_t$, where A_{st} are smooth functions on M. Now, the exterior derivative gives $0=d\psi(e_a,I_se_a,\xi_t)\;=\;2A_{st}$. $\square$

The assumption in the next Lemma is a kind of $\partial\bar{\partial}_H$-lemma result, which we do not know how to prove at the moment, but we believe that it is true. We show how it implies the converse of Theorem 6.20.

Lemma 6.23. *Suppose, for $i=1,2,3$,*

$$DD_{I_i}f\equiv dd_if+d_jd_kf=\sum_{s=1}^{3}p^i_s\omega_s\quad \mathit{mod}\ \eta$$

implies

$$dd_if-dd_jA_i\;=\;2\sum_{s=1}^{3}r^i_s\,\omega_s\qquad \mathit{mod}\ \eta \tag{6.33}$$

for some function A_i on a QC manifold of dimension $(4n+3)>7$. With this assumption, if $DD_{I_i}f=\sum_{s=1}^3p^i_s\omega_s\quad mod\ \eta$, $i=1,2,3$, then f is a real part of an anti-CRF-function.

Proof of Lemma 6.23. Consider the closed 2-forms

$$\Omega_i=d(d_if-d_jA_i-\textstyle\sum_{s=1}^3r^i_s\eta_s).$$

We have $d\Omega_i=0$ and $\Omega_{i_{|_H}}=0$ due to (6.33) and (3.1). Applying Lemma 6.22 we conclude $\Omega_i=0$, after which the Poincare lemma yields

$$d_if-d_jA_i-dB_i=0\quad mod\quad \eta$$

for some smooth functions A_1,A_2,A_3,B_1,B_2,B_3. The latter implies

$$df+d_1(A_2+B_1)+d_2(B_2-A_3)+d_3(A_1-B_3)=0\quad mod\quad \eta.$$

Setting $w=-A_2-B_1,\quad u=A_3-B_2,\quad v=B_3-A_1$ proves the claim. $\square$

Corollary 6.24. *Let $f:M\to\mathbb{R}$ be a smooth real function on a (4n+3)-dimensional $(n>1)$ 3-Sasakian manifold (M,η). If f is the real part of an anti-CRF function $f+iw+ju+kv$ then we have:*

i) equation (6.26) holds true;

ii) for any $X,Y\in H$ we have the equality

$$(\nabla_Xdf)(Y)+(\nabla_{I_1X}df)(I_1Y)+(\nabla_{I_2X}df)(I_2Y)+(\nabla_{I_3X}df)(I_3Y)=\lambda g(X,Y).$$

The function λ is determined in (6.30).

Corollary 6.25. *Let $f:G(\mathbb{H})\to\mathbb{R}$ be a smooth real function on the (4n+3)-dimensional $(n>1)$ quaternionic Heisenberg group endowed with the standard flat quaternionic contact structure and $\{T_a,X_a,Y_a,Z_a,\quad a=1,\dots,4n\}$ be ∇-parallel basis on $G(\mathbb{H})$. If f is the real part of an anti-CRF function $f+iw+ju+kv$ then the following equivalent conditions hold true.*

i) The equation (6.26) holds.

ii) The horizontal Hessian of f is given by

$$T_bT_af + X_bX_af + Y_bY_af + Z_aZ_b = \lambda g(T_b, T_a).$$

iii) The function f satisfies the following second order differential equation

$$D_{T_b}\overline{D}_{T_a}f = \lambda(g - i\omega_1 - j\omega_2 - k\omega_3)(T_b, T_a);$$

The function λ is given by (6.30).

Proposition 6.2, Corollary 6.24 and Example 4.12 imply the next Corollary.

COROLLARY 6.26. *Let (M, η) be a (4n+3)-dimensional $(n > 1)$ 3-Sasakian manifold, $f : M \to \mathbb{R}$ a positive smooth real function. Then the conformally 3-Sasakian QC structure $\bar{\eta} = f\eta$ is qc-pseudo Einstein if and only if the operators $DD_{I_s}f, s = 1, 2, 3$ satisfy (6.26). In particular, if f is real part of anti CRF function then the conformally 3-Sasakian qc structure $\bar{\eta} = f\eta$ is qc-pseudo Einstein.*

CHAPTER 7

Infinitesimal Automorphisms

7.1. 3-contact manifolds

We start with the more general notion of 3-contact manifold (M, H), where H is an orientable codimension three distribution on M. Let $E \subset TM^*$ be the canonical bundle determined by H, i.e., the bundle of 1-forms with kernel H. Hence, M is orientable if and only if E is also orientable, i.e. E has a global non-vanishing section vol_E locally given by $vol_E = \eta_1 \wedge \eta_2 \wedge \eta_3$. Denote by $\eta = (\eta_1, \eta_2, \eta_3)$ the local 1 -form with values in $\mathbb{R}^3$. Clearly $H = Ker\ \eta = \cap_{i=1}^3 \eta_i$.

Definition 7.1. *A* $(4n+3)$*-dimensional orientable smooth manifold* $(M, \eta, H = Ker\ \eta)$ *is said to be a* 3-contact manifold *if* H *is a codimnesion three distribution and the restriction of each of the 2-forms* $d\eta_i$ *to* H *is non-degenerate, i.e.,*

$$d\eta_i^{2n} \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 = u_i\ vol_M \tag{7.1}$$

for some smooth functions $u_i > 0$, $i = 1, 2, 3$.

We shall denote by Ω_i the restriction of $d\eta_i$ to H, $\Omega_i = d\eta_{i_{|_H}}, i = 1, 2, 3$. The condition (7.1) is equivalent to

$$\Omega_i^{2n} \neq 0, \quad i = 1, 2, 3$$

and the forms Ω_i^{2n} define the same orientation of H.

We remark that the notion of 3-contact structure is more general than the notion of QC structure. For example, any real hypersurface M in $\mathbb{H}^{n+1}$ with non-degenerate second fundamental form carries 3-contact structure defined in Section 6.3 (cf. Proposition 6.12 and Definition 6.13 where this structure is QC if and only if (6.11) holds, or equivalently, the second fundamental form is, in addition, invariant with respect to the hypercomplex structure on $\mathbb{H}^{n+1}$). Another examples of 3-contact structure is the so called quaternionic CR structure introduced in [**KN**] and the so called weak QC structures considered in [**D1**]. Note that in these examples the 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ are globally defined.

On any 3-contact manifold (M, η, H) there exists a unique triple (ξ_1, ξ_2, ξ_3) of vector fields transversal to H determined by the conditions

$$\eta_i(\xi_j) = \delta_{ij}, \quad (\xi_i \lrcorner d\eta_i)_{|_H} = 0.$$

We refer to such a triple as fundamental vector fields or Reeb vector fields and denote $V = span\{\xi_1, \xi_2, \xi_3\}$. Hence, we have the splitting $TM = H \oplus V$.

The 3-contact structure (η, H) and the vertical space V are determined up to an action of $GL(3, \mathbb{R})$, namely for any $GL(3, \mathbb{R})$ matrix Φ with smooth entries the structure $\Phi \cdot \eta$ is again a 3-contact structure. Indeed, it is an easy algebraic fact that the condition (7.1) also holds for $\Phi \cdot \eta$. The Reeb vector field are transformed with the matrix with entries the adjunction quantities of Φ, i.e. with the inverse matrix Φ^{-1}. This leads to the next

Definition 7.2. *A diffeomorphism ϕ of a 3-contact manifold (M, η, H) is called a* 3-contact automorphism*if ϕ preserves the 3-contact structure η, i.e.,*

$$\phi^* \eta = \Phi \cdot \eta, \tag{7.2}$$

for some matrix $\Phi \in GL(3, \mathbb{R})$ with smooth functions as entries and $\eta = (\eta_1, \eta_2, \eta_3)^t$ is considered as an $\mathbb{R}^3$-valued one-form.

The infinitesimal versions of these notions lead to the following definition.

Definition 7.3. *A vector field Q on a 3-contact manifold (M, η, H) is an infinitesimal generator of a 3-contact automorphism (3-contact vector field) if its flow preserves the 3-contact structure, i.e.*

$$\mathcal{L}_Q \, \eta = \phi \cdot \eta, \qquad \phi \in gl(3, \mathbb{R}).$$

We show that any 3-contact vector field on a 3-contact manifold depend on 3-functions which satisfy certain differential relations. We begin with describing infinitesimal automorphisms of the 3-contact structure η i.e., vector fields Q whose flow satisfies (7.2). Our main observation is that 3-contact vector fields on a 3-contact manifold are completely determined by their vertical components in the sense of the following

Proposition 7.4. *Let (M, η, H) be a 3-contact manifold. For a smooth vector field Q on M consider the functions $f_i = \eta_i(Q), i = 1, 2, 3$. A smooth vector field Q is a 3-contact vector field if and only if its horizontal part*

$$Q_H \stackrel{def}{=} Q - f_1\xi_1 - f_2\xi_2 - f_3\xi_3,$$

is the horizontal 3-contact hamiltonian field of (f_1, f_2, f_3) defined on H by

$$Q_H \lrcorner \Omega_i = [-df_i - f_j(\xi_j \lrcorner d\eta_i) - f_k(\xi_k \lrcorner d\eta_i)]_{|_H}, \quad i = 1, 2, 3. \tag{7.3}$$

Proof. A direct calculation gives (cf. (3.10))

$$d\eta_i = \Omega_i + \sum_s \eta_s \wedge (\xi_s \lrcorner d\eta_i)$$
$$-d\eta_i(\xi_j, \xi_k)\eta_j \wedge \eta_k - d\eta_i(\xi_k, \xi_i)\eta_k \wedge \eta_i - d\eta_i(\xi_i, \xi_j)\eta_i \wedge \eta_j$$

Furthermore, we compute

$$\begin{aligned} \mathcal{L}_Q \, \eta_i &= Q \lrcorner d\eta_i + d(Q \lrcorner \eta_i) \\ &= Q_H \lrcorner \Omega_i + [d(\eta_i(Q)) + \eta_i(Q)\xi_i \lrcorner d\eta_i + \eta_j(Q)\xi_j \lrcorner d\eta_i + \eta_k(Q)\xi_k \lrcorner d\eta_i]_{|_H} \\ &\quad + [\xi_i(\eta_i(Q)) + d\eta_i(Q, \xi_i)]\eta_i \\ &\quad + [\xi_j(\eta_i(Q)) + d\eta_i(Q, \xi_j)]\eta_j + [\xi_k(\eta_i(Q)) + d\eta_i(Q, \xi_k)]\eta_k. \\ \\ &= Q_H \lrcorner \Omega_i + [df_i + f_j(\xi_j \lrcorner d\eta_i) + f_k(\xi_k \lrcorner d\eta_i)]_{|_H} \\ &\quad + [\xi_i(f_i) - f_j d\eta_i(\xi_i, \xi_j) - f_k d\eta_i(\xi_i, \xi_k)]\eta_i + [\xi_j(f_i) + d\eta_i(Q, \xi_j)]\eta_j \\ &\quad + [\xi_k(f_i) + d\eta_i(Q, \xi_k)]\eta_k. \end{aligned} \tag{7.4}$$

Suppose Q is a 3-contact vector field. Then (7.4) implies that f_i and Q_H necessarily satisfy (7.3). The converse follows from (7.4) and the conditions of the proposition. $\square$

The last Proposition implies that the space of 3-contact vector fields is isomorphic to the space of triples consisting of smooth function f_1, f_2, f_3 satisfying the compatibility conditions (7.3).

COROLLARY 7.5. *Let (M, η) be a 3-contact manifold. Then*

a) If Q is a horizontal 3-contact vector field on M then Q vanishes identically.

b) The vector fields $\xi_i, i = 1, 2, 3$ are 3-contact vector fields if and only if

$$\xi_i \lrcorner d\eta_j|_H = 0, \quad i, j = 1, 2, 3.$$

PROOF. a) With the notation of Proposition 7.4, we have $f_i = \eta_i(Q) \equiv 0$, $Q = Q_H$. Hence $Q_H \lrcorner \Omega_i = 0$ and since Ω_i is a non-degenerate it follows $Q_H = 0$.

b) The necessary and sufficient conditions are given by Proposition 7.4. For $Q = \xi_s$, $s = 1, 2, 3$ and $Q_H = 0$ equation (7.3) becomes

$$0 = \eta_j(\xi_s) d\eta_i(\xi_j, X) + \eta_k(\xi_s) d\eta_i(\xi_k, X) = d\eta_i(\xi_s, X), \quad i = 1, 2, 3, \ X \in H,$$

which proves the claim. $\square$

7.2. QC vector fields

Suppose $(M, g, \mathbb{Q})$ is a quaternionic contact manifold.

DEFINITION 7.6. *A diffeomorphism ϕ of a QC manifold $(M, [g], \mathbb{Q})$ is called a* conformal quaternionic contact automorphism (conformal qc-automorphism) *if ϕ preserves the QC structure, i.e.*

$$\phi^* \eta = \mu \Psi \cdot \eta,$$

for some positive smooth function μ and some matrix $\Psi \in SO(3)$ with smooth functions as entries and $\eta = (\eta_1, \eta_2, \eta_3)^t$ is a local 1-form considered as an element of $\mathbb{R}^3$.

In view of the uniqueness of the possible associated almost complex structures, see Lemma 2.2, a quaternionic contact automorphism will preserve also the associated (if any) almost complex structures, $\phi^* \mathbb{Q} = \mathbb{Q}$ and consequently, it will preserve the conformal class [g] on H. We note that conformal QC diffeomorphisms on S^{4n+3} are considered in [**Kam**]. The infinitesimal versions of these notions lead to the following definition.

DEFINITION 7.7. *A vector field Q on a QC manifold $(M, [g], \mathbb{Q})$ is an infinitesimal generator of a conformal quaternionic contact automorphism (*QC vector field *for short) if its flow preserves the QC structure, i.e.,*

$$\mathcal{L}_Q \eta = (\nu I + O) \cdot \eta, \tag{7.5}$$

where ν is a smooth function and $O \in so(3)$ with smooth entries.

In view of the discussion above a QC vector field on a QC manifold $(M, \eta, \mathbb{Q})$ satisfies the conditions.

$$\mathcal{L}_Q g = \nu g, \tag{7.6}$$

$$\mathcal{L}_Q I = O \cdot I, \qquad O \in so(3), \quad I = (I_1, I_2, I_3)^t, \tag{7.7}$$

If the flow of a vector field Q is a conformal diffeomorphism of the horizontal metric g, i.e. (7.6) holds, we shall call it *infinitesimal conformal isometry.* If the function $\nu = 0$ then Q is said to be *infinitesimal isometry.*

A QC vector field on a QC manifold is a 3-contact vector field of special type. Indeed, let $\sharp$ be the musical isomorphism between T^*M and TM with respect to the fixed Riemannian metric g on TM and recall that the forms α_j were defined in (3.29). (3.30). We have

PROPOSITION 7.8. *Let $(M, g, \mathbb{Q})$ be a quaternionic contact manifold. The vector field Q is a QC vector field if and only if*

$$Q \;=\; -\frac{1}{2}\big(f_j\, I_i\, \alpha_k^\sharp \;-\; f_k\, I_i\, \alpha_j^\sharp \;-\; I_i\,(df_i)^\sharp\big) \;+\; \sum_{s=1}^{3} f_s\, \xi_s, \tag{7.8}$$

for some functions f_1, f_2 and f_3 such that for any positive permutation (i, j, k) of $(1, 2, 3)$ we have

$$\begin{aligned} f_j\, d\eta_i(\xi_j, \xi_i) + \; f_k\, d\eta_i(\xi_k, \xi_i) \;+\; \xi_i\, f_i \\ = \; f_k\, d\eta_j(\xi_k, \xi_j) + \; f_i\, d\eta_j(\xi_i, \xi_j) \;+\; \xi_j\, f_j, \end{aligned} \tag{7.9}$$

$$\begin{aligned} f_i\, d\eta_i(\xi_i, \xi_j) + \; f_k\, d\eta_i(\xi_k, \xi_j) \;+\; \xi_j\, f_i \\ = \; -\, f_j\, d\eta_j(\xi_j, \xi_i) \;-\; f_k\, d\eta_j(\xi_k, \xi_i) \;-\; \xi_i\, f_j, \end{aligned} \tag{7.10}$$

$$\begin{aligned} f_j\, I_i(\alpha_k)^\sharp \;-\; f_k\, I_i(\alpha_j)^\sharp \;-\; I_i(df_i)^\sharp \\ = \; f_i\, I_k(\alpha_j)^\sharp \;-\; f_j\, I_k(\alpha_i)^\sharp \;-\; I_k(df_k)^\sharp \end{aligned} \tag{7.11}$$

Conversely, any three smooth functions satisfying the compatibility conditions (7.9), (7.10) and (7.11) determine a QC vector field by (7.8).

PROOF. Notice that (7.8) implies $f_i = \eta_i(Q)$. By Cartan's formula (7.5) is equivalent to

$$Q \lrcorner d\eta_i \;+\; df_i \;=\; \nu\, \eta_i \;+\; o_{is}\, \eta_s.$$

In other words, both sides must be the same when evaluated on ξ_t, $t = 1, 2, 3$ and also when restricted to the horizontal bundle. Let $Q = Q_H + \sum_{s=1}^{3} f_s \xi_s$. Consider first the action on the vertical vector fields. Pairing with ξ_t and taking successively $t = i, j, k$ gives

$$\begin{aligned} f_j\, d\eta_i(\xi_j, \xi_i) + \; f_k\, d\eta_i(\xi_k, \xi_i) \;+\; \xi_i\, f_i \;&=\; \nu \;+\; o_{ii} \\ \alpha_k(Q_H) \;+\; f_i\, d\eta_i(\xi_i, \xi_j) + \; f_k\, d\eta_i(\xi_k, \xi_j) \;+\; \xi_j\, f_i \;&=\; o_{ij} \\ -\alpha_j(Q_H) \;+\; f_i\, d\eta_i(\xi_i, \xi_k) + \; f_j\, d\eta_i(\xi_j, \xi_k) \;+\; \xi_k\, f_i \;&=\; o_{ik}. \end{aligned} \tag{7.12}$$

Equating the restrictions to the horizontal bundle, i.e., $d\eta_i(Q, .)|_H \;+\; df_i|_H \;=\; 0$, gives

$$\Big(f_j\, d\eta_i(\xi_j, .) \;+\; f_k\, d\eta_i(\xi_k, .) \;+\; d\eta_i(Q_H, .) \;+\; df_i\Big)|_H \;=\; 0.$$

Since $g(A, .)|_H \;=\; 0 \Leftrightarrow A \;=\; \sum_{s=1}^{3} \eta_s(A) \xi_s$, the last equation is equivalent to

$$-f_j \alpha_k^\sharp + f_k \alpha_j^\sharp + 2 I_i Q_H + (df_i)^\sharp \;=\; \sum_{s=1}^{3} \big(-f_j \alpha_k(\xi_s) + f_k \alpha_j(\xi_s) + \xi_s f_i\big)\, \xi_s. \tag{7.13}$$

Acting with I_i determines $2Q_H \;=\; f_j\, I_i(\alpha_k)^\sharp - f_k\, I_i(\alpha_j)^\sharp - I_i(df_i)^\sharp$, which implies (7.8). In addition we have

$$\alpha_j(Q_H) \;=\; -\frac{1}{2}\Big(f_j\, \alpha_j(I_i(\alpha_k)^\sharp) \;-\; f_k\, \alpha_j(I_i(\alpha_j)^\sharp) \;-\; \alpha_j(I_i(df_i)^\sharp)\Big)$$

On the other hand, $o \in so(3)$ is equivalent to o being a skew symmetric which is equivalent to (7.9) and (7.10), by the above computations. Therefore, if we are given three functions $f_1,\ f_2,\ f_3$ satisfying (7.9), (7.10) and (7.11), then we define Q by (7.8). Using (7.12) we define ν and o with $o \in so(3)$ with smooth entries. With these definitions Q is a QC vector field. $\square$

Using the formulas in Example 4.13 we obtain from Proposition 7.8 the following '3-hamiltonian' form of a QC vector field on 3-Sasakian manifold.

Corollary 7.9. *Let (M, η) be a 3-Sasakian manifold. Any QC vector field Q has the form*

$$Q = Q_H + f_1\xi_1 + f_2\xi_2 + f_3\xi_3,$$

where the smooth functions f_1, f_2, f_3 satisfy the conditions

$$d_i f_i = d_j f_j, \quad \xi_i(f_i) = \xi_j(f_j), \quad \xi_i(f_j) = -\xi_j(f_i), \quad i, j = 1, 2, 3,$$

and the horizontal part $Q_H \in H$ is determined by

$$Q_H \lrcorner d\eta_i = d_i f_i, \quad i \in \{1, 2, 3\}.$$

The matrix in (7.5) has the form

$$\nu I_{d_3} + O = \begin{pmatrix} \xi_1(\eta_1(Q)) & -\xi_1(\eta_2(Q)) - 2\eta_3(Q) & -\xi_1(\eta_3(Q)) + 2\eta_2(Q) \\ \xi_1(\eta_2(Q)) + 2\eta_3(Q) & \xi_1(\eta_1(Q)) & -\xi_2(\eta_3(Q)) - 2\eta_1(Q) \\ \xi_1(\eta_3(Q)) - 2\eta_2(Q) & \xi_2(\eta_3(Q)) + 2\eta_1(Q) & \xi_1(\eta_1(Q)) \end{pmatrix}.$$

In particular, the Reeb vector fields ξ_1, ξ_2, ξ_3 are infinitesimal isometries.

We note that on any QC structure homothetic to a 3-Sasakian structure, the Reeb vector fields are also infinitesimal isometries, i..e, (7.6) with $\nu = 0$ and (7.7) hold for $Q = \xi_i, i = 1, 2, 3$. This follows easily from Corollary 7.9. Our next goal is to characterize QC structures for which the Reeb vector fields are QC vector fields. It turns out that the just mentioned setting is the only possible. More precisely, we have the following Theorem.

Theorem 7.10. *Let $(M, g, \mathbb{Q})$ be a QC manifold with positive qc-scalar curvature, assumed constant in dimension seven. The following conditions are equivalent.*

i) Each of the Reeb vector fields is a QC vector field.

ii) The QC structure is homothetic to a 3-Sasakian structure. In particular, the Reeb vector fields are infinitesimal isometries.

Proof. We note that Corollary 7.5 shows that on a QC manifold the Reeb vector fields ξ_1, ξ_2, ξ_3 are 3-contact exactly when the connection 1-forms vanish on H. By Lemma 4.18, Theorem 4.9 together with the made assumptions on the qc-scalar curvature we see that the qc-scalar curvature is a positive constant. Now, Corollary 4.17 shows that the given QC stricture is homothetic to a 3-Sasakian structure. The converse direction was already explained before the statement of the Theorem. $\square$

For the remaining of this section we prove other useful properties of QC vector fields. The next three Lemmas are of independent interest. Given a vector field Q, we define the symmetric tensor T^0_Q and the skew-symmetric tensor u_Q

$$T^0_Q \ = \sum_{s=1}^{3} \eta_s(Q)\, T^0_{\xi_s}, \qquad u_Q \ = \sum_{s=1}^{3} \eta_s(Q)\, I_s u, \tag{7.14}$$

respectively, such that, $\quad T(Q,X,Y) \;=\; g(T^0_Q X,Y) \;+\; g(u_Q X,Y),$

LEMMA 7.11. *The tensors T^0_Q and u_Q lie in the $[-1]$ component associated to the operator $\dagger$ cf.* (2.8) *and* (2.7).

PROOF. By the definition of u_Q, we have $g(u_Q I_1 X, I_1 Y) \;=\; \sum_{s=1}^3 \eta_s(Q)\, g(I_s u X,Y)$ and after summing we find

$$\sum_{j=1}^3 g(u_Q I_j X, I_j Y) \;= \sum_{j=1}^3 \eta_j(Q)\, g(I_j u X,Y) \;=\; -g(u_Q X,Y).$$

We turn to the second claim. Recall that $T^0_{\xi_j}$ anti-commutes with I_j, see (2.14). Hence,

$$\begin{aligned}
g(T^0_Q I_1 X, I_1 Y) \;&=\; -\eta_1(Q)\, g(T^0_{\xi_1} X,Y) - \eta_2(Q)\, [g({T^0_{\xi_2}}^{--+} X,Y) - g({T^0_{\xi_2}}^{+--} X,Y)]\\
&\quad -\; \eta_3(Q)\, [g({T^0_{\xi_2}}^{-+-} X,Y) \;-g({T^0_{\xi_3}}^{+--} X,Y)],\\
g(T^0_Q I_2 X, I_2 Y) \;&=\; -\eta_2(Q)\, g(T^0_{\xi_2} X,Y) - \;\eta_1(Q)\, [g({T^0_{\xi_1}}^{--+} X,Y) \;- g({T^0_{\xi_1}}^{-+-} X,Y)]\\
&\quad -\; \eta_3(Q)\, [g({T^0_{\xi_3}}^{+--} X,Y) \;- g({T^0_{\xi_3}}^{-+-} X,Y)],\\
g(T^0_Q I_3 X, I_3 Y) \;&=\; -\eta_3(Q)\, g(T^0_{\xi_3} X,Y) - \;\eta_1(Q)\, [g({T^0_{\xi_1}}^{-+-} X,Y) \;- g({T^0_{\xi_2}}^{--+} X,Y)]\\
&\quad -\; \eta_2(Q)\, [g({T^0_{\xi_2}}^{+--} X,Y) \;- g({T^0_{\xi_2}}^{--+} X,Y)].
\end{aligned}$$

Summing the above three equations we come to

$$\sum_{j=1}^3 g(T^0_Q I_j X, I_j Y) \;=\; -\sum_{j=1}^3 g(Q,\xi_j)\, g(T^0_{\xi_j} X,Y) \;=\; -g(T^0_Q X,Y),$$

which finishes the proof of Lemma 7.11. ☐

LEMMA 7.12. *If Q is a QC vector field then the next two equalities hold*

$$g(\nabla_X Q,Y) \;+\; g(\nabla_Y Q,X) \;+\; 2g(T^0_Q X,Y) = \nu\, g(X,Y), \tag{7.15}$$

$$\begin{aligned}
3g(\nabla_X Q,Y) - \sum_{s=1}^3 g(\nabla_{I_s X} Q, I_s Y) + 4g(T^0_Q X,Y) + 4g(u_Q X,Y)&\\
= -2\sum_{(ijk)} L_{ij}(Q)\omega_k(X,Y),&
\end{aligned} \tag{7.16}$$

where the sum is over all even permutation of (1,2,3) and

$$\begin{aligned}
L_{ij}(Q) = -L_{ji}(Q) = \xi_j(\eta_i(Q)) - \eta_j(Q) d\eta_j(\xi_i,\xi_j)&\\
+\frac{1}{2}\eta_k(Q)\left(\frac{Scal}{8n(n+2)} + d\eta_j(\xi_k,\xi_i) - d\eta_i(\xi_j,\xi_k) - d\eta_k(\xi_i,\xi_j)\right)&.
\end{aligned} \tag{7.17}$$

PROOF. In terms of the Biquard connection (7.6) reads exactly as (7.15). Furthermore, from (7.7), (7.12) and (3.28) it follows

$$\begin{aligned}
&o_{ij} I_j X + o_{ik} I_k X = \;(\mathcal{L}_Q\, I_i)(X) =\\
&=\; -\nabla_{I_i X} Q \;+ I_i \nabla_X Q - \alpha_j(Q) I_k X + \alpha_k(Q) I_j X - \;T(Q, I_i X) \;+\; I_i T(Q,X).
\end{aligned} \tag{7.18}$$

A use of (7.12), (3.29) and (3.30) allows us to write the last equation in the form

$$\begin{aligned} g(\nabla_X Q, Y) \ - \ g(\nabla_{I_i X} Q, I_i Y) + \ T(Q, X, Y) \ - \ T(Q, I_i X, I_i Y) \\ = (o_{ij} - \alpha_k(Q))\omega_k(X, Y) - (o_{ik} + \alpha_j(Q))\omega_j(X, Y) \\ = -L_{ij}(Q)\omega_k(X, Y) + L_{ik}(Q)\omega_j(X, Y), \end{aligned}$$

where $L_{ij}(Q)$ satisfy (7.17). Summing the above identities for the three almost complex structures and applying Lemma 7.11, we obtain (7.16), which completes the proof of Lemma 7.12. □

COROLLARY 7.13. *Let $(M, g, \mathbb{Q})$ be a QC manifold with positive qc-scalar curvature, assumed to be constant in dimension seven. The following conditions are equivalent*

i) There exists a local 3-Sasakian structure compatible with $H = Ker\ \eta$;
ii) There are three linearly independent vertical QC-vector fields.

PROOF. Let $\gamma_1, \gamma_2, \gamma_3$ be linearly independent vertical QC vector fields. Then (7.15) for $Q = \gamma_i$ yields $T^0_{\gamma_i} = 0, i = 1, 2, 3, \quad \nu = 0$ since T_{γ_i} is trace-free. Thus, for any cyclic permutation (i, j, k) of $(1, 2, 3)$, (7.16) imply $u_{\gamma_i} = 0, L_{ij}(\gamma_s) = 0$ by comparing the trace and the trace-free part. Hence, we get $T_{\xi_s} = u_{\xi_s} = 0$ since γ_s are linearly independent vertical vector fields. Now, Theorem 1.3 shows that the given QC stricture is homothetic to a 3-Sasakian structure. □

In the particular case when the vector field Q is the gradient of a function defined on the manifold M, we have the following formulas.

COROLLARY 7.14. *If h is a smooth real valued function on M and $Q = \nabla h$ is a QC vector field, then for any horizontal vector fields X and Y we have*

a) $[(\nabla dh)]_{[3][0]}(X, Y) \ = \ 0$
b) $[\nabla dh\ ]_{[sym][-1]}(X, Y) \ = \ -T^0_Q(X, Y)$ (*cf.* (7.14))
c) $u_Q(X, Y) \ = \ 0$ (*cf.* (7.14)), $\qquad L_{ij}(\nabla h) = 0.$

PROOF. Use (7.15) and (5.4) to get

$$2\nabla dh(X, Y) \ + \ 2dh(\xi_j)\,\omega_j(X, Y) \ + \ 2g(T^0_Q X, Y) \ = \ \nu g(X, Y).$$

Decomposing in the $[-1]$ and $[3]$ components completes the proof of a) and b), taking into account (7.16) and Lemma 7.11. The skew-symmetric part of (7.16) gives $2u_Q \ + \ \sum_{(ijk)} L_{ij}(\nabla h)\omega_k \ = \ 0$, where the sum is over all even permutations of $(1, 2, 3)$. Hence, c) follows by comparing the trace and trace-free parts of the last equality. □

We finish the section with another useful observation.

LEMMA 7.15. *Let $(M, [g], \mathbb{Q})$ be QC manifold and Q be a QC vector field determined by (7.5) and (7.12). The next equality hold*

$$d\eta_i([Q, I_i X]^{\perp}, Y) + d\eta_i(I_i X, [Q, Y]^{\perp}) = 0$$

Proof. We have using (7.5) that

$$
\begin{aligned}
\mathcal{L}_Q\, d\eta_i(I_iX,Y) &= 2(\mathcal{L}_Q\,\omega_i)(I_iX,Y) - d\eta_i([Q,I_iX]^{\perp},Y) \\
&\quad - d\eta_i(I_iX,[Q,Y]^{\perp}) = -2(\mathcal{L}_Q\, g)(X,Y) + 2g((\mathcal{L}_Q\, I_i)I_iX,Y) \\
&\quad - d\eta_i([Q,I_iX]^{\perp},Y) - d\eta_i(I_iX,[Q,Y]^{\perp}) = (d\mathcal{L}_Q\,\eta_i)(I_iX,Y) \\
&= (d\nu\wedge\eta_i + \nu d\eta_i + do_{ij}\wedge\eta_j + o_{ij}d\eta_j + do_{ik}\wedge\eta_k + o_{ik}d\eta_k)(I_iX,Y) \\
&= -2\nu g(X,Y) - 2o_{ij}\omega_k(X,Y) + 2o_{ik}\omega_j(X,Y),
\end{aligned}
\tag{7.19}
$$

where o_{st} are the entries of the matrix O given by (7.12). An application of (7.6) and (7.7) to (7.19) gives the assertion. $\square$

CHAPTER 8

Quaternionic contact Yamabe problem

8.1. The Divergence Formula

Let (M, η) be a quaternionic contact manifold with a fixed globally defined contact form η. For a fixed $j \in \{1, 2, 3\}$ the form

$$Vol_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \omega_j^{2n} \tag{8.1}$$

is a volume form. Note that Vol_η is independent of j. We define the (horizontal) divergence of a horizontal vector field/one-form $\sigma \in \Lambda^1(H)$ by

$$\nabla^* \sigma \ = tr|_H \nabla\sigma = \sum_{a=1}^{4n} (\nabla_{e_\alpha}\sigma)(e_\alpha). \tag{8.2}$$

Clearly the horizontal divergence does not depend on the basis and is an $Sp(n)Sp(1)$-invariant. For any horizontal 1-form $\sigma \in \Lambda^1(H)$ we denote with $\sigma^\#$ the corresponding horizontal vector field via the horizontal metric defined with the equality $\sigma(X) = g(\sigma^\#, X)$. It is justified to call the function $\nabla^* \sigma$ divergence of σ in view of the following Proposition.

PROPOSITION 8.1. *Let (M, η) be a quaternionic contact manifold of dimension $(4n+3)$ and $\eta \wedge \omega_s^{2n-1} \overset{def}{=} \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \omega_s^{2n-1}$. For any horizontal 1-form $\sigma \in \Lambda^1(H)$ we have*

$$d(\sigma^\# \lrcorner (Vol_\eta)) \ = \ -(\nabla^* \sigma)\, \eta \wedge \omega^{2n} \ .$$

Therefore, if M is compact,

$$\int_M (\nabla^* \sigma)\, \eta \wedge \omega^{2n} \ = \ 0.$$

PROOF. We work in a qc-normal frame at a point $p \in M$ constructed in Lemma 4.5. Since σ is horizontal, we have $\sigma^\# = g(\sigma^\#, e_a)e_a$. Therefore, we calculate

$$\sigma^\# \lrcorner (Vol_\eta) = \sum_{a=1}^{4n} (-1)^a g(\sigma^\#, e_a) \eta \wedge e_{a_1}^\# \wedge \cdots \wedge \hat{e_a^\#} \wedge \cdots \wedge e_{4n}^\#,$$

where $\hat{e_a^\#}$ means that the 1-form $e_a^\#$ is missing in the above wedge product. The exterior derivative of the above expression gives

$$d(\sigma^\# \lrcorner (Vol_\eta)) = -\sum_{a=1}^{4n} e_a g(\sigma^\#, e_a) Vol_\eta = -(\nabla^* \sigma)\, Vol_\eta.$$

Indeed, since the Biquard connection preserves the metric, the middle term is calculated as follows $e_a g(X, e_a) = g(\nabla_{e_a} X, e_a) + g(X, \nabla_{a_a} e_a)$ which evaluated at the point p gives

$$e_a g(X, e_a)_{|_p} = g(\nabla_{e_a} X, e_a)_{|_p}) = ((\nabla_{e_a}\sigma)e_a)_{|_p}.$$

In order to obtain the last equality we also used the definition of the Reeb vector fields, (2.10), and the following sequence of identities

$$de_b^{\#}(e_b, e_a)_{|_p} = e_b^{\#}([e_b, e_a])_{|_p} = e_b^{\#}(\nabla_{e_b} e_a - \nabla_{e_a} e_b - T(e_b, e_a))_{|_p} = 0$$

since $T(e_b, a_a)$ is a vertical vector field. This proves the first formula. If the manifold is compact, then Stoke's theorem completes the proof. □

We note that the integral formula of the above theorem was essentially proved in [[W], Proposition 2.1].

8.2. Partial solutions of the qc-Yamabe problem

Given a QC structure $\tilde{\eta}$, the qc-Yamabe problem is to find all qc-structures η that are qc-conformal to $\tilde{\eta}$ and have constant qc-scalar curvature. The relation between the two qc-scalar curvatures is given by the qc-Yamabe equation (5.8) and the problem is to find all solutions of this equation. In this Section we shall present a partial solution of the qc-Yamabe problem on the quaternionic sphere. Equivalently, using the Cayley transform this provides a partial solution of the qc-Yamabe problem on the quaternionic Heisenberg group. The extra assumption under which we classify the solutions of the qc-Yamabe equation consists of assuming that the "new" quaternionic structure has an integrable vertical space. The change of the vertical space is given by (5.1). Of course, the standard quaternionic contact structure has an integrable vertical distribution. A note about the Cayley transform is in order. We shall define below the explicit Cayley transform for the considered case, but one should keep in mind the more general setting of groups of Heisenberg type [**CDKR1**]. In that respect, the solutions of the qc-Yamabe equation on the quaternionic Hesenberg group, which we describe, coincide with the solutions on the groups of Heisenebrg type [**GV1**].

As in Section 5 we are considering a conformal transformation $\tilde{\eta} = \frac{1}{2h}\eta$, where $\tilde{\eta}$ represents a fixed quaternionic contact structure and η is the "new" structure conformal to the original one. In fact, eventually, $\tilde{\eta}$ will stand for the standard quaternionic contact structure on the quaternionic sphere. In this case the qc-Yamabe problem , up to a homothety, is to find all structures η, which are conformal to $\tilde{\eta}$ and have constant scalar curvature equal to $16n(n+2)$, see Corollary 4.13.

Proposition 8.2. *Let $(M, \bar{\eta})$ be a compact qc-Einstein manifold of dimension $(4n+3)$. Let $\bar{\eta} = \frac{1}{2h}\eta$ be a conformal transformation of the qc-structure $\bar{\eta}$ on M. Suppose η has constant scalar curvature.*

a) If $n > 1$, then any one of the following two conditions implies that η is a qc-Einstein structure:
   i) the vertical space of η is integrable;
   ii) the QC structure η is qc-pseudo Einstein.
b) If $n = 1$ and the vertical space of η is integrable than η is a qc-Einstein structure.

Proof. The proof follows the steps of the solution of the Riemannian Yamabe problem on the standard unit sphere, see [**LP**]. Theorem 1.3 shows that $\tilde{\eta}$ is a qc-Einstein structure. Theorem 3.12 and equations (5.7), (5.5), and (5.6) imply

$$\begin{aligned}(8.3)\quad [Ric_0]_{[-1]}(X,Y) &= (2n+2)T^0(X,Y)\\ &= -(2n+2)h^{-1}[\nabla dh]_{[sym][-1]}(X,Y)\end{aligned}$$

$$\begin{aligned}(8.4)\quad [Ric_0]_{[3]}(X,Y) &= 2(2n+5)U(X,Y)\\ &= -(2n+5)h^{-1}[\nabla dh - 2h^{-1}dh\otimes dh]_{[3][0]}(X,Y).\end{aligned}$$

Furthermore, when the scalar curvature of η is a constant then Theorem 4.8 gives

$$\nabla^* T^0 = (n+2)\mathbb{A}, \qquad \nabla^* U = \frac{(1-n)}{2}\mathbb{A}. \tag{8.5}$$

If $n > 1$ and either the vertical space of η is an integrable distribution or η is qc-pseudo Einstein $U = 0$, then (8.5) shows that $\mathbb{A} = 0$ and the divergences of T^0 and U vanish $\nabla^* T^0 = 0$ and $\nabla^* U = 0$. The same conclusion can be reached in the case $n = 1$ assuming the integrability of the vertical space (recall that always $U = 0$ when $n = 1$). We shall see that, in fact, T^0 and U vanish, i.e., η is also qc-Einstein. Consider first the $[-1]$ component. Taking norms, multiplying by h and integrating, the divergence formula gives

$$\begin{aligned}\int_M h\,|\,[Ric_o]_{[-1]}\,|^2\,\eta\wedge\omega^{2n} &= (2n+2)\int\langle [Ric_o]_{[-1]}, \nabla dh]\rangle\,\eta\wedge\omega^{2n}\\ = (2n+2)&\int_M\langle\nabla^*\ [Ric_o]_{[-1]}, \nabla h]\rangle\,\eta\wedge\omega^{2n} = 0.\end{aligned}$$

Thus, the $[-1]$ component of the qc-Einstein tensor vanishes $|\,[Ric_o]_{[-1]}\,| = 0$. Define $h = \frac{1}{2u}$, inserting (6.2) into (8.4) one gets

$$[Ric_0]_{[3]} = 2(2n+5)U = -(2n+5)[\nabla du]_{[3][0]},$$

from where, arguing as before we get $[Ric_0]_{[3]} = 0$. Theorem 1.3 completes the proof. ∎

Corollary 8.3. *Let $\bar\eta = \frac{1}{2h}\eta$ be a conformal transformation of a compact qc-Einstein manifold of dimension $(4n+3)$ and suppose $\bar\eta$ has constant qc-scalar curvature.*

- *i) If $n > 1$ and either the gradient ∇h or the gradient $\nabla(\frac{1}{h})$ is a QC vector fields then h is a constant.*
- *ii) If $n = 1$ and the gradient $\nabla(\frac{1}{h})$ is a QC vector fields then h is a constant.*

Proof. Suppose ∇h is a QC-vector field. Corollary 7.14, b) yields

$$[\nabla dh]_{[sym][-1]} = 0$$

since the torsion of Biquard connection vanishes due to Proposition 4.2. Then Proposition 8.2 and a) in Corollary 7.14 imply that on H we have

$$dh\otimes dh + d_1h\otimes d_1h + d_2h\otimes d_2h + d_3h\otimes d_3h = \tfrac{|dh|^2}{n}g.$$

If $n > 1$ then $dh_{|H} = 0$, which implies $dh = 0$ using the bracket generating condition.

Suppose $\nabla(\frac{1}{h})$ is a QC vector field. Then Proposition 8.2, (6.2) combined with b) in Corollary 7.14 show that on H we have

$$3dh\otimes dh - d_1h\otimes d_1h - d_2h\otimes d_2h - d_3h\otimes d_3h = 0.$$

Define $X = I_1X, Y = I_1Y$ etc. to get $dh\otimes dh = d_1h\otimes d_1h = d_2h\otimes d_2h = d_3h\otimes d_3h$. Hence, $dh_{|H} = 0$ since dim $Ker\,dh = 4n-1$ and $dh = 0$ as above. ∎

8.3. Proof of Theorem 1.2

Proof. We start the proof with the observation that from Proposition 8.2 and Corollary 6.26 the new structure η is also qc-Einstein. Next we bring into consideration the quaternionic Heisenberg group. Let us identify $G(\mathbb{H})$ with the boundary Σ of a Siegel domain in $\mathbb{H}^n \times \mathbb{H}$,

$$\Sigma \;=\; \{(q',p') \in \mathbb{H}^n \times \mathbb{H} \;:\; \Re\, p' \;=\; |q'|^2\},$$

by using the map $(q',\omega') \mapsto (q',|q'|^2-\omega')$. The standard contact form, written as a purely imaginary quaternion valued form, is given by (cf. (5.13)) $2\tilde{\Theta} = (d\omega \;-\; q'\cdot d\bar{q}' \;+\; dq'\cdot\bar{q}')$, where $\cdot$ denotes the quaternion multiplication. Since $dp' \;=\; q'\cdot d\bar{q}' \;+\; dq'\cdot\bar{q}' \;-\; d\omega'$, under the identification of $G(\mathbb{H})$ with Σ we have also $2\tilde{\Theta} \;=\; -dp' \;+\; 2dq'\cdot\bar{q}'$. Taking into account that $\tilde{\Theta}$ is purely imaginary, the last equation can be written also in the following form

$$4\,\tilde{\Theta} \;=\; (d\bar{p}' \;-\; dp') \;+\; 2dq'\cdot\bar{q}' \;-\; 2q'\cdot d\bar{q}'.$$

Now, consider the Cayley transform as the map $\mathcal{C} : S \mapsto \Sigma$ from the sphere $S \;=\; \{|q|^2+|p|^2=1\} \subset \mathbb{H}^n \times \mathbb{H}$ minus a point to the Heisenberg group Σ, with $\mathcal{C}$ defined by

$$(q',p') \;=\; \mathcal{C}\Big((q,p)\Big), \qquad q' \;=\; (1+p)^{-1}\,q, \qquad p' \;=\; (1+p)^{-1}\,(1-p)$$

and with an inverse map $(q,p) \;=\; \mathcal{C}^{-1}\Big((q',p')\Big)$ given by

$$q \;=\; 2(1+p')^{-1}\,q', \qquad p \;=\; (1+p')^{-1}\,(1-p').$$

The Cayley transform maps S minus a point to Σ since

$$\Re\, p' \;=\; \Re\frac{(1+\bar{p})(1-p)}{|1+p\,|^2} \;=\; \Re\frac{1-|p|}{|1+p\,|^2} \;=\; \frac{|q|^2}{|1+p\,|^2} \;=\; |q'|^2.$$

Writing the Cayley transform in the form $(1+p)q' \;=\; q, \quad (1+p)p' \;=\; 1-p$, gives

$$dp\cdot q' \;+\; (1+p)\cdot dq' \;=\; dq, \qquad\qquad dp\cdot p' \;+\; (1+p)\cdot dp' \;=\; -dp,$$

from where we find

$$\begin{aligned} dp' \;&=\; -2(1+p)^{-1}\cdot dp\cdot(1+p)^{-1}\\ dq' \;&=\; (1+p)^{-1}\cdot[dq \;-\; dp\cdot(1+p)^{-1}\cdot q]. \end{aligned} \tag{8.6}$$

The Cayley transform is a conformal quaternionic contact diffeomorphism between the quaternionic Heisenberg group with its standard quaternionic contact structure $\tilde{\Theta}$ and the sphere minus a point with its standard structure $\tilde{\eta}$, a fact which can be seen as follows. Equations (8.6) imply the following identities

$$
\begin{aligned}
(8.7)\quad 2\,\mathcal{C}^*\,\tilde{\Theta} \;&=\; -(1+\bar p)^{-1}\cdot d\bar p\cdot(1+\bar p)^{-1} \;+\; (1+p)^{-1}\cdot dp\cdot(1+p)^{-1}\\
&\quad +\; (1+p)^{-1}\,[dq \;-\; dp\cdot(1+p)^{-1}\cdot q]\cdot\bar q\cdot(1+\bar p)^{-1}\\
&\quad -\; (1+p)^{-1}\,q\cdot[d\bar q \;-\; \bar q\cdot(1+\bar p)^{-1}\cdot d\bar p\,]\cdot(1+\bar p)^{-1}\\
&= (1+p)^{-1}\Big[dp\cdot(1+p)^{-1}\cdot(1+\bar p) \;-\; |q|^2\,dp\cdot(1+p)^{-1}\Big](1+\bar p)^{-1}\\
&\quad +\; (1+p)^{-1}\Big[-(1+p)\cdot(1+\bar p)^{-1}\cdot d\bar p \;+\; |q|^2\,(1+p)^{-1}d\bar p\,\Big](1+\bar p)^{-1}\\
&\quad +(1+p)^{-1}\Big[dq\cdot\bar q \;-\; q\cdot d\bar q\Big](1+\bar p)^{-1} \;=\; \frac{1}{|1+p\,|^2}\,\lambda\,\tilde\eta\,\bar\lambda,
\end{aligned}
$$

where $\lambda \;=\; |1+p\,|\,(1+p)^{-1}$ is a unit quaternion and $\tilde\eta$ is the standard contact form on the sphere,

$$
\tilde\eta \;=\; dq\cdot\bar q \;+\; dp\cdot\bar p \;-\; q\cdot d\bar q - \; p\cdot d\bar p. \tag{8.8}
$$

Since $|1+p| \;=\; \frac{2}{|1+p'|}$ we have $\lambda \;=\; \frac{1+p'}{|1+p'\,|}$ equation (8.7) can be put in the form

$$
\lambda\;\cdot(\mathcal{C}^{-1})^*\,\tilde\eta\;\cdot\bar\lambda \;=\; \frac{8}{|1+p'\,|^2}\,\tilde\Theta.
$$

We see that up to a constant multiplicative factor and a quaternionic contact automorphism the forms $(\mathcal{C}^{-1})^*\tilde\eta$ and $\tilde\Theta$ are conformal to each other. It follows that the same is true for $(\mathcal{C}^{-1})^*\eta$ and $\tilde\Theta$.

In addition, $\tilde\Theta$ is qc-Einstein by definition, while η and hence also $(\mathcal{C}^{-1})^*\eta$ are qc-Einstein as we observed at the beginning of the proof. Now we can apply Theorem 1.1 according to which up to a multiplicative constant factor the forms $(\mathcal{C}^{-1})^*\tilde\eta$ and $(\mathcal{C}^{-1})^*\eta$ are related by a translation or dilation on the Heisenebrg group. Hence, we conclude that up to a multiplicative constant, η is obtained from $\tilde\eta$ by a conformal quaternionic contact automorphism, see Definition 7.6. $\square$

Let us note that the Cayley transform defined in the setting of groups of Heisenberg type is also a conformal transformation on H, see cf. [**ACD**, Lemma 2.5]. One can write the above transformation formula in this more general setting.

Bibliography

[AK] Alekseevsky, D. & Kamishima, Y., *Pseudo-conformal quaternionic CR structure on* $(4n+3)$*-dimensional manifold*, Ann. Mat. Pura Appl. (4) **187** (2008), no. 3, 487–529 8

[AM] Alekseevsky, D. & Marciafava, S., *Quaternionic structures on a manifold and subordinated structures*, Ann. Mat. Pura Appl., **171** (1996), 205–273. 30

[A1] Alesker, S., *Non-commutative linear algebra and plurisubharmonic functions of quaternionic variables*, Bull. Sci. Math., **127** (2003), 1–35. 51

[A2] ———, *Quaternionic Monge-Ampére equations*, J. Geom. Anal., **13** (2003), 183–216. 51

[A3] ———, *Quaternionic plurisubharmonic functions and their applications to convexity*, Algebra i Analiz, **19** (1) (2007), 5–22. 51

[AV] Alesker, S. & Verbitsky, M., *Plurisubharmonic functions on hypercomplex manifolds and HKT-geometry*, J. Geom. Anal., **16** (2006), 375–399. 51

[ACD] Astengo, F., Cowling, M. & Di Blasio, B., *The Cayley transform and uniformly bounded representations*, J. Funct. Anal., **213** (2004), 241–269. 75

[BS] Banos, B., & Swann, A., *Potentials for Hyper-Kähler Metrics with Torsion*, Classical Quantum Gravity, **21** (2004), no. 13, 3127–3135. 52

[Bes] Besse, A., *Einstein manifolds* , Ergebnisse der Mathematik, vol. 3, Springer-Verlag, Berlin, 1987. 33

[Biq1] Biquard, O., *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **265** (2000). 1, 7, 8, 10, 11, 17, 18, 29, 38

[Biq2] Biquard, O., *Quaternionic contact structures*, Quaternionic structures in mathematics and phisics (Rome, 1999), 23–30 (electronic), Univ. Studi Roma "La Sapienza", Roma, 1999. 1, 7

[BG] Boyer, Ch. & Galicki, K., *3-Sasakian manifolds*, Surveys in differential geometry: essays on Einstein manifolds, 123–184, Surv. Differ. Geom., **VI**, Int. Press, Boston, MA, 1999. 28

[BGN] Boyer, Ch., Galicki, K. & Mann, B., *The geometry and topology of* 3*-Sasakian manifolds* J. Reine Angew. Math., **455** (1994), 183–220. 28

[CSal] Capria, M. & Salamon, S., *Yang-Mills fields on quaternionic spaces* Nonlinearity **1** (1988), no. 4, 517–530. 9, 51

[CL1] Chen, J. & Li, J., *Quaternionic maps between hyperkähler manifolds*, J. Diff. Geom. **55** (2000), no. 2, 355–384. 4, 50

[CL2] ———, *Quaternionic maps and minimal surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), no. 3, 375–388. 4, 50

[CDKR2] Cowling, M., Dooley, A. H., Korányi, A. & Ricci, F., *An approach to symmetric spaces of rank one via groups of Heisenberg type.*, J. Geom. Anal., **8** (1998), 199–237. 1

[CDKR1] ———, *H-type groups and Iwasawa decompositions*, Adv. Math., **87** (1991), 1–41. 1, 2, 72

[D1] Duchemin, D., *Quaternionic contact hypersurfaces*, math.DG/0604147. 53, 63

[D] ———, *Quaternionic contact structures in dimension 7*, Ann. Inst. Fourier, **56** (4) (2006), 851–885. 10

[F] Fueter, R., *Über die analische Darstellung der regulären Funktionen einer Quaternionen variablen*, Comm. Math. Helv., **8** (1936), 371–378. 48

[FG] Fefferman, Ch. & Graham, C. R., *Conformal invariants. The mathematical heritage of Élie Cartan (Lyon, 1984).* Astérisque 1985, Numero Hors Serie, 95–116. 1

[Fo] G. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Math., **13** (1975), 161–207. 2

[FSt] G. B. Folland & E. M. Stein, *Estimates for the* $\bar{\partial}_b$ *Complex and Analysis on the Heisenberg Group*, Comm. Pure Appl. Math., **27** (1974), 429–522. 2

[GV1] Garofalo, N., & Vassilev, D., *Symmetry properties of positive entire solutions of Yamabe type equations on groups of Heisenberg type*, Duke Math J, **106** (2001), no. 3, 411–449. 2, 3, 45, 72

[GV2] ———, *Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups*, Math Ann., **318** (2000), no. 3, 453–516. 2

[GL] Graham, C. R., & Lee, J. M., *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math., **87** (1991), 186–225. 1

[GP] Grantcharov, G., & Poon, Y. S., *Geometry of hyper-Kähler connections with torsion*, Comm. Math. Phys., **213** (2000), 19–37. 52

[HP] Howe, P.S., & Papadopoulos, G., *Twistor spaces for hyper-Kähler manifolds with torsion*, Phys. Lett., **B 379** (1996), 80–86. 50, 51, 52

[Iv] Ivanov, S., *Geometry of Quaternionic Kähler connections with torsion* J. Geom. Phys., **41** (2002) 235–257 30

[IMV] Ivanov, S., Minchev, I., & Vassilev, D., *Extremals for the Sobolev inequality on the seven dimensional quaternionic Heisenberg group and the quaternionic contact Yamabe problem*, J. Eur. Math. Soc., **12** (2010), pp. 1041–1067. 3

[JL1] Jerison, D., & Lee, J., *A subelliptic, nonlinear eigenvalue problem and scalar curvature on CR manifolds*, Contemporary Math., **27** (1984), 57–63. 1

[JL3] ———, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc., **1** (1988), no. 1, 1–13. 1, 2

[JL4] ———, *Intrinsic CR normal coordinates and the CR Yamabe problem*, J. Diff. Geom., **29** (1989), no. 2, 303–343. 1, 2

[JL2] ———, *The Yamabe problem on CR manifolds*, J. Diff. Geom., **25** (1987), 167–197. 1, 2

[Joy] Joyce, D., *Hypercomplex algebraic geometry*, Quart. J. Math. Oxford **49** (1998), 129–162. 48, 52

[KN] Kamada, H.,& Nayatani, S., *Quaternionic analogue of CR geometry*, Séminaire de théorie spectrale et géométrie de Gronoble, **19** (2001), 41–52. 63

[Kam] Kamishima, Y., *Geometric rigidity of spherical hypersurfaces in quaternionic manifolds*, Asian J. Math., **3** (1999), 519–556. 65

[Kas] Kashiwada, T., *A note on Riemannian space with Sasakian 3-structure*, Nat. Sci. Reps. Ochanomizu Univ., **22** (1971), 1–2. 28

[KoNo] Kobayashi, Sh., & Nomizu, K., *Foundations of differential geometry. Vol. II.* Interscience Tracts in Pure and Applied Mathematics, No. 15 Vol. II Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney 1969 xv+470 pp. 34

[K] Korányi, A., *Kelvin transform and harmonic polynomials on the Heisenberg group*, Adv.Math. **56** (1985), 28–38. 1

[L1] Lee, J., *Pseudo-einstein structures on CR manifolds*, Amer. J. Math., **110** (1988), 157–178. 47

[LP] Lee, J. M. & Parker, T., *The Yamabe Problem*, Bull Am. Math. Soc. **17** (1987), no. 1, 37–91. 1, 2, 72

[LZ] Li, J. & Zhang, X., *Quaternionic maps between quaternionic Kähler manifolds*, Math. Z. **250** (2005), no. 3, 523–537. 4, 50

[MO] Matzeu, P. & Ornea, L., *Local almost contact 3-structures*, Publ. Math. Debrezen, **57** (2000), 499–508. 33

[MS] Michelson, J. & Strominger, A., *The geometry of (super) conformal quantum mechanics*, Comm. Math. Phys. **213** (2000), 1–17. 52

[On] O'Neill, B., *Semi-Riemannian geometry*, Academic Press, New York, 1883. 33

[P] Pansu, P., *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. Math. (2) 129 (1989), no. 1, 1–60. 1

[Per1] Pertici, D., *Funzioni regolari di piú variabili quaternioniche*, Ann. Mat. Pura Appl., Serie IV, **CLI** (1988), 39–65. 48, 52, 56

[Per2] ———, *Trace theorems for regular functions of several quaternionic variables*, Forum Math. **3** (1991), 461-478. 48, 52, 56

[Q] Quillen, D., *Quaternionic algebra and sheaves on the Riemann sphere*, Quart. J. Math. Oxford, **49** (1998), 163–198. 52

[Sal2] Salamon, S., *Differential geometry of quaternionic manifolds*, Ann. scient. És. Norm. Sup., (4) 19 (1986), no. 1, 31–55 51

[Sal1] ———, *Quaternionic Kähler manifolds*, Invent. math., **67** (1982), 143–171. 51

[Sti] Stiefel, E., *On Cauchy-Riemann Equations in Higher Dimensions*, J. Res. Nat. Bureu of Standarts, **48** (1952), 395–398. 50

[S] Sudbery, A., *Quaternionic Analysis*, Math. Proc. Camb. Phil. Soc., **85** (1979), 199–225. 4, 48

[T] Tanaka, N., *A differential geometric study on strongly pseudo-convex manifolds*, Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 9. Kinokuniya Book-Store Co., Ltd., Tokyo, 1975 1

[Va1] Vassilev, D., *Regularity near the characteristic boundary for sub-laplacian operators*, Pacific J Math, **227** (2006), no. 2, 361–397 2

[Va2] ———, *Yamabe type equations on Carnot groups*, Ph. D. thesis Purdue University, 2000. 2

[V] Verbitsky, M., *Hyper Kähler manifolds with torsion, Supersymmetry and Hodge theory*, Asian J. Math., **6** (2002), 679–712. 51

[W] Wang, W., *The Yamabe problem on quaternionic contact manifolds*, Ann. Mat. Pura Appl., **186** (2006), no. 2, 359–380. 2, 72

[We] Webster, S. M., *Pseudo-hermitian structures on a real hypersurface*, J.Diff. Geom., **13** (1979), 25–41. 1

[Wu] Wu, H.-H., *The Bochner technique in differential geometry*, Math. Rep., **3** (1988), no. 2, i–xii and 289–538. 24