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Overdetermined Boundary Value Problems, Quadrature Domains and Applications

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Abstract. We discuss an overdetermined problem in planar multiply connected domains Ω. This problem is solvable in Ω if and only if Ω is a quadrature domain carrying a solid-contour quadrature identity for analytic functions. At the same time the existence of such quadrature identity is equivalent to the solvability of a special boundary value problem for analytic functions. We give a complete solution of the problem in some special cases and discuss some applications concerning the shape of electrified droplets and small air bubbles in a fluid flow.

Keywords. Overdetermined boundary value problem, quadrature domain, analytic function, quadratic differential.

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1. Introduction

The goal of this paper is to study several connected overdetermined boundary value problems in the complex plane $\mathbb{C}$. In Section 2 we formulate an overdetermined torsion problem related to the celebrated theorem of Serrin [S]. In Section 3 we consider quadrature identities for analytic and harmonic functions, which are equivalent to solvability of the overdetermined problems under consideration. The results of Sections 2 and 3 have analogs in $\mathbb{R}^n$ for any dimension n.

In Section 4 we reduce the overdetermined problem to a certain boundary value problem for analytic functions. From that point on our methods are essentially complex analytic. In Section 5 we once more reformulate the problem as a boundary value problem for the inverse of the Koebe circular function, i.e. for the conformal mapping from a canonical circular domain onto a given multiply connected domain. In the same section we discuss some special cases of the

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problem. Our main applications are contained in Section 6. Here we confine ourselves to simply connected domains in $\overline{\mathbb{C}}$ but allow the "free" solution f to have poles, i.e. we consider meromorphic solutions. In particular, we study shapes of droplets of perfectly conducting fluid in the presence of an analytic potential. One special case also gives a shape of the "extremal" air bubble in a fluid flow subject to some mild natural restrictions, found earlier by E. McLeod [M1]. The last section contains some further examples.

2. Overdetermined torsion problem

Consider the overdetermined problem:

$$\triangle u = -1 \quad \text{in } \Omega \tag{2.1a}$$

$$u = c_j \quad \text{on } \gamma_j \tag{2.1b}$$

$$\frac{\partial u}{\partial n} = a_j \quad \text{on } \gamma_j \tag{2.1c}$$

where Ω is a finitely connected planar domain bounded by $n+1$ smooth Jordan curves γ_j, $0 \leq j \leq n$, (γ_0 is the outer boundary) and c_j, a_j denote real constants. Here and below $\partial/\partial n$ denotes the inner normal derivative on $\partial\Omega$. Clearly the solution of the above problem (if it exists) is unique.

Problem 2.1. Find all domains Ω on $\mathbb{C}$ of a given connectivity $n+1$ and all possible boundary values c_j and a_j, $0 \leq j \leq n$, for which problem (2.1) is solvable.

The Dirichlet problem (2.1a)–(2.1b) is often called *the torsion* problem since the solution u describes the stress in a homogenous cylindrical beam. Another reason to study Problem 2.1 comes from hydrodynamics. For a Newtonian fluid with velocity $\vec{\mathbf{v}}$ and viscosity μ the force $\vec{\mathbf{A}}$ exerted on the walls is given by

$$\vec{\mathbf{A}} = \left(p - \frac{4}{3}\mu \operatorname{div} \vec{\mathbf{v}}\right)\vec{\mathbf{n}} + \mu\vec{\mathbf{n}} \wedge \operatorname{curl} \vec{\mathbf{v}},$$

where p is the pressure and $\vec{\mathbf{n}}$ the outer normal, see [Be]. This gives a natural decomposition of the action on the wall into a normal pressure orthogonal to the boundary and a shear (stress) which is tangential to the wall. If the fluid is incompressible, i.e. $\operatorname{div} \vec{\mathbf{v}} = 0$ and the flow is laminary along the z-axis, i.e. $\vec{\mathbf{v}} = v(x,y)\vec{\mathbf{k}}$, where $\vec{\mathbf{k}}$ is the unit vector along the $z-$axis, we find

$$\vec{\mathbf{n}} \wedge \operatorname{curl} \vec{\mathbf{v}} = -\frac{\partial v}{\partial n}\vec{\mathbf{k}}$$

and thus

$$\vec{\mathbf{A}} = p\vec{\mathbf{n}} - \mu\frac{\partial v}{\partial n}\vec{\mathbf{k}}.$$

The motion of a fluid with density ρ and external force $\vec{\mathbf{F}}$ is governed by the Navier-Stokes equations:

$$\rho \frac{d\vec{\mathbf{v}}}{dt} = \rho\vec{\mathbf{F}} - \nabla p + \mu \triangle \vec{\mathbf{v}}.$$

Assuming further that the flow is steady, i.e. $\frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{0}}$, in the absence of external force the above system takes the form

$$\mu \triangle v - p_z = 0.$$

Denoting the pressure gradient by $P = -p_z$ and taking into account the boundary conditions we obtain

$$\mu \triangle v = -P, \qquad -\mu \frac{\partial v}{\partial n} = \text{shear on } \partial\Omega, \qquad v = v_j \text{ on } \gamma_j,$$

where v_j equals the speed of the corresponding wall (non-slipping condition), which is exactly the overdetermined boundary value Problem 2.1 provided that the shear on $\partial\Omega$ is constant.

The study of the overdetermined Problem 2.1 begins with Serrin's seminal paper [S] in which he employed the Alexandrov moving plane method to show that for a simply connected domain, solvability of Problem 2.1 in $\mathbb{R}^N$ with constant boundary conditions, $a_j = a$, $c_j = c$, forces the simply connected domain Ω to be a ball and the solution u to be radially symmetric.

In a short note following Serrin's paper, Weinberger [W] gave another proof of Serrin's result mentioned above based on the strong Maximum Principle for the auxiliary function

$$\varphi = |\nabla u|^2 + u$$

and a Rellich type identity.

Both Serrin's and Weinbereger's methods were extended to more general differential equations and more general boundary conditions. In general, the methods allow the specification of a free boundary in some of the cases when the solution of Problem 2.1 does not achieve local extrema inside Ω. We mention only the following theorem, which combines the results of [S], [A], [WGS], [R] and [Si] when applied to Problem 2.1 in the case of a domain in $\mathbb{R}^N$.

Theorem 2.2. *Let $c_0 = 0$, $a_0 \geq 0$, and $c_j > 0$, $a_j \leq 0$ for $0 \leq j \leq n$. Then the overdetermined boundary value Problem 2.1 is solvable for a finitely connected domain Ω in $\mathbb{R}^N$ if and only if Ω is a ball or a spherical shell. In either case u is a radial function.*

For a disc or an annulus problem (2.1a)–(2.1c) is elementary. As an instructive example, consider the case of an annulus $A = A(r, R) = \{z : r < |z| < R\}$ in the complex plane. Let

$$v(z) = \frac{|z|^2}{4},$$

so that $\triangle v = 1$. Now if u solves (2.1a)–(2.1c) in A then $h(z) = u(z) + v(z)$ is harmonic and has constant boundary values $b_0 = c_0 + (1/4)R^2$ on γ_0 and $b_1 = c_1 + (1/4)r^2$ on γ_1. Therefore,

$$h(z) = (b_0 - b_1)\frac{\log(|z|/r)}{\log(R/r)} + b_1,$$

and for $z \in A$,

$$u(z) = \left(c_0 - c_1 + \frac{1}{4}(R^2 - r^2)\right)\frac{\log(|z|/r)}{\log(R/r)} + c_1 + \frac{1}{4}(r^2 - |z|^2).$$

An easy calculation gives

$$R^2 - r^2 = 2(a_0 R + a_1 r) > 0.$$

The latter inequality shows that at least one of the values a_0 and a_1 is positive. (This also follows from Hopf's Lemma.)

We stress that even for a planar doubly-connected domain Problem 2.1 remains open when the (inner) normal derivatives are positive constants, i.e. the solution achieves a maximum inside the domain.

3. Dual quadrature identities

The so-called *duality method* developed in [B, PS] reduces problem (2.1a)–(2.1c) to a quadrature identity for harmonic or analytic functions.

Theorem 3.1. *The overdetermined boundary value problem* (2.1a)–(2.1c) *is solvable in* Ω *if and only if the quadrature identity*

$$\int_\Omega f(z)\,dA = \sum_{j=0}^{n} a_j \int_{\gamma_j} f(z)\,ds \tag{3.1}$$

holds for all f *analytic in* Ω *and continuous on* $\bar{\Omega}$.

Proof. Suppose the quadrature formula (3.1) holds for all functions f analytic in Ω and continuous on $\bar{\Omega}$. Let v denote the solution of the Dirichlet problem $\triangle v = -1$ with zero boundary conditions $v = 0$ on $\partial\Omega$. Using Green's identity we obtain

$$\sum_{j=0}^{n} a_j \int_{\gamma_j} f\,ds = \int_\Omega f\,dA = -\int_\Omega f \triangle v\,dA = \int_{\partial\Omega} f\frac{\partial v}{\partial n}\,ds.$$

Hence for every f analytic in Ω and continuous on $\bar{\Omega}$

$$\int_{\partial\Omega} f\left(\frac{\partial v}{\partial n} - \sum_{j=0}^{n} a_j\omega_j\right)ds = 0,$$

where $\omega_j = \omega(z, \gamma_j, \Omega)$ denotes the harmonic measure of γ_j with respect to Ω, i.e. $\omega_j(z) = \int_{\gamma_j} d\omega_z(w)$, see Section 4. Therefore, as in the proof of Theorem 3.2 [GK], also cf. [Kh3],

$$\frac{\partial v}{\partial n} - \sum_{j=0}^{n} a_j \omega_j$$

annihilates all analytic functions in Ω and hence

$$\frac{\partial v}{\partial n} - \sum_{j=0}^{n} a_j \omega_j = -\sum_{j=0}^{n} c_j \frac{\partial \omega_j}{\partial n} \qquad \text{on } \partial\Omega \tag{3.2}$$

for some constants c_j. By defining $u = v + \sum_{j=0}^{n} c_j \omega_j$ we obtain a solution of the overdetermined boundary value Problem 2.1.

To prove the converse, suppose that u is a solution of the overdetermined problem (2.1). Set $v = u - \sum_{j=0}^{n} c_j \omega_j$. Then

$$\triangle v = -1, \qquad v = 0 \text{ on } \partial\Omega.$$

Furthermore, the boundary conditions (2.1b) and (2.1c) imply that v satisfies (3.2) on $\partial\Omega$.

Let f be analytic in Ω and continuous on $\bar{\Omega}$. Then applying Green's identity (twice) we obtain:

$$\begin{aligned}
\sum_{j=0}^{n} a_j \int_{\gamma_j} f\, ds &= \int_{\partial\Omega} f \sum_{j=0}^{n} a_j \omega_j\, ds \\
&= \int_{\partial\Omega} f \left(\frac{\partial v}{\partial n} + \sum_{j=0}^{n} c_j \frac{\partial \omega_j}{\partial n} \right) ds \\
&= \sum_{j=0}^{n} c_j \int_{\partial\Omega} f \frac{\partial \omega_j}{\partial n}\, ds + \int_{\partial\Omega} \frac{\partial f}{\partial n} v\, ds + \int_{\Omega} (v \triangle f - f \triangle v)\, dA \\
&= \sum_{j=0}^{n} c_j \int_{\partial\Omega} f \frac{\partial \omega_j}{\partial n}\, ds + \int_{\Omega} f\, dA \\
&= \sum_{j=0}^{n} c_j \int_{\gamma_j} \omega_j \frac{\partial f}{\partial n}\, ds + \int_{\Omega} f\, dA \\
&= \int_{\Omega} f\, dA.
\end{aligned}$$

The last equality in this chain follows from the fact that f is a single-valued analytic function on Ω and therefore all of its boundary periods $\int_{\gamma_j} (\partial f / \partial n)\, ds$ are zero. ■

An equivalent form of Theorem 3.1 states that problem (2.1a)–(2.1c) is solvable if and only if the quadrature identity (3.1) holds true for all harmonic functions which are real parts of functions analytic in Ω and continuous on $\bar{\Omega}$.

The following theorem, whose proof is similar to that of Theorem 3.1 and hence is omitted, shows that if f in (3.1) is allowed to be an arbitrary harmonic function, the latter forces the boundary values (2.1b) to be the same.

Theorem 3.2. *The overdetermined boundary value problem* (2.1a)–(2.1c) *is solvable in* Ω *with constant boundary values* $c_k = 0$ *for all* $k = 0, \dots, n$ *if and only if the quadrature identity* (3.1) *holds for all* f *harmonic in* Ω *and continuous in* $\bar{\Omega}$.

If problem (2.1a)–(2.1c) is solvable in a domain Ω, then the boundary values c_j and a_j are related to geometric characteristics of the domain. Indeed, let

$$l_k = \int_{\gamma_k} ds, \qquad s_k = \int_{\Omega} \omega_k \, dA, \qquad p_{j,k} = \int_{\gamma_j} \frac{\partial \omega_k}{\partial n} \, ds.$$

Thus, l_k is the length of γ_k, s_k is the L^1-norm of the harmonic measure ω_k with respect to area, and $p_{j,k}$ is the period of the harmonic conjugate of ω_k around γ_j. Let $P(\Omega)$ and $A(\Omega)$ denote the perimeter of Ω and its area, respectively. If u solves Problem 2.1, then applying Green's identity we find:

$$A(\Omega) = -\int_{\Omega} \triangle u \, dA = \int_{\partial\Omega} \frac{\partial u}{\partial n} \, ds = \sum_{k=0}^{n} a_k l_k.$$

With this notation, we have:

$$\sum_{k=0}^{n} l_k = P(\Omega), \qquad \sum_{k=0}^{n} s_k = \sum_{k=0}^{n} a_k l_k = A(\Omega), \qquad \sum_{k=0}^{n} p_{j,k} = 0. \tag{3.3}$$

The last equation in (3.3) holds for all $0 \le j \le n$, see [N, p. 41]. Integrating (3.2) over γ_k, we obtain

$$\int_{\gamma_k} \frac{\partial v}{\partial n} \, ds - a_k l_k = \sum_{j=1}^{n} c_j p_{k,j}.$$

Applying Green's identity, we find

$$\int_{\gamma_k} \frac{\partial v}{\partial n} \, ds = -\int_{\Omega} (\omega_k \triangle v - v \triangle \omega_k) \, dA = \int_{\Omega} \omega_k \, dA = s_k.$$

One of the constants c_k may be prescribed; so let $c_0 = 0$. Then combining the previous results, we conclude that the parameters c_k and a_k satisfy a system of equations:

$$\sum_{j=1}^{n} c_j p_{k,j} = s_k - a_k l_k, \qquad k = 1, \dots, n. \tag{3.4}$$

It is well known that the determinant of the matrix of periods $p_{j,k}$, $j, k = 1, \dots, n$, is non-zero, see [N, p. 40]. Therefore for every choice of a_k, s_k, and l_k the system (3.4) has at most one solution $c_1, \dots, c_n$. Now system (3.4) contains n linearly independent equations. It follows from (3.3) that the $(n+1)$-th equation, namely, $\sum_{j=1}^{n} c_j p_{0,j} = s_0 - a_0 l_0$, depends on the others.

In the special case when all the normal derivatives a_k equal to the same constant λ, we have

$$\lambda = \frac{A(\Omega)}{P(\Omega)}.$$

4. BVP for analytic functions

There is another approach to the problems under consideration, see [Kh2], which requires minimal a priori regularity of the boundary. First we introduce necessary terminology.

Let Ω be a finitely connected domain on $\bar{\mathbb{C}}$ bounded by $n+1$ Jordan rectifiable curves γ_j, $0 \le j \le n$.

We say that a function f, analytic in the domain Ω, belongs to the (Smirnov) class $E^1 = E^1(\Omega)$ if f has non-tangential boundary values $f^* \in L^1(\partial\Omega)$ a.e. on $\partial\Omega$ and if the Cauchy integral formula holds for f,

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f^*(\zeta)}{\zeta - z} \, d\zeta, \qquad z \in \Omega$$

(cf. [Du1, Chapter 10] and [Kh1] for a basic account on Smirnov classes).

Let $\zeta = \psi(z)$ be a Koebe circular function mapping Ω conformally onto a domain G bounded by circles $C_k = \{\zeta : |\zeta - \zeta_k| = r_k\}$, where $C_k = \psi(\gamma_j)$, $0 \le k \le n$. In what follows we always assume that C_0 is the unit circle $\mathbb{T} = \{\zeta : |\zeta| = 1\}$ and C_1 is a circle $\{\zeta : |\zeta| = \rho\}$ for some $0 < \rho < 1$. Then for $2 \le k \le n$, C_k are boundaries of disjoint discs in the annulus $A(\rho, 1)$. Since $\partial\Omega$ is rectifiable, the inverse Koebe function $\varphi = \psi^{-1}$ is continuous on ∂G and φ' has boundary values a.e. on ∂G.

A domain Ω with a rectifiable boundary is called a *Smirnov domain* if for $\zeta \in G$,

$$\log|\varphi'(\zeta)| = \frac{1}{2\pi} \int_{\partial G} \log|\varphi'(w)| \frac{\partial g_G(w,\zeta)}{\partial n} \, |dw|, \tag{4.1}$$

that is, if φ' can be recovered from the boundary values of $|\varphi'|$ via the Poisson formula for G. Here $g_G(w,\zeta)$ denotes the positive Green function of G having singularity at ζ. The measure

$$d\omega_\zeta(w) = \frac{1}{2\pi} \frac{\partial g_G(w,\zeta)}{\partial n} \, |dw|$$

on ∂G is the harmonic measure at the point $\zeta \in G$.

It is worth noting that in the case of a Smirnov domain Ω the boundary values f^* of functions f in $E^1(\Omega)$ coincide with the space $L^1_a(\partial\Omega)$ — the L^1-closure of the set of rational functions with poles outside $\bar\Omega$, cf. [Du1, Chapter 10].

The following known form of the Reflection Principle will be used.

Lemma 4.1. *For $f \in E^1(\Omega)$ and some function h analytic on an arc $\sigma_j \subset C_j$, let $g(\zeta) = f(\varphi(\zeta))\varphi'(\zeta)h(\zeta)$, where φ is the inverse Koebe function. If* $\operatorname{Im} g(\zeta) = 0$ *a.e. on σ_j, then g can be continued analytically across σ_j.*

Proof. Since $\partial\Omega$ is rectifiable and since h is analytic on C_j, it follows that $g = (f \circ \varphi) \cdot \varphi' \cdot h$ is in the Hardy space $H^1(A)$, where A is a thin annulus in G with one boundary component being C_j. Now since $\operatorname{Im} g = 0$ on σ_j a.e., by a standard Reflection Principle g extends analytically across σ_j. ■

Theorem 4.2. *Let Ω be a Smirnov domain bounded by $n+1$ Jordan curves γ_j, $0 \le j \le n$. Then the quadrature identity* (3.1) *holds for all functions f analytic in Ω and continuous on $\bar\Omega$ if and only if there exists a function F analytic in Ω such that $F \in H^\infty(\Omega)$ and for $j = 0, \ldots, n$,*

$$F(z) = \bar z + 2ia_j\dot{\bar z} \qquad \text{a.e. on } \gamma_j. \tag{4.2}$$

Here and below $\dot z = dz/ds$ stands for the unit tangent in the positive direction on $\partial\Omega$ and $\dot{\bar z}$ denote the conjugate of $\dot z$.

Proof. Let f be analytic in Ω and continuous on $\bar\Omega$. Applying complex Green's identity and (3.1) we obtain

$$\int_{\partial\Omega} \bar z f\, dz = -2i\int_\Omega f\, dA = -2i\sum_{j=0}^n a_j \int_{\gamma_j} f\, ds = -2i\sum_{j=0}^n a_j \int_{\gamma_j} f\dot{\bar z}\, dz. \tag{4.3}$$

Therefore,

$$\int_{\partial\Omega} f\left(\bar z + 2i\sum_{j=0}^n a_j\omega_j\dot{\bar z}\right) dz = 0. \tag{4.4}$$

Now by a variant of F. and M. Riesz Theorem for finitely connected domains proved by G. Tumarkin and S. Khavinson, see [TKh] or [Kh4], there exists a function F in $L^1_a(\partial\Omega)$ such that $F(z) = \bar z + 2i\sum_{j=0}^n a_j\omega_j\dot{\bar z}$ a.e. on $\partial\Omega$. Since Ω is a Smirnov domain and $F|_\Gamma \in L^\infty(\Gamma)$ we conclude that $F \in H^\infty(\Omega)$.

Since Cauchy's Theorem holds for the class E^1, the converse is also true. Indeed, if F in $E^1(\Omega)$ satisfies (4.2), then (4.4) holds for all f analytic in Ω and continuous on $\bar\Omega$. Now (4.3) implies the quadrature identity (3.1). ■

Equation (4.2) first appeared in [Kh2] in connection with the so-called *analytic content* of Ω defined by

$$\lambda = \lambda(\Omega) = \inf \|\bar z - f(z)\|_\infty,$$

were the infimum is taken over all functions f analytic in Ω and continuous on $\bar{\Omega}$. So, $\lambda(\Omega)$ characterizes how well $\bar{z}$ can be approximated in L^∞-norm on Ω by analytic functions. The analytic content admits nice upper and lower bounds:

$$2A(\Omega)/P(\Omega) \le \lambda(\Omega) \le (A(\Omega)/\pi)^{1/2}. \tag{4.5}$$

The upper bound in (4.5) was established by Alexander [Al], with the equality holding only for discs. The lower bound was found in [Kh2]. It is also sharp, equality in the left inequality in (4.5) holds for discs and circular annuli. The question concerning a complete list of extremal domains for this inequality first raised in [Kh2] still remains open. It was conjectured in [Kh2] that

$$\lambda(\Omega) = 2A(\Omega)/P(\Omega)$$

if and only if Ω is a disc of radius λ or an annulus, also see [Kh3].

The following theorem, which is an easy extension of the results in [Kh2], where $\partial\Omega$ was assumed to be real analytic, links the question on the extremal domains for the analytic content with equation (3.1) and therefore via quadrature identity (3.2) with the question on solvability of the overdetermined problem (2.1a)–(2.1c).

Theorem 4.3. *Let Ω be a Smirnov domain on $\mathbb{C}$, then*

$$\lambda(\Omega) = \frac{2A(\Omega)}{P(\Omega)}$$

if and only if there is F in $H^\infty(\Omega)$ such that

$$F(z) = \bar{z} - i\lambda\dot{\bar{z}} \qquad a.e.\ on\ \partial\Omega.$$

5. The mapping function and quadratic differentials

Equation (4.2) suggests a slightly more general question.

Problem 5.1. Let $p_j, \tau_j \in \mathbb{R}$ and $c_j \in \mathbb{C}$, $0 \le j \le n$, be given numbers, such that ${p_j}^2 + {\tau_j}^2 \ne 0$. Find all Smirnov domains $\Omega \subset \overline{\mathbb{C}}$ of connectivity $n+1$ such that Ω is bounded by $n+1$ curves γ_j and there exists a function F analytic (or meromorphic with prescribed poles and orders) in Ω and $F \in E^1$ near the boundary, satisfying the boundary condition

$$F(z) = p_j\bar{z} + 2i\tau_j\dot{\bar{z}} + c_j \qquad \text{for a.e. } z \in \gamma_j. \tag{5.1}$$

We shall also consider meromorphic functions in some special yet interesting cases in the next section. Here we will assume that F is an analytic function, which makes the problem very closely related to the original overdetermined problem, see Theorem 4.2. The constants in (5.1) cannot be arbitrary. Suppose Ω is bounded and γ_0 is the outer boundary. Let $P(\gamma_j)$ and $A(\gamma_j)$ denote the length of γ_j and the area bounded by γ_j, respectively. Integrating (5.1) along $\partial\Omega$ we obtain

$$-p_0 A(\gamma_0) + \sum_{j=1}^{n} p_j A(\gamma_j) = \sum_{j=0}^{n} \tau_j P(\gamma_j).$$

With obvious modifications, a similar identity holds if Ω is unbounded.

Our goal is to solve Problem 5.1 in certain cases. From the geometric point of view our discussion is reduced to classifying domains which allow analytic quadratic differentials that are real on the boundary.

Suppose $\partial\Omega \in \mathcal{C}^2$. Differentiating (5.1) along γ_j with respect to arc length we obtain

$$F'(z)\, dz^2 = (p_j + 2\tau_j k_j(z))\, (ds)^2, \tag{5.2}$$

where k_j is the signed curvature of γ_j at z. This shows that the quadratic differential $F'(z)\, dz^2$ is real on $\partial\Omega$. We refer the reader to [J] and [G] for the basic account on quadratic differentials.

Changing variables via $z = \varphi(\zeta)$, where φ is the inverse of the Koebe circular function ψ defined in Section 4, we obtain the quadratic differential

$$Q(\zeta)\, d\zeta^2 = F'(\varphi(\zeta))\varphi'^2(\zeta)\, d\zeta^2, \tag{5.3}$$

which is analytic in the circular domain G and real on ∂G. The space T_G of all such quadratic differentials on G has real dimension $3n-3$ if $n \geq 2$; it has real dimension 0 if $n = 0$ and real dimension 1 if $n = 1$, see [G, Theorem 6, p. 27] (the second exceptional case however is missing there). If $n \geq 1$, every $Q(\zeta)\, d\zeta^2$ in T_G has $2n-2$ zeros on G counting multiplicity (boundary zeros are counted with half of their multiplicities), cf. [J, Lemma 3.2].

This shows, in particular, that $Q(\zeta)\, d\zeta^2$ is identically zero if G is the unit disc ($n = 0$) and that

$$Q(\zeta)\, d\zeta^2 = \frac{C}{\zeta^2}\, d\zeta^2$$

with some real constant C if G is an annulus ($n = 1$).

The signed curvature at $z \in \gamma_j$ can be expressed in terms of the inverse Koebe function φ as follows, cf. [Du2, p. 74]:

$$k_j(z) = \frac{1}{r_j|\varphi'(\zeta)|} \operatorname{Re}\left(1 + \frac{(\zeta - \zeta_j)\varphi''(\zeta)}{\varphi'(\zeta)}\right). \tag{5.4}$$

Combining these results yields the following necessary condition for the solvability of problem (2.1a)–(2.1c).

Theorem 5.2. *Let Ω be a domain of connectivity $n+1$ in $\mathbb{C}$ with a $\mathcal{C}^2$-boundary. If the problem* (5.1)*, and therefore the overdetermined problem* (2.1a)–(2.1c)*, is* *solvable with a function F analytic in Ω then the inverse Koebe function φ from the circular domain G onto Ω satisfies the equation*

$$(5.5)\qquad |\varphi'(\zeta)|^2 \left| p_j + \frac{2\tau_j}{r_j|\varphi'(\zeta)|}\operatorname{Re}\left(1+\frac{(\zeta-\zeta_j)\varphi''(\zeta)}{\varphi'(\zeta)}\right)\right| = |Q(\zeta)|, \qquad \zeta\in C_j,$$

with some $Q(\zeta)$ analytic in Ω and such that the quadratic differential $Q(\zeta)\,d\zeta^2$ is real on ∂G.

Note nevertheless that even using Theorem 5.2 we still have not been able to solve Problem 5.1 for doubly-connected Ω under the assumptions of analyticity of F. When G is an annulus and $p_0 = p_1 = 1$ equation (5.5) becomes

$$|\varphi'(\zeta)|^2 + \frac{2\tau_j|\varphi'(\zeta)|}{r_j}\operatorname{Re}\left(1+\frac{\zeta\varphi''(\zeta)}{\varphi'(\zeta)}\right) = Cr_j^{-2} \qquad \text{if } |\zeta| = r_j,\ j=0,1.$$

This is the simplest case when the problem still remains unsolved.

Three distinguished special cases of Problem 5.1 are the following:

$$(5.6)\qquad \text{(i) } F(z) = p_j\bar z \qquad \text{(ii) } F(z) = 2i\tau_j\dot{\bar z} \qquad \text{(iii) } F(z) = p_j\bar z + c_j$$

for a.e. $z\in\gamma_j$, $j=0,\dots,n$. It is worth noting that the case (iii) with $p_j = 0$ for all j was studied by D. Aharonov and H. Shapiro in connection with quadrature identities for analytic functions, see [Sh] and references therein.

First we deal with the case (i) of (5.6) and Ω a bounded multiply-connected Smirnov domain. Here we have the following result.

Theorem 5.3. *If the boundary conditions are*

$$F(z) = p_j\bar z \qquad \text{a.e. on } \gamma_j, \quad j=0,\dots,n,$$

for given constants $p_j\in\mathbb{R}$ with $p_0 > 0$ then the existence of $F\in E^1(\Omega)$ with the above boundary values is possible if and only if the domain is the annulus $A = \{z : r < |z| < R\}$ such that $(R/r)^2 = p_1/p_0$.

Proof. The function $zF(z)$ is in $E^1(\Omega)\cap L^\infty(\partial\Omega)$ and since Ω is a Smirnov domain, $zF(z)\in H^\infty(\Omega)$. Since it is real valued on $\partial\Omega$ it is therefore a constant [Du1]. Hence, $F(z) = \text{const}/z$ and on each γ_j, $j=0,\dots,n$ we have $|z| = \text{const}/p_j$, so Ω is either a disc or an annulus centered at the origin. Since $F(z) = \text{const}/z$ is not analytic in the disc we are left with the annulus $A = \{z : r < |z| < R\}$. Now the boundary conditions imply that $(R/r)^2 = p_1/p_0$. ■

Remark 5.4. In general, even in Smirnov domains there exist non-constant E^1 functions (not bounded!) with real boundary values (cf. [Kh1]).

Next let us observe that problem (5.6) (ii) can be reduced to problem (5.6) (iii). This can be done by defining the function

$$g(z) = \int F^2(z)\,dz.$$

It is easy to see from (ii) that g is a single-valued analytic function in Ω and there are constants c_j such that $g(z) = -4\tau_j^2\bar{z} + c_j$ a.e. on γ_j, $j = 0, \dots, n$.

We should emphasize that problem (ii) of (5.6) is related to a special case of the boundary value problem, for which there is an enormous literature. We mention only a classical monograph of I. N. Vekua [V]. It follows from the results of Section V.5 of this book that for some real constants $c_1, \dots, c_n$

$$F(z) = c_1 \frac{\partial \omega_1(z)}{\partial z} + \cdots + c_n \frac{\partial \omega_n(z)}{\partial z}, \tag{5.7}$$

where ω_j denotes the harmonic measure of γ_j with respect to Ω and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Since the boundary curves are level sets of the harmonic measures we have

$$\frac{\partial \omega_k}{\partial z} = -i\dot{\bar{z}}\frac{\partial \omega_k}{\partial n}$$

a.e. on the boundary of Ω. Taking into account (ii) of (5.6), we find

$$F(z) = -i\sum_{k=1}^{n} c_k \dot{\bar{z}} \frac{\partial \omega_k}{\partial n} = i\sum_{j=0}^{n} \tau_j \dot{\bar{z}} \omega_j, \qquad \text{a.e. on } \partial\Omega. \tag{5.8}$$

Now (5.7) and (5.8) imply that the function

$$u = -\sum_{k=1}^{n} c_k \omega_k$$

satisfies the following overdetermined boundary value problem:

$$\begin{aligned} \triangle u &= 0 && \text{in } \Omega \\ u &= -c_j && \text{on } \gamma_j \\ \frac{\partial u}{\partial n} &= \tau_j && \text{on } \gamma_j \end{aligned}$$

where $j = 0, \dots, n$ and $c_0 = 0$. At this point we arrive at the overdetermined problem for harmonic functions in multiply connected planar domains. To our knowledge even for this simplest differential equation the problem is open.

Equation (5.7) expresses the function F in terms of harmonic measures of boundary components of an unknown domain Ω. Next we will find the Koebe function φ from the canonical circular domain G onto Ω defined in Section 4. We will consider problem (5.6) (ii) above, working under the assumption that Ω is a Smirnov domain. Now we have

$$-F^2(z)\, dz^2 = 4{\tau_j}^2\, (ds)^2 > 0 \qquad \text{a.e. on } \gamma_j. \tag{5.9}$$

Consider the quadratic differential

$$Q(\zeta)\, d\zeta^2 = -\left(F(\varphi(\zeta))\varphi'(\zeta)\right)^2\, d\zeta^2, \tag{5.10}$$

which is analytic in G. Since $\partial\Omega$ is rectifiable, the function

$$g(\zeta) = iF(\varphi(\zeta))\varphi'(\zeta)(\zeta - \zeta_k)$$

is defined a.e. on C_k, $0 \le k \le n$, and has real boundary values on C_k in view of (5.9). Invoking Lemma 4.1 we see that $Q(\zeta)\,d\zeta^2$ can be extended to an analytic quadratic differential on G which is positive on ∂G, cf. [J].

For $1 \le \nu \le n-1$, let $g_G(\zeta, \zeta_\nu)$ denote the Green function of G with a pole at ζ_ν, where ζ_ν, $1 \le \nu \le n-1$, denote zeros (each one of even order) of the quadratic differential (5.10). It follows from (4.1), (5.9), and (5.10) that

$$\log|\varphi'(\zeta)|^2 = -\log 4 + \log|Q(\zeta)| - \sum_{j=0}^{n} \omega_j(\zeta)\log {\tau_j}^2 + 2\sum_{\nu=1}^{n-1} g_G(\zeta, \zeta_\nu).$$

This implies

$$\varphi'(\zeta) = \frac{1}{2} Q^{1/2}(\zeta)\exp\left(\sum_{\nu=1}^{n-1} \tilde{g}_G(\zeta, \zeta_\nu) - \sum_{j=0}^{n} \tilde{\omega}_j(\zeta)\log|\tau_j|\right). \tag{5.11}$$

Here $\tilde{g}_G = g_G + ig_G^*$ and $\tilde{\omega}_j = \omega_j + i\omega_j^*$, where g_g^* and ω_j^* denote harmonic conjugates of g_G and ω_j, respectively. Integrating (5.11) we obtain

$$\varphi(\zeta) = \frac{1}{2}\int Q^{1/2}(\zeta)\exp\left(\sum_{\nu=1}^{n-1} \tilde{g}_G(\zeta, \zeta_\nu) - \sum_{j=0}^{n} \tilde{\omega}_j(\zeta)\log|\tau_j|\right) d\zeta. \tag{5.12}$$

To be single-valued the function $\varphi'(\zeta)$ must have zero periods around all contours C_j, $j = 0, \dots, n$. Hence, for $j = 1, \dots, n$,

$$\sum_{\nu=1}^{n-1} \int_{C_j} \tilde{g}'_G(\zeta, \zeta_\nu)\,d\zeta - \sum_{k=0}^{n} \log|\tau_k| \int_{C_j} \tilde{\omega}'_k(\zeta)\,d\zeta = 0 \mod 2\pi i \tag{5.13}$$

and a similar equation for $j = 0$ follows from equations (5.13). A standard calculation, cf. [N, Section I.10], reveals that (5.13) reduces to

$$\sum_{\nu=1}^{n-1} \omega_j(\zeta_\nu) - \sum_{k=0}^{n} p_{jk}\log|\tau_k| = 0 \mod 1, \qquad j = 1, \dots, n.$$

As before, p_{jk} denotes the period of ω_k with respect to γ_j, i.e.

$$p_{jk} = -\frac{1}{2\pi}\int_{\gamma_j} \frac{\partial\omega_k}{\partial n}\,ds.$$

Since φ is single-valued we have additional n constraints, for $1 \le j \le n$,

$$\int_{\gamma_j} Q^{1/2}(\zeta)\exp\left(\sum_{\nu=1}^{n-1} \tilde{g}_G(\zeta, \zeta_\nu) - \sum_{j=0}^{n} \tilde{\omega}_j(\zeta)\log|\tau_j|\right) d\zeta = 0.$$

Finally we have one more crucial restriction — the function φ should be univalent on G.

It is unlikely that all these conditions can be satisfied in connectivity greater or equal to 3. However, for doubly-connected domains they are consistent and the above argument leads to a complete solution of problem (5.6) (ii).

Theorem 5.5. *Let Ω be a doubly-connected bounded Smirnov domain. Then (a slightly modified) problem* (5.6) (ii) *has a solution, i.e. there exists a function $F \in E^1(\Omega)$ such that*

$$F(z) = \tau_j \dot{\bar{z}} \qquad \text{a.e. on } \gamma_j, \quad j = 0, 1,$$

for given constants $\tau_j \in \mathbb{R}$ with $\tau_0 > 0$ if and only if after a translation Ω becomes an annulus $A = \{z : r < |z| < R\}$ such that $R/r = \tau_1/\tau_0$ and

$$F(z) = \frac{iR\tau_0}{z}.$$

Proof. Since Ω is doubly-connected, G is an annulus $\{z : \rho < |z| < 1\}$ for some $0 < \rho < 1$. In this case the quadratic differential (5.10) has the form $Q(\zeta)\, d\zeta^2 = -C^2/\zeta^2\, d\zeta^2$ with some $C > 0$ and therefore it has no zeros in Ω. Since $\omega_1(\zeta) = 1 - \omega_0(\zeta) = \log|\zeta|/\log\rho$, equation(5.12) becomes

$$\varphi(\zeta) = \frac{iC}{2} \int \zeta^{\log(\tau_0/(\rho|\tau_1|))/\log\rho}\, d\zeta,$$

which is univalent and maps $\mathbb{T}$ onto an outer boundary component γ_0 if and only if $\tau_0/|\tau_1| = \rho$. Then

$$\varphi(\zeta) = c_1\zeta + c_2$$

for some constants $c_1 \neq 0$ and c_2 and thus φ is a linear map. An easy calculation then shows that Ω is an annulus $\{z : r < |z - z_0| < R\}$ with some $z_0 \in \mathbb{C}$ such that $R/r = \tau_1/\tau_0$ and F is of the required form. ■

We conclude this section by quoting some results of [Ga] and [Kh2], which we consider important for understanding equations (4.2) and (5.1). Assuming analyticity of the boundary, let $S_j(z)$ denote the Schwarz function of γ_j, that is a function analytic in a neighborhood of γ_j such that

$$S_j(z) = \bar{z} \qquad \text{on } \gamma_j.$$

Now equation (5.1) can be written as

$$(5.14) \qquad F(z) = p_j S_j(z) + 2i\tau_j S_j'(z)\dot{z} = p_j S_j(z) + 2i\tau_j \sqrt{S_j'(z)}, \qquad z \in \gamma_j,$$

where the branch of the square root is chosen such that

$$\sqrt{S_j'(z)} = \dot{\bar{z}} \qquad \text{on } \gamma_j.$$

Let

$$u_j = \sqrt{S_j'(z)}.$$

Then (5.14) becomes the Riccati equation for u_j:

$$(5.15) \qquad p_j u_j^2 + i\tau_j u_j' = g, \qquad \text{where } g = F'.$$

A standard substitution $u_j = (\tau_j/p_j)iv'/v$ transforms (5.15) into a homogeneous linear equation

$$v'' + \frac{g}{p_j{\tau_j}^2}v = 0. \tag{5.16}$$

As is well known (cf. [L]) solutions of (5.16) are locally analytic in Ω. Hence u_j and S_j are meromorphic in Ω. Furthermore, u_j and S_j can have only simple poles and all the residues of S_j are equal to $(\tau_j/p_j)^2$.

The above consideration leads to another quadratic differential related to F. Let $\Phi_j(z) = -(F(z) - p_jS_j(z))^2$. Then

$$\Phi_j(z)\,dz^2 = -(F(z) - p_jS_j(z))^2\,dz^2 = 4\tau_j^2\,(ds)^2 > 0 \qquad \text{on } \gamma_j.$$

Thus, $\Phi_j(z)\,dz^2$ is a meromorphic quadratic differential on Ω (multiple-valued in general) which is single-valued and positive on γ_j (but not necessarily on γ_k if $k \neq j$). In addition, all the poles of $\Phi_j(z)\,dz^2$ are of second order such that $\Phi_j(z)\,dz^2$ has the circular structure of trajectories near the poles. Thus, it seems to support once more the conjecture that annuli and discs are the only domains in which (5.1) has a non-trivial solution.

6. The shape of droplets

As we already mentioned discs are the only simply connected domains, for which problem (2.1a)–(2.1c) is solvable or, equivalently, equation (4.2) is solvable in the class of analytic functions. In this section we consider Problem 5.1 in simply connected domains but allow F to be meromorphic rather than analytic.

Let z_ν, $0 \le \nu \le n$, be distinct distinguished points on $\mathbb{C}$, n_ν be positive integers, p and τ be real numbers at least one of which is non-zero, and let $\dot z$ be the unit tangent vector as in (4.2).

Problem 6.1. Find all simply connected Smirnov domains $\Omega \subset \overline{\mathbb{C}}$ such that $z_\nu \in \Omega$, $0 \le \nu \le n$ and there exists a function F analytic in Ω, except poles of order n_ν at the points z_ν, and $F \in E^1$ near the boundary, satisfying the boundary condition

$$F(z) = p\bar z + i\tau\bar{\dot z}, \qquad \text{a.e. on } \partial\Omega. \tag{6.1}$$

When Ω is an unbounded domain we assume that $z_0 = \infty$ is among the distinguished points and that F is regular or meromorphic at z_0; so $n_0 \ge 0$.

Let us observe that when Ω is a bounded domain and $p = 0$, Problem 6.1 is solved, see [Sh, p. 45] and [Gu]. Our motivation to study Problem 6.1 is its relation with overdetermined problem (2.1a)-(2.1c), quadrature identities, and approximation theory. Yet the same boundary condition (6.1) describes the shape of a planar droplet of a perfectly conducting fluid bounded by a Jordan curve γ that separates it from vacuum with p representing the fluid pressure on the free boundary and τ equalling the surface tension, see [Ga] and [GMV].

More precisely, let Ω be the unbounded component of $\bar{\mathbb{C}} \setminus \gamma$. Let E be an electrostatic field in Ω with an analytic potential ζ given by

$$\zeta = g(z) = E_\infty z + \alpha_0 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \cdots ,$$

where E_∞ stands for the electric field at $z = \infty$. Rotating the z-plane if necessary we may assume that $E_\infty > 0$. Then the field is uniform and horizontal at infinity, and g maps Ω conformally onto the exterior of a vertical slit. Translating the ζ-plane we may assume that the center of this vertical slit is at the origin. Changing the scale in the ζ-plane we can assume $E_\infty = 1$. Finally, using translation in z-plane, we can assume $\alpha_0 = 0$. Thus in what follows we will work with an analytic potential having the form

$$\zeta = g(z) = z + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \cdots . \tag{6.2}$$

An electrical force E and the fluid pressure p act outward on the interface γ between the droplet and the vacuum, and are balanced by an inward force $\tau\kappa$ due to surface tension, where τ denotes a constant and κ is the curvature of γ. Assuming in addition that the droplet is in equilibrium, the latter conditions imply (cf. [Ga]) that the free boundary of the droplet must assume a shape such that the boundary condition (6.1) is satisfied for the function

$$F(z) = \int (g'(z))^2\, dz = z + \frac{2\alpha_1}{z} + \cdots , \tag{6.3}$$

which is analytic in Ω except for a simple pole at ∞. From now on we will refer to F as *the integrated analytic potential.*

Motivated by the above consideration, any simply connected domain D bounded by a Jordan closed curve γ such that the boundary value problem (6.1) with some $p, \tau \geq 0$ admits a solution F analytic in $\Omega = \bar{\mathbb{C}} \setminus \bar{D}$ except for a simple pole at ∞, where it has expansion (6.3) will be called *a mathematical droplet.* It is clear that every physical droplet is at the same time a mathematical droplet. For the converse to be true, the analytic potential $g(z)$ recovered from F and therefore its derivative $g'(z) = \sqrt{F'(z)}$ must be single-valued functions in Ω. We will refer to the parameters p and τ as *the pressure* and *surface tension* of a droplet. At present there are no available methods to find all possible mathematical droplets. Nevertheless, in Section 7 we give a one-parameter family of mathematical droplets, i.e. domains Ω for which Problem 6.1 is solvable with non-zero pressure $p > 0$ and surface tension $\tau > 0$. To our knowledge, no examples of such domains were known previously.

If the surface tension τ is very large then the forces due to the fluid pressure become negligible, cf. [Ga]. Therefore equation (6.1) can be simplified to

$$F(z) = i\tau\dot{\bar{z}}. \tag{6.4}$$

Note that the last equation characterizes also the boundary of some small bubbles in a fluid flow, cf. [Ga, M1]. So every closed Jordan curve γ for which the problem (6.3) and (6.4) can be solved, with g given by (6.2), describes a shape of an electrified droplet or a small bubble in a fluid flow.

In this section we deal with the following question: describe the shapes of mathematical droplets corresponding to the integrated analytic potential (6.3), which satisfies the boundary condition (6.4) for a given value $\tau > 0$. To characterize the free boundary of a droplet we will use a slightly relaxed form of equation (6.4), namely, we assume that (6.4) is satisfied a.e. on γ, which we assume to be a Jordan curve such that the unbounded component Ω is a Smirnov domain, and $F(z) \in E^1$ near γ. Thus, a priori, we allow non-smooth shapes for the droplets.

Let $z = \varphi(w)$ be a conformal mapping from the unit disc $\mathbb{D}$ onto Ω normalized by conditions $\varphi(0) = \infty$, $\mathrm{Res}[\varphi, 0] > 0$. If γ surrounds a physical droplet then the function $\zeta = g(\varphi(w))$ maps $\mathbb{D}$ onto a plane slit along a vertical segment. Hence, $g(\varphi(w)) = R\,(w^{-1} - w)$ with some $R > 0$ and therefore

$$\varphi(w) = R\left(\frac{1}{w} + a_1 w + a_2 w^2 + \cdots\right), \tag{6.5}$$

where

$$a_1 = -\left(1 + \frac{\alpha_1}{R^2}\right) \tag{6.6}$$

and α_1 is the coefficient of (6.2). We will see in the proof of Theorem 6.2 that the same normalization (that is with $a_0 = 0$) holds for mathematical droplets as well. The leading coefficient $R = R(\Omega)$, responsible for the scale of Ω, is a significant characteristic of Ω — the so-called *outer radius of* Ω. It is well known that the outer radius coincides with the logarithmic capacity of $\gamma = \partial\Omega$, i.e. $R(\Omega) = \mathrm{cap}\,\gamma$.

By the Area Theorem (cf. [Du2]), $|a_1| \le 1$. Moreover, in the case of equality $|a_1| = 1$ the boundary $\gamma = \partial\Omega$ is a straight line segment and therefore γ cannot enclose a droplet. In our next theorem we find a range of the logarithmic capacity $R(\Omega)$ and the surface tension τ for which the boundary value problem (6.4) has a solution and describe the shape of the corresponding mathematical droplets γ.

Theorem 6.2. *Suppose γ is a Jordan curve whose exterior Ω is a Smirnov domain with logarithmic capacity $R = R(\Omega)$, $R > 0$. Suppose further that the boundary condition* (6.4) *with the surface tension τ, such that,*

$$\tau > \frac{3 + 2\sqrt{3}}{3} R, \tag{6.7}$$

holds a.e. on γ for a function $F \in E^1$, which is analytic in $\Omega \setminus \{\infty\}$ and has an expansion of the form (6.3) *at $z = \infty$. Then $\gamma = \{z \in \mathbb{C} : (z/R) \in \gamma_\lambda\}$, where $\lambda = R/\tau$ and γ_λ is the image of the unit circle under the conformal mapping*

$$z = \varphi_\lambda(w) = \frac{1}{w} - 2\lambda w - \frac{\lambda^2}{3} w^3 \tag{6.8}$$

from the unit disc onto $\Omega_\lambda = \varphi_\lambda(\mathbb{D})$. *The integrated analytic potential* (6.3) *has the form*

$$F(z) = \tau \frac{w^2+\lambda}{w(1+\lambda w^2)}, \qquad \text{where } w = \varphi_\lambda^{-1}(z/R). \tag{6.9}$$

For any given logarithmic capacity $R = R(\Omega) > 0$ *and surface tension* τ *such that* $0 \le \tau \le R(3+2\sqrt{3})/3$, *the problem* (6.4) *has no solutions in the class of mathematical droplets with a Jordan boundary.*

Proof. Let

$$z = \varphi(w) = R\left(\frac{1}{w} + a_0 + a_1 w + \cdots\right) \tag{6.10}$$

be a conformal mapping from $\mathbb{D}$ onto Ω. Since the curve γ is rectifiable the function $h(w) = F(\varphi(w))\varphi'(w)w$ is defined a.e. on $\mathbb{T}$. Equation (6.4) shows that $\operatorname{Im} h(w) = 0$ a.e. on $\mathbb{T}$. Now by Lemma 4.1, $h(w)$ can be continued analytically across $\mathbb{T}$. This implies that the quadratic differential

$$Q(w)\,dw^2 = -F^2(\varphi(w))(\varphi'(w))^2\,dw^2 \tag{6.11}$$

is analytic on $\mathbb{D}\setminus\{0\}$ and positive on $\mathbb{T}$. Therefore, $Q(w)\,dw^2$ can be continued to a meromorphic quadratic differential on the whole Riemann sphere by the reflection with respect to the unit circle. Since $F(z)$ has a simple pole at $z=\infty$, $Q(w)\,dw^2$ has a pole of order six at $w=0$. Furthermore, by reflection, the point at infinity $\zeta=\infty$ is also a pole of order six. Since for a meromorphic quadratic differential on the Riemann sphere the difference between the number of poles and the number of zeros counted with their multiplicities equals four cf. [J], it follows that $Q(w)\,dw^2$ has eight zeros in the plane and the set of zeros is symmetric with respect to the unit circle. From (6.11) it is evident that each zero is of even multiplicity. In summary, there is a constant $C>0$ such that $Q(w)\,dw^2$ has the form

$$Q(w)\,dw^2 = -C^2 w^{-6}(w-A)^2(w-B)^2(1-\bar{A}w)^2(1-\bar{B}w)^2\,dw^2, \tag{6.12}$$

where $|A|<1$, $|B|<1$. Now (6.4), (6.11), and (6.12) yield

$$|w^2\varphi'(w)| = \frac{C}{\tau}|1-\bar{A}w|^2|1-\bar{B}w|^2 \qquad \text{a.e. on } |w|=1. \tag{6.13}$$

Since Ω is a Smirnov domain and $\varphi'(z)\ne 0$ in $\mathbb{D}$, equation (6.13) implies that for $w\in\mathbb{D}$ we have

$$\varphi'(w) = C_1 w^{-2}(1-\bar{A}w)^2(1-\bar{B}w)^2, \qquad \text{where } |C_1| = \frac{C}{\tau}. \tag{6.14}$$

Hence

$$\varphi(w) = C_1\Big(- w^{-1} - 2(\bar{A}+\bar{B})\log w + (\bar{A}^2+\bar{B}^2+4\bar{A}\bar{B})w - \bar{A}\bar{B}(\bar{A}+\bar{B})w^2 + \frac{\bar{A}^2\bar{B}^2}{3}w^3 + c_0\Big)$$

with some $c_0 \in \mathbb{C}$. Since φ is single-valued, we must have $\bar{A}+\bar{B} = 0$. In addition, it follows from the normalization (6.10) that $C_1 = -R$. Therefore, denoting $\lambda = -\bar{A}^2$, we obtain

$$\varphi(w) = R\left(\frac{1}{w} - 2\lambda w - \frac{\lambda^2}{3}w^3 + c_0\right). \tag{6.15}$$

Now from (6.11), (6.12), (6.14), and (6.15) we find

$$F(z) = \tau\frac{w^2+\bar{\lambda}}{w(1+\lambda w^2)} = \tau\bar{\lambda}\left(\frac{1}{w} + \mathcal{O}(w)\right) = \frac{\tau\bar{\lambda}}{R}\left(z - Rc_0 + \mathcal{O}(w)\right).$$

Comparing this with the expansion (6.3), we find $\lambda = R/\tau$ and $c_0 = 0$. This proves (6.9) and (6.8).

The function φ must also be univalent on $\mathbb{D}$. This gives a constraint on the range of λ. Indeed, the function (6.15) is univalent if and only if the function φ_λ defined by (6.8) with $\lambda = R/\tau$ is univalent. Let $w = e^{it}$. Then the curve γ_λ is given by $\gamma_\lambda = \{x(t) + iy(t) : -\pi \le t \le \pi\}$ with

$$x(t) = (2\lambda+1)\cos t - \frac{\lambda^2}{3}\cos 3t, \qquad y(t) = (2\lambda-1)\sin t - \frac{\lambda^2}{3}\sin 3t,$$

where $-\pi \le t \le \pi$. Since $x(t) = x(-t)$ and $y(t) = -y(-t)$, we see that γ possesses symmetry with respect to the coordinate axes. Since γ_λ is a Jordan curve which is symmetric with respect to the real axis, we necessarily have $y(t) \neq 0$ for $0 < t < \pi$. This yields

$$(2\lambda-1)\sin t - \frac{\lambda^2}{3}\sin 3t \neq 0, \qquad 0 < t < \pi,$$

which is equivalent to

$$\sin^2 t \neq \frac{3}{4}\frac{\lambda^2 - 2\lambda + 1}{\lambda^2}, \quad 0 < t < \pi.$$

Therefore, for φ_λ to be univalent we must have $3(\lambda^2 - 2\lambda + 1) > 4\lambda^2$, or

$$0 \le \lambda < 2\sqrt{3} - 3. \tag{6.16}$$

For λ in this interval we have

$$x'(t) = -(2\lambda+1)\sin t + \lambda^2\sin 3t = \sin t(-1 - 2\lambda + 3\lambda^2 - 4\lambda\sin^2 t) < 0,$$

since $-1 - 2\lambda + 3\lambda^2 < 0$ for $0 \le \lambda < 2\sqrt{3} - 3$. Thus, $\operatorname{Re}\varphi(e^{it})$ is decreasing on $0 < t < \pi$ and therefore φ_λ is univalent on $\mathbb{T}$. This shows that the function (6.15) maps $\mathbb{D}$ univalently onto a Jordan domain if and only if λ satisfies (6.16). Hence the problem under consideration is solvable in the class of Jordan droplets if and only if R and τ satisfy (6.7). The proof is now complete. ■

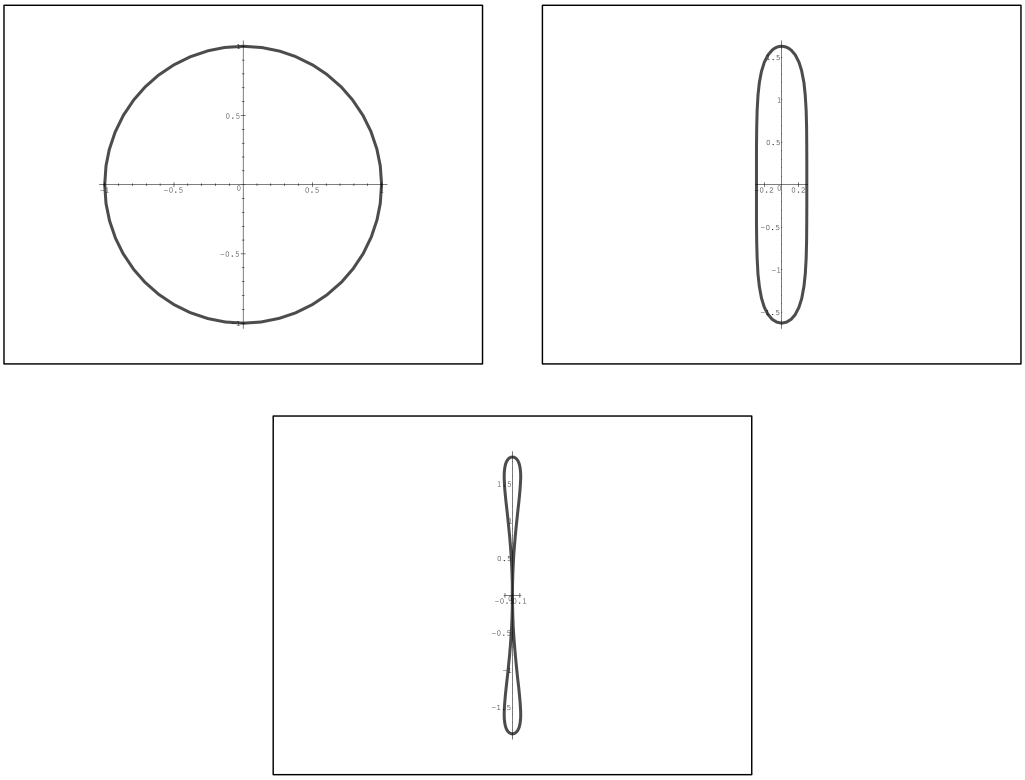

FIGURE 1. Droplets for $\tau = \infty$, $\tau = 3$, and $\tau = (3 + 2\sqrt{3})/3$.

Figure 1 displays the evolution of the curves γ_λ with surface tension $\tau = 1/\lambda$ decreasing from $\tau = \infty$ to $\tau = (3 + 2\sqrt{3})/3$. In particular, this reflects the following qualitative features of the shape change of mathematical droplets, which also agrees with physical intuition.

1. All the curves γ_λ possess double symmetry — we cannot observe droplets in equilibrium with integrated analytic potential (6.3) which are non-symmetric.
2. If the surface tension dominates, i.e. $\tau \gg 1(= R)$, then the mathematical droplets become almost perfect circles.
3. If the surface tension decreases to the critical value $\tau_0 = (3 + 2\sqrt{3})/3$, the free boundary of the mathematical droplet will be destroyed first at a single "weakest" point.
4. The curves γ_λ, being convex for large values of $\tau = 1/\lambda$, lose this property when τ drops below a certain critical value τ_1.

To find τ_1, we compute the curvature $k_\lambda(z)$ of γ_λ at the point $z = \varphi_\lambda(w)$, where $\lambda = 1/\tau$. Using formulas (5.4) and (6.8), we obtain

$$k_\lambda(z) = \frac{1}{|\varphi'_\lambda(w)|}\operatorname{Re}\left(1 + \frac{w\varphi''_\lambda(w)}{\varphi'_\lambda(w)}\right) = \frac{1}{|\varphi'_\lambda(w)|}\operatorname{Re}\left(1 - 2\frac{1-\lambda w^2}{1+\lambda w^2}\right),$$

which is non-positive for all $|w| = 1$ if and only if $0 \le \lambda \le 1/3$. Since $\lambda = 1/\tau$, this shows that mathematical droplets γ_λ are convex if and only if $\tau \ge \tau_1 = 3$.

It is worth mentioning that the critical value of the surface tension $\tau_1 = 3$ corresponds to McLeod's solution [M1] of the problem in Theorem 6.2. Indeed, recovering the analytic potential g defined by (6.2) from the integrated analytic potential F we get

$$g'(z) = \sqrt{\frac{\tau}{R}}\,\frac{\sqrt{\lambda w^4 - (1-3\lambda^2)w^2 + \lambda}}{(1+\lambda w^2)^2}, \qquad \text{where } w = \varphi_\lambda^{-1}(z/R).$$

The latter function is single-valued if and only if the polynomial under the radical has no zeros in $\mathbb{D}$. Then $\lambda = 1/3$ and $g(\varphi(w)) = R(w - 1/w)$. This shows that the series of mathematical droplets given by Theorem 6.2 contains the only one (up to scaling) physical droplet discovered by E. B. McLeod [M1].

McLeod [M2] also employed Schiffer's variational method to show that the same shape (a rotation of the curve $\gamma_{1/3}$) minimizes the ratio $(P(\Omega))^2/|\operatorname{Re}\alpha_1|$ among all Jordan rectifiable curves γ such that the outer radius $R(\Omega)$ of the exterior Ω of γ equals 1. Here $P(\Omega)$ stands for the perimeter of Ω and $\alpha_1 = \alpha_1(\Omega)$ denotes the coefficient of z in the expansion (6.2) of the analytic potential. (The quantity $M(\Omega) = 2\pi\operatorname{Re}\alpha_1(\Omega)$, called *the hydrodynamical mass of* Ω, plays an important role in hydrodynamics. In particular, $M(\Omega) = M_v(\gamma) + A(\gamma)$, where $M_v(\gamma)$ is the virtual mass of the flow about $\gamma = \partial\Omega$ and $A(\gamma)$ is the mass of the bubble bounded by γ assuming unit density.)

Let $\varphi(w)$ be the univalent map of the unit disc $\mathbb{D}$ onto Ω defined by (6.6) with $R = 1$. Thus, denoting $\alpha = a_1$, we have

$$z = \varphi(w) = w^{-1} + \alpha w + a_2 w^2 + \cdots. \tag{6.17}$$

Let $\alpha_1 = \alpha_1(\Omega)$ be the coefficient of z^{-1} in the expansion (6.2) of the analytic potential in the exterior of γ. From (6.6) we have $\alpha_1 = -(1+\alpha)$. Hence $|\operatorname{Re}\alpha_1| = 1 + \operatorname{Re}\alpha$ since $|\alpha| \le 1$ by the Area Theorem. Thus, using notation Ω_λ from Theorem 6.2, McLeod's result gives the inequality

$$\frac{P^2(\Omega)}{1+\operatorname{Re}\alpha} \ge \frac{3}{5}P^2(\Omega_{1/3}) = \frac{80\pi^2}{27}. \tag{6.18}$$

It is interesting to note that the same inequality is true for any locally univalent function φ in $\mathbb{D}$ (not necessarily univalent) having expansion (6.17) at $w = 0$. For a non-univalent function φ, $P(\Omega)$ will denote the perimeter of the image $\Omega = \varphi(\mathbb{D})$ on the Riemann surface of φ^{-1}. To show this, we first note that transformation $e^{i\beta}\varphi(e^{i\beta}w)$ with $\beta \in \mathbb{R}$ preserves the perimeter of the image and the modulus of the coefficient of w. Therefore we may assume that $\alpha \geq 0$. Since φ is locally univalent, $\varphi'(w) \neq 0$. Therefore the function

$$h(w) = \sqrt{\varphi'(w)} = iw^{-1}\left(1 - (\alpha/2)w^2 + \sum_{k=2}^{\infty} \beta_k w^k\right)$$

is well-defined and analytic in $\mathbb{D}$. Then

$$\begin{aligned} P(\Omega) &= \int_0^{2\pi} |\varphi'(e^{i\theta})|\, d\theta = \int_0^{2\pi} |h(e^{i\theta})|^2\, d\theta \\ &= 2\pi\left(1 + \frac{\alpha^2}{4} + \sum_{k=2}^{\infty} |\beta_k|^2\right) \geq 2\pi\left(1 + \frac{\alpha^2}{4}\right). \end{aligned} \tag{6.19}$$

Equality holds in the above if and only if $h(w) = iw^{-1}(1 - (\alpha/2)w^2)$, or equivalently, if and only if $\varphi(w) = \psi_\alpha(w)$, where

$$\psi_\alpha(w) = w^{-1} + \alpha w - \frac{\alpha^2}{12} w^3. \tag{6.20}$$

Let $\Omega(\alpha) = \psi_\alpha(\mathbb{D})$ and $p(\alpha) = P^2(\Omega(\alpha))/(4\pi^2\alpha)$. Then

$$p(\alpha) = \frac{(1 + (\alpha^2/4))^2}{1 + \alpha} \qquad \text{and} \qquad p'(\alpha) = \frac{(4 + \alpha^2)(3\alpha^2 + 4\alpha - 4)}{16(1 + \alpha)^2}.$$

Since $p'(2/3) = 0$, $\alpha = 2/3$ is a critical value of $p(\alpha)$ which minimizes $p(\alpha)$. Thus, the domain $\Omega(2/3)$ minimizes the functional $P^2(\Omega)/(1 + \operatorname{Re}\alpha)$. Since $\Omega(1/3)$ is a rotation of the domain $\Omega_{1/3}$ of Theorem 6.2, this proves inequality (6.18) for locally univalent functions.

Although the proof of (6.18) given above only uses elementary complex analysis, much deeper methods, such as Schiffer's variational method, are needed to deal with even slightly modified problems. We would like to mention two of them.

(I) Minimal perimeter problem. For a given α, $0 \leq \alpha < 1$, find

$$P(\alpha) = \min P(\varphi(\mathbb{D})),$$

where the minimum is taken over all conformal mappings having expansion (6.17). Inequality (6.19) shows that the function ψ_α, defined by (6.20), is the unique extremal when it is univalent, i.e. for $0 \leq \alpha \leq 4\sqrt{3} - 6$. But for the so-called non-trivial range: $4\sqrt{3} - 6 < \alpha < 1$, when ψ_α is not univalent, there is no available technique to attack the problem. Recently two similar problems in conformal mapping concerning minimal area rather than minimal perimeter were solved in [AShS1] and [AShS2].

(II) The second question is to find

$$\min \frac{P(\varphi(\mathbb{D}))}{|\alpha|}$$

over the same class of conformal mappings. Let $p_0(\alpha) = P(\Omega(\alpha))/(2\pi\alpha)$, where $\Omega(\alpha)$ is the domain defined above. Then

$$p_0(\alpha) = \frac{1}{\alpha} + \frac{\alpha}{4}, \qquad p_0'(\alpha) = -\frac{1}{\alpha^2} + \frac{1}{4}.$$

Since $p_0'(2) = 0$, the function $\psi_2(w)$ minimizes the considered ratio in the class of all analytic functions φ such that $\sqrt{\varphi'}$ is well-defined. Since ψ_2 is not univalent in $\mathbb{D}$, extremal functions for this problem will be among the "non-trivial" extremal functions of problem (I).

We should note that the study in [M1] and [Ga] were restricted to the case of convex bubbles and droplets. Our Theorem 6.2 describes all possible shapes when the whole boundary of a mathematical droplet is free. Nevertheless, this does not cover the range of all possible values for the ratio $\tau : R$ (surface tension : logarithmic capacity). In the remaining part of the range, when $0 \le \tau/R \le (3 + 2\sqrt{3})/3$ all mathematical droplets must have a non-free part of the boundary.

Inequality (6.7) gives a sharp lower bound for the surface tension τ, for which there are Jordan mathematical droplets in equilibrium for the simplified boundary value problem (6.4). Physical evidence, cf. [Ga], suggests that a similar positive lower bound should exist for the original problem (6.1) as well. At least, physical droplets cannot exist in the absence of surface tension, i.e. if $\tau = 0$. We do not know any numerical bounds for τ, which guarantee the existence or non-existence of physical droplets in presence of significant pressure p. In contrast, we will see in the next section that mathematical droplets do exists for a zero surface tension and any positive pressure p.

A similar approach as in the proof of Theorem 6.2 can be used in some other problems. To give a little perspective we mention two of them.

Theorem 6.3 (cf. [EKS]). *Suppose γ is a rectifiable Jordan curve such that its exterior Ω is a Smirnov domain. Suppose further that there is a function $F \in E^1(\Omega)$ analytic in $\Omega \bigcup \{\infty\}$ and such that*

$$F(z) = \dot{\bar{z}} \qquad \text{a.e. on } \gamma. \tag{6.21}$$

Then γ is a circle $\{z : |z - z_0| = R$ with some $z_0 \in \mathbb{C}$ and $R > 0$ and

$$F(z) = \frac{-iR}{z - z_0}.$$

Under an additional assumption $F(\infty) = 0$ this theorem was proved in [EKS]. The argument below shows that the assumption $F(\infty) = 0$ is redundant.

Proof. Let $F \in E^1(\Omega)$ satisfies equation (6.21) and

$$\varphi(w) = aw^{-1} + a_0 + a_1 w + \cdots$$

be a conformal mapping from $\mathbb{D}$ onto Ω. Let $Q(w) = F^2(\varphi(w))(\varphi'(w))^2$. Arguing as in the proof of Theorem 6.2, we find that the quadratic differential $Q(w)\, dw^2$

can be continued by symmetry with respect to $\mathbb{T}$ to a quadratic differential positive on $\mathbb{T}$ and analytic on $\mathbb{D}$ except at a pole of order four at $w = 0$. Then $Q(w)\,dw^2$ has the form

$$Q(w)\,dw^2 = C_1^2 w^{-4}(w - A)^2(1 - \bar{A}w)^2\,dw^2$$

with $C_1 > 0$ and $|A| < 1$. Hence φ satisfies the equation

$$|w^2\varphi'(w)| = C_1|1 - \bar{A}w|^2 \qquad \text{a.e. on } \mathbb{T}.$$

Since Ω is a Smirnov domain this implies that $\varphi'(w) = Cw^{-2}(1 - \bar{A}w)^2$ with some C such that $|C| = C_1$. Therefore

$$\varphi(w) = -C\left(\frac{1}{w} + 2\bar{A}\log w - \bar{A}^2 w\right) + z_0,$$

where $z_0 \in \mathbb{C}$. Since φ is single-valued, $A = 0$ and $\varphi(w) = -C/w + z_0$. Then γ is a circle and F has the desired form. ∎

Similar result holds also if F has a single pole at a prescribed finite point z_0 under an additional constraint $F(\infty) = 0$. The proof follows the lines of the above argument and is therefore omitted.

Theorem 6.4. *Suppose that γ, Ω, and F satisfy conditions of Theorem 6.3, but F has a simple pole at a given finite point $z_0 \in \Omega$ and $F(\infty) = 0$. Then*

$$\gamma = \left\{ z = a\zeta + z_0 : \left|\zeta - \frac{p}{1-p^2}\right| = \frac{p^2}{1-p^2} \right\}$$

with some $a \in \mathbb{C} \setminus \{0\}$ and $0 < p < 1$.

Theorem 6.3 shows that the boundary condition (6.21) forces the analytic function F to vanish at ∞. This phenomenon has a general nature: overdetermined boundary conditions require a special type of contour and special behavior of a solution function. In particular, it reflects the fact that some logarithmic terms in the expansion of a Riemann mapping function must disappear after integration. In Section 7 we will explore this property to construct some solutions of Problem 6.1.

7. Remarks and examples

In this section we will give examples of unbounded domains Ω, for which Problem 6.1 admits a solution F having a polar singularity at ∞. First we find mathematical droplets in the absence of surface tension. Notice that equation (6.1) with $\tau = 0$ defines the Schwarz function of γ. It follows from the known results about Schwarz functions that γ must be an ellipse, cf. [D, Chapter 5] and [Sh, Section 3.3]. To deduce this using an approach of this paper, we differentiate equation (6.1) with $\tau = 0$ with respect to the arc length. Then we obtain a quadratic differential of the type (5.3):

$$Q(w)\,dw^2 = F'(\varphi(w))\,(\varphi'(w))^2\;dw^2,$$

where $z = \varphi(w)$ is the Riemann mapping from $\mathbb{D}$ onto Ω normalized by conditions (6.5). Now $Q(w)\,dw^2$ is positive on $\mathbb{T}$ and has in $\mathbb{D}$ a single pole of order four at $w = 0$. Therefore

$$Q(w)\,dw^2 = -C^2\frac{(w-A)(w-B)(1-\bar{A}w)(1-\bar{B}w)}{w^4}\,dw^2$$

with some $C > 0$, $|A| < 1$, and $|B| < 1$. Recovering φ from the latter two equations and taking into account normalization (6.5), we obtain:

$$\varphi(w) = R\left(\frac{1}{w} + (\bar{A}+\bar{B})\log w - \bar{A}\bar{B}w\right).$$

Since φ is single-valued we must have $B = -A$. Since scaling does not change a shape we put $R = 1$. Then taking $A = c$, $0 < c < 1$ we obtain a family of functions $\varphi(w) = w^{-1} + c^2 w$, which gives a series of examples of mathematical droplets — the ellipses with foci at the points $z = \pm 2c$. The corresponding integrated analytic potential F has the derivative

$$F'(z) = -p\frac{w^2-c^2}{1-c^2w^2}$$

with $p = c^{-2}$. This gives an analytic potential g with a single-valued derivative $g'(z) = \sqrt{F'(z)}$ only in two limit cases $c = 0$ and $c = 1$. Therefore neither of the mathematical droplets in this set, which exist if the pressure exceeds the logarithmic capacity, i.e. $p > 1 = R$, can be realized as a physical droplet.

In retrospect, we may observe a common feature of all problems with known solutions, cf. Theorems 6.2, 6.3, and 6.4 — all solution curves are algebraic. We think that for "most" non-zero values of p and τ the solution curves of the boundary value Problem 6.1 are transcendental and in general cannot be expressed as a finite combination of elementary functions. This might be a reason why it is so difficult to describe them. But it is reasonable to expect that for particular non-zero values of the parameters p and τ the solution curves are algebraic. This is indeed the case as we will see next.

Example 7.1. For $0 \le \lambda < 2\sqrt{3}-3$, let γ_λ be the curve defined in Theorem 6.2. The function

$$\varphi_\lambda(w) = \frac{1}{w} - 2\lambda w - \frac{\lambda^2}{3}w^3$$

maps $\mathbb{D}$ conformally onto $\Omega_\lambda = \varphi_\lambda(\mathbb{D})$. Let $S_\lambda(z)$ be the Schwarz function of γ_λ. Since γ_λ is symmetric with respect to the real axis, $\varphi_\lambda(\bar{w}) = \overline{\varphi_\lambda(w)}$. This symmetry property implies that

$$S_\lambda(z) = \varphi_\lambda(1/\varphi_\lambda^{-1}(z)) = w - 2\lambda\frac{1}{w} - \frac{\lambda^2}{3}\frac{1}{w^3},$$

where $w = \varphi_\lambda^{-1}(z)$. This shows, in particular, that S_λ has a pole of order three at ∞.

In conclusion, the function $f(z) = pS_\lambda(z) + (s/\tau)F(z)$ with $s \geq 0$ and F defined by (6.9) is analytic in Ω except at a pole of order three at $z = \infty$ and satisfies the boundary condition $f(z) = p\bar{z} + is\dot{\bar{z}}$ on γ_λ. Thus, for any real p and s, $f(z)$ solves problem (6.1) in the class of meromorphic functions having a single pole of order three at ∞.

Example 7.2. A similar construction leads to a one-parameter family of mathematical droplets, i.e. domains Ω, for which Problem 6.1 admits solutions having a single simple pole at ∞. Suppose that γ, Ω, and F satisfy the assumptions of Theorem 6.4 with $z_0 = 0 \in \Omega$, but $F(\infty) \neq 0$. Let φ be a conformal mapping from $\mathbb{D}$ onto Ω such that $\varphi(0) = \infty$ and $\varphi(c) = 0$ for some $0 < c < 1$. Now the quadratic differential $Q(w)\,dw^2 = F^2(\varphi(w))(\varphi'(w))^2\,dw^2$ can be continued by symmetry with respect to $\mathbb{T}$ to a quadratic differential on $\bar{\mathbb{C}}$, which has the form

$$Q(w)\,dw^2 = -C^2\frac{(w-A)^2(w-B)^2(1-\bar{A}w)^2(1-\bar{B}w)^2}{w^4(w-c)^2(1-cw)^2}\,dw^2,$$

where $C > 0$, $|A| < 1$, and $|B| < 1$. Then

$$|w^2\varphi'(w)| = C\frac{|(1-\bar{A}w)(1-\bar{B}w)|^2}{|1-cw|^2} \qquad \text{a.e. on } \mathbb{T}.$$

Since Ω is a Smirnov domain the latter equation allows us to recover φ' and then φ:

$$\varphi(w) = C_1\left(\frac{1}{w} - \alpha_1\log w - \frac{\alpha_2}{1-cw} + \alpha_3\log(1-cw) - \alpha_4 w\right) + C_0. \tag{7.1}$$

Here $|C_1| = C$, $C_0 \in \mathbb{C}$, and the coefficients are given by the following formulas

$$\begin{aligned}
\alpha_1 &= 2(c - (\bar{A}+\bar{B})),\\
\alpha_2 &= c^{-3}(c^4 - 2c^3\bar{A} - 2c^3\bar{B} + c^2\bar{A}^2 + c^2\bar{B}^2\\
&\qquad -2\bar{A}^2\bar{B}c - 2\bar{A}\bar{B}^2c + 4c^2\bar{A}\bar{B} + \bar{A}^2\bar{B}^2),\\
\alpha_3 &= -2c^{-3}(c^4 - \bar{A}c^3 - \bar{B}c^3 + \bar{A}^2\bar{B}c + \bar{A}\bar{B}^2c - \bar{A}^2\bar{B}^2),\\
\alpha_4 &= \bar{A}^2\bar{B}^2c^{-2}.
\end{aligned}$$

To have a single-valued function φ, we must satisfy $\alpha_1 = 0$ or, equivalently, $A + B = c$. This simplifies the coefficients as follows:

$$\alpha_2 = \frac{\bar{A}^2\bar{B}^2}{(\bar{A}+\bar{B})^3}, \qquad \alpha_3 = -\frac{2\bar{A}\bar{B}(\bar{A}^2+\bar{A}\bar{B}+\bar{B}^2)}{(\bar{A}+\bar{B})^3}, \qquad \alpha_4 = \frac{\bar{A}^2\bar{B}^2}{(\bar{A}+\bar{B})^2}.$$

Since we are looking for particular examples we are not interested in finding all possible values of the parameters for which φ is univalent. Instead, we consider the special case $C_1 = 1$ and $A^2 + AB + B^2 = 0$. The latter together with the equation $A+B=c$ gives $A=ce^{i\pi/3}$ and $B=ce^{-i\pi/3}$. Now φ has no logarithmic terms and (7.1) reduces to

$$\varphi(w)=\frac{1}{w}-\frac{c}{1-cw}-c^2w+C_0. \tag{7.2}$$

The normalization $\varphi(c)=0$ implies

$$C_0=-\frac{1-2c^2-c^4+c^6}{c(1-c^2)}.$$

After some work one can show that φ is univalent if and only if

$$0<c<c_0=\frac{\sqrt{5}-1}{2}=0.618\ldots.$$

For a fixed c, $0<c<c_0$, let l_c denote the image of the unit circle $\mathbb{T}$ under the mapping (7.2). Figure 2 displays some typical shapes of the curves l_c. Since $\varphi(\bar{w})=\overline{\varphi(w)}$, the Schwarz function of γ has the form:

$$S(z)=\varphi(1/\varphi^{-1}(z))=w-\frac{cw}{w-c}-\frac{c^2}{w}+C_0,$$

where $w=\varphi^{-1}(z)$. In particular, $S(z)$ has simple poles at $z=\infty$ and $z=0$ with the residues $\mathrm{Res}[S,\infty]=1$ and

$$\mathrm{Res}[S,0]=\frac{(1-c^2+c^4)^2}{(1-c^2)^2}.$$

In terms of $w=\varphi^{-1}(z)$, the solution F can be expressed as

$$F(z)=\frac{\sqrt{Q(w)}}{\varphi'(w)}=-i\frac{(w^2-cw+c^2)(1-cw)}{(1-cw+c^2w^2)(w-c)}.$$

The function F is analytic in Ω except for a simple pole at $z=0$ with the residue

$$\mathrm{Res}[F,0]=-i\frac{1-c^2+c^4}{1-c^2}.$$

Finally, the function

$$G_c(z)=(1-c^2)S(z)+i(1-c^2+c^4)F(z)$$

is analytic in $\Omega=\varphi(\mathbb{D})$ except for a simple pole at ∞. Moreover, G_c satisfies the boundary condition:

$$G_c(z)=p\bar{z}+i\tau\dot{\bar{z}}$$

with $p=1-c^2$ and $\tau=1-c^2+c^4$.

Thus, the family of curves l_c, $0<c<c_0$ gives a series of mathematical droplets, i.e. domains Ω for which the boundary value problem (6.1) is solvable in the class of meromorphic functions having a single simple pole at ∞. A tedious computation of the function $\sqrt{G_c'(z)}$ shows that none of these mathematical droplets can be realized as a physical droplet.

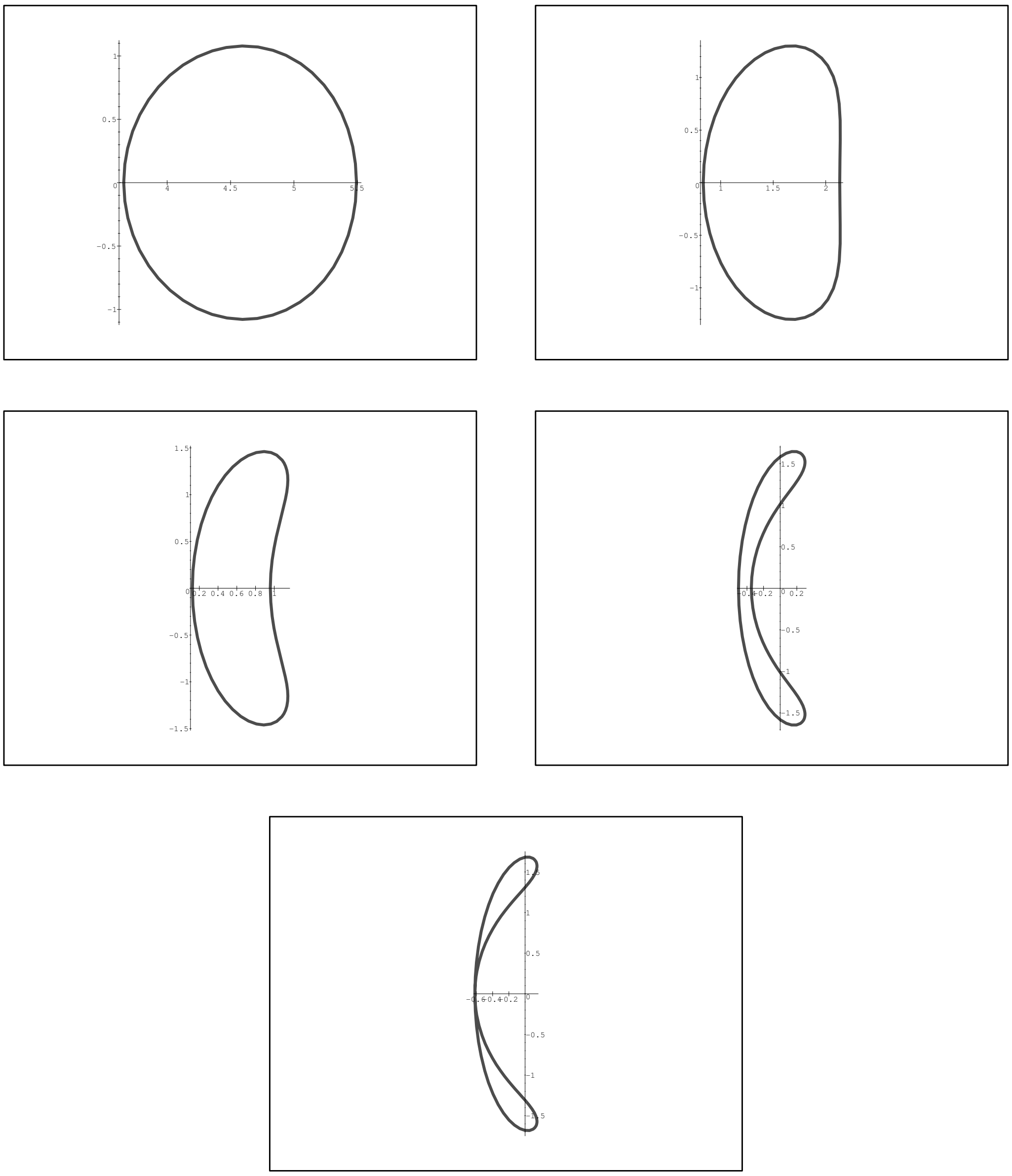

FIGURE 2. Droplets l_c for $c = 0.09$, $c = 0.4$, $c = 0.5$, $c = 0.6$ and $c_0 = 0.618\dots$.

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Added after proofreading. After this paper was accepted we have become aware of further papers relevant to the subject of our work. In particular, the problem on the shape of two-dimensional *physical bubbles* was studied both numerically and analytically in [MVK] and [Shan]. We stress once more that our Theorem 6.2 gives a complete description of *mathematical droplets* for *all possible values* of the surface tension and logarithmic capacity.

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