THE LICHNEROWICZ AND OBATA FIRST EIGENVALUE THEOREMS AND THE OBATA UNIQUENESS RESULT IN THE YAMABE PROBLEM ON CR AND QUATERNIONIC CONTACT MANIFOLDS

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ABSTRACT. We report on some aspects and recent progress in certain problems in the sub-Riemannian CR and quaternionic contact (QC) geometries. The focus are the corresponding Yamabe problems on the round spheres, the Lichnerowicz-Obata first eigenvalue estimates, and the relation between these two problems. A motivation from the Riemannian case highlights new and old ideas which are then developed in the settings of Iwasawa sub-Riemannian geometries.

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1. Introduction

As the title suggests, the goal of this paper is to report on some aspects of certain problems in the sub-Riemannian CR and quaternionic contact (QC) geometries. It seems appropriate in lieu of an extensive Introduction to begin with a section about the corresponding problems in the Riemannian case. Besides an introduction to the discussed problems we give key steps of the proofs of some well known results highlighting ideas which can be used, although with a considerable amount of extra analysis in the sub-Riemannian setting. In the later sections we show the difficulties and current state of the art in the corresponding results on CR and quaternionic contact manifolds. However, this article is not designed to be a complete survey of the subjects, especially in the case of the Yamabe problem, but rather a collection of particular results with which we have been involved directly while giving references to important works in the area, some of which are covered in this volume.

Convention 1.1. *A convention due to traditions: when considering eigenvalue problems, it is more convenient to use the non-negative (sub-)Laplacian. Correspondingly, $\triangle u = -tr^g(\nabla^2 u)$ for a function u and metric g. On the other hand the (sub-)Laplacian appearing in the Yamabe problem is the "usual" negative (sub-)Laplacian $\triangle u = tr^g(\nabla^2 u)$.*

2. Background - The Riemannian problems

The only new result here is Proposition 2.4. This fact is exploited later in a new simplified proof of the Obata type theorem in the qc-setting. Interestingly, the CR case presents another type of behavior.

2.1. The Lichnerowicz and Obata first eigenvalue theorems.

The relation between the spectrum of the Laplacian and geometric quantities has been a topic of continued interest. One such relation was given by Lichnerowcz [175] who showed that on a compact Riemannian manifold (M, g) of dimension n for which the Ricci curvature satisfies $Ric(X, X) \geq (n-1)g(X, X)$ the first positive eigenvalue λ_1 of the (positive) Laplace operator $\triangle f = -tr^g \nabla^2 f$ satisfies the inequality $\lambda_1 \geq n$. Here ∇ is the Levi-Civita connection of g. In particular, n is the smallest eigenvalue of the Laplacian on compact Einstein spaces of scalar curvature equal to $n(n-1)$-the scalar curvature of the round unit sphere. Subsequently, Obata [189] proved that equality is achieved iff the Riemannian manifold is isometric to the round unit sphere. It should be noted that the smallest possible value n is achieved on the round unit sphere by the restrictions of the linear functions to the unit sphere (spherical harmonics of degree one), which give the associated eigenspace. Later, Gallot [95] generalized these results to statements involving the higher eigenvalues and corresponding eigenfunctions of the Laplace operator.

The above described results of Lichnerowicz and Obata we want to discuss in detail are summarized in the next theorem.

Theorem 2.1. *Suppose (M, g) is a compact Riemannian manifold of dimension n which satisfies a positive lower Ricci bound*

$$Ric(X, X) \geq (n-1)g(X, X). \tag{2.1}$$

a) If λ is a non-zero eigenvalue of the (positive) Laplacian, $\triangle f = \lambda f$, then $\lambda \geq n$, see [175].
b) If there is $\lambda = n$, then (M, g) is isometric with the round sphere $S^n(1)$, see [188]

Let us briefly sketch the proof of Theorem 2.1 including a new observation, Proposition 2.4, which will be exploited in the sub-Riemannian setting. The key to Lichnerowicz' inequality is Bochner's identity ($\triangle \geq 0$),

$$-\frac{1}{2}\triangle|\nabla f|^2 = |\nabla df|^2 - g(\nabla(\triangle f), \nabla f) + Ric(\nabla f, \nabla f). \tag{2.2}$$

After an integration over the compact manifold we find

$$0=\int_M |(\nabla df)_0|^2+\frac{1}{n}(\triangle f)^2-g(\nabla(\triangle f),\nabla f)+Ric(\nabla f,\nabla f)\,dvol_g.$$

Let us assume at this point the inequality $Ric(\nabla f,\nabla f)\geq (n-1)|\nabla f|^2$ for any eigenfunction f, $\triangle f=\lambda f$. We obtain then the inequality

$$\begin{aligned}0=\int_M |(\nabla df)_0|^2+\frac{1}{n}\lambda|\nabla f|^2-\lambda|\nabla f|^2+Ric(\nabla f,\nabla f)\,dvol_g\\ =\int_M |(\nabla df)_0|^2\,dvol_g+\int_M Ric(\nabla f,\nabla f)-\frac{n-1}{n}\lambda|\nabla f|^2\,dvol_g\\ \geq\int_M |(\nabla df)_0|^2\,dvol_g+\frac{n-1}{n}\int_M (n-\lambda)|\nabla f|^2\,dvol_g.\end{aligned}$$

Hence $(\nabla df)_0=0$ and $0\geq n-\lambda$, which proves Lichnerowicz' estimate. Furthermore, if the lowest possible eigenvalue is achieved then the trace-free part of the Riemannian Hessian of an eigenfunction f with eigenvalue $\lambda=n$ vanishes, i.e., it satisfies the system

$$\nabla^2 f=-fg. \tag{2.3}$$

Obata's result which describes the case of equality was preceded by several results where the case of equality was characterized under the additional assumption that g is Einstein [231] or has constant scalar curvature [119]. It turns out that besides Obata's proof these assumptions can also be removed as we found in Proposition 2.4. Nevertheless, even under the assumption that g is Einstein the proof that $(M,g)=S^n$ requires further delicate analysis involving geodesics and the distance function from a point. Furthermore, Obata showed in fact a more general result, namely, on a *complete* Riemannian manifold (M,g) equation (2.3) above allows a non-constant solution iff the manifold is isometric to the round unit sphere S^n.

Remark 2.2. *A good reference for Hessian equations characterizing the spaces of constant curvature is* [155]. *For example, if (M,g) is compact Riemannian manifold admitting a non-constant solution to $\nabla^2 f=\frac{\triangle f}{n}g$ then (M,g) is conformally diffeomorphic to the unit round sphere. Furthermore, if the scalar curvature of (M,g) is constant then (M,g) is isometric to a Euclidean sphere of certain radius.*

Thanks to the Bonnet-Myers and S.-Y. Cheng's improved Toponogov theorems we can sketch the proof of this fact as described in details in [53, Chapter III.4]. First, we note that assuming (M,g) is complete and satisfies (2.1) we have

(i) (Bonnet-Myers) M is compact, the diameter $d(M)\leq\pi$ and $\pi_1(M)$ is finite;
(ii) (improved Toponogov theorem) $d(M)=\pi$ iff M is isometric to $S^n(1)$, [56].

The Hessian equation (2.3) implies that if $\gamma(t)$ is unit speed geodesics we have $(f\circ\gamma)''+f\circ\gamma=0$, hence $f(\gamma(t))=A\cos t+B\sin t$ for some constants A and B. Let $p\in M$ be such that $f(p)=\max_M f$ which exists since M is compact. For any unit tangent vector $\xi\in T_p(M)$ the unit speed geodesic $\gamma_\xi(t)$ from p in the direction of ξ satisfies $f(\gamma_\xi(t))=f(p)\cos t$ sinc the derivative at $t=0$ is zero. Therefore, $f(\gamma_\xi(t))$ is injective for $0\leq t\leq\pi$ which implies $d(M)\geq\pi$. This shows that $d(M)=\pi$ and by Cheng's theorem we conclude $M=S^n$.

Remark 2.3. *We remark explicitly that the above approach to Obata's theorem cannot be used in sub-Riemannian setting in which case both (i) and (ii) are very challenging open problems with the exception of some results generalizing (i) in some special cases, see Section 2.4.*

We turn to our result mentioned in context above.

Proposition 2.4. *Suppose (M,g) is a compact Riemannian manifold of dimension n which satisfies (2.1). If the lowest possible eigenvalue is achieved, $\triangle f=nf$ for some function f, then (M,g) is an Einstein space.*

Proof. The proof follows from several calculations and a use of the divergence formula. By the proof of Lichnerowicz' estimate the eigenfunction f satisfies (2.3). Differentiating (2.3) and using Ricci's identity $\nabla^3 f(X,Y,Z)-\nabla^3 f(Y,X,Z)=-R(X,Y,Z,\nabla f)$ we find the next formula for the curvature tensor

$$R(X,Y,Z,\nabla f)=df(X)g(Y,Z)-df(Y)g(X,Z). \tag{2.4}$$

Taking a trace in the above formula we see

$$Ric(X, \nabla f) = (n-1)df(X). \tag{2.5}$$

A differentiation of (2.5) and another use of (2.3) gives

$$(\nabla_Z Ric)(Y, \nabla f) = fRic(X, Y) - (n-1)fg(X, Y). \tag{2.6}$$

On the other hand, taking the covariant derivative of (2.4) and then using (2.3) for $\nabla_V(\nabla f)$, we obtain

$$(\nabla_V R)(Z, X, Y, \nabla f) = fR(Z, X, Y, V) - fg(V, Z)g(X, Y) + fg(V, X)g(Z, Y).$$

Therefore, taking a trace, it follows.

$$(\nabla^* R)(Y, X, \nabla f) = fRic(X, Y) - (n-1)fg(X, Y). \tag{2.7}$$

A substitution of (2.7) in the formula $(\nabla_Z Ric_0) = (\nabla_Z Ric)(X, Y) - \frac{1}{n}dS(Z)g(X, Y)$ with $Z = \nabla f$ gives the key identity

$$(\nabla_{\nabla f} Ric_0)(X, Y) = 2fRic_0(X, Y) - \frac{2S}{n}g(X, Y) - 2(n-1)fg(X, Y) - \frac{1}{n}dS(\nabla f)g(X, Y). \tag{2.8}$$

Hence, $L_{\nabla f}|Ric_0|^{2k} = 4kf|Ric_0|^{2k}$. Integrating over compact manifold M with respect to the Riemannian volume we obtain

$$\int_M |Ric_0|^{2k} f^2\, dvol_g = \frac{1}{n}\int_M g(\nabla |Ric_0|^{2k} f, \nabla f)\, dvol_g$$
$$= \frac{1}{n}\int_M |Ric_0|^{2k}|\nabla f|^2\, dvol + \frac{4k}{n}\int_M |Ric_0|^{2k} f^2\, dvol_g.$$

Therefore,

$$(n-4k)\int_M |Ric_0|^{2k} f^2\, dvol_g = \int_M |Ric_0|^{2k}|\nabla f|^2\, dvol_g,$$

hence choosing $k > n/4$ it follows $Ric_0 = 0$, i.e., g is an Einstein metric. □

It should be noted that (2.3) is obtained in connection with infinitesimal conformal transformations on Einstein spaces, see 2.12. Thus the unit sphere is characterized as the only complete Einstein Riemannian manifold of scalar curvature $n(n-1)$ admitting a non-homothetic conformal transformation, [119], [189], [218], [231], see also later in this Section for relations with the Yamabe problem on the Euclidean sphere. In addition, this result can be considered as a characterization of the unit sphere as a compact Einstein space which admits an eigenfunction belonging to the possible smallest eigenvalue of the Laplacian for compact Einstein spaces.

We remark that it is natural to consider characterizations of the unit sphere by its second eigenvalue $2(n+l)k$ of the Laplacian. In this case, the gradients of the corresponding eigenfunctions are infinitesimal projective transformations, which also gives a system of differential equations of order three satisfied by the divergence of an infinitesimal projective transformation on an Einstein space. Furthermore, it is shown that the complete Riemannian manifold admitting a non-trivial solution for the system is isometric to the unit sphere provided that the manifold is simply connected, , [80], [98], [27], [190], [189]. There are results in the Kähler case where an infinitesimal holomorphically projective transformation plays a role similar to that of projective one on Riemannian manifolds, [190]. We shall seek a characterization of the model CR and qc unit spheres through the first eigenfunctions of the respective sub-Laplacians

2.2. **Conformal transformations.** Let (M, g) and (M', g') be two Riemannian manifold of dimension n. A smooth map $F : M \to M'$ is called a conformal map if $F^*g' = \phi^{-2}\, g$ for some smooth positive function ϕ. For our goals we shall consider $(M, g) = (M', g')$, F a diffeomorphism, and let $\bar{g} = F^*g'$. In this case, we say that F is a conformal diffeomorphism while the metrics g and $\bar{g}$ are called (point-wise) conformal to each other. For $n \geq 3$, we shall need the following well known formulas relating the traceless Ricci and scalar curvatures of the metrics g and $\bar{g}$,

$$\overline{Ric}_0 = Ric_0 + (n-2)\phi^{-1}(\nabla^2\phi)_0 \tag{2.9}$$

$$\bar{S} = \phi^2 S + 2(n-1)\phi\triangle\phi - n(n-1)|\nabla\phi|^2, \tag{2.10}$$

where ∇ is the Levi-Civita connection of g, $\triangle\phi = tr^g\nabla^2\phi$, and $(\nabla^2\phi)_0$ is the traceless part of the Hessian of ϕ.

A conformal vector field on a compact Riemannian manifold (M, g) is a vector field X whose flow consists of conformal transformations (diffeomorphisms). In the case the flow is a one-parameter group of isometries the vector field X is called a Killing field. For M compact, the algebra $\mathfrak{c}(M, g)$ of conformal vector fields is exactly the group of conformal diffeomorphisms $C(M, g)$ of (M, g). It is worth recalling [230] and [175] that X is a conformal vector field iff

$$\mathcal{L}_X g = \frac{2}{n}(div_g X)g, \tag{2.11}$$

where $\mathcal{L}_X$ is the Lie derivative operator and $div_g X$ is the divergence operator $(div_g X)\, vol_g = \mathcal{L}_X vol_g$ defined with the help of Riemannian volume element vol_g associated to g. In particular, a gradient vector field $X = \nabla\phi$ is infinitesimal conformal vector iff

$$(\nabla^2\phi)_0 = 0. \tag{2.12}$$

A short calculation, see [230], [175] or [232, (1.11)], shows that if X is a conformal vector field then

$$\triangle(div\, X) = -\frac{1}{n-1}(div\, X)S - \frac{n}{2(n-1)}X(S). \tag{2.13}$$

2.3. **The Yamabe problem - Obata's uniqueness theorem.** The Yamabe type equation has its origin in both geometry and analysis. Yamabe [228] considered the question of finding a conformal transformations of a given Riemannian metric on a compact manifold to one with a constant scalar curvature, see also [219] and [10, 11, 12, 9]. When the ambient space is the Euclidean space $\mathbb{R}^n$, G. Talenti [214] and T. Aubin [10, 11, 12] described all positive solutions of a more general equation, that is the Euler - Lagrange equation associated with the best constant, i.e., the norm, in the L^p Sobolev embedding theorem. With the help of the stereographic projection, which is a conformal transformation, Yamabe's question for the standard round sphere turns into the L^2 case of Talenti's question. The solution of these special cases is an important step in solving the general Yamabe problem, the solution of which in the case of a compact Riemannian manifold was completed in the 80's after the work of T. Aubin and R. Schoen [10, 11, 12, 9, 206, 207, 208], see also [167]. It should be noted that the "solution" used the positive mass theorem of R. Schoen and S.-T. Yau [210]. An alternative approach was developed by A. Bahri [13] where solutions of the Yamabe equation were obtained through "higher" energies of the Yamabe functional. As well known, in general, there is no uniqueness of the metric of constant scalar curvature within a fixed conformal class. However, with the exception of the round sphere, according to Obata's theorem uniqueness (up to isometry) holds in a conformal class containing an Einstein metric.

2.3.1. *The Yamabe problem and functional.* Let $(M^n,\, g)$ be a compact Riemannian manifold of dimension n. The Yamabe problem is to find a metric $\bar{g}$ point-wise conformal to the Riemannnian metric g of constant scalar curvature $\bar{S}$. Clearly, this is a type of uniformization problem which is one generalization of the classical surface case.

2.3.2. *Riemann surfaces - the 2-D case.* In the 2-D case, where we are dealing with the uniformization of a closed orientable surface, if we set $\bar{g} = e^{\phi}g$, then the equation which needs to be solved is

$$\triangle\phi - K = -\bar{K}e^{\phi},$$

for some constant $\bar{S}$, where S is the Gaussian curvature of g. By the Gauss-Bonnet formula

$$2\pi\chi(M) = \int_M K\, dv_g,$$

which determines the sign of $\bar{K}$. By the *uniformization theorem* of the universal cover $\hat{M}$ of M, M is biholomorphic to $\hat{M}/G$ for some G-properly discontinuous subgroup of $Aut(\hat{M})$. Thus, depending on the sign, (M, g) is hyperbolic, parabolic, or elliptic Riemann surface, i.e., it is conformal to one of constant Gauss curvature $-1,\ 0,\ 1$. Explicitly, depending on its genus, M is conformal (in fact biholomorphic) to a surface in one of next three cases:

(1) $\mathbb{H}/\Gamma$, for a properly discontinuous Γ subgroup of $PSL(2, R)$-the automorphism group of the unit disc $\mathbb{D}$, when the genus of M at least two;
(2) $\mathbb{C}/\Lambda$-elliptic curve, corresponding to a lattice $\Lambda = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in Z\}, \omega/\omega_2 \notin \mathbb{R}$ when the genus of M is one;
(3) S^2 of genus 0.

2.3.3. *The higher dimensional cases.* For $n \geq 3$ such a complete picture is not possible. It is customary to take the conformal factor in a way which is best suited for the problem, accordingly we begin with the form exhibiting the relation the critical Sobolev exponent. As well known, if we write the conformal factor in the form $\bar{g} = u^{4/(n-2)}\, g$ then the Yamabe problem becomes an existence problem for a positive solution to the Yamabe equation, see (2.10),

$$4\frac{n-1}{n-2}\triangle u - S\cdot u \;=\; -\overline{S}\cdot u^{2^*-1}. \tag{2.14}$$

where $\triangle u \;=\; tr^g(\nabla du)$, S and $\overline{\text{S}}$ are the scalar curvatures of g and $\bar{g}$, and $2^* \;=\; \frac{2n}{n-2}$ is the Sobolev conjugate exponent.

The Yamabe problem 2.3.1 is of variational nature as we remind next. The critical points of the Einstein-Hilbert (total scalar curvature) functional

$$\Upsilon(\bar{g}) = \Big(\int_M \bar{S}\, dv_{\bar{g}}\Big) / \Big(\int_M dv_{\bar{g}}\Big)^{2/2^*}$$

are Einstein metrics. The *Yamabe functional* is obtained by restricting $\Upsilon(\bar{g})$ to the conformal class $[g] = \{\bar{g} = u^{4/(n-2)}g \mid 0 < u \in \mathcal{C}^\infty(M)\}$ and defining (a conformally invariant functional)

$$\Upsilon_g(u) = \Big(\int_M 4\frac{n-1}{n-2}\,|\nabla u|^2 \;+\; S\,u^2\,dv_g\Big) / \Big(\int_M u^{2^*}\,dv_g\Big)^{2/2^*}. \tag{2.15}$$

The critical points, i.e., the solutions of $\frac{d}{dt}\Upsilon(u+t\phi)_{|t=0} \;=\; 0, \quad \phi \in \mathcal{C}^\infty(M)$, are metrics of constant scalar curvature (Yamabe metrics) since they are given by the solutions of (2.14) with $\bar{S}$ the corresponding "critical" energy level. The Yamabe constant of (M, g) is

$$\Upsilon(M,[g]) \equiv \Upsilon([g]) = \;\inf\{\Upsilon_g(u):\; u > 0\}.$$

The Yamabe invariant is the supremum $\lambda(M) = \sup_{[g]}\Upsilon([g])$.

According to the result of Aubin and Talenti, for the round unit sphere $\Upsilon(S^n,[g_{st}]) = n(n-1)\omega_n^{2/n}$. The existence of a Yamabe metric is the content of the next theorem, which collects a number of remarkable results, see for example [167] for a full account,

Theorem 2.5 (N. Trudinger, Th. Aubin, R. Schoen; A. Bahri)**.** *Let (M^n, g), $n \geq 3$, be a compact Riemannian manifold. There is a $\bar{g} \in [g]$, s.t., $\bar{S} = const.$*

The main steps in proof of the above theorem are as follows, see [167, 9] for a full account.

- We have $\Upsilon([g]) \leq \Upsilon(S^n, st)$. The Yamabe problem can be solved on any compact manifold M with $\Upsilon([g]) < \Upsilon(S^n,[g_{st}])$, see [228], [219], and [11].
- If $n \geq 6$ then $\Upsilon(S^n,[g_{st}]) - \Upsilon([g]) \geq c\|W^g\|^2$, hence the Yamabe problem can be solved if $n \geq 6$ and M is not locally conformally flat, see [11].
- If $3 \leq n \leq 5$, or if M is locally conformally flat, then the Yamabe problem has a solution since $\Upsilon(S^n,[g_{st}]) - \Upsilon([g]) \geq cm_o$, where m_o is the mass of a one point blow-up (stereographic projection) of M, see [206].
- If M is locally conformally flat a critical point of the Yamabe functional exists (which may be of higher than $\Upsilon(M,[g])$ energy), see [13].

Given the above existence of a Yamabe metric on every compact Riemannian manifold it is natural to study the question of uniqueness. When $\Upsilon(M,[g]) \leq 0$ the Yamabe metric is unique in its conformal class as implied by the maximum principle. However, for $\Upsilon(M,[g]) > 0$ this is no longer true. An example of non-uniqueness is provided by the round unit sphere as described next in 2.3.4. Another example was given by [208] in $S^1(R)\times S^n$. Remarkably, in the case of the sphere the set of solutions is non-compact (in the $\mathcal{C}^2$ topology), see further below, and it was conjectured in [208] that this is the only case, which became known as the compactness conjecture, see the review [34] for further details and references. In short, the conjecture is true in the following cases:

- for a locally conformally flat manifold (different from the round sphere) [204] and [205];
- for $n \leq 7$, see [173] and [180];
- for $8 \leq n \leq 24$ provided that the positive mass theorem holds (this covers all cases of a spin manifold), see [173], [172] for $8 \leq n \leq 11$ and [147] for $12 \leq n \leq 24$.

Furthermore, the conjecture is not true for $n \geq 25$, see [33] and [35]. Putting the compactness conjecture aside we turn to the uniqueness result of Obata.

2.3.4. *Obata's uniqueness theorem for the Yamabe problem.* The main result here is that the conformal class of an Einstein metric on a connected compact Riemannian manifold $(M, \bar{g})$ contains a unique Yamabe metric unless M is the round unit sphere S^n in $\mathbb{R}^{n+1}$. It should be noted that if M is not conformal to the round sphere the Yamabe metrics are nondegenerate global minima of the Einstein-Hilbert functional. The structure of the set of Yamabe metrics in conformal classes near a nondegenerate constant scalar curvature metric was considered in the smooth case in [150]. An extension of Obata's result to a local uniqueness result for the Yamabe problem in conformal classes near to that of a nondegenerate solution was established in [75]. After this short background we turn to Obata's theorem.

Theorem 2.6 ([188] and [187])**.** *a) Let $(M, \bar{g})$ be a connected compact Riemannian manifold which is Einstein and $\bar{g} = \phi^{-2}g$. If $\bar{S} = S = n(n-1)$, then $\phi = 1$ unless $(M, \bar{g}) = (S^n, g_{st})$.*

b) If g is a Riemannian metric conformal to g_{st}, $g_{st} = \phi^{-2}g$, with scalar curvature $S = n(n-1)$, then g is obtained from g_{st} by a conformal diffeomorphism of the sphere, i.e., there is $\Phi \in Diff(S^n)$ such that $g = \Phi^* g_{st}$ and up to an additive constant ϕ is an eigenfunction for the first eigenvalue of the Laplacian on the round sphere. In particular, $\nabla\phi$ is a gradient conformal field and for some t we have $\Phi = exp(t\nabla\phi)$-the one parameter group of diffeomorphisms generated by $\nabla\phi$.

Proof. In the proof of part a) we use the argument of [29] and [167] which is very close to Obata's argument but uses the "new" metric as a background metric rather than the given Einstein metric. Suppose $\bar{g}$ is Einstein, hence by (2.9) we have

$$0 = \overline{Ric}_o = Ric_o + \frac{n-2}{\phi}(\nabla^2\phi)_0.$$

Therefore, $(\nabla^2\phi)_0 = -\frac{\phi}{n-2}Ric_0$. From the contracted Bianchi identity and S=const we have $\nabla^* Ric = \frac{1}{2}\nabla S = 0$, hence $\nabla^*\left(Ric(\nabla\phi, .)\right) = (\nabla^* Ric)(\nabla\phi) + g(Ric, \nabla^2\phi) = \frac{1}{2}g(\nabla S, \nabla\phi) - \frac{\phi}{n-2}|Ric_o|^2$. Integration over M and an application of the divergence theorem shows that g is *also an Einstein metric*, $Ric_o = 0$. This implies $(\nabla^2\phi)_0 = 0$, hence $\nabla\phi$ is a gradient conformal vector field, see (2.12). Now, from (2.13) taking into account $S = n(n-1)$ it follows $\triangle(\triangle\phi + n\phi) = 0$, hence by the maximum principle we have $\triangle u = -nu$, where $u = \phi + a$ for some constant a. Notice that we also have $(\nabla^2 u)_0 = 0$. Hence by Obata's result in the eigenvalue Theorem 2.1 either u =const or g is isometric to g_{st} and u is a restriction of a linear function to S^n, $u = (a_o x_o + \cdots + a_n x_n)|_{S^n}$, which implies the claimed form of ϕ. $\square$

We note that in the case of the round sphere, once we proved that g is also Einstein, we can conclude that g is isometric to g_{st} since it is Einstein and conformally flat, $W = 0$, see [155], [156], [181], [148] for further details and references on conformal transformations between Einstein spaces in a variety of spaces. Thus, there is an isometry $\Phi : (S^n, g) \to (S^n, g_{st})$, $\Phi^* g_{st} = \phi^{-2} g_{st}$ hence $F \in C(S^n, g_{st})$. Part b) of the above theorem shows that Φ belongs to the largest connected subgroup of $C(S^n, g_{st})$ and determines the exact form of ϕ. The same conclusion can be reached with the help of the stereographic projection and relates the analysis to the Liuoville's theorem and the best constant in the L^2 Sobolev embedding theorem in Euclidean space. In fact, using the stereographic projection we can reduce to a conformal map of the Euclidean space, which sends the Euclidean metric to a conformal to it Einstein metric. By a purely local argument, see [36], the resulting system can be integrated, in effect proving also Liuoville's theorem, which gives the form of ϕ after transferring the equations back to the unit sphere. Such argument was used in the quaternionic contact setting [120] to classify all qc-Einstein structures on the unit $4n+3$ dimensional sphere (quaternionic Heisenberg group) conformal to the standard qc-structure on the unit sphere. We will come back to the qc-Liouville theorem later in the paper, see Section 8.5.

In any case, the key point here which will be used in the sub-Riemannian CR or QC setting is that Obata's argument shows the validity of a system of partial differential equations, namely, $(\nabla^2\phi)_0 = 0$ assuming (2.10) holds with $\bar{g}$ being Einstein and g of constant scalar curvature. On the other hand, using the stereographic projection, Yamabe's equation on the round sphere turns into (2.14) for the Euclidean Laplacian with $S = 0$ and $\bar{S}$=const after interchanging the roles of g and $\bar{g}$, i.e., assuming that g is the "background" standard constant curvature metric and $\bar{g}$ is the "new" conformal to g metric of constant scalar curvature. This is nothing but the equation characterizing the extremals of the variational problem associated to the L^2 Sobolev embedding theorem. An alternative to Obata's argument is then the symmetrization argument (described briefly in Section 3).

2.4. **Sub-Riemannian comparison results and Yamabe type problems - a summary.** The interest in relations between the spectrum of the Laplacian and geometric quantities justified the interest in Lichnerowicz-Obata type theorems in other geometric settings such as Riemannian foliations (and the eigenvalues of the basic Laplacian) [164, 163], [141] and [194], to CR geometry (and the eigenvalues of the sub-Laplacian) [109], [15], [48, 46, 47], [49], [61], [169], and to general sub-Riemannian geometries, see [16] and [117]. Complete results have been achieved in the settings of (strictly pseudoconvex) CR, [109],[46, 47], [48], [61],[170, 171],[131, 132], and QC, [127, 126, 128], geometries which shall be covered in Sections 7 and 8.

As far as other comparison results are concerned we mention

(i) [202] for a Bonnet-Myers type theorem on general 3-D CR manifolds;

(ii) [118], where a Bonnet-Myers type theorem on a three dimensional Sasakian was proved.

Both of the above papers use analysis of the second-variation formula for sub-Riemannian geodesics.

(iii) [52] for an isoperimetric inequalities and volume comparison theorems on CR manifolds.

(iv) [16], [17], [18], [19], [111, 110] where an extension to the sub-Riemannian setting of the Bakry-Emery technique on curvature-dimension inequalities are used to obtain Myers-type theorems, volume doubling, Li-Yau, Sobolev and Harnack inequalities, Liouville theorem. Such inequalities are obtained usually under a transverse symmetry assumption. The latter means that we are actually dealing with a Riemannian manifold with bundle like metrics which are foliated by totally geodesic leaves. Thi scondition equivalent to vanishing torsion in the QC setting (qc-Einstein) and is not very far from the Sasakian case (vanishing torsion) in the CR case.

(v) [1], [3], [2], where sub-Riemannian geodesics and measure-contraction properties are used to establish for Sasakian manifolds results such as a Bishop comparison theorem, Laplacian and Hessian comparison, volume doubling, Poncaré and Harnack inequalities, and Liuoville theorem. comparison results in the Sasakian case.

(vi) [117] for Lichnerowicz type estimates and a Bonnet-Myers theorems in some special sub-Riemannian geometries.

A variant of the Yamabe problem in the setting of a compact strictly pseudoconvex pseudohermitian manifold (called here simply CR manifold) is the CR Yamabe problem where one seeks in a fixed pseudoconformal class of pseudo-Hermitian structures on a compact CR manifold one with constant scalar curvature (of the canonical Tanaka-Webster connection). After the works of D. Jerison & J. Lee [137] - [140] and N.Gamara & R. Yacoub [96], [97] the CR Yamabe problem on a compact manifold is complete. The case of the standard CR structure on the unit sphere in $\mathbb{C}^n$ is equivalent to the problem of determining the best constant in the L^2 Folland & Stein [90] Sobolev type embedding inequality on the Heisenberg group. The best constant in the L^2 Folland & Stein inequality together with the minimizers were determined recently using a different from [139] method by Frank & Lieb [92], see also [32]. Nevertheless this simpler approach *does not* yield the uniqueness result of D. Jerison & J. Lee. A positive mass theorem in the three dimensional case was proven recently in [58].

In the other case of interest, the qc-Yamabe problem was studied in [120, 121, 122] and [225]. According to [225] the Yamabe constant of a compact qc manifold is less than or equal to that of the standard qc sphere. Furthermore, if the constant is strictly less than the corresponding constant of the sphere, the qc-Yamabe problem has a solution, i.e., there is a conformal 3-contact form for which the qc-scalar curvature is constant. The Yamabe constant of the standard qc structure on the unit $(4n+3)$-dimensional sphere was determined in [122] with the help of a clever center of mass argument following in the footsteps of the CR case [92] and [32]. However, due to the limitations of the method [122] does not exclude the possibility that in the qc-conformal class of the standard qc structure there are qc Yamabe metrics of higher energies. The seven dimensional case was settled completely earlier in [121]. A conformal curvature tensor was found in [129], which should prove useful in establishing existence of a solution to the qc-Yamabe problem in the qc locally non-flat case.

Finally, we mention [62, 63] where the sharp Hardy-Littlewood-Sobolev inequalities in the quaternion and octonian versions of the approach found by Frank and Lieb was developed. In particular, at this point the sharp constants in the Hardy-Littlewood-Sobolev inequalities on all groups of Iwasawa type are known.

3. The Folland-Stein inequality on groups of Iwasawa type

We start by recalling the following embedding theorem due to Folland and Stein [90]. Let $\boldsymbol{G}$ be a Carnot group $\boldsymbol{G}$ of homogeneous dimension Q, fixed metric g on the the "horizontal" bundle spanned by the first layer and Haar measure dH. For any $1 < p < Q$ there exists $S_p = S_p(\boldsymbol{G}) > 0$ such that for $u \in C_o^\infty(\boldsymbol{G})$ we have

$$\left(\int_{\boldsymbol{G}} |u|^{p^*} \, dH \right)^{1/p^*} \leq S_p \left(\int_{\boldsymbol{G}} |Xu|^p \, dH \right)^{1/p}, \tag{3.1}$$

where $|Xu| = \sum_{j=1}^m |X_j u|^2$ with $X_1, \dots, X_m$ denoting an orthonormal basis of the first layer of $\boldsymbol{G}$ and $p^* = \frac{pQ}{Q-p}$. In the case $\boldsymbol{G} = \mathbb{R}^n$ this embedding is nothing but the Sobolev embedding theorem. We insist on $X_1, \dots, X_m$ denoting an orthonormal basis of the first layer in order to have a well defined constant which obviously depends on the chosen (left invariant) metric. For the sake of brevity we do not give the definition of a Carnot group since our focus is in the particular case of groups of Iwasawa type, in which case there is a natural metric. Also, the case $p = 1$ which we did not include above, is the isoperimetric inequality, see [41] for the proof in a much wider setting, which as well known [82, 182], see also [214] and [203], implies the whole range of inequalities (3.1).

The most basic fact of the above inequality is its invariance under translations and dilations. The latter fact determines the relation between the exponents p and p^* appearing in both sides. For a function $u \in C_o^\infty(\boldsymbol{G})$ we let

$$\tau_h u \overset{def}{=} u \circ \tau_h, \qquad h \in \boldsymbol{G}, \tag{3.2}$$

where $\tau_h : \boldsymbol{G} \to \boldsymbol{G}$ is the operator of left-translation $\tau_h(g) = hg$, and also

$$u_\lambda \equiv \lambda^{Q/p^*} \delta_\lambda u \overset{def}{=} \lambda^{Q/p^*} u \circ \delta_\lambda, \qquad \lambda > 0. \tag{3.3}$$

Here, δ_λ is the non-isotropic dilation

$$\delta_\lambda(g) = \exp \circ \Delta_\lambda \circ \exp^{-1}(g), \quad g \in \boldsymbol{G}, \tag{3.4}$$

where $\exp : \mathfrak{g} \to \boldsymbol{G}$ is the exponential map and $\Delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ is defined by $\Delta_\lambda(\xi_1 + \dots + \xi_r) = \lambda \xi_1 + \dots + \lambda^r \xi_r$. It is easy to see that the norms in the two sides of the Folland-Stein inequality are invariant under the translations (3.2) and the rescaling (3.3).

Let S_p be the best constant in the Folland-Stein inequality, i.e., the smallest constant for which (3.1) holds. The equality is achieved on the space $\overset{o}{\mathcal{D}}{}^{1,p}(\boldsymbol{G})$, where for a domain $\Omega \subset \boldsymbol{G}$ the space $\overset{o}{\mathcal{D}}{}^{1,p}(\Omega)$ is defined as the closure of $C_o^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overset{o}{\mathcal{D}}{}^{1,p}(\Omega)} = \left(\int_\Omega |Xu|^p dH \right)^{1/p}. \tag{3.5}$$

This fact was proved in [221] with the help of P.L. Lions' method of concentration compactness. The question of determining the norm of the embedding, i.e., the value of the best constant is open with the exception of the Euclidean case and the $p = 2$ case on the (two step Carnot) groups of Iwasawa type. In the Euclidean case, a symmetrization argument involving symmetric decreasing rearrangement, see [215], can be used to show that equality is achieved for radial functions which can be determined explicitly. As of now there is no such argument in the non-Euclidean setting which to a large degree is the reason for the much more sophisticated analysis in the sub-Riemannian setting. However, the recently found approach [92] and [32] based on the center of mass argument allows the determination of the sharp constant (in fact in the Hardy-Littlewood-Sobolev inequality) in the geometric setting of groups of Iwasawa type when $p = 2$. This analysis exploits the Cayley transform and the conformal invariance of the associated Euler-Lagrange equation which is the Yamabe equation on the corresponding Iwasawa group,

$$\triangle u = -u^{\frac{Q+2}{Q-2}}, \qquad u \in \overset{o}{\mathcal{D}}{}^{1,2}(\boldsymbol{G}), \quad u \geq 0. \tag{3.6}$$

Of course, in order to give a geometric meaning of the equation one needs to use the relevant geometries and their "canonical" connections which we do Section 4.2. In the Euclidean and CR cases these are just the well known Levi-Civita and Tanaka-Webster connections. In the quaternionic and octonian case the geometric picture emerged only after the work of Biquard [25, 26]. The goal of this section is to give some ideas surrounding the analysis of the Yamabe equation as a partial differential equation and some of the known results on the optimal constants which largely belong to the area of analysis. The key results on the optimal constants are summarized in the following two theorems in which m is the dimension of the first layer, while k is the dimension of the center of the Iwasawa algebra.

Theorem 3.1 ([139],[92],[121, 122],[62]). *Let $\boldsymbol{G}$ be a group of Iwasawa type. For every $u \in D^{1,2}(\boldsymbol{G})$ one has the Folland-Stein inequality* (3.1) *with*

$$S_2 = \frac{1}{\sqrt{m(m+2(k-1))}}\, 4^{\frac{k}{m+2k}}\, \pi^{-\frac{m+k}{2(m+2k)}} \left(\frac{\Gamma(m+k)}{\Gamma\left(\frac{m+k}{2}\right)}\right)^{\frac{1}{m+2k}}. \tag{3.7}$$

An extremal is given by the function

$$F(g) = \gamma(m,k)\ \left[(1+|x(g)|^2)^2 + 16|y(g)|^2)\right]^{-(Q-2)/4}, \tag{3.8}$$

where

$$\gamma(m,k) = \left[4^k\, \pi^{-(m+k)/2(m+2k)}\, \frac{\Gamma(m+k)}{\Gamma((m+k)/2)}\right]^{(m+2(k-1))/2(m+2k)}.$$

Any other non-negative extremal is obtained from F by (3.2) *and* (3.3)*.*

We remark that (3.8) is a solution to the Yamabe equation on any group of Heisenberg type [100], see also [143, Proposition 2] for an equation related to the case of Iwasawa groups. It also should be noted that [139] and [121] actually determine all critical points of the associated to (3.1) variational problem rather than only the functions with lowest energy. In fact, [139] solves completely the Yamabe equation (3.6) on the Heisenberg group while [121] achieves this on the seven dimensional quaternionic Heisenberg group (the higher dimensional cases are still open). We report on the ideas behind these proofs in Sections 5.1 and 6.1.2 which involve ideas inspired by Theorem 2.6. In the remaining cases of Iwasawa type groups the partial result in the next Theorem 3.2 supports the general agreement that (3.8) gives all solutions.

Theorem 3.2 ([99]). *All partially symmetric solutions of the Yamabe equation on a group of Iwasawa type are given* (3.8) *up to dilation and translation.*

For the definition of partially symmetric solution we refer to Section 3.4.

3.1. **Groups of H-type and the Iwasawa groups.** Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra equipped with a scalar product $< .,. >$ for which $\mathfrak{n} = V_1 \oplus V_2$-an orthogonal direct sum, V_2 is the center of $\mathfrak{n}$. Consider the map $J : V_2 \to End(V_1)$ defined by

$$< J(\xi_2)\xi_1', \xi_1'' > = < \xi_2, [\xi_1', \xi_1''] >, \quad \text{for } \xi_2 \in V_2 \ \text{ and } \ \xi_1', \xi_1'' \in V_1. \tag{3.9}$$

By definition we have that $J(\xi_2)$ is skew-symmetric. Adding the additional condition that it is actually an almost complex structure on V_1 when ξ_2 is of unit length [142] motivates the next definitions. A 2-step nilpotent Lie algebra $\mathfrak{n}$ is said to be of *Heisenberg type* if for every $\xi_2 \in V_2$, with $|\xi_2| = 1$, the map $J(\xi_2) : V_1 \to V_1$ is orthogonal. A simply connected connected Lie group $\boldsymbol{G}$ is called of Heisenberg type (or H-type) if its Lie algebra $\mathfrak{n}$ is of Heisenberg type. We shall use the exponential coordinates and regard $G = exp\, \mathfrak{n}$, so that the product of two elements of N is

$$(\xi_1, \xi_2)\cdot(\xi_1', \xi_2') = (\xi_1 + \xi_1', \xi_2 + \xi_2' + \frac{1}{2}[\xi_1, \xi_1']), \tag{3.10}$$

taking into account the Baker-Campbell-Hausdorff formula. Correspondingly we shall use V_i, $i = 1, 2$ to also denote the sub-bundle of left invariant vector fields which coincides with the given V_i at the identity element. In [142] Kaplan found the explicit form of the fundamental solution of the sub-Laplacian on every group of H-type, where the sub-Laplacian is the operator

$$\triangle\ = \sum_{j=1}^m X_j^2, \tag{3.11}$$

for vector fields X_j, $j = 1, \dots, m$ which are an orthonormal basis of V_1.

On a group N of Heisenberg type there is a very important homogeneous norm (gauge) given by

$$N(g) = \left(|\xi_1(g)|^4 + 16|\xi_2(g)|^2\right)^{1/4}, \tag{3.12}$$

which induces a left-invariant distance. Kaplan proved in [142] that in a group of Heisenberg type, in particular in every Iwasawa group, the fundamental solution Γ of the sub-Laplacian $\mathcal{L}$, see (3.11), is given by the formula

$$\Gamma(g,h)\ =\ C_Q\, N(h^{-1}g)^{-(Q-2)}, \qquad g, h \in N, g \neq h, \tag{3.13}$$

where C_Q is a suitable constant.

Remark 3.3. *It is known that the distance induced by the gauge* (3.12) *is the Gromov limit of a one parameter family of Riemannian metrics on the group* N [152]*, see also* [23] *and* [43].

Kaplan and Putz [144], see also [[152], Proposition 1.1], observed that the nilpotent part N in the Iwasawa decomposition $\boldsymbol{G} = NAK$ of every semisimple Lie group $\boldsymbol{G}$ of real rank one is of Heisenberg type. We shall refer to such a group as *Iwasawa group* and call the corresponding Lie algebra *Iwasawa algebra*.

The Heisenberg type groups allowed for the generalization of many important concepts in harmonic analysis and geometry, see [144], [145], [152], [74] and the references therein, in addition to the above cited papers. Another milestone was achieved in [65], which allowed for avoiding the classification rank one symmetric spaces and the heavy machinery of the semisimple Lie group theory, when studying the non-compact symmetric spaces of real rank one. Specifically, in [65] the authors considered the H-type algebras satisfying the so called J^2 condition defined in [65], see also [66].

Definition 3.4. *We say that the H-type algebra* $\mathfrak{n}$ *satisfies the* J^2 *condition if for every* $\xi_2, \xi_2' \in V_2$ *which are orthogonal to each other,* $<\xi_2, \xi_2'>= 0$*, there exists* $\xi_2'' \in V_2$ *such that*

$$J(\xi_2)J(\xi_2') = J(\xi_2''). \tag{3.14}$$

A noteworthy result here is the following Theorem of [65], see also [64], which can be used to show that if N is an H-type group, then the Riemannian space $S = NA$ is symmetric iff the Lie algebra $\mathfrak{n}$ of N satisfies the J^2 condition, see [[65], Theorem 6.1].

Theorem 3.5. *If* $\mathfrak{n}$ *is an H-type algebra satisfying the* J^2*-condition, then* $\mathfrak{n}$ *is an Iwasawa type algebra.*

This fundamental result has many consequences among them allowing a unified proof of some classical results on symmetric spaces, in addition to some beautiful properties of extensions of the classical Cayley transform, inversion and Kelvin transform, which are of a particular importance for our goals.

From a geometric point of view, the above Iwasawa groups can be seen as the nilpotent part in the Iwasawa decomposition of the isometry group of the non-compact symmetric spaces M of real rank one. Such a space can be expressed as a homogeneous space G/K where G is the identity component of the isometry group of M, i.e., one of the simple Lorentz groups $SO_o(n,1)$, $SU(n,1)$, $Sp(n,1)$ or $F_{4(-20)}$, and K is a maximal compact subgroup of G, see [114], namely, $K = SO(n)$, $SU(n)$, $Sp(n)Sp(1)$, or $Spin(9)$, respectively, see for example [227, Theorem 8.12.2] or [114]. Thus $M = H^n_{\mathbb{K}}$ is one of the hyperbolic spaces over the real, complex, quaternion or Cayley (octonion) numbers, respectively. As well known, these spaces carry canonical Riemannian metrics with sectional curvature $k = -1$ for $\mathbb{K} = \mathbb{R}$ and $-1 < k < -1/4$ in the remaining cases cases. Here, $\mathbb{K}$ denotes one of the real division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, or the octonions $\mathbb{O}$.

Writing $G = NAK$ and letting $S = NA$, A-one-dimensional Abelian subalgebra, we have that S is a closed subgroup of G, which is isometric with the hyperbolic space M, thus giving the corresponding hyperbolic space a Lie group structure. The nilpotent part N is isometrically isomorphic to $\mathbb{R}^n$ in the degenerate case when the Iwasawa group is Abelian or to one of the Heisenberg groups $\boldsymbol{G}\,(\mathbb{K}) \ = \ \mathbb{K}^n \times \operatorname{Im}\mathbb{K}$ with the group law given by

$$(q_o, \omega_o) \circ (q, \omega) \ = \ (q_o \ + \ q, \omega \ + \ \omega_o \ + \ 2\operatorname{Im} q_o\, \bar{q}), \tag{3.15}$$

where $q,\ q_o \in \mathbb{K}^n$ and $\omega, \omega_o \in \operatorname{Im}\mathbb{K}$. In particular, in the non-Euclidean case the Lie algebra $\mathfrak{n}$ of N has center of dimension $\dim V_2 = 1$, 3, or 7.

Iwasawa groups are distinguished also by the properties of the sphere product $S_1(R_1)\times S_2(R_2)$, where, for $j = 1, 2$, $S_j(R_j)$ is the sphere of radius R_j in V_j-the two layers of the 2-step nilpotent Lie algebra. In fact, for a group of Iwasawa type the Kostant double-transitivity theorem shows that the action of $A(N)$ is transitive, where as before $A(N)$ stands for the orthogonal automorphisms of N, see [66, Proposition 6.1]. This fact points to the importance of the bi-radial or cylindrically symmetric functions. Notice that both the fundamental solution of the sub-Laplacian and the known solutions of the Yamabe equation have such symmetry, see (3.13) and (3.8).

Motivated by the way the Iwasawa type groups appear as "boundaries" of the hyperbolic spaces, Damek [72] introduced a generalization of the hyperbolic spaces as follows. For a group N of H-type consider a semidirect product with a one dimensional Abelian group, i.e., take the multiplicative group $A = \mathbb{R}^+$ acting on an H-type group by dilations given in exponential coordinates by the formula $\delta_a(\xi_1,\xi_2)=(a^{1/2}\xi_1, a\xi_2)$ and define $S=NA$ as the corresponding semidirect product. Thus, the Lie algebra of S is $\mathfrak{s}=V_1\oplus V_2\oplus\mathfrak{a}$, $\mathfrak{n}=V_1\oplus V_2$, with the bracket extending the one on $\mathfrak{n}$ by adding the rules

$$[\zeta,\xi_1]=\frac{1}{2}\xi_1,\quad [\zeta,\xi_2]=\xi_2\quad \xi_i\in V_i, \tag{3.16}$$

where ζ is a unit vector in $\mathfrak{a}$, so that S is the connected simply connected Lie group with Lie algebra $\mathfrak{s}$. In the coordinates $(\xi_1,\xi_2,a)=\exp(\xi_1+\xi_2)\exp(\log a\zeta)$, $a>0$, which parameterize $S=exp\,\mathfrak{s}$, the product rule of S is given by the formula

$$(\xi_1,\xi_2,a)\cdot(\xi_1',\xi_2',a')=(\xi_1+a^{1/2}\xi_1',\,\xi_2+a\xi_2'+\frac{1}{2}a^{1/2}[\xi_1,\xi_1'],\,aa'), \tag{3.17}$$

for all (ξ_1,ξ_2,a), $(\xi_1',\xi_2',a')\in\mathfrak{n}\times\mathbb{R}^+$. Notice that S is a solvable group. We equip the Lie algebra $\mathfrak{s}$ with the inner product

$$<(\xi_1,\xi_2,a),(\tilde\xi_1,\tilde\xi_2,\tilde a)>=<(\xi_1,\xi_2),(\tilde\xi_1,\tilde\xi_2)>+a\tilde a \tag{3.18}$$

using the fixed inner product on $\mathfrak{n}$ and then define a corresponding translation invariant Riemannian metric on S. The main result of [72] is that the group of isometries $Isom(S)$ of S is as small as it can be and equals $A(S)\ltimes S$ with S acting by left translations, unless N is one of the Heisenberg groups (3.15), i.e., S is one of the classical hyperbolic spaces. Here, $A(S)$ denotes the group of automorphisms of S (or $\mathfrak{s}$) that preserve the left-invariant metric on S. The spaces constructed in this manner became known as Damek-Ricci spaces, see [24] for more details. It was shown in [73] that the just described solvable extension of H-type groups, which are not of Iwasawa type, provide noncompact counter-examples to a conjecture of Lichnerowicz, which asserted that harmonic Riemannian spaces must be rank one symmetric spaces.

3.2. **The Cayley transform.** In this section we focus on the Cayley transform, of which we shall make extensive use later. Here, we give the well known abstract definition valid in the setting of groups of H-type. Other explicit formulas will be given in the CR and QC cases in Sections 4.3 and 4.5.4. Starting from an H-type group, its solvable extension S defined above has the following realizations, [74], [65] and [66].

First, consider the "Siegel domain" or an upper-half plane model of the hyperbolic space

$$D=\{p=(\xi_1,\xi_2,a)\in\mathfrak{s}=V_1\oplus V_2\oplus\mathfrak{a}:\ a>\frac{1}{4}|\xi_1|^2\}. \tag{3.19}$$

Consider the map $\Theta:S\to S$,

$$\Theta(\xi_1,\xi_2,a)=(\xi_1,\xi_2,a+\frac{1}{4}|\xi_1|^2), \tag{3.20}$$

which is injective map of S into itself. Here we use a to denote the element $a\zeta\in A$, ζ defined after (3.16), and we regard D as a subset of S using the exponential coordinates. Thus, the group S acts simply transitively on D by conjugating left multiplication in the group S by Θ, $s\cdot p=\Theta s\cdot(\Theta^{-1}p)$ for $s\in S$ and $p\in D$, while N acts simply transitively on the level sets of $h=a-\frac{1}{4}|\xi_1|^2$. In particular, we can define an invariant metric on D by pulling via Θ the left-invariant metric (3.18) of S to D, thus making Θ an isometry, cf. [[66], (3.3)].

Second, there is the "ball" model of S,

$$B=\{(\xi_1,\xi_2,a)\in\mathfrak{s}=V_1\oplus V_2\oplus\mathfrak{a}:\ |\xi_1|^2+|\xi_2|^2+a^2<1\}, \tag{3.21}$$

equipped with the metric obtained from D via the inverse of the so called *Cayley transform* $\mathcal{C}:B\to D$ defined by $\mathcal{C}(\xi_1,\xi_2,a)=({\xi_1}',{\xi_2}',a')$, where

$$\begin{aligned}\xi_1'&=\frac{2}{(1-a)^2+|\xi_2|^2}\left((1-a)\xi_1+J(\xi_2)\xi_1\right),\\ \xi_2'&=\frac{2}{(1-a)^2+|\xi_2|^2}\,\xi_2,\qquad a'=\frac{1-a^2-|\xi_2|^2}{(1-a)^2+|\xi_2|^2}.\end{aligned} \tag{3.22}$$

The inverse map $\mathcal{C}^{-1} : D \to B$ is given by $\mathcal{C}^{-1}(\xi_1', \xi_2', a') = (\xi_1, \xi_2, a)$, where

$$\begin{aligned} \xi_1 &= \frac{2}{(1+a')^2 + |\xi_2'|^2} \left((1+a')\xi_1' - J(\xi_2')\xi_1' \right), \\ \xi_2 &= \frac{2}{(1+a')^2 + |\xi_2'|^2} \, \xi_2', \qquad a = \frac{-1 + a'^2 - |\xi_2'|^2}{(1+a')^2 + |\xi_2'|^2}. \end{aligned} \tag{3.23}$$

For other versions of the Cayley transform see [[83], Chapter X]. The Jacobian of $\mathcal{C}$ and its determinant were computed in [74]. The latter is given by the formula $\det \mathcal{C}'(\xi_1, \xi_2, a) = 2^{m+k+1} \left((1-a)^2 + |\xi_2|^2 \right)^{-(m+2k+2)/2}$, where, as before, $m = \dim V_1$, $k = \dim V_2$.

It is very important and we shall make use of the fact that the Cayley transform can be extended by continuity to a bijection (denoted by the same letter!)

$$C : \partial B \setminus \{(0,0,1)\} \to \partial D, \tag{3.24}$$

where $(0,0,1)$ (referred to as "ζ" for short) is the point on the sphere where $\xi_1 = \xi_2 = 0$ and the third component is ζ in agreement with out notation set after equation (3.20). The boundaries of the ball and Siegel domain models are, respectively,

$$\Sigma \equiv \partial D = \{ p = (\xi_1', \xi_2', a') \in \mathfrak{s} = V_1 \oplus V_2 \oplus \mathfrak{a} : \ a' = \frac{1}{4} |\xi_1'|^2 \} \tag{3.25}$$

and

$$\partial B = \{ (\xi_1, \xi_2, a) \in \mathfrak{s} = V_1 \oplus V_2 \oplus \mathfrak{a} : \ |\xi_1|^2 + |\xi_2|^2 + a^2 = 1 \}, \tag{3.26}$$

The group of Heisenberg type N can be identified with Σ via the map

$$(\xi_1', \xi_2') \mapsto (\xi_1', \xi_2', \frac{1}{4} |\xi_1'|^2). \tag{3.27}$$

With this identification we obtain the form of the Cayley transform (stereographic projection) identifying the sphere minus the point "ζ" and the H-type group, $\mathcal{C} : \partial B \setminus \{(0,0,1)\} \to N$ defined by $\mathcal{C}(\xi_1, \xi_2, a) = (\xi_1', \xi_2')$, where

$$\begin{aligned} \xi_1' &= \frac{2}{(1-a)^2 + |\xi_2|^2} \left((1-a)\xi_1 + J(\xi_2)\xi_1 \right), \\ \xi_2' &= \frac{2}{(1-a)^2 + |\xi_2|^2} \, \xi_2. \end{aligned} \tag{3.28}$$

Later, we shall make use of this "boundary" Cayley transform in the case of the Heisenberg and quaternionic Heisenberg group in which place we shall give some other explicit formulas. In particular, we shall use that the Cayley transform is a pseudoconformal map in the CR case and quaternionic contact conformal transformation in the QC case. The Cayley transform is also a 1-quasiconformal map [14], see also [8]. The definition of the "horizontal" space in the tangent bundle of the sphere and the distance function on the sphere require a few more details for which we refer to [66] and [14]. Multicontact maps and their rigidity in Carnot groups have been studied in [196], [198], [153], [69], [70], [40], [76], [191], [192], [193].

3.3. **Regularity of solutions to the Yamabe equation.** In order for the geometric analysis to proceed we need the next regularity result for the Euler-Lagrange equation associated to the problem of the optimal constant in (3.1).

Theorem 3.6. *Let Ω be an open set in a Carnot group $\boldsymbol{G}$. Suppose $u \in \overset{o}{\mathcal{D}}{}^{1,p}(\Omega)$ is a weak solution to the equation*

$$\sum_{i=1}^{m} X_i(|Xu|^{p-2} X_i u) \ = \ - \, V \, u^{p-1} \qquad \text{in} \quad \Omega. \tag{3.29}$$

a) If $u \geq 0$ and $V \in L^t(\Omega)$ for some $t > \frac{Q}{p}$, then u satisfies the Harnack inequality: for any Carnot-Carathédory (or gauge) ball $B_{R_0}(g_0) \subset \Omega$ there exists a constant $C_0 > 0$ such that

$$\operatorname*{esssup}_{B_R} u \leq C_0 \operatorname*{essinf}_{B_R} u, \tag{3.30}$$

for any Carnot-Carathédory (or gauge) ball $B_R(g)$ such that $B_{4R}(g) \subset B_{R_0}(g_0)$.

b) If $u \in \overset{o}{\mathcal{D}}{}^{1,p}(\Omega)$ is a weak solution to (3.29) and $V \in L^t(\Omega) \cap L^{Q/p}(\Omega)$, then $u \in \Gamma_\alpha(\Omega)$ for some $0 < \alpha < 1$.

c) If $u \in \overset{o}{\mathcal{D}}^{1,2}(\Omega)$ is a non-negative solution of the Yamabe equation on the domain Ω,

$$\triangle u \;=\; -u^{2^*-1}, \tag{3.31}$$

then either $u > 0$ and $u \in \mathcal{C}^\infty\,(\Omega)$ or $u \equiv 0$.

The Hölder regularity of weak solutions of equation (3.29) follows from a suitable adaptation of the classical De Giorgi-Nash-Moser result. The higher regularity when $p = 2$ follows by an iteration argument based on sub-elliptic regularity. A detailed proof of Theorem 3.6 can be found in [130, Theorem 1.6.9]. It is simply a combination of the fundamental Harnack's inequality of [42, Theorem 3.1] (valid for Hörmander type operators), the boundedness of the weak solution [221, Theorem 4.1], the regularity of [42, Theorem 3.35], and the sub-elliptic regularity result concerning Hörmander type operators acting on non-isotropic Sobolev or Lipschitz spaces of [90, 88], see also [89] for a general overview and further details. Note that these results together with the idea of [90] to "osculate" with the Heisenberg group carry over to obtain $\mathcal{C}^\infty$ regularity in the CR and QC settings, see [137, 138] and [225] for details.

3.4. **Solution of the Yamabe type equation with partial symmetry.** By Theorem 3.6 any weak solution of the Yamabe equation is actually a smooth bounded function which is everywhere strictly positive, $u > 0$ and $u \in \mathcal{C}^\infty\,(\Omega)$. The symmetries we are concerned are the following.

Definition 3.7. *Let $\boldsymbol{G}$ be a Carnot group of step two with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$. We say that a function $U : \boldsymbol{G} \to \mathbb{R}$ has* partial symmetry *(with respect to a point $g_o \in \boldsymbol{G}$) if there exists a function $u : [0,\infty) \times V_2 \to \mathbb{R}$ such that for every $g = \exp(x(g) + y(g)) \in \boldsymbol{G}$ one has $\tau_{g_o}\, U(g) = u(|x(g)|, y(g))$. A function U is said to have* cylindrical symmetry *(with respect to $g_o \in \boldsymbol{G}$) if there exists $\phi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ for which $\tau_{g_o}\, U(g) = \phi(|x(g)|, |y(g)|)$, for every $g \in \boldsymbol{G}$.*

The proof of Theorem 3.2 due to [99] consists of two steps, first one shows that any entire solution with partial symmetry is cylindrically symmetric and then that all entire solutions with cylindrical symmetries.

The proof of the first result relies an adaption of the method of moving hyper-planes due to Alexandrov [4] and Serrin [211]. The moving plane technique was developed further in the two celebrated papers [104], [105] by Gidas, Ni and Nirenberg to obtain symmetry for semi-linear equations with critical growth in $\mathbb{R}^n$ or in a ball. In our proof we incorporate some important simplification of the proof in [105] due to Chen and Li [54]. We mention that a crucial role is played by the knowledge of the explicit solutions (3.8) and also by the inversion and the related Kelvin transform introduced by Korányi for the Heisenberg group [151], and subsequently generalized to groups of Heisenberg type in [68], [65], see also [100] for properties of the Kelvin transform.

The proof of the second main result has been strongly influenced by the approach of Jerison and Lee for the Heisenberg group, see Theorem 7.8 in [138]. After a change in the dependent variable, which relates the Yamabe equation to a new non-linear pde in a quadrant of the Poincaré half-plane, one is led to prove that the only positive solutions of the latter are quadratic polynomials of a certain type.

Besides ideas from Jerison and Lee's paper, the proof has some features of the method of the so-called P*-functions* introduced by Weinberger in [222]. Given a solution u of a certain partial differential equation, such method is based on the construction of a suitable non-linear function of u and $grad\ u$, a P-function, which is itself solution (or sub-solution) to a related partial differential equation, and therefore satisfies a maximum principle. In fact, starting with a cylindrical solution U of the Yamabe equation 3.6, the function $\phi = v^{-4/(Q-2)}$ where $v = \left(\frac{Q-2}{4}\right)^{-(Q-2)/2} U$ satisfies

$$\mathcal{L}\phi = (\frac{Q-2}{4} + 1)\,\frac{|X\phi|^2}{\phi} + \frac{Q-2}{4}. \tag{3.32}$$

By the cylindrical symmetry assumption Φ is a function of the variables

$$y = \frac{|\xi_1|^2}{4}, \qquad x = |xi_2|, \tag{3.33}$$

which satisfies the equation

$$\Delta\phi = \frac{n+2}{2}\,\frac{|\nabla\phi|^2}{\phi} - \frac{a}{x}\,\phi_x - \frac{b}{y}\,\phi_y + \frac{n}{2y}, \tag{3.34}$$

in $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x>0, y>0\}$ with $a = k-1 \ \geq \ 0$, $b = \frac{m}{2} \ \geq \ 1$ and $n = a+b \ \geq 1$. The case $k=1$ corresponds to the Heisenberg group $\mathbb{H}^n$, and it was considered earlier in [138]. A long calculation shows that with $h = x^a y^b \phi^{-(n+1)}$,

$$F = 2 < \nabla\phi, \nabla\phi_x > -2\frac{n}{2b}\,\phi_{xy} - \phi_x\,\frac{|\nabla\phi|^2}{\phi} \quad \text{and} \quad G = -2 < \nabla\phi, \nabla\phi_y > +2\frac{n}{2b}\,\phi_{yy} + (\phi_y - \delta)\,\frac{|\nabla\phi|^2}{\phi},$$

the following identity holds true

$$(hF)_x - (hG)_y = \ h\left\{\left[2\,||\nabla^2\phi||^2 - (\Delta\phi)^2\right] + \ \frac{n+2}{n}\ \left(\Delta\phi - \frac{|\nabla\phi|^2}{\phi}\right)^2 + \frac{2ab}{n}\left(\frac{\phi_x}{x} - (\frac{\phi_y}{y} - \frac{n}{2by})\right)^2\right\}.$$

An integration over the first quadrant, noting that the integrals are finite as a consequence of the properties of the Kelvin transform on a group of Iwasawa type, we obtain

$$2\,||\nabla^2\phi||^2 = (\Delta\phi)^2, \quad \Delta\phi - \frac{|\nabla\phi|^2}{\phi} = 0, \quad \frac{\phi_x}{x} = \frac{\phi_y}{y} - \frac{n}{2by}. \tag{3.35}$$

We remark that the Kelvin transform allows us to find the asymptotic behavior of every solution of the Yamabe equation, including all its derivatives. The behaviour at infinity of a finite energy solution can be found in more general settings with the method of [159]. From the first two equations in (3.35) we conclude (see, e.g., [222] or also [138]) that ϕ must be of form

$$\phi(x,y) = A^2\,(x^2+y^2) + 2A\alpha x + 2B\beta y + \alpha^2 + \beta^2 \tag{3.36}$$

for some numbers A, B, α and β, with $A^2 = B^2$. On the other hand, the third equation in (3.35) implies that $\alpha = 0$ and $\beta = \frac{n}{4bB}$. Recalling that $x = |\xi_2|, y = |\xi_1|^2/4$ one easily concludes from the above that

$$\phi(|\xi_1|, |\xi_2|) = \frac{A^2}{16}\ \left[(\frac{a+b}{bA^2} + |\xi_1|^2)^2 + 16|\xi_2|^2\right] \tag{3.37}$$

for some $A \neq 0$, hence

$$\phi(|\xi_1|, |\xi_2|) = \frac{Q-2}{16m\epsilon^2}\ [(\epsilon^2 + |\xi_1|^2)^2 + 16|\xi_2|^2] \tag{3.38}$$

where $\epsilon^2 = \frac{Q-2}{mA}$. Finally, the relation between Φ and U, we obtain

$$U(g) = C_\epsilon\,((\epsilon^2 + |x(g)|^2)^2 + 16|y(g)|^2)^{-(Q-2)/4},$$

with $C_\epsilon = [m(Q-2)\epsilon^2]^{(Q-2)/4}$. All other cylindrically symmetric solutions are obtained from this one by left-translation, which completes the proof of Theorem 3.2.

We remark that (3.34) was used in [220], see also [179], to establish the sharp constant and the extremals in a L^2 Hardy-Sobolev inequality involving distance to a lower dimensional subspace.

3.5. **The best constant in the L^2 Folland-Stein inequality on the quaternionic Heisenberg groups.** In this section we explain the ideas behind the proof of Theorem 3.1.

The proof relies on the realization made in [32] and used more recently in [92] that the "center of mass" idea of Szegö [213] and Hersch [115] can be used to find the sharp form of (logarithmic) Hardy-Littlewood-Sobolev type inequalities on the Heisenberg group. This method does not give all solutions of the Yamabe equation on the Iwasawa group, but is enough to determine the best constant.

The Cayley transform and the conformal nature of the problem are crucial for its solution. Another key is Theorem 4.1 which will be used to see that the constants are the only minimizers on the sphere among all positive local minimizers which viewed as densities place the center of mass of the sphere at the origin. *We shall focus here on the qc case* [122] *but the argument is valid in any of the groups of Iwasawa type using the just mentioned facts.*

Let $\tilde\eta$, cf. (4.29), be the standard qc structure on the unit sphere S^{4n+3}. Szegö and Hersch's center of mass method suggests the following lemma.

Lemma 3.8. *For every $v \in L^1(S^{4n+3})$ with $\int_{S^{4n+3}} v\, Vol_{\tilde\eta} = 1$ there is a quaternionic contact conformal transformation $\psi : (S^{4n+3}, \tilde\eta) \to (S^{4n+3}, \tilde\eta)$ such that*

$$\int_{S^{4n+3}} \psi\, v\, Vol_{\tilde\eta} = 0.$$

Proof. Fix a point $P \in S^{4n+3}$ on the quaternionic sphere and denote by N its antipodal point and consider the local coordinate system near P defined by the Cayley transform $\mathcal{C}_N$ from N, see (4.5.4). We know that $\mathcal{C}_N$ is a quaternionic contact conformal transformation between $S^{4n+3} \setminus N$ and the quaternionic Heisenberg group, cf. (4.31). Notice that in this coordinate system P is mapped to the identity of the group. For every r, $0 < r < 1$, let $\psi_{r,P}$ be the qc conformal transformation of the sphere, which in the fixed coordinate chart is given on the group by a dilation with center the identity by a factor δ_r. If we select a coordinate system in $\mathbb{R}^{4n+4} = \mathbb{H}^n \times \mathbb{H}$ so that $P = (1, 0)$ and $N = (-1, 0)$. Applying the Cayley transform (4.5.4) to $(q^*, p^*) = \psi_{r,P}(q, p)$ we have

$$q^* = 2r\left(1 + r^2(1+p)^{-1}(1-p)\right)^{-1}(1+p)\,q$$
$$p^* = \left(1 + r^2(1+p)^{-1}(1-p)\right)^{-1}\left(1 - r^2(1+p)^{-1}(1-p)\right), i.e,$$

Consider the map $\Psi : B \to \bar{B}$, where B ($\bar{B}$) is the open (closed) unit ball in $\mathbb{R}^{4n+4}$, by the formula

$$\Psi(rP) = \int_{S^{4n+3}} \psi_{1-r,P}\, v\, Vol_{\tilde{\eta}}.$$

Notice that Ψ can be continuously extended to $\bar{B}$ since for any point P on the sphere, where $r = 1$, we have $\psi_{1-r,P}(Q) \to P$ when $r \to 1$. In particular, $\Psi = id$ on S^{4n+3}. Since the sphere is not a homotopy retract of the closed ball it follows that there are r and $P \in S^{4n+3}$ such that $\Psi(rP) = 0$, i.e., $\int_{S^{4n+3}} \psi_{1-r,P}\, v\, Vol_{\tilde{\eta}} = 0$. Thus, $\psi = \psi_{1-r,P}$ has the required property. $\square$

In the next step one proves that there is a minimizer of the Folland-Stein inequality which satisfies the zero center of mass condition. A number of well known invariance properties of the Yamabe functional are exploited. For the rest of the Section, given a qc form η and a function u we will denote by $\nabla^\eta u$ the horizontal gradient of u.

We shall call a (positive) function u on the sphere a *well centered* function when viewing u^{2^*} as a density it places the center of mass of the sphere at the origin, i.e.,

$$\int_{S^{4n+3}} P\, u^{2^*}(P)\, Vol_{\tilde{\eta}} = 0, \qquad P \in \mathbb{R}^{4n+4} = \mathbb{H}^n \times \mathbb{H}. \tag{3.39}$$

For the next Lemma recall the functionals $\mathcal{E}_{\tilde{\eta}}$ and $\mathcal{N}_{\tilde{\eta}}$ introduced in (4.6).

Lemma 3.9. *Let v be a smooth positive function on the sphere with $\mathcal{N}_{\tilde{\eta}}(u) = 1$. There is a well centered smooth positive function u such that $\mathcal{E}_{\tilde{\eta}}(u) = \mathcal{E}_{\tilde{\eta}}(v)$ and $\mathcal{N}_{\tilde{\eta}}(u) = 1$. In particular, the Yamabe constant (4.7) is achieved for a positive function u which is well centered, i.e., for a function u satisfying (3.39).*

Proof. Given a positive function v on the sphere $\int_{S^{4n+3}} v^{2^*}\, Vol_{\tilde{\eta}} = 1$, consider the function

$$u = \phi^{-1}(v \circ \psi^{-1}), \tag{3.40}$$

where ψ is the qc conformal map of Lemma 3.8, $\eta \equiv (\psi^{-1})^* \tilde{\eta}$, and ϕ is the corresponding conformal factor of ψ. The claim of the Lemma follows directly from the conformal invariance (4.10). $\square$

The next step shows that a well centered minimizer has to be constant.

Lemma 3.10. *If u is a well centered local minimum of the problem (4.7) for $M = (S^{4n+3}, \tilde{\eta})$, then $u \equiv const$.*

Proof. Let ζ be a smooth function on the sphere S^{4n+3}. Recalling (4.6), with the help of the divergence formula (4.4) we obtain the formula

$$\mathcal{E}(\zeta u) = \int_{S^{4n+3}} \zeta^2 \Big(4\frac{Q+2}{Q-2}\, |\nabla^{\tilde{\eta}} u|^2 + \tilde{S}\, u^2\Big)\, Vol_{\tilde{\eta}} - 4\frac{Q+2}{Q-2} \int_{S^{4n+3}} u^2 \zeta\, \tilde{\triangle}\zeta\, Vol_{\tilde{\eta}}. \tag{3.41}$$

At this point we let ζ be an eigenfunction corresponding to the first eigenvalue of the sub-Laplacian $\tilde{\triangle}$ associated to $\tilde{\eta}$, $\tilde{\triangle}\zeta = -\lambda_1 \zeta$. Remarkably, the first eigenspace of the standard sub-Laplacian is spanned by restrictions to the sphere of the linear (coordinate functions) in $R^{4n+4} = \mathbb{H}^n \times \mathbb{H}$, see Theorem 4.1.

Computing the second variation $\delta^2 \Upsilon(u) v = \frac{d^2}{dt^2} \Upsilon(u + tv)_{|t=0}$ of $\Upsilon(u)$ we see that the local minimum condition $\delta^2 \Upsilon(u) v \geq 0$ implies

$$\mathcal{E}(v) - (2^* - 1)\mathcal{E}(u) \int_{S^{4n+3}} u^{2^*-2} v^2\, Vol_{\tilde{\eta}} \ \geq 0$$

for any function v such that $\int_{S^{4n+3}} u^{2^*-1} v\, Vol_{\tilde\eta} = 0$. Therefore, for ζ being any of the coordinate functions in $\mathbb{H}^n \times \mathbb{H}$ we have (taking $v = \zeta u$ and recalling that u is well centered)

$$\mathcal{E}(\zeta u) - (2^* - 1)\mathcal{E}(u) \int_{S^{4n+3}} u^{2^*} \zeta^2\, Vol_{\tilde\eta} \geq 0,$$

which after a summation over all coordinate functions and a use of (3.41) gives

$$\mathcal{E}(u) - (2^* - 1)\mathcal{E}(u) + 4\lambda_1(2^* - 1) \int_{S^{4n+3}} u^2\, Vol_{\tilde\eta} \geq 0,$$

which implies, recall $2^* - 1 = (Q + 2)/(Q - 2)$,

$$0 \leq 4(2^* - 1)\,(2^* - 2) \int_{S^{4n+3}} |\nabla^{\tilde\eta} u|^2\, Vol_{\tilde\eta} \leq \left(4\lambda_1(2^* - 1) - (2^* - 2)\,\tilde S\right) \int_{S^{4n+3}} u^{2^*}\, Vol_{\tilde\eta}.$$

By Theorem a 4.1 we have actually equality $\lambda_1 = \tilde S/(Q + 2)$, hence $|\nabla^{\tilde\eta} u| = 0$, which completes the proof. □

After these preliminaries we turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. Let F be a minimizer (local minimum) of the Yamabe functional $\mathcal{E}$ on $\boldsymbol{G}\,(\mathbb{H})$ and f the corresponding function on the sphere defined with the help of the Cayley transform by

$$f = \mathcal{C}^*(F\Phi^{-1}), \tag{3.42}$$

where Φ is a solution of the Yamabe equation on $\boldsymbol{G}\,(\mathbb{H})$ defined in (4.32). By the conformality of the qc structures on the group and the sphere we have by (4.8) $Vol_\Theta = \Phi^{2^*} Vol_{\tilde\Theta}$, hence $F^{2^*} Vol_{\tilde\Theta} = f^{2^*}\phi^{-2^*} Vol_{\tilde\eta}$, where $\phi = \mathcal{C}^*(\Phi)$. This, together with the Yamabe equation implies that the Yamabe integral is preserved

$$\int_{\boldsymbol{G}\,(\mathbb{H})} a|\nabla^{\tilde\Theta} F|^2\, Vol_{\tilde\Theta} \;=\; \int_{S^{4n+3}} \left(a|\nabla^{\tilde\eta} f|^2 + \tilde S f^2\right) Vol_{\tilde\eta}, \tag{3.43}$$

where $a = 4(Q + 2)/(Q - 2)$. By Lemma 3.9 and (3.40) the function $f_0 = \phi^{-1}(f \circ \psi^{-1})$ will be well centered minimizer (local minimum) of the Yamabe functional Υ on S^{4n+3}. The latter claim uses also the fact that the map $v \mapsto u$ of equation (3.40) is one-to-one and onto on the space of smooth positive functions on the sphere. Now, from Lemma 3.10 we conclude that $g_o = const$. Looking back at the corresponding functions on the group we see that

$$F_0 = \gamma\left[(1 + |q'|^2)^2 + |\omega'|^2\right]^{-(Q-2)/4}$$

for some $\gamma = const. > 0$. Furthermore, the proof of Lemma 3.8 shows that F_0 is obtained from F by a translation (3.2) and dilation (3.3).

Remark 3.11. *We remark that the above argument shows that any local minimum of the Yamabe functional Υ on the sphere (or the Iwasawa group) has to be a global one.*

The Yamabe constant of the sphere is calculated immediately by taking a constant function in the Yamabe functional and a use of (4.34).

The remaining part of the proof (the value of the best constant S_2) is quite straightforward. Since it involves mainly calculations depending on the chosen normalization of the contact form we refer to [130, Section 6.7] for the details. This completes the proof of Theorem 3.1. □

Remark 3.12. *One should keep in mind that the the standard basis (4.27) is not an orthonormal basis which turns the group $\boldsymbol{G}\,(\mathbb{H})$ into a group of H-type, cf. also (4.16) and the paragraph above it. The two constants differ by a multiple of 4^{-k} in the general case of a group of Iwasawa type with center of dimension k. For more details on the relation between the Haar measure and the volume form associated to the contact form, as well as the exact relation between the best constants computed with respect to different bases see* [130, p. 188–189].

4. SUB-RIEMANNIAN GEOMETRY AS CONFORMAL INFINITIES

4.1. **Riemannian case.** Let (N, h) be a Riemannian manifold with boundary $M = \partial N$ with defining function $r > 0$ on interior of N which vanishes of order one on M. Suppose that $r^2 \cdot h$ extends continuously to M thus defining a "conformal structure" on the boundary M. Fefferman & Graham [84] reversed the construction and used "canonical asymptotically hyperbolic (AH) filling" metrics to obtain conformal invariants. This is of interest also because of the AdS/CFT correspondence in physics relating gravitational theories on N with conformal theories on M. More specifically, if one can associate to a conformal class on M a canonical AH filling, then the Riemannian invariants for the interior metric give conformal invariants of the boundary structure.

For a basic example, consider on the open unit ball B in $\mathbb{R}^n$ the hyperbolic metric

$$h = \frac{4}{\rho^2} g_{euc}, \qquad \rho = 1 - |x|^2.$$

The conformal infinity is the conformal class of $g_{euc}|_{\partial B}$ - the standard metric on the unit sphere. Graham & Lee [107] gave the first general examples of AH Einstein metrics. The idea has been very useful especially due to its relation with the AdS/CFT correspondence [178] in physics.

4.2. **Conformal Infinities and Iwasawa Sub-Riemannian geometries.** The main references here are [25, 26] where the sub-Riemannian structures and geometries on the spheres at infinity of the hyperbolic spaces were used as model spaces for a wide class of sub-Riemannian structures which we shall call Iwasawa Sub-Riemannian geometries. As a motivation we start with a few examples based on the real, complex and quaternion hyperbolic cases which. An explicit description of the octonian hyperbolic plane and the ball model can be found in [185] while [25] is the reference for the corresponding conformal infinity.

On the open unit ball B in $\mathbb{C}^{n+1}$ consider the Bergman metric

$$h \;=\; \frac{4}{\rho} g_{euc} \;+\; \frac{1}{\rho^2}\left((d\rho)^2 + (Id\rho)^2\right), \qquad \rho \;=\; 1 - |x|^2.$$

Notice that as $\rho \to 0$ we have that $\rho \cdot h$ is finite only on H-the so called *horizontal space*, $H \;=\; Ker\,(I\, d\rho)$, which is the kernel of the contact form $\theta \;=\; I\, d\rho$. The conformal infinity of $\rho{\cdot}h$ is the conformal class of a pseudohermitian CR structure defined by H and θ. If we look for Kähler-Einstein deformations Cheng & Yau [59] showed that any smooth (in fact C^2) strictly pseudoconvex domain in $\mathbb{C}^{n+1}$ admits a unique complete Kähler-Einstein metric of constant holomorphic sectional curvature -1 which is asymptotic to the CR-structure of the boundary, see also [183] for an extension to an arbitrary bounded domain of holomorphy.

In the quaternion case, consider the open unit ball B in $\mathbb{H}^{n+1}$ consider the hyperbolic metric $h \;=\; \frac{4}{\rho} g_{euc} + \frac{1}{\rho^2}\left((d\rho)^2 + (I_1 d\rho)^2 + (I_2 d\rho)^2 + (I_3 d\rho)^2\right)$. The conformal infinity is the conformal class of a quaternionic contact structure. In fact, $\rho{\cdot}h$ defines a conformal class of degenerate metrics with kernel

$$H = \cap_{j=1}^3 Ker\,(I_j\, d\rho).$$

Biquard showed that the infinite dimensional family [160] of complete quaternionic-Kähler deformations of the quaternion hyperbolic metric have conformal infinities which provide an infinite dimensional family of examples of qc structures. Conversely, according to [25] every real analytic qc structure on a manifold M of dimension at least eleven is the conformal infinity of a unique quaternionic-Kähler metric defined in a neighborhood of M.

Finally, [25] considered CR and qc structures as boundaries of infinity of Einstein metrics rather than only as boundaries at infinity of Kähler-Einstein and quaternionic-Kähler metrics, respectively. In fact, [25] showed that in each of the three cases (complex, quaternionic, octoninoic) any small perturbation of the standard Carnot-Carathéodory structure on the boundary is the conformal infinity of an essentially unique Einstein metric on the unit ball, which is asymptotically symmetric. Various explicit examples of qc structures were constructed in [6].

In the above examples the geometry at the conformal infinity is asymptotic to the hyperbolic geometry of the corresponding symmetric space of noncompact type of real rank one G/K, see paragraphs after Theorem 3.5. The corresponding geometries at infinity are conformal metrics, CR structures, quaternionic-contact structures or octonionic contact structures as we define below following [25, 26]. The symmetric case belongs to the parabolic geometries modelled on G/P, where P is a minimal parabolic subgroup of G, see [38]. We mention another class of asymptotic geometries considered in [7] which are no longer asymptotic to a symmetric space, but the model at infinity is a homogeneous Einstein space, which m*ay vary from point to point* on the boundary at infinity. Such a construction is motivated by Heber [113] who showed that every deformation of the solvable group $S = NA$ carries a unique homogeneous Einstein metric. Thus, deformations of the nilpotent Lie algebra $\mathfrak{n}$ give a homogeneous Einstein metric on the corresponding solvable group S.

Leaving the real case aside, we turn to the precise definition of the general (sub-Riemannian) geometric setting of the above constructions. Let G be one of the groups $U(n)$, $Sp(n)Sp(1)$ or $Spin(7)$, corresponding to the complex, quaternionic or octonionic cases, respectively, recalling the homogeneous models of the corresponding boundary spheres of the hyperbolic space, namely, $S^{2n+1} = U(n+1)/U(n)$, $S^{4n+3} = Sp(n+1)Sp(1)/Sp(n)Sp(1)$ or $S^7 = Spin(9)/Spin(7)$. Let M be a manifold with a 1-form η with values in $\mathbb{R}, \mathbb{R}^3$ or $\mathbb{R}^7$, respectively, whose kernel $H = Ker\,\eta$ - the so called *horizontal distribution* - is of co-dimension $k = 1,\ 3$ or 7, respectively. Following Biquard [25], a Carnot-Carathéodory metric (positive definite symmetric two tensor) compatible with $d\eta$ is defined to be a metric g on H such that:

i) in the complex case, the restriction $\omega = \frac{1}{2}d\eta|_H$ is a symplectic form on H compatible with g, i.e., $\omega(\cdot,\cdot) = g(I\cdot,\cdot)$ where I is an almost complex structure on H;
ii) in the quaternionic case, the three 2-forms $\omega_i = \frac{1}{2}d\eta_i|_H$, $i = 1,2,3$, on H are the fundamental forms of a quaternionic structure compatible with g, i.e., $\omega_i(\cdot,\cdot) = g(I_i\cdot,\cdot)$ for almost complex structures I_i satisfying the quaternionic commutation relations;
iii) in the octonionic case, the seven 2-forms $\omega_i = \frac{1}{2}d\eta_i|_H$, $i = 1,\dots,7$ on H provide a $Spin(7)$ structure compatible with g, i.e., $\omega_i(\cdot,\cdot) = g(I_i\cdot,\cdot)$ for almost complex structures I_i satisfying the octonionic commutation relations.

We shall call the geometric structures above *Iwasawa sub-Riemannian geometries* since at every point the osculating nilpotent group [90], [201] is isomorphic to the corresponding (non-degenerate) Iwasawa group. For simplicity, the above definition of CR, qc and octonionic contact structures requires the existence of a global 1-form defining H. The obstructions to global existence of such a form in the CR and qc cases are the first Stiefel-Whitney class and the first Pontryagin class of M, respectively.

The complex case defines a strictly pseudoconvex almost CR manifold with a fixed pseudo-Hermitian structure, which is a CR structure when the integrability condition $[IX, IY] - [X, Y] \in H$ for $X, Y \in H$ holds. In the quaternionic and octonionic cases, a distribution H for which a Carnot-Carathéodory H-metric exists will be called a quaternionic contact structure or an octonionic contact structure. The focus here will be mainly the CR and qc cases. Note that the topological dimensions of these manifolds are $2n+1$ and $4n+3$, respectively. The so called *homogeneous dimension* of M is $Q = m + 2k$ where $m =$ dim H and $k =$ codim H. We shall denote with Vol_η the volume forms

$$Vol_\eta = \eta \wedge (d\eta)^n \qquad \text{and} \qquad Vol_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \Omega^n,$$

in the CR ($n = m/2$) and qc ($n = m/4$) case respectively, where $\Omega = \omega_1\wedge\omega_1 + \omega_2\wedge\omega_2 + \omega_3\wedge\omega_3$ is the *fundamental 4-form*. There is a Riemannian metric on M obtained by extending in a natural way the horizontal metric g to a true Riemannian metric, denoted by h, explicitly given by

$$h = g + \sum_{i=1}^{k}(\eta_i)^2. \tag{4.1}$$

The Riemannian volume form is up to a constant multiple the just defined volume form Vol_η.

For each of the considered geometries there is a canonically defined connection $\nabla = \nabla^\eta$ with torsion T. In the integrable CR case this is the Tanaka-Webster connection, see [215]and [223]. In the qc and octonionic cases this is the Biquard connection, see [25, 26] and [78]. The curvature tensor of the corresponding canonical connection ∇ and the associated (0,4) tensor, which is denoted with the same letter, are

$$R(A,B)C = [\nabla_A, \nabla_B]C - \nabla_{[A,B]}C, \qquad R(A,B,C,D) \overset{def}{=} h(R(A,B)C,D). \tag{4.2}$$

Let $\{e_1,\dots,e_m\}$, $m = \dim H$, be a local orthonormal basis of the *horizontal space* H, $g(e_a,e_b) = \delta_{ab}$. The Ricci type and scalar curvature tensors are obtained by taking *horizontal traces*

$$Ric(A,B) = \sum_{b=1}^{m} R(e_b, A, B, e_b), \quad S = \sum_{a,b=1}^{m} R(e_b, e_a, e_a, e_b),\ A, B \in T(M), \tag{4.3}$$

which manifests the sub-Riemannian nature of these tensors. In the qc case these tensors are called *qc-Ricci tensor* and *qc-scalar curvature* tensor of the Biquard connection.

The (horizontal) divergence of a horizontal vector field/one-form $\sigma \in \Lambda^1(H)$ defined by $\nabla^*\sigma = tr^g\nabla\sigma = \nabla\sigma(e_a, e_a)$ supplies the "integration by parts" over compact M formula, [215], [120], see also [225],

$$\int_M (\nabla^*\sigma)\, Vol_\eta \;=\; 0. \tag{4.4}$$

4.2.1. *The first eigenvalue on the sphere.*

Theorem 4.1. *The eigenspaces of the first eigenvalue of the sub-Laplacian of the canonical Iwasawa sub-Riemannian structures on the spheres at infinity of the hyperbolic spaces are the restrictions of all real-linear functions in the corresponding Eucliden space to the sphere.*

The exact value of the eigenvalue depends on the normalization of the "standard" form η, which will be made explicit later. Of course, in the real case the eigenspace is the space of spherical harmonics of order one. Various proofs of Theorem 4.1 are possible. The simplest proof is a direct computation based on the explicit definition of the corresponding sub-Laplacians, see [92, 32] for the complex case, [122, Lemma 2.3] for the quaternion, and [62] for the octonian cases. Alternatively, one can relate the sub-Laplacian to the corrsponding Laplace-Beltrami operator on the sphere, see (7.17) and (8.1) for the complex and quaternion case. Finally, the result follows from an abstract approach as in [8] where the corresponding "spherical harmonics" are studied.

4.2.2. *The Yamabe problem on Iwasawa sub-Riemannian manifolds.*

Definition 4.2. *The "conformal" class of $[\eta]$ consists of all 1-forms $\bar\eta = \phi^{4/(Q-2)}\Psi\eta$ for a smooth positive function ϕ and $\Psi \in SO(k)$ with smooth functions as entries.*

In the CR case $\Psi \equiv 1$, while in the QC case Ψ is an $SO(3)$ matrix with entries smooth functions. We note that the canonical connection is independent of Ψ, but depends on ϕ, which brings us to the Yamabe type problems. The Yamabe functional is

$$\Upsilon_{[\eta]}(\phi) = \left(\int_M \Big(c_n|\nabla\phi|^2 + S\,\phi^2\Big) Vol_\eta\right) \Big/ \left(\int_M \phi^{2^*}\, Vol_\eta\right)^{2/2^*}, \tag{4.5}$$

where c_n is a constant, $2^* = 2Q/(Q-2)$, $\nabla = \nabla^\eta$ is the connection of η, S is the scalar curvature (4.3) of (M, η) and $|\nabla\phi| = \left(\sum_{a=1}^m (d\phi(e_a))^2\right)^{1/2}$ is the length of the horizontal gradient. In the QC case $c_n = 4\frac{n+2}{n+1}$ while in the CR case $c_n = \frac{4(n+1)}{n}$. It will be useful to introduce the functionals

$$\mathcal{E}_\eta(\phi) \overset{def}{=} \int_M \Big(c_n\,|\nabla^\eta\phi|^2 \;+\; S\,\phi^2\Big)\, Vol_\eta, \qquad \mathcal{N}_\eta(\phi) = \left(\int_M \phi^{2^*}\, Vol_\eta\right)^{2/2^*}, \tag{4.6}$$

hence the Yamabe functional can be written as $\Upsilon(\phi){=}\mathcal{E}(\phi)/N(\phi)$ (dropping the subscript η when there is no confusion). The *Yamabe constant* is

$$\Upsilon(M,[\eta]) = \inf\{\Upsilon_{[\eta]}(\phi) : \phi \in \mathcal{C}^\infty(M)\} = \inf\{\mathcal{E}_\eta(\phi) : \mathcal{N}_\eta(\phi) = 1,\ \phi \in \mathcal{C}^\infty(M)\}. \tag{4.7}$$

In the above notation we tacitly introduced $[\eta]$ as a subscript. The reason for this notation is that the Yamabe functional is conformally invariant, which follows from the formulas relating the (sub-Riemannian) scalar curvatures of the associated to η and $\bar\eta$ connections, see below, together with the formula for the change of the volume form,

$$Vol_\eta = \phi^{2^*}\, Vol_{\bar\eta}. \tag{4.8}$$

Finding the Yamabe constant in the case of the standard Iwasawa sub-Riemannian structures on the unit spheres is equivalent to the problem of determining the best constant in the L^2 Folland & Stein [90] Sobolev type embedding inequality on the corresponding Heisenberg group. As noted earlier the best constant in the L^2 Folland & Stein inequality together with the minimizers were determined recently in [92, 122, 62] by the method of Frank & Lieb [92], see also [32]. Nevertheless this simpler approach does not yield the conjectured uniqueness (up to an automorphism) in the case of the spheres.

Finding the Yamabe constant is closely related to the *Yamabe problem* which seeks all Iwasawa sub-Riemannian structures of constant scalar curvature conformal to a given structure η. In fact, taking the conformal factor in the form

$\bar\eta = \phi^{4/(Q-2)}\eta$ as we did above, a calculation (done separately for each of the cases) gives the following *Yamabe equation*,

$$\mathcal{L}\phi \overset{def}{=} c_n \triangle\phi - S\,\phi = -\overline{S}\,\phi^{2^*-1}, \tag{4.9}$$

where $\triangle$ is the horizontal sub-Laplacian, $\triangle\phi = tr^g_H(\nabla du)$, S and $\overline{S}$ are the scalar curvatures corresponding to the associated to η and $\bar\eta$ canonical connections.

A natural question is to find all solutions of the Yamabe equation (4.9). As usual the two fundamental problems are related by noting that on a compact manifold M with a fixed conformal class $[\eta]$ the Yamabe equation characterizes the non-negative extremals of the Yamabe functional. The operator $\mathcal{L}\phi$ in (4.9) is the so called conformal sub-Laplacian. Using the divergence formula (4.4) we can write equation (4.9) in the form

$$\phi^{-1}v\,\bar{\mathcal{L}}(\phi^{-1}v)\,Vol_{\bar\eta} = v\mathcal{L}(v)\,Vol_\eta, \tag{4.10}$$

for any $v \in C^\infty(M)$, which makes explicit the conformal invariance. Here $\bar{\mathcal{L}}$ denotes the conformal sub-Laplacian associated to the canonical connection $\overline{\nabla}$ of $\bar\eta$.

4.3. **CR Manifolds.** A CR manifold is a smooth manifold M of real dimension 2n+1, with a fixed n-dimensional complex sub-bundle $\mathcal{H}$ of the complexified tangent bundle $\mathbb{C}TM$ satisfying $\mathcal{H}\cap\overline{\mathcal{H}} = 0$ and $[\mathcal{H},\mathcal{H}] \subset \mathcal{H}$. If we let $H = Re\,\mathcal{H}\oplus\overline{\mathcal{H}}$, the real sub-bundle H is equipped with a formally integrable almost complex structure J. We assume that M is oriented and there exists a globally defined compatible contact form η such that the *horizontal space* is given by $H = Ker\,\eta$. In other words, the hermitian bilinear form $g(X,Y) = 1/2\,d\eta(JX,Y)$ is non-degenerate. The CR structure is called strictly pseudoconvex if g is a positive definite tensor on H. For brevity we shall frequently use the term *CR manifold* to refer to a strictly pseudoconvex pseudohermitian manifold. In other words, unless specified otherwise a CR manifold will be an *integrable strictly pseudoconvex CR manifold with a fixed pseudohermitian stricture*.

The almost complex structure J is formally integrable in the sense that $([JX,Y]+[X,JY]) \in H$ and the Nijenhuis tensor vanishes $N^J(X,Y) = [JX,JY]-[X,Y]-J[JX,Y]-J[X,JY] = 0$. A CR manifold (M,η,g) with a fixed compatible contact form θ is called *a pseudohermitian manifold*. In this case the 2-form $d\eta_{|_H} \overset{def}{=} 2\omega$ is called the fundamental form. The contact form whose kernel is the horizontal space H is determined up to a conformal factor, i.e., $\bar\theta = \nu\theta$ for a positive smooth function ν, defines another pseudohermitian structure called pseudo-conformal to the original one.

A Riemannian metric is defined in the usual way, written with a slight imprecision as $h = g+\eta^2$. The vector field ξ dual to η with respect to g satisfying $\xi\lrcorner d\eta = 0$ is called the Reeb vector field.

4.3.1. *Invariant decompositions.* As usual any endomorphism Ψ of H can be decomposed with respect to the complex structure J uniquely into its $U(n)$-invariant $(2,0)+(0,2)$ and $(1,1)$ parts. In short we will denote these components correspondingly by $\Psi_{[-1]}$ and $\Psi_{[1]}$. Furthermore, we shall use the same notation for the corresponding two tensor, $\Psi(X,Y) = g(\Psi X,Y)$. Explicitly, $\Psi = \Psi = \Psi_{[1]} + \Psi_{[-1]}$, where

$$\Psi_{[1]}(X,Y) = \frac{1}{2}\left[\Psi(X,Y)+\Psi(JX,JY)\right], \qquad \Psi_{[-1]}(X,Y) = \frac{1}{2}\left[\Psi(X,Y)-\Psi(JX,JY)\right]. \tag{4.11}$$

The above notation is justified by the fact that the $(2,0)+(0,2)$ and $(1,1)$ components are the projections on the eigenspaces of the operator $\Upsilon = J\otimes J$, $\quad (\Upsilon\Psi)(X,Y) \overset{def}{=} \Psi(JX,JY)$ corresponding, respectively, to the eigenvalues -1 and 1. Note that both the metric g and the 2-form ω belong to the [1]-component, since $g(X,Y) = g(JX,JY)$ and $\omega(X,Y) = \omega(JX,JY)$. Furthermore, the two components are orthogonal to each other with respect to g.

The Tanaka-Webster connection [215, 223, 222] is the unique linear connection ∇ with torsion T preserving a given pseudohermitian structure, i.e., it has the properties that the almost complex structure J and the contact form are parallel, $\nabla\xi = \nabla J = \nabla\eta = \nabla g = 0$, and the torsion tensor is of pure type, i.e., for X, $Y \in H$ we have

$$\begin{aligned} &T(X,Y) = d\eta(X,Y)\xi = 2\omega(X,Y)\xi,\\ &T(\xi,X)\in H, g(T(\xi,X),Y) = g(T(\xi,Y),X) = -g(T(\xi,JX),JY). \end{aligned} \tag{4.12}$$

The (Webster) torsion A of the pseudohermitian manifold is the symmetric tensor $A \overset{def}{=} T(\xi,.) : H\to H$. Clearly, equation (4.12) shows that $A \in \Psi_{[-1]}$. It is also well known [215] that A is the obstruction for a pseudohermitian manifold to be Sasakian. We recall that a contact manifold (M, η) is Sasakian if its Riemannian cone $C = M \times \mathbb{R}^+$ with metric $t^2 h + dt^2$ is Kähler (see e.g. [27, 31]).

4.3.2. *Curvature tensors of the Tanaka-Webster connection.* The curvature tensors are defined in a standard fashion using (4.2) and (4.3), noting again that traces are taken only on the horizontal space. The Ricci 2-form is defined by $\rho(A,B) = \frac{1}{2}\, R(A,B,e_a,Je_a)$.The horizontal part of the Ricci 2-form is (1,1)-form with respect to J and the first Bianchi identity implies

$$\rho(X,JY) = \frac{1}{2}\, R(e_a,Je_a,X,JY). \tag{4.13}$$

The tensor $\rho(X,JY) \in \Psi_{[1]}$ is a symmetric tensor and is frequently also called the Webster Ricci tensor. The CR Ricci tensor has the following type decomposition in J invariant and skew-invariant forms [215], [223], see also [77] and [130, Chapter 7]

$$Ric(X,Y) = \rho(JX,Y) + 2(n-1)A(JX,Y). \tag{4.14}$$

It is well known that a pseudohermitian manifold with a flat Tanaka-Webster connection is locally isomorphic to the (complex) Heisenberg group. For $n > 1$ the vanishing of the horizontal part of the Tanaka-Webster connection implies the vanishing of the whole curvature. If $n = 1$, in addition to the vanishing of the horizontal part of the curvature one needs also the vanishing of the pseudohermitian torsion to have zero curvature.

4.3.3. *The Heisenberg group.* Given the ubiquitous role of the Heisenberg group $\boldsymbol{G}\,(\mathbb{C}) \equiv \mathbb{C}^n \times \mathbb{R}$ in analysis and being the flat model of the considered CR geometries since the Tanaka-Webster connection coincides with the invariant flat connection on the group. We shall write explicitly a number of formulas in this special setting, which also will be made explicit in Section 4.5 for quaternionic contact structures in which case the quaternionic Heisenberg group will play the role of the flat model space. The group $\boldsymbol{G}\,(\mathbb{C})$ arises as the nilpotent part in the Iwasawa decomposition of the complex hyperbolic space. Thus, $\boldsymbol{G}\,(\mathbb{C})$ is a Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ with group law given by (3.15) where for $z, z' \in \mathbb{C}^n$ we let $z \cdot z' = \sum_{j=1}^n z_j z_j'$. A (real) basis for the Lie algebra of left-invariant vector fields on $\boldsymbol{G}\,(\mathbb{C})$ is given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad X_{n+j} \equiv Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad \xi = 2\frac{\partial}{\partial t}, \quad j = 1, ..., n, \tag{4.15}$$

with corresponding contact form

$$\tilde{\Theta} = \frac{1}{2}dt + \sum_{j=1}^n (x_j dy_j - y_j dx_j) = \frac{1}{2}dt + \sum_{j=1}^n \Im(\bar{z}_j dz_j)$$

Here, we have identified $z = x + iy \in \mathbb{C}^n$, with the real vector $(x,y) \in \mathbb{R}^{2n}$. Since $[X_j, Y_k] = -4\delta_{jk}\frac{\partial}{\partial t}$, the Lie algebra is generated by the system $X = \{X_1, ..., X_{2n}\}$. The sub-Laplacian is $\mathcal{L} = \sum_{j=1}^{2n} X_j^2$ which is the real part of the Kohn complex Laplacian. In this case the exponential map is the identity and, as for any group of step two, we have the the parabolic dilations $\delta_\lambda(z,t) = (\lambda z, \lambda^2 t)$. The corresponding homogeneous dimension is $Q = 2n + 2$.

In regards to the theory of groups of Heisenberg type, cf. Section 3, some care has to be taken when defining the scalar product which turns $\boldsymbol{G}\,(\mathbb{C})$ into a group of Heisenberg type. For example, the standard inner product of $\mathbb{C}^n \times \mathbb{R}$, i.e., the inner product in which the basis of left invariant vector fields given in (4.15) is an orthonormal basis will not make the Heisenberg group $\boldsymbol{G}\,(\mathbb{C})$ a group of Heisenberg type. An orthonormal basis of an H-type compatible metric is given by, $j = 1, ..., n$,

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t},\ X_{n+j} \equiv Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},\ T = \frac{1}{4}\frac{\partial}{\partial t} \tag{4.16}$$

and homogeneous gauge (3.12) given by $N(z,t) = (|z|^4 + 16t^2)^{1/4}$, $|z| = \left(\sum_{j=1}^n (x_j^2 + y_j^2)\right)^{1/2}$.

4.4. **The CR sphere and the Cayley transform.** The simplest CR manifolds are the three hyperquadrics in complex space, [60]

$$Q_+ :\ r = |z_1|^2 + \cdots + |z_n|^2 + |w|^2 = 1, \qquad Q_- :\ r = |z_1|^2 + \cdots + |z_n|^2 - |w|^2 = -1 \tag{4.17}$$

$$Q_0 :\ r = |z_1|^2 + \cdots + |z_n|^2 - \Im(w) = 0, \tag{4.18}$$

where $(z_1, z_2, \dots z_n, w) \in \mathbb{C}^n \times \mathbb{C}$ with corresponding contact forms $\tilde\eta \overset{def}{=} \eta_+, \eta_-$ and $\tilde\Theta{=}\eta_0$ equal to $-i\partial r$ which define strictly pseudoconvex pseudohermitian structures. Of course, these are the "standard" (up to a multiplicative factor depending on the reference) pseudohermitian structures on the sphere S^{2n+1}, hyperboloid, and Heisenberg group $\mathbb{H}^n$, the latter identified with the boundary of the Siegel domain via the map $(z,t) \mapsto (z, t + i|z|^2)$. A transformation mapping Q_- onto Q_+ minus a curve is given by $w = 1/w'$ and $z_j = z'_j/w'$. On the other hand, the transformation

$$\mathcal{C}(Z,W) = \Big(\frac{iZ}{1-W}, i\frac{1+W}{1-W}\Big), \qquad \text{with inverse} \qquad \mathcal{C}^{-1}(z,w) = \Big(\frac{2z}{i+w}, \frac{w-i}{w+i}\Big),$$

maps the sphere $S^{2n+1} \setminus (0,0,...,0,1)$ onto $\mathbb{H}^n$. The map $\mathcal{C}$ is the Cayley transform (with a pole at $(0,0,...,0,1)$). These transformations clearly preserve the CR structure since they are restrictions of holomorphic maps, but do not preserve the contact forms and are in fact pseudoconformal pseudohermitian maps

$$\tilde\Theta = \frac{1}{2}dw - i\bar z \cdot dz = \frac{1}{|1-W|^2}\tilde\eta, \qquad \tilde\eta = -i\left(\overline{W}dW + \bar Z \cdot dZ\right).$$

4.4.1. *CR conformal flatness.* A fundamental fact characterizing CR conformal flatness is the Cartan-Chern-Moser theorem [44, 60, 222]. A proof based on the classical approach used by H. Weyl in Riemannian geometry (see e.g. [79]) can be found in [130], see also [133].

Theorem 4.3. *[[44, 60, 222]] Let (M,θ,g) be a 2n+1-dimensional non-degenerate pseudo-hermitian manifold. If $n > 1$ then (M,η,g) is locally pseudoconformaly equivalent to a hyperquadric in $\mathbb{C}^{n+1}$ if and only if the Chern-Moser tensor vanishes, $\mathcal{S} = 0$,* [60, 222]. *In the case $n = 1$, (M,η,g) is locally pseudoconformaly equivalent to a hyperquadric in $\mathbb{C}^2$ if and only the tensor F^{car} given below vanishes, $F^{car} = 0$,* [44].

Here, the Chern-Moser tensor $\mathcal{S}$ is a pseudoconformaly invariant tensor, i.e., if ϕ is a smooth positive function and $\tilde\eta = \phi\eta$, then $\mathcal{S}_{\tilde\eta} = \phi\mathcal{S}_\eta$. The Chern-Moser tensor $\mathcal{S}$ [60] is determined completely by the (1,1)-part of the curvature and the Ricci 2-form,

$$\begin{aligned}
&S(X,Y,Z,V) = \frac{1}{2}\Big[R(X,Y,Z,V) + R(JX,JY,Z,V)\Big] \\
&-\frac{S}{4(n+1)(n+2)}\Big[g(X,Z)g(Y,V) - g(Y,Z)g(X,V) + \Omega(X,Z)\Omega(Y,V) - \Omega(Y,Z)\Omega(X,V) + 2\Omega(X,Y)\Omega_s(Z,V)\Big] \\
&\quad - \frac{1}{2(n+2)}\Big[g(X,Z)\rho(Y,JV) - g(Y,Z)\rho(X,JV) + g(Y,V)\rho(X,JZ) - g(X,V)\rho(Y,JZ)\Big] \\
&\quad - \frac{1}{2(n+2)}\Big[\Omega(X,Z)\rho(Y,V) - \Omega(Y,Z)\rho(X,V) + \Omega(Y,V)\rho(X,Z) - \Omega(X,V)\rho(Y,Z)\Big] \\
&\qquad\qquad - \frac{1}{n+2}\Big[\Omega(X,Y)\rho(Z,V) + \Omega(Z,V)\rho(X,Y)\Big].
\end{aligned} \tag{4.19}$$

For $n = 1$ the tensor S vanishes identically and the Cartan condition can be expressed by the vanishing of the *the Cartan tensor* $[-1]$ type tensor F^{car} defined on H by ([133, 130]

$$\begin{aligned}
F^{car}(X,Y) &= \nabla^2 S(X,JY) + \nabla^2 S(Y,JX) + 16(\nabla^2_{Xe_a}A)(Y,e_a) + 16(\nabla^2_{Ye_a}A)(X,e_a) + 36SA(X,Y) \\
&\quad + 48(\nabla^2_{e_aJe_a}A)(X,JY) + 3g(X,Y)\nabla^2 S(e_a,Je_a).
\end{aligned} \tag{4.20}$$

4.5. **Quaternionic Contact Structures.** Following Biquard [25, 26], a 4n+3 dimensional manifold M^{4n+3} is quaternionic contact (qc) if we have:

i) a co-dimension three distribution H, which locally is the intersction of the kernels of three 1-forms on M, $H = \bigcap_{s=1}^3 Ker\,\eta_s$, $\eta_s \in \Gamma(M : T^*M)$;

ii) a 2-sphere bundle $\mathbb{Q}$ of "almost complex structures" locally generated by $I_s : H \to H, \quad I_s^2 = -1$, satisfying $I_1 I_2 = -I_2 I_1 = I_3$;
iii) a "metric" tensor g on H, such that, $g(I_s X, I_s Y) = g(X,Y)$, and $2g(I_s X, Y) = d\eta_s(X,Y)$, for all $X, Y \in H$.

We let $\omega_s(X,Y) = g(I_s X, Y)$ be the associated 2-forms. The "canonical" Biquard connection is the unique linear connection defined by the following theorem [25].

Theorem 4.4 ([25]). *If (M,η) is a qc manifold and $n > 1$, then there exists a unique complementary to H subbundle V, $TM = H \oplus V$ and a linear connection ∇ with the properties that V, H, g and the 2-sphere bundle $\mathbb{Q}$ are parallel and the torsion T of ∇ satisfies: a) for $X, Y \in H, \quad T(X,Y) = -[X,Y]|_V \in V$; b) for $\xi \in V$, $X \in H$, $T_\xi(X) \equiv T(\xi, X) \in H$ and $T_\xi \in (sp(n) + sp(1))^\perp$.*

Biquard also showed that the "vertical" space V is generated by the Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ determined by $\eta_s(\xi_k) = \delta_{sk}, \quad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \quad (\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}$. If the dimension of M is seven, $n = 1$, the Reeb vector fields might not exist. D. Duchemin [78] showed that if we assume their existence, then there is a connection as before. Henceforth, by a qc structure in dimension 7 we mean a qc structure satisfying the Reeb conditions.

Note that the extended Riemannian metric h given by (4.1) as well as the Biquard connection do not depend on the action of $SO(3)$ on V, but both change if η is multiplied by a conformal factor.

4.5.1. *Invariant decompositions.* As usual any endomorphism Ψ of H can be decomposed with respect to the hypercomplex complex structure I_s, $s = 1,2,3$ uniquely into its two $Sp(n)Sp(1)$-invariant parts. In short we will denote these components correspondingly by $\Psi_{[-1]}$ and $\Psi_{[3]}$. Furthermore, we shall use the same notation for the corresponding two tensor, $\Psi(X,Y) = g(\Psi X, Y)$. Explicitly, $\Psi = \Psi = \Psi_{[3]} + \Psi_{[-1]}$, where

$$
\begin{aligned}
\Psi_{[3]}(X,Y) &= \frac{1}{4}\left[\Psi(X,Y) + \Psi(I_1X, I_1Y) + \Psi(I_2X, I_2Y) + \Psi(I_3X, I_3Y)\right], \\
\Psi_{[-1]}(X,Y) &= \frac{1}{4}\left[3\Psi(X,Y) - \Psi(I_1X, I_1Y) - \Psi(I_2X, I_2Y) - \Psi(I_3X, I_3Y)\right].
\end{aligned}
\tag{4.21}
$$

The above notation is justified by the fact that the [3] and $[-1]$ components are the projections on the eigenspaces of the operator $\Upsilon = \ I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3, \quad (\Upsilon\Psi)(X,Y) \overset{def}{=} \Psi(I_1X, I_1Y) + \Psi(I_2X, I_2Y) + \Psi(I_3X, I_3Y)$ corresponding, respectively, to the eigenvalues 3 and -1. Note that the metric g belong to the [3]-component, while and the 2-forms ω_s, $s = 1,2,3$ belong to the [-1]-component since. Furthermore, the two components are orthogonal to each other with respect to g. For $n = 1$ the [3]-component is 1-dimensional, $\Psi_{[3]} = \frac{tr\Psi}{4} g$.

4.5.2. *Curvature of a Quaternionic Contact Structure.* The curvature tensors are defined in a standard fashion using (4.2) and (4.3). In addition we define the qc Ricci 2-forms

$$
\rho_s(A,B) = \frac{1}{4n} R(A,B,e_a,I_s e_a), \quad s = 1,2,3.
\tag{4.22}
$$

Biquard [25] showed that the map $T_{\xi_j} = T^0_{\xi_j} + I_j U$, where $T^0_{\xi_j}$ is symmetric while $I_j U$ is a skew-symmetric map and $U \in \Psi_{[3]}$, $I_s U = U I_s$, $s = 1,2,3$. Further properties were found in [120]. A remarkable fact, [120, Theorem 3.12], is that (unlike the CR case!) the torsion endomorphism determines the (horizontal) qc-Ricci tensor and the (horizontal) qc-Ricci forms of the Biquard connection. For this we also need the torsion type tensor $T^0 \overset{def}{=} T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3 \in \Psi_{[-1]}$ introduced in [120].

Theorem 4.5 ([120]). *On a QC manifold (M,η) we have*

$$
\begin{aligned}
Ric(X,Y) &= (2n+2)T^0(X,Y) + (4n+10)U(X,Y) + \frac{S}{4n} g(X,Y) \\
\rho_s(X, I_s Y) &= -\frac{1}{2}\Big[T^0(X,Y) + T^0(I_s X, I_s Y)\Big] - 2U(X,Y) - 8n(n+2)Sg(X,Y).
\end{aligned}
\tag{4.23}
$$

We say that M is a *qc-Einstein manifold* if the horizontal Ricci tensor is proportional to the horizontal metric g,

$$
Ric(X,Y) = \frac{S}{4n} g(X,Y)
\tag{4.24}
$$

which taking into account (4.23) is equivalent to $T^0 = U = 0$. Furthermore, by [120, Theorem 4.9] and [123, Theorem 1.1] any qc-Einstein structure has constant qc-scalar curvature. It should be mentioned that qc-Einstein structures have proved useful in the construction of metrics with special holonomy [6] and heterotic string theory, see Section 9 for some details on the latter. Such applications are possible due to the following properties/ characterization of the qc-Einstein structures, see [130, Theorem 1.3] and [132, Theorem 4.4.4] for $S \neq 0$ and [123, Theorem 5.1] for $S = 0$ cases.

Theorem 4.6. *Let M be a qc manifold. The following conditions are equivalent:*

a) M is a qc-Einstein manifold;

b) locally, the given qc-structure is defined by 1-form (η_1, η_2, η_3) such that for some constant S we have

$$d\eta_i = 2\omega_i + \frac{S}{8n(n+2)}\eta_j \wedge \eta_k; \tag{4.25}$$

c) locally, the given qc-structure is defined by 1-form (η_1, η_2, η_3) such that the corresponding connection 1-forms vanish on H, in fact, $\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j$, $\nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j$ with $\alpha_s = -\frac{S}{8n(n+2)}\eta_s$.

In particular, in the positive scalar curvature case the qc-Einstein manifold are exactly the locally 3-Sasakian manifolds, i.e, for every $p \in M$ there exist an open neighborhood U of p and a matrix $\Psi \in \mathcal{C}^\infty(U : SO(3))$, s.t., $\Psi \cdot \eta$ is 3-Sasakian. A (4n + 3)-dimensional (pseudo) Riemannian manifold (M, g) is 3-Sasakian if the cone metric is a (pseudo) hyper-Kähler metric [30, 31]. We note explicitly that in some questions it is useful to define 3-Sasakian manifolds in the wider sense of positive (the usual terminology) or negative 3-Sasakian structures, cf. [130, Section 2] and [132, Section 4.4.1] where the "negative" 3-Sasakian term was adopted in the case when the Riemannian cone is hyper-Kähler of signature $(4n, 4)$. As well known, a positive 3-Sasakian manifold is Einstein with a positive Riemannian scalar curvature [146] and, if complete, it is compact with finite fundamental group due to Myers theorem. The negative 3-Sasakian structures are Einstein with respect to the corresponding pseudo-Riemannian metric of signature $(4n, 3)$ [146, 217]. In this case, by a simple change of signature, we obtain a positive definite nS metric on M, [217, 136, 149].

4.5.3. *The quaternionic Heisenberg Group $\boldsymbol{G}(\mathbb{H})$.* The basic example of a qc manifold is provided by the quaternionic Heisenberg group $\boldsymbol{G}(\mathbb{H})$ on which we introduce coordinates by regarding $\boldsymbol{G}(\mathbb{H}) = \mathbb{H}^n \times \mathrm{Im}\mathbb{H}$, $(q, \omega) \in \boldsymbol{G}(\mathbb{H})$ so that the multiplication takes the form (3.15).

The "*standard*" qc contact form in quaternion variables is $\tilde{\Theta} = (\tilde{\Theta}_1, \tilde{\Theta}_2, \tilde{\Theta}_3) = \frac{1}{2}(d\omega - q \cdot d\bar{q} + dq \cdot \bar{q})$ or, using real coordinates,

$$\tilde{\Theta}_1 = \frac{1}{2}\, dx - x^\alpha dt^\alpha + t^\alpha dx^\alpha - z^\alpha dy^\alpha + y^\alpha dz^\alpha, \quad \tilde{\Theta}_2 = \frac{1}{2}\, dy - y^\alpha dt^\alpha + z^\alpha dx^\alpha + t^\alpha dy^\alpha - x^\alpha dz^\alpha,$$
$$\tilde{\Theta}_2 = \frac{1}{2}\, dz - z^\alpha dt^\alpha - y^\alpha dx^\alpha + x^\alpha dy^\alpha + t^\alpha dz^\alpha. \tag{4.26}$$

The left-invariant horizontal vector fields are

$$\begin{aligned} T_\alpha &= \frac{\partial}{\partial t_\alpha} + 2x^\alpha\frac{\partial}{\partial x} + 2y^\alpha\frac{\partial}{\partial y} + 2z^\alpha\frac{\partial}{\partial z}, \ X_\alpha = \frac{\partial}{\partial x_\alpha} - 2t^\alpha\frac{\partial}{\partial x} - 2z^\alpha\frac{\partial}{\partial y} + 2y^\alpha\frac{\partial}{\partial z}, \\ Y_\alpha &= \frac{\partial}{\partial y_\alpha} + 2z^\alpha\frac{\partial}{\partial x} - 2t^\alpha\frac{\partial}{\partial y} - 2x^\alpha\frac{\partial}{\partial z}, \ Z_\alpha = \frac{\partial}{\partial z_\alpha} - 2y^\alpha\frac{\partial}{\partial x} + 2x^\alpha\frac{\partial}{\partial y} - 2t^\alpha\frac{\partial}{\partial z}, \end{aligned} \tag{4.27}$$

with corresponding sub-Laplacian

$$\triangle_{\tilde{\Theta}} u = \sum_{\alpha=1}^{n}\left(T_\alpha^2 u + X_\alpha^2 u + Y_\alpha^2 u + Z_\alpha^2 u\right). \tag{4.28}$$

The (left-invariant vertical) Reeb fields ξ_1, ξ_2, ξ_3 are $\xi_1 = 2\frac{\partial}{\partial x}$, $\quad \xi_2 = 2\frac{\partial}{\partial y}$, $\quad \xi_3 = 2\frac{\partial}{\partial z}$.On $\boldsymbol{G}(\mathbb{H})$, the left-invariant flat connection is the Biquard connection, hence $\boldsymbol{G}(\mathbb{H})$ is a flat qc structure. It should be noted that the latter property characterizes (locally) the qc structure $\tilde{\Theta}$ by [120, Proposition 4.11], but in fact vanishing of the curvature on the horizpntal space is enogh [129, Propositon 3.2]. Thus, by [129, Propositon 3.2], a quaternionic contact manifold is locally isomorphic to the quaternionic Heisenberg group exactly when the curvature of the Biquard connection restricted to H vanishes, $R_{|_H} = 0$.

4.5.4. *Standard qc-structure on 3-Sasakain sphere and the qc Cayley transform.* The second example is the Standard qc-structure on the 3-Sasakain sphere. The "standard" qc 3-form on the sphere $S^{4n+3} = \{|q|^2+|p|^2 = 1\} \subset \mathbb{H}^n \times \mathbb{H}$, is

$$\tilde{\eta} \;=\; dq\cdot\bar{q} \;+\; dp\cdot\bar{p} \;-\; q\cdot d\bar{q} - \; p\cdot d\bar{p}. \tag{4.29}$$

We identify $\boldsymbol{G}\,(\mathbb{H})$ with the boundary Σ of a Siegel domain in $\mathbb{H}^n\times\mathbb{H}$, $\Sigma \;=\; \{(q',p')\in \mathbb{H}^n\times\mathbb{H}\;:\; \operatorname{Re} p' \;=\; |q'|^2\}$, by using the map $(q',\omega')\mapsto (q',|q'|^2-\omega')$. The Cayley transform, $\mathcal{C}: S\setminus\{\text{pt.}\}\to\Sigma$, is

$$(q',p') \;=\; \mathcal{C}\,\Big((q,p)\Big) = ((1+p)^{-1}\,q, (1+p)^{-1}\,(1-p)).$$

By [120, Section 8.3] we have on $\boldsymbol{G}\,(\mathbb{H})$

$$\Theta \;\overset{def}{=}\; \lambda\,\cdot(\mathcal{C}^{-1})^*\,\tilde{\eta}\;\cdot\bar{\lambda} \;=\; \frac{8}{|1+p'\,|^2}\,\tilde{\Theta}. \tag{4.30}$$

where $\lambda \;= |1+p\,|\,(1+p)^{-1}$ is a unit quaternion. Alternatively, on the sphere this can be written as

$$\eta \overset{def}{=} \mathcal{C}^*\,\tilde{\Theta} \;=\; \frac{1}{2\,|1+p\,|^2}\,\lambda\,\tilde{\eta}\,\bar{\lambda}, \tag{4.31}$$

where λ is a unit quaternion. In any case, the above formulas show the Cayley transforms is a conformal quaternionic contact map. In addition, we can use it to determine the qc scalar curvature of the sphere $(S^{4n+3},\tilde{\eta})$ and find a solution of the Yamabe equation on $\boldsymbol{G}\,(\mathbb{H})$. For $(q',p')\in\Sigma\subset\mathbb{H}^n\times\mathbb{H}$, $\quad p'=|q'|^2+\omega'$, consider the function

$$\begin{aligned} h &= \frac{1}{16}|1+p'|^2 = \frac{1}{16}\left[(1+|q'|^2)^2+|\omega'|^2\right],\\ \Phi &= (2h)^{-(Q-2)/4} = 8^{(Q-2)/4}\left[(1+|q'|^2)^2+|\omega'|^2\right]^{-(Q-2)/4}, \end{aligned} \tag{4.32}$$

so that we have

$$\Theta = \frac{1}{2h}\tilde{\Theta} = \Phi^{4/(Q-2)}\tilde{\Theta}.$$

A small calculation shows that the sub-laplacian of h with respect to $\tilde{\Theta}$ is given by $\triangle_{\tilde{\Theta}}h = \frac{Q-6}{4}+\frac{Q+2}{4}|q'|^2$ and thus Φ is a solution of the qc Yamabe equation on the Heisenberg group,

$$\triangle_{\tilde{\Theta}}\Phi = -K\,\Phi^{2^*-1}, \qquad K=(Q-2)(Q-6)/8. \tag{4.33}$$

Denoting with $\mathcal{L}$ and $\mathcal{L}_{\tilde{\Theta}}$ the conformal sub-laplacians of Θ and $\tilde{\Theta}$, respectively, we have (see also (4.10))

$$\Phi^{-1}\mathcal{L}(\Phi^{-1}u) = \Phi^{-2^*}\mathcal{L}_{\tilde{\Theta}}u.$$

Taking $u=\Phi$ we come to $\mathcal{L}(1)=\Phi^{1-2^*}\triangle_{\tilde{\Theta}}\Phi$, since the qc structure $\tilde{\Theta}$ is flat, which shows

$$S_{\tilde{\eta}} = S_{\Theta} = 4\frac{Q+2}{Q-2}K = 8n(n+2) \tag{4.34}$$

using that the two structures are isomorphic via the diffemorphism $\mathcal{C}$, or rather its extension, since we can consider $\mathcal{C}$ as a quaternionic contact conformal transformation between the whole sphere S^{4n+3} and the compactification $\hat{\Sigma}\cup\infty$ of the quaternionic Heisenberg group by adding the point at infinity, cf. [121, Section 5.2].

4.5.5. *QC conformal flatness* [129]. A QC manifold (M,η) is called locally qc conformally flat if there is a local diffeomorphisms $F:\boldsymbol{G}\,(\mathbb{H})\to M$, such that $F^*\eta=\phi\Psi\tilde{\Theta}$ for some positive function ϕ.

The qc-conformal flatness of a manifold is characterized by the vanishing of the *qc-conformal curvature* tensor W^{qc} found in [129],

$$W^{qc}(X,Y,Z,V) = \frac{1}{4}\Big[R(X,Y,Z,V)+\sum_{s=1}^{3}R(I_sX,I_sY,Z,V)\Big]-\frac{1}{2}\sum_{s=1}^{3}\omega_s(Z,V)\Big[T^0(X,I_sY)-T^0(I_sX,Y)\Big]$$

$$+\frac{S}{32n(n+2)}\Big[(g\owedge g)(X,Y,Z,V)+\sum_{s=1}^{3}(\omega_s\owedge\omega_s)(X,Y,Z,V)\Big]+(g\owedge U)(X,Y,Z,V)+\sum_{s=1}^{3}(\omega_s\owedge I_sU)(X,Y,Z,V),$$

where $(A \owedge B)$ denotes the Kulkarni-Nomizu product of two tensor, i.e.,

$$(A \owedge B)(X,Y,Z,V) = A(X,Z)B(Y,V) - A(Y,Z)(B(X,V) + A(Y,V)B(X,Z) - A(X,V)(B(Y,Z).$$

Theorem 4.7 ([129]). *a)W^{qc} is qc-conformal invariant, i.e., if $\bar\eta = \kappa\Psi\eta$ then $W^{qc}_{\bar\eta} = \phi\, W^{qc}_{\eta}$, where $0 < \kappa \in \mathcal{C}^\infty(M)$, and $\Psi \in \mathcal{C}^\infty(M : SO(3))$.*

b) A qc-structure is locally qc-conformal to the standard flat qc-structure on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$ if and only if $W^{qc} = 0$.

Taking into account the qc Cayley transform we also have the quaternionic sphere S^{4n+3} if and only if the qc conformal curvature vanishes, $W^{qc} = 0$.

We end this section with the remark that unlike the CR case the realization of qc manifolds as hypersurfaces in a hyper-Kähler manifold is very restrictive. For example, it was shown in [124] that if M is a connected qc-hypersurface in the flat quaternion space $\mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$ then, up to a quaternionic affine transformation of $\mathbb{H}^{n+1}$, M is contained in one of the following three hyperquadrics:

$(i)\ \ |q_1|^2 + \cdots + |q_n|^2 + |p|^2 = 1, \qquad (ii)\ \ |q_1|^2 + \cdots + |q_n|^2 - |p|^2 = -1, \qquad (iii)\ \ |q_1|^2 + \cdots + |q_n|^2 + \mathbb{R}e(p) = 0.$

Here $(q_1, q_2, \dots q_n, p)$ denote the standard quaternionic coordinates of $\mathbb{H}^{n+1}$. In particular, if M is a compact qc-hypersurface of $\mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$ then, up to a quaternionic affine transformation of $\mathbb{H}^{n+1}$, M is the standard 3-Sasakian sphere. For other results and more details we refer to [124].

5. The CR Yamabe problem and the CR Obata theorem

The CR Yamabe problem seeks pseudoconformal pseudohermitian transformation of a compact CR pseudohermitian manifold which lead to constant scalar curvature of the canonical Tanaka-Webster connection, see Section 4.2.2. After the works of D. Jerison & J. Lee [137, 138, 139, 140] and N. Gamara & R. Yacoub [96], [97] the solution of the CR Yamabe problem on a compact manifold is complete.

Let (M^{2n+1}, η) be a strictly pseudoconvex CR manifold andf $\Upsilon(M, [\eta])$ be the CR-Yamabe constant (cf. (4.7)). The CR-Yamabe constant $\Upsilon(M, [\eta])$ depends only on the CR structure of M, not of the choice of η.

The solution of the CR-Yamabe problem is outlined in the next fundamental results.

Theorem 5.1 ([138, 140, 96, 97]). *Let (M^{2n+1}, η) be a strictly pseudoconvex CR manifold. The CR-Yamabe constant satisfies the ineguality $\Upsilon(M, [\eta]) \le \Upsilon(S^{2n+1}, \bar\eta)$, where $S^{2n+1} \subset \mathbb{C}^{n+1}$ is the sphere with its standard CR structure $\bar\eta$.*

a) If $\Upsilon([M,\eta]) < \Upsilon(S^{2n+1}, [\bar\eta])$, then the Yamabe equation has a solution. [138]

b) If $n \ge 2$ then the Yamabe constant satisfies

$$\Upsilon(M, [\eta_\epsilon]) = \begin{cases} \Upsilon(S^{2n+1}, [\bar\eta])\left(1 - c_n |S(q)|^2 \epsilon^4\right) + \mathcal{O}(\epsilon^5), & n \ge 2; \\ \Upsilon(S^5, \bar\eta])\left(1 + c_2 |S(q)|^2 \epsilon^4 \ln \epsilon\right) + \mathcal{O}(\epsilon^4), & n = 2. \end{cases}$$

Thus, if M is not locally CR equivalent to $(S^{2n+1}, \bar\eta)$, then $\Upsilon(M, [\eta]) < \Upsilon(S^{2n+1}, [\bar\eta])$. [140]

c) If $n = 1$ or M is locally CR equivalent to S^{2n+1}, then the Yamabe equation has a solution. [96, 97]

5.1. **Solution of the CR Yamabe problem on the sphere and Heisenberg group.** The CR version of the Obata theorem was proved by D. Jerison and J. Lee [139].

Theorem 5.2 ([139]). *If η is the contact form of a pseudo-Hermitian structure proportional to the standard contact form $\bar\eta$ on the unit sphere in $\mathbb{C}^{n+1}$ and the pseudohermitian scalar curvature S_η =const, then up to a multiplicative constant $\eta = \Phi^*\,\bar\eta$ with Φ a CR automorphism of the sphere.*

A key step of the proof consists of showing that a CR structure with a constant pseudohermitian scalar curvature is pseudoconformal to the standard pseudo-Einstein torsion-free structure on the CR sphere iff it is a pseudo-Einstein with vanishing Webster torsion. It is well known that a strictly pseudoconvex torsion-free CR manifold is Sasakian. In addition, if the CR space is pseudo Einstein then it is not hard to observe that it is a Sasaki-Einstein space with respect to the associated Riemannian metric h. Indeed, the Ricci tensors Ric^g and Ric of the Levi-Civita and the Tanaka-Webster connection, respectively, of a torsion-free CR space are connected by [77] $Ric^g(X,Y) = Ric(X,Y) - 2g(X,Y)$, $Ric^g(\xi,\xi) = 2n$. Because of the second identity, a Sasaki-Einstein space has Riemannian scalar curvature $S^g =$

$2n(2n+1)$. Hence, a torsion-free pseudo Einstein CR manifold is a Sasaki-Einstein space if the pseudohermitian scalar curvature is equal to $S = 4n(n+1)$ and the Jerison-Lee theorem can be stated as follows.

Theorem 5.3 ([139]). *If a compact Sasaki-Einstein manifold $(M, \bar\eta)$ is psedoconformal to a CR manifold $(M, \eta = 2h\bar\eta)$ with constant positive pseudohermmitian scalar curvature $S = 4n(n+1)$ then (M, η) is again a Sasaki-Einstein space.*

The proof follows trivially from the divergence formula discovered in [139] which we state in real coordinates in Theorem 5.4. First, we need some definitions. Let $h > 0$ be a smooth function on a pseudohermitian manifold (M, η, g) and $\bar\eta = \frac{1}{2h}\eta$ be a pseudoconformal to η contact form. We will denote the connection, curvature and torsion tensors of $\bar\eta$ by over-lining the same object corresponding to η. The new Reeb vector field $\bar\xi$ is $\bar\xi = 2h\,\xi + 2h\ J\nabla h$,where ∇h is the horizontal gradient, $g(\nabla h, X) = dh(X)$. The Webster torsion and the pseudohermitian Ricci tensors of η and $\bar\eta$ are related by [166],

$$4h\bar A(X, JY) = 4hA(X, JY) + \nabla^2 h(X, Y) - \nabla^2 h(JX, JY) \tag{5.1}$$

$$\begin{aligned}\bar\rho(X, JY) = \rho(X, JY) &- (4h)^{-1}(n+2)[\nabla^2 h(X, Y) + \nabla^2 h(Y, X) + \nabla^2 h(JX, JY) + \nabla^2 h(JY, JX)]\\ &+ (2h^2)^{-1}(n+2)[dh(X)dh(Y) + dh(JX)dh(JY)] - (2h)^{-1}\left(\triangle h - h^{-1}(n+2)|\nabla h|^2\right) g(X, Y),\end{aligned} \tag{5.2}$$

where $\triangle h = \nabla^* dh = \sum_{a=1}^{2n} \nabla^2 h(e_a, e_a)$ is the sublaplacian. The pseudohermitian scalar curvatures changes according to [166],

$$\overline{S} = 2hS - 2(n+1)(n+2)h^{-1}|\nabla h|^2 + 4(n+1)\triangle h. \tag{5.3}$$

Let B be the traceless part of ρ, $B(X, JY) \overset{def}{=} \rho(X, JY) + \frac{S}{2n} g(X, Y)$ since by (4.14), we have $\rho(e_a, Je_a) = -Ric(e_a, e_a) = -S$. The above formulas imply

$$\begin{aligned}\bar B(X, JY) = B(X, JY) &- \frac{n+2}{4h}[\nabla^2 h(X, Y) + \nabla^2 h(Y, X) + \nabla^2 h(JX, JY) + \nabla^2 h(JY, JX)]\\ &+ \frac{n+2}{2h^2}[dh(X)dh(Y) + dh(JX)dh(JY)] + \frac{n+2}{2n}\left(\frac{\triangle h}{h} - \frac{1}{h^2}|\nabla h|^2\right) g(X, Y).\end{aligned} \tag{5.4}$$

Suppose $\bar\theta$ is Sasaki-Einstein structure, i.e., $\bar A = \bar B = 0$ and both pseudo-Hermitain structures are of constant pseudohermitian scalar curvatures $\bar S = S = 4n(n+1)$. With these assumptions (5.3) becomes

$$\triangle h = n - 2nh + \frac{(n+2)}{2h}|\nabla h|^2. \tag{5.5}$$

At this point we recall the Ricci identities for the Tanaka-Webster connection [166] (see e.g. [133, 130] for these and other expressions in real coordinates),

$$\begin{aligned}\nabla^2 h(X, Y) - \nabla^2 h(Y, X) = -2\omega(X, Y)dh(\xi), \qquad \nabla^2 h(X, \xi) - \nabla^2 h(\xi, X) = A(X, \nabla h),\\ \nabla^3 h(X, Y, Z) - \nabla^3 h(Y, X, Z) = -R(X, Y, Z, \nabla h) - 2\omega(X, Y)\nabla^2 h(\xi, Z).\end{aligned} \tag{5.6}$$

The contracted Bianchi identities for Tanaka-Webster connection [166] are

$$\begin{aligned}dS(X) = 2\sum_{a=1}^{2n}(\nabla_{e_a} Ric)(e_a X) = -2\sum_{a=1}^{2n}(\nabla_{e_a}\rho)(e_a, JX) + 4(n-1)\sum_{a=1}^{2n}(\nabla_{e_a} A)(e_a, JX);\\ Ric(\xi, X) = \sum_{a=1}^{2n}(\nabla_{e_a} A)(e_a, Z); \quad dS(\xi) = 2\sum_{a,b=1}^{2n}(\nabla^2_{e_a e_b} A)(e_a, e_b).\end{aligned} \tag{5.7}$$

When $\bar A = 0$, (5.1) takes the form

$$4hA(X, JY) = -\Big[\nabla^2 h(X, Y) - \nabla^2 h(JX, JY)\Big]. \tag{5.8}$$

Differentiating (5.8) using the equation $\nabla J = 0$, taking the trace in the obtained equality and applying the Ricci identities (5.6), (4.14) and the CR Yamabe equation (5.3), we find the next formula for the divergence of A,

$$\nabla^* A(JX) = -2\rho(JX,\nabla h) - \frac{n+2}{h}[\nabla^2(\nabla h, X) - 2dh(JX)dh(\xi)] + 2ndh(X) + \frac{n+2}{2h^2}|\nabla h|^2 dh(X) - (2n+4)\nabla^2(JX,\xi), \tag{5.9}$$

where the divergence of a 1-form α is $\nabla^*\alpha = \sum_{a=1}^{2n}(\nabla_{e_a}\alpha)e_a$. A substitution of (5.5) into (5.4) and a use of the Ricci identities together with $\bar{B} = 0$ give

$$B(X,JY) = \frac{n+2}{2h}\Big[\nabla^2 h(Y,X) + \nabla^2 h(JY,JX) - 2\omega(X,Y)dh(\xi)\Big] - \frac{n+2}{2h^2}[dh(X)dh(Y) + dh(JX)dh(JY)] - \frac{n+2}{2}\left(\frac{1}{h} - 2 + \frac{1}{2h^2}|\nabla h|^2\right)g(X,Y). \tag{5.10}$$

From $\rho(X,JY) = B(X,JY) - 2(n+1)g(X,Y)$ and (5.10) it follows

$$\rho(X,J\nabla h) = \frac{n+2}{2h}\Big[\nabla^2 h(\nabla h,X) + \nabla^2 h(J\nabla h,JX) - 2dh(\xi)dh(JX) - \frac{3}{2h}|\nabla h|^2 dh(X) - dh(X)\Big] - ndh(X). \tag{5.11}$$

Substituting equation (5.11) into equation (5.9) shows

$$\nabla^* A(JX) = \frac{n+2}{4}\Big[\frac{\nabla^2 h(J\nabla h,JX)}{h^2} - \frac{|\nabla h|^2}{h^3}dh(X) - \frac{1}{h^2}dh(X) - \frac{2}{h}\nabla^2 h(JX,\xi)\Big]. \tag{5.12}$$

With the help of (5.8), (5.10) and (5.12) we define the following 1-forms

$$\begin{aligned} d(X) &= -4h^{-1}A(X,J\nabla h) = \frac{\nabla^2 h(\nabla h,X) - \nabla^2 h(J\nabla h,JX)}{h^2};\\ e(X) &= \frac{2}{n+2}h^{-1}B(X,J\nabla h) = \frac{\nabla^2 h(\nabla h,X) + \nabla^2 h(J\nabla h,JX)}{h^2} - \frac{2dh(\xi)dh(JX)}{h^2}\\ &\quad - \Big(\frac{1}{h^2} - \frac{2}{h} + \frac{3|\nabla h|^2}{2h^3}\Big)dh(X);\\ u(X) &= \frac{4}{n+2}\nabla^* A(JX) = \frac{\nabla^2 h(J\nabla h,JX)}{h^2} - \frac{2}{h}\nabla^2 h(JX,\xi) - \Big(\frac{|\nabla h|^2}{h^3} + \frac{1}{h^2}\Big)dh(X). \end{aligned} \tag{5.13}$$

We obtain easily from (5.13) the next identity

$$u(X) = \frac{e(X) - d(X)}{2} - \frac{2\nabla^2 h(JX,\xi)}{h} - \frac{1}{h^2}\Big(\frac{1}{2} + h + \frac{|\nabla h|^2}{4h}\Big)dh(X) + \frac{dh(\xi)dh(JX)}{h^2}. \tag{5.14}$$

Define the following tensors

$$\begin{aligned} D(X,Y) &= -4A(X,Y), \quad D^h(X,Y,Z) = h^{-1}\left[D(.,Z)dh(.)\right]_{[1]},\\ E(X,Y) &= \frac{2}{n+2}B(X,Y), \quad E^h(X,Y,Z) = h^{-1}\left[E(X,.)dh(.)\right]_{[-1]}. \end{aligned} \tag{5.15}$$

In other words, we have

$$\begin{aligned} D^h(X,Y,Z) &= \frac{1}{2h}\left[dh(X)D(Y,Z) + dh(JX)D(JY,Z)\right],\\ E^h(X,Y,Z) &= \frac{1}{2h}\left[dh(Z)E(X,Y) - dh(JZ)E(X,JY)\right]. \end{aligned}$$

At this point we can state one of the main results of [139].

Theorem 5.4 ([139]). *Let $(M,\bar{\eta})$ be a Sasaki-Einstein manifold pseudoconformally equivalent to a CR manifold $(M,\eta,\bar{\eta} = \frac{1}{2h}\eta)$ of constant pseudohermitian scalar curvature so that $\bar{S} = S = 4n(n+1)$. For*

$$f = \frac{1}{2} + h + \frac{|\nabla h|^2}{4h}, \tag{5.16}$$

we have

$$\nabla^*\Big(f[d+e]-dh(\xi)Jd+dh(\xi)Je-6dh(\xi)Ju\Big) \\ =\frac{1}{2}\left(\frac{1}{2}+h\right)\left(|D|^2+|E|^2\right)+\frac{h}{8}|E^h{+}D^h|^2+Q(d,e,u). \tag{5.17}$$

where $Q(d,e,u)$ is non-negative quadratic form of the vector fields (d,e,u).

Proof. We recall, that for a horizontal 1-form α the 1-form $J\alpha$ is defined by $J\alpha(X)=-\alpha(JX)$. The divergences of the involved vector fields are calculated using (5.13) and the Bianchi identies (5.7). Since $S=4n(n+1)$ the Bianchi identities (5.7) take the form

$$\nabla^*B(JX)=2(n-1)\nabla^*A(JX)=\frac{(n+2)(n-1)}{2}u(X),\quad \nabla^*Ju=\frac{4}{n+2}\sum_{a,b=1}^{2n}(\nabla^2_{e_ae_b}A)(e_a,e_b)=0. \tag{5.18}$$

A direct computation gives

$$\nabla^*d=\sum_{a=1}^{2n}(\nabla_{e_a}D)(e_a)=-h^{-1}D(\nabla h)-(n+2)h^{-1}u(J\nabla h)+\frac{1}{2}|D|^2. \tag{5.19}$$

Using the properties of A, we calculate

$$\nabla^*(Jd)=h^{-1}d(J\nabla h)+(n+2)h^{-1}u(J\nabla h), \tag{5.20}$$

taking into account $\sum_{a,b=1}^{2n}A(e_a,Je_b)\nabla^2h(Je_a,e_b)=0$ due to (5.8). Similarly, we calculate

$$\nabla^*e=(n-1)h^{-1}u(\nabla h)+\frac{1}{2}|E|^2 \tag{5.21}$$

after using the equality $\frac{2}{n+2}h^{-1}\sum_{a,b=1}^{2n}B(e_a,Je_b)\nabla^2h(e_a,e_b)=h^{-1}e(\nabla h)+\frac{1}{2}|E|^2$ follows from (5.10) and (5.15). Finally, we have

$$\nabla^*(Je)=(n-1)h^{-1}u(J\nabla h) \tag{5.22}$$

since $B(JX,JX)=0$ and $\sum_{a,b=1}^{2n}B(e_a,Je_b)\nabla^2h(Je_a,e_b)=0$ due to (5.10). We obtain from (5.16) after applying the Ricci identities and (5.13) the identity

$$df(X)=\frac{h}{2}\Big(u(X)+d(X)\Big)+\nabla^2h(JX,\xi)+h^{-1}fdh(X)-h^{-1}dh(\xi)dh(JX). \tag{5.23}$$

At this point we are ready to calculate the divergence formula using (5.19),(5.20),(5.21),(5.22), (5.23) and (5.14) which give

$$\nabla^*\Big(f[d+e]-dh(\xi)Jd+dh(\xi)Je-6dh(\xi)Ju\Big) \\ =\frac{1}{2}\left(\frac{1}{2}+h+\frac{|\nabla h|^2}{4h}\right)\Big[|D|^2+|E|^2\Big]+\frac{h}{2}\Big[|d|^2+|e|^2+6|u|^2+4g(d,u)-4g(u,e)\Big] \\ =\frac{1}{2}\left(\frac{1}{2}+h\right)\left(|D|^2+|E|^2\right)+\frac{h}{4}|D^h+E^h|^2+\frac{h}{2}\left(|d|^2+|e|^2+6|u|^2+4g(d,u)-4g(u,e)-2g(d,e)\right) \\ =\frac{1}{2}\left(\frac{1}{2}+h\right)\left(|D|^2+|E|^2\right)+\frac{h}{4}|D^h+E^h|^2+\frac{h}{2}Q(d,e,u) \tag{5.24}$$

with $Q=\begin{bmatrix}1 & -1/2 & 2\\ -1/2 & 1 & -2\\ 2 & -2 & 6\end{bmatrix}$ using the next identity in the last equality

$$\frac{|\nabla h|^2}{4h}\left(|D|^2+|E|^2\right)=\frac{h}{2}|D^h+E^h|^2-hg(d,e). \tag{5.25}$$

Notice that Q has eigenvalues $\frac{15\pm\sqrt{209}}{4}$ and $\frac{1}{2}$, hence it is a positive definite matrix. Finally, the validity of (5.25) can be seen as follows,

$$(5.26)\quad |D^h+E^h|^2=\frac{1}{4h^2}\,|dh(e_a)D(e_b,e_c)+dh(Je_a)D(Je_b,e_c)+dh(e_c)E(e_a,e_b)-dh(Je_c)E(e_a,Je_b)|^2 \\ =\frac{1}{2h^2}|\nabla h|^2\left(|D|^2+|E|^2\right)+\frac{2}{h^2}\left(D(\nabla h,e_a)E(\nabla h,e_a)\right)=\frac{1}{2h^2}|\nabla h|^2\left(|D|^2+|E|^2\right)+2g(d,e).$$

□

5.2. **The uniqueness theorem in a Sasaki-Einstein class.** Motivated by Theorem 2.6 it is natural to investigate the uniqueness of the pseudohermitian structures of constant scalar curvature in the Sasaki-Einstein case, especially in view of Theorem 5.3. The fact that the divergence formula of [139] can be stated as in Theorem 5.3 was observed earlier in [120] which influenced the results [120, 121] in which there is a clear separation of the two steps of Jerison and Lee's argument, the first involving the conformal equivalence of an "Einstein structure" to a structure of "constant scalar curvature" and the second involving the characterization of the conformal equivalence of two (conformally flat) Einstein structures. A corresponding QC version of the Obata uniqueness theorem was formulated by the second author. Clearly, in the CR case Theorem 5.3 addresses the first step, while a part of the second step is contained in [139] where the (suitable) conformal factor is characterized as a pluriharmonic function. For the completion of the second step one can reduce to the result mentioned in Remark 2.2 with an argument employed in [131, Theorem 1.3] (see also the end of Section 7.3) rather than relying on the calculation on the Heisenberg group when in the pseudoconformal class of the Sasaki-Einstein sphere as Jerison & Lee did. For details of this last reduction see [226]. Alternatively, a conceptual proof using again in the first step the Jerison & Lee's divergence formula and as a second step a generalization of (2.13) is possible based on the proof found in the quaternionic contact case in the forthcoming paper [125]. Next, we sketch briefly the obvious adaption of the argument from the quaternionic contact case [125]. First, we use the well known fact that a vector field Q on a CR pseudo-Hermitian manifold is an infinitesimal CR transformation iff there is a (smooth real-valued) function σ such that $Q=-\frac{1}{2}J\nabla\sigma-\sigma\,\xi$ and σ satisfies the second order equation $\mathcal{L}_Q J=0$, see [57]. In fact, decomposing Q in its horizontal and vertical parts $Q=Q_H-\sigma\xi$ it follows that Q_H ("contact Hamiltonian field") is determined by $\eta(Q_H)=0$ and $i_{Q_H}\,d\eta\equiv 0\ \ (mod\,\eta)$ while the preservation of the complex structure gives the second order system $[\nabla^2\sigma]_{[-1]}(X,Y)=-2\sigma A(JX,Y)$. Next, as a consequence of the CR Yamabe equation one obtains a formula as in Lemma 2.13 for an infinitesimal CR automorphism Q on (M,η), namely

$$(5.27)\qquad \Delta(\nabla^*Q_H)\ =\ -\,\frac{n}{2(n+1)}dS(Q)\ -\ \frac{S}{2(n+1)}\nabla^*Q_H,$$

where Q_H is the horizontal part of Q. In our case $A=0$ and $S=4n(n+1)$, hence for $\sigma=dh(\xi)$ it follows from Ricci's identity (7.6) and (5.8) that the vector field Q defined by $Q=-\frac{1}{2}J\nabla dh(\xi)-dh(\xi)\xi$ is an infinitesimal CR vector field unless it vanishes. Now, for f defined in (5.16), from (5.23) it follows $Q=-\frac{1}{2}\nabla f-dh(\xi)\xi$. This implies that $\phi=\triangle f$ either vanishes identically or is an eigenfuction of the sublaplacian realizing the smallest possible eigenvalue on a (pseudo-Einstein) Sasakian manifold. Finally, if $h\neq const$ then the CR Lichnerowicz-Obata theorem [46, 47], see Section 7.2, shows that (M,η) is homothetic to the CR unit sphere, which completes the proof. We note that the above arguments have as a corollary that in Jerison & Lee's identity [139, (3.1)], letting $\phi=\triangle_b Re(f)$ we have $\triangle_b\phi=-2n\phi$.

Thus, a pseudoconformal class of a Sasaki-Einstein pseudohermitian form different from the standard Sasaki-Einstein form on the round sphere contains a unique (up to homothety) pseudohermitian form of constant CR scalar curvature, namely, the Sasaki-Einstein form itself.

6. The qc-Yamabe problem and the Obata type uniqueness theorem

In this section we consider the quaternionic contact version of the Yamabe problem described in Section4.2.2. We begin by quoting the next result of Wang [225] which follows from the known techniques in the Riemannian and CR settings.

Theorem 6.1 ([225])**.** *Let (M,η) be a compact quaternionic contact manifold of real dimension $4n+3$.*

a) *The qc Yamabe constant satisfies the inequality* $\Upsilon(M,[\eta]) \leq \Upsilon(S^{4n+3},([\tilde\eta])$.
b) *If* $\Upsilon(M,[\eta]) < \Upsilon(S^{4n+3},[\tilde\eta])$*, then the Yamabe problem has a solution.*

In view of the Riemannian and CR cases, it is expected that on a compact qc manifods $\Upsilon(M,[\eta]) < \Upsilon(S^{4n+3},[\tilde\eta])$ unless the qc manifold $(M,[\eta])$ is locally qc conformal to $(S^{4n+3},[\tilde\eta])$. Some steps towards the proof of this result include the qc-normal coordinates constructed in [158] and the qc conformal tensor, $|W^{qc}|^2$ [129].

Another known general result is the (local) classification of all (local) qc-conformal transformations of the flat structure on the $\boldsymbol{G}\,(\mathbb{H})$ which are also qc-Einstein. This classification is used as a replacement of the theory of the pluriharmonic functions that appear in the CR case. Some attempts in extending the latter in the quaternionic (contact) case can be found in [120]. However, so far, these extensions have not proven useful in the solution of the Yamabe problem. The following Theorem precises the result of [120, Theorem 1.1] where only the vanishing qc-scalar curvature case was considered.

Theorem 6.2. *Let* $\Theta = \frac{1}{2h}\tilde\Theta$ *be a conformal deformation of the standard qc-structure* $\tilde\Theta$ *on the quaternionic Heisenberg group* $\boldsymbol{G}\,(\mathbb{H})$*. If* Θ *is also qc-Einstein, then up to a left translation the function* h *is given by*

$$h(q,\omega) \;=\; c_0\left[\left(\sigma\;+\;|q+q_0|^2\right)^2\;+\;|\omega\;+\;\omega_o\;+\;2\,\mathit{Im}\,q_o\,\bar q|^2\right], \tag{6.1}$$

for some fixed $(q_o,\omega_o)\in \boldsymbol{G}\,(\mathbb{H})$ *and constants* $c_0>0$ *and* $\sigma\in\mathbb{R}$*. Furthermore,*

$$S_\Theta = 128n(n+2)c_0\sigma \tag{6.2}$$

The proof follows from a careful reading of the proof of [120, Theorem 1.1] and making of the necessary changes. As in [120, Theorem 1.1], h satisfies a system of partial differential equations whose solution is a family of polynomial of degree four.

The final known general result concerns the seven dimensional case and is still open in the higher dimensions.

Theorem 6.3 ([121])**.** *If a quaternionic contact structure* (M^7,η) *is conformal to a qc-Einstein structure* $(M^7,\bar\eta)$*,* $\bar\eta \;=\; \frac{1}{2h}\,\eta$ *so that* $S=\bar S = 16n(n+2)$*, then* (M^7,η) *is also qc-Einstein.*

The above results lead to a complete solution of the qc Yamabe problem on the standard qc *seven dimensional* sphere and quaternionic Heisenberg groups. In particular, as conjectured in [99], all solutions of the qc Yamabe equations are given by those that realize the Yamabe constant of the sphere or the best constant in the Folland-Stein inequality.

Theorem 6.4 ([121])**.** *a) Let* $\tilde\eta = \frac{1}{2h}\eta$ *be a conformal deformation of the standard qc-structure* $\tilde\eta$ *on the quaternionic unit sphere* S^7*. If* η *has constant qc-scalar curvature, then up to a multiplicative constant* η *is obtained from* $\tilde\eta$ *by a conformal quaternionic contact automorphism. In particular,* $\Upsilon(S^7) \;=\; 48\,(4\pi)^{1/5}$ *and this minimum value is achieved only by* $\tilde\eta$ *and its images under conformal quaternionic contact automorphisms.*
b) On the the seven dimensional quaternionic Heisenberg group the only solutions of the qc-Yamabe equation, up to translations (3.2) *and dilations* (3.3)*, are those given in* (3.8)*.*

The proof of Theorem 6.4 relies on Theorem 6.2 and Theorem 6.3 and will be sketched near the end of the Section.

6.1. **The Yamabe problem on a 7-D qc-Einstein manifold. Proof of Theorem 6.3.** In this section we give some details on the proof of Theorem 6.3. The analysis involves a number of intrinsic to the structure vector fields/ 1-forms which are defined in any dimension, so here $n\geq 1$. We shall consistently keep the notation introduced in [121], which can be consulted for details.

6.1.1. *Intrinsic vector fields and their divergences.* We begin by defining the horizontal 1-forms A_s, also letting $A=A_1+A_2+A_3$,

$$A_i(X) = \omega_i([\xi_j,\xi_k],X). \tag{6.3}$$

The contracted Bianchi identity on a $(4n+3)$-dimensional qc manifold with constant qc-scalar curvature reads [120], Theorem 4.8],

$$\nabla^*T^0 = (n+2)A, \qquad \nabla^*U = \frac{1-n}{2}A. \tag{6.4}$$

Let h be a positive smooth function on a qc manifold (M, g, η) and $\bar\eta = \frac{1}{2h}\eta$ be a conformal deformation of the qc structure η. As usual, the objects related to $\bar\eta$ will be denoted by an over-line. Thus,

$$d\bar\eta = -\frac{1}{2h^2}dh\wedge\eta + \frac{1}{2h}d\eta, \quad \bar g = \frac{1}{2h}g.$$

The new Reeb vector fields $\{\bar\xi_1, \bar\xi_2, \bar\xi_3\}$ are $\xi_s = 2h\xi_s + I_s\nabla h, s = 1, 2, 3$. The components T^0, U of the Biquard connection and the qc scalar curvatures change as follows [121]

$$\bar T^0(X,Y) = T^0(X,Y) + \frac{1}{4h}\Big(3\nabla^2 h(X,Y) - \sum_{s=1}^3 \nabla^2 h(I_sX, I_sY)\Big) - \frac{1}{2h}\sum_{s=1}^3 dh(\xi_s)\omega_s(X,Y). \tag{6.5}$$

$$\begin{aligned}\bar U(X,Y) = U(X,Y) &+ \frac{1}{8h}\Big(\nabla^2 h(X,Y) + \sum_{s=1}^3 \nabla^2 h(I_sX, I_sY)\Big)\\ &- \frac{1}{4h^2}\Big(dh(X)dh(Y) + \sum_{s=1}^3 dhI_sX)dh(I_sY)\Big) - \frac{1}{8h}\Big(\triangle h - \frac{2}{h}|\nabla h||^2\Big)g(X,Y),\end{aligned} \tag{6.6}$$

$$\bar S = 2hS + 8(n+2)\triangle h - 8(n+2)^2 h^{-1}|\nabla h|^2. \tag{6.7}$$

Suppose $\bar\eta$ is a positive 3-Sasakian structure, i.e. $\bar T^0 = \bar U = 0, \bar S = 16n(n+2)$. Then (6.7) takes the form

$$2n = 4nh + \triangle h - (n+2)h^{-1}|\nabla h|^2, \tag{6.8}$$

which is the qc Yamabe equation. We also have the formulas

$$\begin{aligned}A_1(X) \;=\; &-\frac{1}{2}h^{-2}dh(X) \;-\; \frac{1}{2}h^{-3}|\nabla h|^2 dh(X) - \;\frac{1}{2}h^{-1}\Big(\;\nabla dh(I_2X, \xi_2) \;+\; \nabla dh(I_3X, \xi_3)\Big)\\ &+\;\frac{1}{2}h^{-2}\Big(dh(\xi_2)\,dh(I_2X) \;+\; dh(\xi_3)\,dh(I_3X)\Big) + \;\frac{1}{4}h^{-2}\Big(\;\nabla dh(I_2X, I_2\nabla h) \;+\; \nabla dh(I_3X, I_3\nabla h)\Big).\end{aligned} \tag{6.9}$$

The expressions for A_2 and A_3 can be obtained from the above formula by a cyclic permutation of $(1, 2, 3)$. Thus, we obtain

$$\begin{aligned}A(X) \;=\; &-\frac{3}{2}h^{-2}dh(X) \;-\; \frac{3}{2}h^{-3}|\nabla h|^2 dh(X) - \;h^{-1}\sum_{s=1}^3\;\nabla dh(I_sX, \xi_s)\\ &+\;h^{-2}\sum_{s=1}^3 dh(\xi_s)\,dh(I_sX) \;+\; \frac{1}{2}h^{-2}\sum_{s=1}^3\;\nabla dh(I_sX, I_s\nabla h).\end{aligned} \tag{6.10}$$

We need the divergences of various vector/1-forms defined above in addition to a few more. We recall that an orthonormal frame $\{e_1, e_2 = I_1e_1, e_3 = I_2e_1, e_4 = I_3e_1, \dots, e_{4n} = I_3e_{4n-3}, \xi_1, \xi_2, \xi_3\}$ is called qc-normal frame at a point of a qc manifold if the connection 1-forms of the Biquard connection vanish at that point. As shown in [120], see also [130, Lemma 6.2.1], a qc-normal frame exists at each point of a qc manifold. If σ is a horizontal 1-form, then with respect to a qc-normal frame, the divergence of $I_s\sigma$, ($I_s\sigma(X) = -\sigma(I_sX)$) is given by

$$\nabla^*(I_s\sigma) \;=\; -\sum_{a=1}^{4n}(\nabla_{e_a}\sigma)(I_se_a).$$

With some calculations using (6.10), (6.9) and the properties of the torsion and curvature of the Biquard connection, we obtain

$$\begin{aligned}\nabla^*\Big(\sum_{s=1}^3 dh(\xi_s)I_sA_s\Big) &= \sum_{s=1}^3\sum_{a=1}^{4n}\;\nabla dh\,(I_se_a, \xi_s)A_s(e_a),\\ \nabla^*\Big(\sum_{s=1}^3 dh(\xi_s)I_sA\Big) &= \sum_{s=1}^3\sum_{a=1}^{4n}\;\nabla dh\,(I_se_a, \xi_s)A(e_a).\end{aligned} \tag{6.11}$$

We define the following one-forms for $s = 1, 2, 3$,

$$D_s(X) = -\frac{1}{2h}\Big[T^0(X,\nabla h) + T^0(I_sX, I_s\nabla h)\Big], D = -\frac{1}{h}T^0(X,\nabla h), F_s(X) = -\frac{1}{h}T^0(X, I_s\nabla h). \tag{6.12}$$

Using the fact that the tensor T^0 belongs to the [-1]-component we obtain from (6.12)

$$D = D_1 + D_2 + D_3, \qquad F_i(X) = -D_i(I_iX) + D_j(I_iX) + D_k(I_iX), \tag{6.13}$$

where (ijk) is a cyclic permutation of (1,2,3). As a consequence of (6.4), (6.9), (6.10), the qc Yamabe equation (6.8) and (6.5) taken with $\bar{A} = 0$, we obtain after some calculations, see [121] for details, the following theorem.

Lemma 6.5 ([121]). *Suppose (M,η) is a quaternionic contact manifold with constant qc-scalar curvature $S = 16n(n+2)$. Suppose $\bar\eta = \frac{1}{2h}\eta$ has vanishing $[-1]$-torsion component $\overline{T}^0 = 0$. Then we have*

$$D(X) = \frac{1}{4}h^{-2}\Big(3\,\nabla dh(X,\nabla h) - \sum_{s=1}^{3}\nabla dh(I_sX, I_s\nabla h)\Big) + h^{-2}\sum_{s=1}^{3}dh(\xi_s)\,dh(I_sX).$$

The divergence of D is given by $\nabla^* D = |T^0|^2 - h^{-1}\sum_{a=1}^{4n}dh(e_a)D(e_a) - h^{-1}(n+2)\sum_{a=1}^{4n}dh(e_a)A(e_a)$, while the divergence of $\sum_{s=1}^3 dh(\xi_s)F_s$ is

$$\nabla^*\Big(\sum_{s=1}^{3}dh(\xi_s)F_s\Big) = \sum_{s=1}^{3}\sum_{a=1}^{4n}\Big[\nabla dh\,(I_se_a,\xi_s)F_s(I_se_a)\Big] + h^{-1}\sum_{s=1}^{3}\sum_{a=1}^{4n}\Big[dh(\xi_s)dh(I_se_a)D(e_a) + (n+2)\,dh(\xi_s)dh(I_se_a)\,A(e_a)\Big].$$

6.1.2. *Solution of the qc-Yamabe equation in 7-D.* At this point we restrict our considerations to the 7-dimensional case turn to the proof of a key divergence formula motivated by the Riemannian and CR cases of the considered problem. As in the CR case [139], the Bianchi identities [120, Theorem 4.8] are not enough for the proof, unlike what happens in the Riemannian case as we saw in the proof of Theorem 2.6.

In fact, the proof of Theorem 6.3 follows by an integration of the following divergence formula (6.14), which implies $T^0 = 0$. In dimension seven the tensor U vanishes identically, $U = 0$, and (4.23) yields the claim. Thus, the crux of the proof of Theorem 6.3 is the next formula, in which for $f = \frac{1}{2} + h + \frac{1}{4}h^{-1}|\nabla h|^2$, the following identity holds true

$$\nabla^*\Big(fD + \sum_{s=1}^{3}dh(\xi_s)\,F_s + 4\sum_{s=1}^{3}dh(\xi_s)I_sA_s - \frac{10}{3}\sum_{s=1}^{3}dh(\xi_s)\,I_sA\Big) = f||T^0||^2 + hVLV^t. \tag{6.14}$$

Here, L is the following positive semi-definite matrix

$$L = \begin{bmatrix} 2 & 0 & 0 & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} \\ 0 & 2 & 0 & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} \\ 0 & 0 & 2 & -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} \\ \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{22}{3} \end{bmatrix}$$

and $V = (D_1, D_2, D_3, A_1, A_2, A_3)$ with A_s, D_s defined, correspondingly, in (6.3) and (6.12).

We sketch the proof of (6.14). Recall that in dimension seven, $n=1$, the [3]-part of the Biquard torsion vanishes identically, $U=0$. Then (6.6) together with the Yamabe equation (6.8) imply

$$\nabla^2 h(X,\nabla h)+\sum_{s=1}^{3}\nabla^2 h(I_sX,I_s\nabla h)-(2-4h+3h^{-1}|\nabla h|^2)dh(X)=0. \tag{6.15}$$

Combining (6.9), (6.11), (6.15) and formulas in Lemma 6.5 it is easy to check the formula of the theorem. It is not hard to see that the eigenvalues of L are given by

$$\{0,\quad 0,\quad 2\,(2+\sqrt{2}),\quad 2\,(2-\sqrt{2}),\quad 10,\quad 10\},$$

which shows that L is a non-negative matrix.

6.1.3. *The 7-D qc Yamabe problem on the sphere and qc Heisenberg group.* At this point we are ready to complete the proof of Theorem 6.4. Recall, that the Cayley transform (4.31) is a conformal quaternionic contact diffeomorphism, hence up to a constant multiplicative factor and a quaternionic contact automorphism the forms $\mathcal{C}_*\tilde\eta$ and $\tilde\Theta$ are conformal to each other. It follows that the same is true for $\mathcal{C}_*\eta$ and $\tilde\Theta$. In addition, $\tilde\Theta$ is qc-Einstein by definition, while η and hence also $\mathcal{C}_*\eta$ are qc-Einstein as we already observed. According to Theorem 6.2, up to a multiplicative constant factor, the forms $\mathcal{C}_*\tilde\eta$ and $\mathcal{C}_*\eta$ are related by a translation or dilation on the Heisenberg group. Hence, we conclude that up to a multiplicative constant, η is obtained from $\tilde\eta$ by a conformal quaternionic contact automorphism which proves the first claim of Theorem 6.4. From the conformal properties of the Cayley transform and the existence Theorem [221] it follows that the minimum $\Upsilon(S^{4n+3},[\tilde\eta])$ is achieved by a smooth 3-contact form, which due to the Yamabe equation is of constant qc-scalar curvature. This completes the proof of Theorem 6.4 a). The proof of part b) is reduced to part a) by "lifting" the analysis to the sphere via the Cayley transform. A point, which requires some analysis is that we actually obtain qc-structures on the whole sphere. This follows from the properties of the Kelvin transform which sends a solution of the Yamabe equation to a solution of the Yamabe equation, see [121, Section 5.2] or [130, Section 6.6] for the details.

A similar argument will be used in the proof of Theorem 8.5.

6.2. **The uniqueness theorem in a 3-Sasakin conformal class.** We mention that similarly similarly to the Riemannian and CR cases it is expected that the class of unit volume qc-Einstein qc-conformal class contains a unique metric of constant scalar curvature, with the exception of the 3-Sasakian sphere, see Section 5.2 for a comparison with the CR case and some background in the sub-Riemannian case. What was problematic was the first step as outlined in Section 5.2, since here Theorem 6.3 supplies the first step in dimension seven while the higher dimensional cases was open. A proof extending the seven dimensional case was found in [125] where the reader can also find a proof of the uniqueness result.

7. The CR Lichneorwicz and Obata theorems

In accordance with Convention 1.1, in this section we use the non-negative definite sub-Laplacian, $\triangle u = -tr^g(\nabla^2 u)$ for a function u on a strictly pseudoconvex pseudohermitian manifold M with a Tanaka-Webster connection ∇. Also, the divergence of a vector field is taken with a "-", hence we have $\triangle u = \nabla^*(\nabla u) = -\sum_{a=1}^{2n}\nabla^2 u(e_a,e_a)$ for an orthonormal basis of the horizontal space.

7.1. **The CR Lichneorwicz first eigenvalue estimate.** From the sub-ellipticity of the sub-Laplacian on a strictly pseudoconvex CR manifold it follows that its spectrum is discrete on a compact manifold. It is therefore natural to ask if there is a sub-Riemannian version of Theorem 2.1. In fact, a CR analogue of the Lichnerowicz theorem was found by Greenleaf [109] for dimensions $2n+1>5$, while the corresponding results for $n=2$ and $n=1$ were achieved later in [169] and [61], respectively. As already observed in Theorem 4.1, the standard Sasakian unit sphere has first eigenvalue equal to 2n with eigenspace spanned by the restrictions of all linear functions to the sphere, hence the following result is sharp.

Theorem 7.1 ([109, 169, 61]). *Let (M,η) be a compact strictly pseudoconvex pseudohermitian manifold of dimension $2n+1$ such that for some $k_0=const>0$ we have the Lichnerowicz-type bound*

$$Ric(X,X)+4A(X,JX)\geq k_0 g(X,X),\qquad X\in H. \tag{7.1}$$

If $n>1$, then any eigenvalue λ of the sub-Laplacian satisfies $\lambda \geq \frac{n}{n+1}k_0$. If $n=1$ the estimate $\lambda \geq \frac{1}{2}k_0$ holds assuming in addition that the CR-Paneitz operator is non-negative , i.e., for any smooth function f we have $\int_M f\cdot Cf\, Vol_\eta \geq 0$, where Cf is the CR-Paneitz operator.

We recall that the fourth-order CR-Paneitz operator written in real coordinates is defined by the formula

$$Cf=\sum_{a,b=1}^{2n}\nabla^4 f(e_a,e_a,e_b,e_b)+\sum_{a,b=1}^{2n}\nabla^4 f(e_a,Je_a,e_b,Je_b)-4n\nabla^*A(J\nabla f)-4n\,g(\nabla^2 f,JA).$$

In view of the prominent role of the CR-Paneitz operator in the geometric analysis on a three dimensional CR manifold we pause for a moment to give an idea of several occurrences. We start with a few definitions, Given a function f we define the one form,

$$P_f(X)=\sum_{b=1}^{2n}\nabla^3 f(X,e_b,e_b)+\sum_{b=1}^{2n}\nabla^3 f(JX,e_b,Je_b)+4nA(X,J\nabla f)$$

so we have $Cf=-\nabla^*P$. The CR Paneitz operator is called non-negative if

$$\int_M f\cdot Cf\, Vol_\eta=-\int_M P_f(\nabla f)\, Vol_\eta \geq 0, \qquad f\in \mathcal{C}^\infty_0(M).$$

In the three dimensional case the positivity condition is a CR invariant since it is independent of the choice of the contact form which follows from the conformal invariance of C proven in [116]. In the case of vanishing pseudohermitian torsion, we have, up to a multiplicative constant, $C=\Box_b\Box_b$, where $\Box_b$ is the Kohn Laplacian, hence the CR Paneitz operator is also non-negative. This property is in fact true for any $n>1$ which can be seen through the relation between the CR Paneitz operator and the $[1]$-component of the horizontal Hessian $(\nabla^2 f)(X,Y)$ found in [165, 108]. For this, consider the tensor $B(X,Y)$ defined by

$$B(X,Y)\equiv B[f](X,Y)=(\nabla^2 f)_{[1]}(X,Y)=\frac{1}{2}\left[(\nabla^2 f)(X,Y)+(\nabla^2 f)(JX,JY)\right]$$

and also the completely traceless part of B,

$$B_0(X,Y)\equiv B_0[f](X,Y)=B(X,Y)+\frac{\triangle f}{2n}g(X,Y)-\frac{1}{2n}g(\nabla^2 f,\omega)\,\omega(X,Y).$$

Then we have the formula [165, 108],

$$\begin{aligned}&\sum_{a=1}^{2n}(\nabla_{e_a}B_0)(e_a,X)=\frac{n-1}{2n}P_f(X),\\ &\int_M |B_0|^2\, Vol_\eta=-\frac{n-1}{2n}\int_M P_f(\nabla f)\, Vol_\eta=\frac{n-1}{2n}\int_M f\cdot(Cf)\, Vol_\eta.\end{aligned} \tag{7.2}$$

In particular, if $n>1$ the CR-Paneitz operator is non-negative. As an application of this result, we recall [165], see also [21] and [20], according to which if $n\geq 2$, a function $f\in\mathcal{C}^3(M)$ is CR-pluriharmonic, i.e, locally it is the real part of a CR holomorphic function, if and only if $B_0[f]=0$. By (7.2) only one fourth-order equation $Cf=0$ suffices for $B_0[f]=0$ to hold. When $n=1$ the situation is more delicate. In the three dimensional case, CR-pluriharmonic functions are characterized by the kernel of the third order operator $P[f]=0$ [165]. However, the single equation $Cf=0$ is enough again assuming the vanishing of the pseudohermitian torsion [108], see also [106]. On the other hand, [45] showed that if the pseudohermitian torsion vanishes the CR-Paneitz operator is essentially positive, i.e., there is a constant $\Lambda>0$ such that

$$\int_M f\cdot(Cf)\, Vol_\eta \geq \Lambda\int_M f^2\, Vol_\eta.$$

for all real smooth functions $f\in(Ker\, C)^\perp$, i.e., $\int_M f\cdot\phi\, Vol_\eta=0$ if $C\phi=0$. In addition, the non-negativity of the CR-Paneitz operator is relevant in the embedding problem for a three dimensional strictly pseudoconvex CR manifold. In the Sasakian case, it is known that M is embeddable, [168], and the CR-Paneitz operator is nonnegative, see [61], [45]. Furthermore, [51] showed that if the pseudohermitian scalar curvature of M is positive and C is non-negative, then M is embeddable in some $\mathbb{C}^n$.

After these preliminaries we are ready to sketch the proof of Theorem 7.1.

7.1.1. *Proof of the CR Lichnerowicz type estimate.* We shall use real coordinates as in [133, 130] and rely on the proof described in details in [131, Section 8.3] valid for $n \geq 1$. Not surprisingly, a key to the solution is the CR Bochner identity due to [109],

$$-\frac{1}{2}\triangle|\nabla f|^2 = |\nabla df|^2 - g(\nabla(\triangle f), \nabla f) + Ric(\nabla f, \nabla f) + 2A(J\nabla f, \nabla f) + 4\nabla df(\xi, J\nabla f). \tag{7.3}$$

The last term can be related to the traces of $\nabla^2 f$, [109],

$$\int_M \nabla^2 f(\xi, J\nabla f)\, Vol_\eta = -\int_M \frac{1}{2n} g(\nabla^2 f, \omega)^2 + A(J\nabla f, \nabla f)\, Vol_\eta$$

and also using the CR-Paneitz operator

$$\int_M \nabla^2 f(\xi, J\nabla f)\, Vol_\eta = \int_M -\frac{1}{2n}(\triangle f)^2 + A(J\nabla f, \nabla f) - \frac{1}{2n} P_f(\nabla f)\, Vol_\eta.$$

Integrating the CR Bochner idenity (for arbitrary function f) and using the last two formulas for the term $\int_M \nabla^2 f(\xi, J\nabla f)\, Vol_\eta$ we find

$$0 = \int_M Ric(\nabla f, \nabla f) + 4A(J\nabla f, \nabla f) - \frac{n+1}{n}(\triangle f)^2\, Vol_\eta$$
$$+ \int_M \left|(\nabla^2 f)\right|^2 - \frac{1}{2n}(\triangle f)^2 - \frac{1}{2n} g(\nabla^2 f, \omega)^2\, Vol_\eta + \int_M \Big[-\frac{3}{2n} P(\nabla f)\Big]\, Vol_\eta.$$

Noticing that $\left\{\frac{1}{\sqrt{2n}} g,\ \frac{1}{\sqrt{2n}}\omega\right\}$ is an orthonormal set in the [1]-space with non-zero traces, we have

$$\left|(\nabla^2 f)_{[0]}\right|^2 \overset{def}{=} \left|(\nabla^2 f)\right|^2 - \frac{1}{2n}(\triangle f)^2 - \frac{1}{2n} g(\nabla^2 f, \omega)^2.$$

Let us assume at this point that $\triangle f = \lambda f$ and the "Ricci" bound (7.1) to obtain the inequality

$$0 \geq \int_M \left(k_0 - \frac{n+1}{n}\lambda\right) |\nabla f|^2\, Vol_\eta + \int_M \left|(\nabla^2 f)_{[0]}\right|^2\, Vol_\eta - \frac{3}{2n}\int_M P_f(\nabla f)\, Vol_\eta,$$

which implies $\lambda \geq \frac{n}{n+1} k_0$ with equality holding iff

$$\nabla^2 f = \frac{1}{2n}(\triangle f)\cdot g + \frac{1}{2n} g(\nabla^2 f, \omega)\cdot \omega \tag{7.4}$$

and $\int_M P_f(\nabla f)\, Vol_\eta = 0$ taking into account the extra assumption for $n = 1$. The proof of Theorem 7.1 is complete.

7.2. The CR Obata type theorem.

Theorem 7.2 ([131]). *Let (M, θ) be a strictly pseudoconvex pseudohermitian CR manifold of dimension $2n+1$ with a divergence-free pseudohermitian torsion, $\nabla^* A = 0$. Assume, further, that M is complete with respect to the associated Riemannian metric (4.1). If $n \geq 2$ and there is a smooth function $f \not\equiv 0$ whose Hessian with respect to the Tanaka-Webster connection satisfies*

$$\nabla^2 f(X, Y) = -f g(X, Y) - df(\xi)\omega(X, Y), \qquad X, Y \in H = Ker\, \theta, \tag{7.5}$$

then up to a scaling of θ by a positive constant (M, θ) is the standard (Sasakian) CR structure on the unit sphere in $\mathbb{C}^{n+1}$. In dimension three, $n = 1$, the above result holds provided the pseudohermitian torsion vanishes, $A = 0$.

This is the best known result for a complete non-compact M unlike the Riemannian and QC cases where the corresponding results are valid without any conditions on the torsion, see the paragraph after (2.3) and Theorem 8.2. It should be noted that besides the Sasakian condition, when $n = 1$, one can invoke assumptions such as the vanishing of the divergence of the torsion, the vanishing of the CR-Paneitz operator or the equality case in (7.1). We insist on the strongest assumption, which avoids a lot of the technicalities which appear when a combination of these assumptions are made while still achieving a (probably) non-optimal result. Results of this nature can be found by combining identities proven in [131]. In the compact case, with the help of a clever integration argument [170, 171] were able to complete the arguments of [131, 132] and remove the assumption of divergence free torsion $\nabla^* A = 0$ for $n \geq 2$, while the case $n = 1$ was completed in [132]. Taking into account (7.4), a consequence of these results is the Obata type theorem characterizing the case of equality in Theorem 7.1. We note that Theorem 7.2 actually shows that in the compact Sasakian case (7.5) characterizes the unit Sasakian sphere. This fact in addition to other results of [131] reappeared in [171].

Theorem 7.3 ([170, 171, 132])**.** *Suppose (M, J, η), $\dim M = 2n+1$, is a compact strictly pseudoconvex pseudohemitian manifold which satisfies the Lichnerowicz-type bound (7.1). If $n \geq 2$, then $\lambda = \frac{n}{n+1}k_0$ is an eigenvalue iff up-to a scaling (M, J, η) is the standard pseudo-Hermitian CR structure on the unit sphere in $\mathbb{C}^{n+1}$. If $n = 1$ the same conclusion holds assuming in addition that the CR-Paneitz operator is non-negative, $C \geq 0$.*

Some earlier papers which contributed to the proof of the above Theorem include S.-C. Chang & H.-L. Chiu who proved the above Theorem in the Sasakian case for $n \geq 2$ in [46] and for $n = 1$ in [47]. The non-Sasakian case, was considered by Chang, S.-C., & C.-T. Wu in [50] assuming $A_{\alpha\beta,\,\bar\beta} = 0$ for $n \geq 2$ and $A_{\alpha\beta,\,\gamma\bar\gamma} = 0$ and $A_{11,\,\bar 1} = 0$ for $n = 1$. $P_1 f = 0$.

Let us give an idea of the proof of Theorem 7.2 following [132]. The first step is to show the vanishing of the Webster torsion A. We shall make clear where the cases $n = 1$ and $n > 1$ diverge. Using the Ricci identity

$$\nabla^3 f(X,Y,\xi) = \nabla^3 f(\xi,X,Y) + \nabla^2 f(AX,Y) + \nabla^2 f(X,AY) + (\nabla_X A)(Y,\nabla f) + (\nabla_Y A)(X,\nabla f) - (\nabla_{\nabla f})A(X,Y). \tag{7.6}$$

in which we substitute the term $\nabla^3 f(\xi,X,Y)$ by its expression obtained after differentiating (7.5) we come to the next equation [131, (3.3)],

$$\nabla^3 f(X,Y,\xi) = -df(\xi)g(X,Y) - (\xi^2 f)\omega(X,Y) - 2fA(X,Y) + (\nabla_X A)(Y,\nabla f) + (\nabla_Y A)(X,\nabla f) - (\nabla_{\nabla f} A)(X,Y), \tag{7.7}$$

With the help of the Ricci identities, (7.5), (4.13) and (4.14) we obtain a formula for $R(X,Y,Z,\nabla f)$, [131, (4.1)]

$$R(Z,X,Y,\nabla f) = \Big[df(Z)g(X,Y) - df(X)g(Z,Y)\Big] + \nabla df(\xi,Z)\omega(X,Y) - \nabla df(\xi,X)\omega(Z,Y) - 2\nabla df(\xi,Y)\omega(Z,X) + A(Z,\nabla f)\omega(X,Y) - A(X,\nabla f)\omega(Z,Y), \tag{7.8}$$

which after taking traces gives identities for $Ric(X,\nabla f)$ and $Ric(JX,J\nabla f)$, [131, (4.2)],

$$\begin{aligned} Ric(Z,\nabla f) &= (2n-1)df(Z) - A(JZ,\nabla f) - 3\nabla df(\xi,JZ)\\ Ric(JZ,J\nabla f) &= df(Z) - (2n-1)A(JZ,\nabla f) - (2n+1)\nabla df(\xi,JZ). \end{aligned} \tag{7.9}$$

Note that when $n = 1$, $Ric(X,Y) = Ric(JX,JY)$, hence the identities for $Ric(X,\nabla f)$ and $Ric(JX,J\nabla f)$ *coincide*, which is the reason for the assumption $n > 1$ when $A \neq 0$. For $n > 1$, taking the $[-1]$ part of $R(.,.,X,Y)$ it follows

$$\nabla^2 f(Y,\xi) = df(JY) + 2A(Y,\nabla f). \tag{7.10}$$

Using the formula for the curvature (7.8) we come to $|\nabla f|^2 A(Y,Z) = df(Y)A(\nabla f,Z) - df(JY)A(\nabla f,JZ)$ found in the proof of [131, Lemma 4.1]. Hence, the Webster torsion is determined by $A(J\nabla f,\nabla f)$ as follows

$$|\nabla f|^4 A(X,Y) = -A(J\nabla f,\nabla f)\left[df(X)df(JY) + df(Y)df(JX)\right], \tag{7.11}$$

which implies in particular, $A(\nabla f,\nabla f) = 0$. On the other hand, from (7.10) we have [131, (4.9)],

$$\nabla^3 f(X,Y,\xi) = -df(\xi)g(X,Y) + f\omega(X,Y) - 2fA(X,Y) - 2df(\xi)A(JX,Y) + 2\nabla A(X,Y,\nabla f). \tag{7.12}$$

For $n > 1$, equations (7.12) and (7.7) imply the identity, see the formula in the proof of [131, Lemma 4.3],

$$2df(\xi)A(JX,Y) - (\nabla_{\nabla f} A)(X,Y) = (\xi^2 f)\omega(X,Y) + f\omega(X,Y) + (\nabla_X A)(Y,\nabla f) - (\nabla_Y A)(X,\nabla f). \tag{7.13}$$

Notice that the left-hand side is symmetric why the right-hand is skew-symmetric, hence they both vanish,

$$(\nabla_{\nabla f} A)(X,Y) = 2df(\xi)A(JX,Y) \qquad \text{and} \qquad (\nabla_X A)(Y,\nabla f) = (\nabla_Y A)(X,\nabla f), \tag{7.14}$$

taking into account $\nabla^2 f(\xi,\xi) = -f + \frac{1}{n}(\nabla^* A)(J\nabla f) = -f$, when $\nabla^* A = 0$, which follows by taking a trace in the (vanishing) right-hand side of (7.13), see [131, Lemma 4.3].

Remark 7.4. *Notice that the first identity implies $g(\nabla f, \nabla|A|^2) = 0$.*

The first equation of (7.14) gives $(\nabla_{\nabla f}A)(J\nabla f,\nabla f) = -2df(\xi)A(\nabla f,\nabla f) = 0$ as we already showed above, see (7.11). Finally, differentiating the identity $A(\nabla f,\nabla f) = 0$ and using (7.5) we obtain $(\nabla_X A)(\nabla f,\nabla f) = 2fA(\nabla f,X) - 2df(\xi)A(J\nabla f,X)$ which shows $(\nabla_\nabla fA)(\nabla f,\nabla f) = -2df(\xi)A(J\nabla f,\nabla f)$. Therefore, $A(J\nabla f,\nabla f) = 0$ which implies $|\nabla f|^4A = 0$. In order to conclude that $A = 0$ we need to know that f cannot be a local constant. For this and other facts we turn to the next step of the proof, where we show that f satisfies an elliptic equation for which we can use a unique continuation argument. We remark that the corresponding sub-elliptic result seems to be unavailable.

Next, we observe that if f satisfies (7.5), then f satisfies an elliptic equation [131, Corollary 4.5 & Lemma 5.1],

$$\triangle^h f = \triangle f - \nabla^2 f(\xi,\xi) = (2n+1)f - \frac{1}{n}(\nabla^* A)(J\nabla f), \qquad \text{if } n > 1, \tag{7.15}$$

$$\triangle^h f = \left(2 + \frac{S-2}{6}\right) f - \frac{1}{12}g(\nabla f,\nabla S) + \frac{1}{3}(\nabla^* A)(J\nabla f), \qquad \text{if } n = 1, \tag{7.16}$$

where $\triangle^h$ is the Riemannian Laplacian associated to the Riemannian metric $h = g + \eta^2$ on M. In particular, f cannot be a local constant. The equations follows from the formula relating the Levi-Civita and the Tanaka-Webster connections, see [77, Lemma 1.3] and [131, (4.15)], which shows

$$-\triangle^h f = -\triangle f + (\xi^2 f). \tag{7.17}$$

When $n > 1$ equation (7.15) follows taking into account (the line after) equation (7.14). The case $n = 1$ requires some further calculations for which we refer to [131, Lemma 5.1].

The final step of the prove of Theorem 7.2 is a reduction to the corresponding Riemannian Obata theorem on a complete Riemannian manifold. In fact, we will show that the Riemannian Hessian computed with respect to the Levi-Civita connection D of the metric h satisfies (2.3) and then apply the Obata theorem to conclude that (M,h) is isometric to the unit sphere. We should mention the influence of [46, 47] where the compact Sasakian case is reduced to Theorem 2.1.

For $n > 1$ where we proved that $A = 0$ we showed the validity of the next two identities

$$\nabla^2 f(\xi,Y) = \nabla^2 f(Y,\xi) = df(JY), \qquad \xi^2 f = -f. \tag{7.18}$$

Next, we show that (7.18) also holds in dimension three when the pseudohermitian torsion vanishes. In the three dimensional case we have $Ric(X,Y) = \frac{S}{2}g(X,Y)$. After a substitution of this equality in (7.9), taking into account $A = 0$, we obtain

$$\nabla^2 f(\xi,Z) = \nabla^2 f(Z,\xi) = \frac{(S-2)}{6}df(JZ). \tag{7.19}$$

Differentiating (7.19) and using (7.5) we find

$$\nabla^3 f(Y,Z,\xi) = \frac{1}{6}\Big[dS(Y)df(JZ) + (S-2)f\omega(Y,Z)\Big] - \frac{1}{6}(S-2)df(\xi)g(Y,Z). \tag{7.20}$$

On the other hand, setting $A = 0$ in (7.7), we have

$$\nabla^3 f(Y,Z,\xi) = -df(\xi)g(Y,Z) - (\xi^2 f)\omega(Y,Z). \tag{7.21}$$

In particular, the function ξf also satisfies (7.5). From (7.16) using again unique continuation $\xi f \neq 0$ almost everywhere since otherwise $\nabla f = 0$ taking into account (7.19), hence $f \equiv 0$, which is not possible by assumption. Now, (7.20) and (7.21) give

$$\frac{S-8}{6}df(\xi)g(Y,Z) - \Big(\xi^2 f + \frac{S-2}{6}\Big)\omega(Y,Z) - \frac{1}{6}dS(Y)df(JZ) = 0, \tag{7.22}$$

which implies $\frac{S-8}{3}df(\xi)|\nabla f|^2 = 0$. Thus, the pseudohermitian scalar curvature is constant, $S = 8$, invoking again the local non-constancy. Equation (7.22) reduces then to $\Big(\xi^2 f + \frac{S-2}{6}\Big)\omega(Y,Z) = 0$ since $dS = 0$ which yields $\xi^2 f = -f$. The latter together with (7.19) and $S = 8$ imply the validity of (7.18) in dimension three.

Finally, we use the relation between D and ∇, [77, Lemma 1.3] which in the case $A = 0$ simplifies to

$$D_B C = \nabla_B C + \theta(B)JC + \theta(C)JB - \omega(B,C)\xi, \quad B, C \in \mathfrak{T}(M), \tag{7.23}$$

where J is extended with $J\xi = 0$. Using (7.23) together with (7.5) and (7.18) we calculate easily that (2.3) holds. The proof of Theorem 7.2 is complete.

7.3. **Proof of the Obata CR eigenvalue theorem in the compact case.** Turning to Theorem 7.3 we mention that when $n = 1$ we need to find an alternative way to the 'missing" equation (7.10), see the remark after (7.9). In fact, in [132, Lemma 5.1]] it was shown that in this case (assuming the Lichnerowicz' type condition), if $\triangle f = 2f$ then we have $A(\nabla f, \nabla f) = 0$ and (7.10) holds true. This is proved with an integration (using the compactness!) of the "vertical Bochner" formula [131, Remark 3.5]

$$- \triangle(\xi f)^2 = 2|\nabla(\xi f)|^2 - 2df(\xi)\cdot\xi(\triangle f) + 4df(\xi)\cdot g(A, \nabla^2 f) - 4df(\xi)(\nabla^* A)(\nabla f). \tag{7.24}$$

At this point, identities (7.9)-(7.14) are available for $n \geq 1$, which imply in particular the identity in Remark 7.4 holds true. We come to the idea of [171, Lemma 4] where integration by parts involving suitable powers of f are used in order to conclude $A = 0$. By Remark 7.4 we have for any $k > 0$

$$g(\nabla f, \nabla |A|^k) = 0, \tag{7.25}$$

while the Lichnerowicz' condition implies *point-wise* the inequality $A(\nabla f, J\nabla f) \leq 0$, hence

$$|\nabla f|^2 |A| = -\sqrt{2} A(\nabla f, J\nabla f). \tag{7.26}$$

Next we shall use an integration by parts argument similar to [171] for the case $n = 1$, i.e., $\triangle f = 2f$. Using (7.25), we have

$$I \overset{def}{=} \int_M |A|^3 f^{2(k+1)}\, Vol_\eta = -\frac{1}{2}\int_M |A|^3 f^{2k+1} \triangle f\, Vol_\eta = \frac{2k+1}{2}\int_M |A|^3 f^{2k} |\nabla f|^2\, Vol_\eta \equiv \frac{2k+1}{2} D.$$

From (7.26) it follows

$$\begin{aligned}\sqrt{2}(2k+1)D \overset{def}{=} -\int_M |A|^2 f^{2k+1} (\nabla^* A)(J\nabla f)\, Vol_\eta &\leq ||\nabla^* A|| \int_M |A|^2 f^{2k+1} |\nabla f|\, Vol_\eta \\ &\leq \frac{||\nabla^* A||}{a}\int_M f^{k+1}\, f^k |\nabla f|\, |A|^3\, Vol_\eta,\end{aligned}$$

assuming $|A| \geq a > 0$ so $|A|^2 \leq \frac{1}{a}|A|^3$. Now, Hölder's inequality gives

$$\begin{aligned}\sqrt{2}(2k+1)D &\leq \frac{||\nabla^* A||}{a}\Big(\int_M |A|^3 f^{2(k+1)}\, Vol_\eta\Big)^{1/2}\Big(\int_M |A|^3 f^{2k}|\nabla f|^2\, Vol_\eta\Big)^{1/2} \\ &= \frac{||\nabla^* A||}{a}\Big(\frac{2k+1}{2}D\Big)^{1/2} D^{1/2} = \frac{||\nabla^* A||}{a}\Big(\frac{2k+1}{2}\Big)^{1/2} D, .\end{aligned}$$

By taking k sufficiently large k we conclude $A = 0$. The assumption $|A| \geq a > 0$ can be removed by employing a suitable cutt-off function, see [171]. Once we know that M is Sasakian one applies [46, 47] where a reduction to Theorem 2.1 is made.

8. The Quaternionic Contact Lichnerowicz and Obata theorems

This section concentrates on the qc versions of the Lichnerowicz and Obata eigenvalue theorems. As in the Riemannian and CR cases we are dealing with a sub-elliptic operator, hence the discreteness of its spectrum on a compact qc manifold. The Lichnerowicz' type result was found in [127] in dimensions grater than seven and in [126] in the seven dimensional case. Remarkably, compare with the CR case, the Obata type theorem characterizing the 3-Sasakian sphere through the horizontal Hessian equation holds under no extra assumptions on the Biquard' torsion when the dimension of the qc manifold is at least eleven as proven in [128]. The general qc Obata result in dimension seven remains open.

We shall use freely the curvature and torsion tensors associated to a given qc structure as defined in Section 4.5. As in the previous sections where eigenvalues were concerned we shall use the non-negative sub-Laplacian, $\triangle u = -tr^g(\nabla^2 u)$.

8.1. **The QC Lichnerowicz theorem.**

Theorem 8.1 ([127, 126]). *Let (M,η) be a compact QC manifold of dimension $4n+3$. Suppose, for $\alpha_n = \frac{2(2n+3)}{2n+1}$, $\beta_n = \frac{4(2n-1)(n+2)}{(2n+1)(n-1)}$ and for any $X \in H$*

$$\mathcal{L}(X,X) \stackrel{def}{=} 2Sg(X,X) + \alpha_n T^0(X,X) + \beta_n U(X,X) \geq 4g(X,X).$$

If $n=1$, assume in addition the positivity of the P-function of any eigenfunction. Then, any eigenvalue λ of the sub-Laplacian $\triangle$ satisfies the inequality $\lambda \geq 4n$.

The 3-Sasakian sphere achieves equality in the Theorem. The eigenspace of the first non-zero eigenvalue of the sub-Laplacian on the unit 3-Sasakian sphere in Euclidean space is given by the restrictions to the sphere of all linear functions by Theorem 4.1.

8.1.1. *The QC P-function.* We turn to the definition of the QC P-function defined in [126]. For a fixed smooth function f we define a one form $P \equiv P_f \equiv P[f]$ on M, which we call the $P-$form of f, by the following equation

$$P_f(X) = \sum_{b=1}^{4n} \nabla^3 f(X,e_b,e_b) + \sum_{t=1}^{3}\sum_{b=1}^{4n} \nabla^3 f(I_tX,e_b,I_te_b) - 4nSdf(X) + 4nT^0(X,\nabla f) - \frac{8n(n-2)}{n-1}U(X,\nabla f).$$

The $P-$function of f is the function $P_f(\nabla f)$. The $C-$operator is the fourth-th order differential operator on M, which is independent of f,

$$f \mapsto Cf = -\nabla^* P_f = \sum_{a=1}^{4n} (\nabla_{e_a} P_f)\,(e_a).$$

We say that the $P-$function of f is non-negative if

$$\int_M f\cdot Cf\, Vol_\eta = -\int_M P_f(\nabla f)\, Vol_\eta \geq 0.$$

If the above holds for any $f \in \mathcal{C}^\infty_o(M)$ we say that the $C-$operator is *non-negative*, $C \geq 0$. Several important properties of the C-operator were found in [126]. The first notable fact is that the $C-$operator is non-negative, $C \geq 0$, for $n > 1$. Furthermore $Cf = 0$ iff $(\nabla^2 f)_{[3][0]}(X,Y) = 0$, where [3][0] denotes the trace-free part of the [3]-part of the Hessian. In this case the $P-$form of f vanishes as well. The key for the last result is the identity $\sum_{a=1}^{4n} (\nabla_{e_a}(\nabla^2 f)_{[3][0]})(e_a,X) = \frac{n-1}{4n}P_f(X)$, hence

$$\frac{n-1}{4n}\int_M f\cdot Cf\, Vol_\eta = -\frac{n-1}{4n}\int_M P_f(\nabla f)\, Vol_\eta = \int_M |(\nabla^2 f)_{[3][0]}|^2\, Vol_\eta,$$

after using the Ricci identities, the divergence formula and the orthogonality of the components of the horizontal Hessian.

In dimension seven, the condition of non-negativity of the $C-$operator is also non-void. For example, [126] showed that on a 7-dimensional compact qc-Einstein manifold with $Scal \geq 0$ the $P-$function of an *eigenfunction* of the sub-Laplacian is non-negative.

The proof relies on several results. First, the qc-scalar curvature of a qc-Einstein is constant [120, 123]. In the higher dimensions this follows from the Bianchi identities. However, the result is very non-trivial in dimension seven where the qc-conformal curvature tensor W^{qc}, see Theorem 4.7, is invoked in the proof. Secondly, on a qc-Einstein manifold we have $\nabla^3 f(\xi_s,X,Y) = \nabla^3 f(X,Y,\xi_s)$, the vertical space is integrable and we have $\nabla^2 f(\xi_k,\xi_j) - \nabla^2 f(\xi_j,\xi_k) = -Sdf(\xi_i)$. Finally, a calculation shows $\int_M |P_f|^2\, Vol_\eta = -(\lambda+4S)\int_M P_f(\nabla f)\, Vol_\eta$, which implies the claim.

At this point we can give the main steps in the proof of the Lichnerowicz' type theorem following the $P-$function approach of [128] which unified the seven and higher dimensional cases of [127, 126] as we did in the CR case in the proof of Theorem 7.1. By the QC Bochner identity established in [127], letting $R_f = \sum_{s=1}^3 \nabla^2 f(\xi_s, I_s\nabla f)$, we have

$$-\frac{1}{2}\triangle|\nabla f|^2 = |\nabla^2 f|^2 - g\left(\nabla(\triangle f),\nabla f\right) + 2(n+2)S|\nabla f|^2 + 2(n+2)T^0(\nabla f,\nabla f) + 2(2n+2)U(\nabla f,\nabla f) + 4R_f.$$

The "difficult" term R_f can be computed in two ways. First with the help of the P-function we have

$$\int_M R_f Vol_\eta = \int_M -\frac{1}{4n}P_n(\nabla f) - \frac{1}{4n}(\triangle f)^2 - S|\nabla f|^2 Vol_\eta + \int_M \frac{n+1}{n-1}U(\nabla f, \nabla f)\, Vol_\eta.$$

On the other hand, using Ricci's identities $g(\nabla^2 f, \omega_s) \overset{def}{=} \sum_{a=1}^{4n} \nabla^2 f(e_a, I_s e_a) = -4ndf(\xi_s)$,we have

$$\int_M R_f Vol_\eta = -\int_M \frac{1}{4n}\sum_{s=1}^{3} g(\nabla^2 f, \omega_s)^2 + T^0(\nabla f, \nabla f) - 3U(\nabla f, \nabla f)\, Vol_\eta.$$

A substitution of a linear combination of the last two identities in the QC Bochner identity shows

$$0 = \int_M |\nabla^2 f|^2 - \frac{1}{4n}\Big[(\triangle f)^2 + \sum_{s=1}^{3}[g(\nabla^2 f, \omega_s)]^2\Big] - \frac{3}{4n}P_n(\nabla f)Vol_\eta$$
$$+ \frac{2n+1}{2}\int_M \mathcal{L}(\nabla f, \nabla f) - \frac{\lambda}{n}|\nabla f|^2\, Vol_\eta.$$

With the Lichnerowicz type assumption, $\mathcal{L}(\nabla f, \nabla f) \geq 4|\nabla f|^2$, it follows

$$0 \geq \int_M |(\nabla^2 f)_0|^2 - \frac{3}{4n}P_n(\nabla f)Vol_\eta + \frac{2n+1}{2n}\int_M \Big(4n - \lambda\Big)|\nabla f|^2 Vol_\eta.$$

For $n = 1$, when $U = 0$ trivially, remove formally the torsion tensor U terms - the formulas are still correct, which completes the proof of Theorem 8.1.

8.2. **The QC Obata type theorem.**

Theorem 8.2 ([128])**.** *Let (M, η) be a quaternionic contact manifold of dimension $4n+3 > 7$ which is complete with respect to the associated Riemannian metric $h = g + (\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2$. There exists a smooth f $\not\equiv$const, such that,*

$$\nabla df(X, Y) = -fg(X, Y) - \sum_{s=1}^{3} df(\xi_s)\omega_s(X, Y).$$

if and only if the qc manifold $(M, \eta, g, \mathbb{Q})$ is qc homothetic to the unit 3-Sasakian sphere.

It should be noted that in dimension seven the problem is still open. The above theorem suffices to charcaterize the cases of equality in Theorem 8.1 for $n > 1$.

Theorem 8.3 ([128])**.** *Let (M, η) be a compact QC manifold of dimension $4n+3$ which satisfies a Lichnerowicz' type bound $\mathcal{L}(X, X) \geq 4g(X, X)$. Then, there is a function f with $\triangle f = 4nf$ if and only if*

- *when $n > 1$, M is qc-homothetic to the 3-Sasakian sphere;*
- *when $n = 1$, and M is qc-Einstein, i.e., $T^0 = 0$, then M is qc-homothetic to the 3-Sasakian sphere.*

Next, we give an outline of the key steps in the proof of Theorem 8.2.

Part 1. The first step is to show that $T^0 = 0$ and $U = 0$, i.e., M is qc-Einstein. This is achieved by the following argument. First we determine the remaining parts of the Hessian of f with respect to the Biquard connection in terms of the torsion tensors. A simple argument shows that $T^0(I_s\nabla f, \nabla f) = U(I_s\nabla f, \nabla f) = 0$. Using the $[-1]$-component of the curvature tensor it follows $T^0(I_s\nabla f, I_t\nabla f) = 0$, $s,\ t \in \{1, 2, 3\}$, $s \neq t$. Then we determine the torsion tensors T^0 and U in terms of ∇f and the tensor $U(\nabla f, \nabla f)$. For example,

$$|\nabla f|^4 T^0(X, Y) = -\frac{2n}{n-1}U(\nabla f, \nabla f)\Big[3df(X)df(Y) - \sum_{s=1}^{3} df(I_s X)df(I_s Y)\Big].$$

Next, we prove formulas of the same type for ∇T^0 and ∇U. In particular we have

$$(\nabla_{\nabla f}U)(X, Y) = \frac{2(n-1)}{n+2}fU(X, Y).$$

Remark 8.4. *We pause for a moment to remark that the last equation shows in particular $L_{\nabla f}|U|^2 = \frac{4(n-1)}{n+2}f|U|^2$* as in the Riemannian case for Ric_0. *Hence, in the compact case we can use an integration as in Proposition 2.4 to see the vanishing of U, hence of T^0 by what we have proved.*

By what we already proved, the crux of the matter is the proof that $U(\nabla f, \nabla f) = 0$ (or $T^0(\nabla f, \nabla f) = 0$). This fact is achieved with the help of the Ricci identities, the contracted Bianchi second identity and many properties of the torsion of a qc-manifolds: $0 = \nabla^3 f(\xi_i, I_i\nabla f, \nabla f) - \nabla^3 f(I_i\nabla f, \nabla f, \xi_i) = \frac{2}{n+2} fU(\nabla f, \nabla f)$. We finish that $|\nabla f| \neq 0$ a.e. using unique continuation by showing that on a qc manifold with $n > 1$, the "horizontal Hessian equation" implies that f satisfies an elliptic partial differential equation,

$$\triangle^h f = (4n+3)f + \frac{n+1}{n(2n+1)}(\nabla_{e_a} T^0)(e_a, \nabla f) + \frac{3}{(2n+1)(n-1)}(\nabla_{e_a} U)(e_a, \nabla f). \tag{8.1}$$

Part 2: By Part 1 it suffices to consider the case of a qc-Einstein structure, in which case we proceed as follows. First, we show that (M, h) (h- Riemannian metric!) is isometric to the unit round sphere by showing that $(\nabla^h)^2 f(X, Y) = -fh(X, Y)$ and using Obata's result, see Remark 2.2 and the paragraph preceding it. Next, we show qc-conformal flatness. For this we use the form of the curvature of the round sphere, $R^h(A, B, C, D) = h(B, C)h(A, D) - h(B, D)h(A, C)$ and the relation between the Riemannian and Biquard curvatures and then the formula for $W^{qc}(X, Y, Z, V)$ which simplifies considerably in the qc-Einstein case. Finally, we employ a standard monodromy argument showing that $(M, g, \eta, \mathbb{Q})$ is qc-conformal to S^{4n+3}, i.e., we have $\eta = \kappa\Psi F^*\tilde{\eta}$ for some diffeomorphism $F : M \to S^{4n+3}$, $0 < \kappa \in \mathcal{C}^\infty(M)$, and $\Psi \in \mathcal{C}^\infty(M : SO(3))$ We conclude the proof of the qc-conformality with the 3-Sasakian sphere by invoking the qc-Liouville theorem 8.5.

Finally, a comparison of the metrics on H show the desired homothety.

Since the above proof used the qc-Liouville theorem and because of its independent interest we devote a short section to it.

8.3. The QC Liouville theorem.

Theorem 8.5. *Every qc-conformal transformation between open subsets of the 3-Sasakian unit sphere is the restriction of a global qc-conformal transformation.*

This result is proved in the more general setting of parabolic geopmetries in [38]. Here we give a relatively self-contained proof of a version of Liouville's theorem in the case of the quaternionic Heisenberg group and the 3-Sasakian sphere equipped with their standard qc structures. The proof is related to the QC Yamabe problem on the 3-Sasakian sphere since a key step is provided by the proof of [120, Theorem 1.1] in which all qc-Einstein structures qc-conformal to the standard qc-structure on the quaternionic Heisenberg group (or sphere) were determined. Thus, our proof of Theorem 8.5 establishes the local Liouville type property in the setting of a sufficiently smooth qc-conformal maps relying only on the qc geometry. A very general version of the Liouville theorem was also proven by Cowling, M., & Ottazzi, A., see [71].

In the Euclidean case Liouville [176], [177] showed that every sufficiently smooth conformal map (C^4 in fact) between two connected open sets of the Euclidean space $\mathbb{R}^3$ is necessarily a restriction of a Möbius transformation. The latter is the group generated by translations, dilations and inversions of the extended Euclidean space obtained by adding an ideal point at infinity. Liouville's result generalizes easily to any dimension $n > 3$. Subsequently, Hartman [112] gave a proof requiring only $\mathcal{C}^1$ smoothness of the conformal map, see also [186], [28], [135], [134] and [91] for other proofs. A CR version of Liouville's result can be found in [215] and [5]. Thus, a smooth CR diffeomorphism between two connected open subsets of the $2n+1$ dimensional sphere is the restriction of an element from the isometry group $SU(n+1, 1)$ of the unit ball equipped with the complex hyperbolic metric. The proof of Alexander [5] relies on the extension property of a smooth CR map to a biholomorphism. Tanaka, see also [197], [44] and [60], in his study of pseudo-conformal equivalence between analytic real hypersurfaces of complex space showed a more general result [215, Theorem 6] showing that any pseudo-conformal homeomorphism between connected open sets of the quadric

$$-\sum_{i=1}^{r} |z_i|^2 + \sum_{i=r+1}^{n} |z_i|^2 = 1, \quad (z_1, \dots, z_n) \in \mathbb{C}^n,$$

is the restriction of a projective transformation of $P^n(\mathbb{C})$.

Another theory began with the introduction of quasiconformal maps [103] and [199], which imposed metric conditions on the maps, and with the works of Mostow [185] and Pansu [195]. In particular, in [195] it was shown that every global 1-quasiconformal map on the sphere at infinity of each of the hyperbolic metrics is an isometry of the corresponding hyperbolic space. The local version of the Liouville's property for 1-quasiconformal map of class C^4

on the Heisenberg group was settled in [154] by a reduction to the CR result. The optimal regularity question for quasiconformal maps was settled later by Capogna [39] in much greater generality including the cases of all Iwasawa type groups, see also [216] and [40].

A closely related property is the so called rigidity property of quasiconformal or multicontact maps, also referred to as Liouville's property, but where the question is the finite dimensionality of the group of (locally defined) quasi-conformal or multicontact maps, see [229], [198], [67], [191], [192],[184], [76], [162].

Besides the Cayley transform, we shall need the generalization of the Euclidean inversion transformation to the qc setting. We recall that in [151] Korányi introduced such an inversion and an analogue of the Kelvin transform on the Heisenberg group, which were later generalized in [68] and [65] to all groups of Heisenberg type. The inversion and Kelvin transform enjoy useful properties in the case of the four groups of Iwasawa type of which $\boldsymbol{G}\,(\mathbb{H})$ is a particular case. For our goals it is necessary to show that the inversion on $\boldsymbol{G}\,(\mathbb{H})$ is a qc-conformal map. In order to prove this fact we shall represent the inversion as the composition of two Cayley transforms, see [121, 130] where the seven dimensional case was used. Let $P_1 = (-1,0)$ and $P_2 = (1,0)$ be correspondingly the 'south' and 'north' poles of the unit sphere $S^{4n+3} = \{|q|^2 + |p|^2 = 1\} \subset \mathbb{H}^n \times \mathbb{H}$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the corresponding Cayley transforms defined, respectively, on $S^{4n+3} \setminus \{P_1\}$ and $S^{4n+3} \setminus \{P_2\}$. Note that $\mathcal{C}_1$ was defined in (4.31), while $\mathcal{C}_2$ is given by $(q',p') = \mathcal{C}_2\left((q,p)\right), q' = -(1-p)^{-1}\,q, \quad p' = (1-p)^{-1}\,(1+p), \qquad (q,p) \in S^{4n+3} \setminus \{P_2\}$The inversion on the quaternionic Heisenberg group (with center the origin) with respect to the unit gauge sphere is the map

$$\sigma = \mathcal{C}_2 \circ \mathcal{C}_1^{-1} : \boldsymbol{G}\,(\mathbb{H}) \setminus \{(0,0)\} \to \boldsymbol{G}\,(\mathbb{H}) \setminus \{(0,0)\}. \tag{8.2}$$

In particular, $\sigma = \mathcal{C}_2 \circ \mathcal{C}_1^{-1}$ is an involution on the group. A small calculation shows that σ is given by the formula (in the Siegel model) $q^* = -p'^{-1}\,q', \qquad p^* = p'^{-1}$,or, equivalently, in the direct product model $\boldsymbol{G}\,(\mathbb{H})$

$$q^* = -(|q'|^2 - \omega')^{-1}\,q', \qquad \omega^* = -\frac{\omega'}{|q'|^4 + |\omega'|^2}.$$

It follows

$$\begin{aligned} \sigma^*\,\Theta &= \frac{1}{|p'|^2}\,\bar{\mu}\,\Theta\,\mu, \qquad \mu = \frac{p'}{|p'|}, \qquad \text{(in the Siegel model)} \\ \sigma^*\,\Theta &= \frac{1}{|q'|^4 + |\omega'|^2}\,\bar{\mu}\,\Theta\,\mu, \qquad \mu = \frac{|q'|^2 + \omega'}{(|q'|^4 + |\omega'|^2)^{1/2}}, \qquad \text{(in the product model)}, \end{aligned} \tag{8.3}$$

which, shows the following fundamental fact.

Lemma 8.6. *The inversion transformation* (8.2) *is a qc-conformal transformation on the quaternionic Heisenberg group.*

As usual, using the dilations and translations on the group, it is a simple matter to define an inversion with respect to any gauge ball.

Turning to the proof of Theorem 8.5 let $\Sigma \neq S^{4n+3}$, noting that in the case $\Sigma = S^{4n+3}$ there is nothing to prove. We shall transfer the analysis to the quaternionic Heisenberg group using the Cayley transform, thereby reducing to the case of a qc-conformal transformation $\tilde{F} : \tilde{\Sigma} \to \boldsymbol{G}\,(\mathbb{H})$ between two domains of the quaternionic Heisenberg group such that $\Theta = \tilde{F}^*\tilde{\Theta} = \frac{1}{2\phi}\tilde{\Theta}$ for some positive smooth function ϕ defined on the open set $\tilde{\Sigma}$. By its definition Θ is a qc-Einstein structure of vanishinq qc-scalar curvature, hence Theorem 6.2 shows $\sigma = 0$ and F is a composition of a translation, cf. (3.15), followed by an inversion and a homothety, cf. Lemma 8.6.

The above analysis implies that F is the restriction of an element of $PSp(n+1,1)$. This completes the proof of Theorem 8.5. Similarly to the Riemannian and CR cases, see [157], [209, Theorem VI.1.6] and [37], Theorem 8.5 and a standard monodromy type argument show the validity of the next

Theorem 8.7. *If (M,η) is a simply connected qc-conformally flat manifold of dimension $4n+3$, then there is a qc-conformal immersion $\Phi : M \to S^{4n+3}$, where S^{4n+3} is the 3-Sasakian unit sphere in the $(n+1)$-dimensional quaternion space.*

9. Heterotic string theory relations

The seven dimensional quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$ has applications in the construction of non-trivial solutions to the so called *Strominger system* in supersymmetric heterotic string theory.

The bosonic fields of the ten-dimensional supergravity which arises as low energy effective theory of the heterotic string are the spacetime metric g, the NS three-form field strength (flux) H, the dilaton ϕ and the gauge connection A with curvature 2-form F^A. The bosonic geometry is of the form $\mathbb{R}^{1,9-d}\times M^d$, where the bosonic fields are non-trivial only on M^d, $d\leq 8$. One considers the two connections $\nabla^{\pm}=\nabla^g\pm\frac{1}{2}H$, where ∇^g is the Levi-Civita connection of the Riemannian metric g. Both connections preserve the metric, $\nabla^{\pm}g=0$ and have totally skew-symmetric torsion $\pm H$, respectively. We denote by $R^g, R^{\pm}$ the corresponding curvature.

The Green-Schwarz anomaly cancellation condition up to the first order of the string constant α' reads

$$dH=\frac{\alpha'}{4}8\pi^2(p_1(\nabla^-)-p_1(E))=\frac{\alpha'}{4}\Big(Tr(R^-\wedge R^-)-Tr(F^A\wedge F^A)\Big), \tag{9.1}$$

where $p_1(\nabla^-)$ and $p_1(E)$ are the first Pontrjagin forms with respect to a connection ∇^- with curvature R and a vector bundle E with connection A.

A heterotic geometry preserves supersymmetry iff in ten dimensions there exists at least one Majorana-Weyl spinor ϵ such that the following Killing-spinor equations hold [212, 22]

$$\nabla^+\epsilon=0,\quad (d\phi-\frac{1}{2}H)\cdot\epsilon=0,\quad F^A\cdot\epsilon=0, \tag{9.2}$$

where $\cdot$ means Clifford action of forms on spinors.

The system of Killing spinor equations (9.2) together with the anomaly cancellation condition (9.1) is known as the *Strominger system* [212]. The last equation in (9.2) is the instanton condition which means that the curvature F^A is contained in a Lie algebra of a Lie group which is a stabilizer of a non-trivial spinor. In dimension 7 the largest such a group is the exceptional group G_2 which is the authomorphism group of the unit imaginary octonions. Denoting by Θ the non-degenerate three-form definning the G_2 structure, the G_2-instanton condition has the form

$$\sum_{k,l=1}^{7}(F^A)^i_j(e_k,e_l)\Theta(e_k,e_l,e_m)=0. \tag{9.3}$$

Geometrically, the existence of a non-trivial real spinor parallel with respect to the metric connection ∇^+ with totally skew-symmetric torsion $T=H$ leads to restriction of the holonomy group $Hol(\nabla^+)$ of the torsion connection ∇^+. In dimension seven $Hol(\nabla^+)$ has to be contained in the exceptional group G_2 [93, 101, 102, 94].

The general existence result [101, 93, 94] states that there exists a non-trivial solution to both dilatino and gravitino Killing spinor equations (the first two equations in (9.2)) in dimension d=7 if and only if there exists a globally conformal co-calibrated G_2-structure (Θ,g) of pure type and the Lee form $\theta^7=-\frac{1}{3}*(*d\Theta\wedge\Theta)=\frac{1}{3}*(*d*\Theta\wedge*\Theta)$ has to be exact, i.e. a G_2-structure (Θ,g) satisfying the equations

$$d*\Theta=\theta^7\wedge*\Theta,\quad d\Theta\wedge\Theta=0,\quad \theta^7=-2d\phi. \tag{9.4}$$

Therefore, the torsion 3-form (the flux H) is given by

$$H=T=-*d\Theta-2*(d\phi\wedge\Theta). \tag{9.5}$$

A geometric model which fits the above structures was proposed in [85] as a certain $\mathbb{T}^3$-bundle over a Calabi-Yau surface. For this, let Γ_i, $1\leq i\leq 3$, be three closed anti-self-dual 2-forms on a Calabi-Yau surface M^4, which represent integral cohomology classes. Denote by ω_1 and by $\omega_2+\sqrt{-1}\omega_3$ the (closed) Kähler form and the holomorphic volume form on M^4, respectively. Then, there is a compact 7-dimensional manifold $M^{1,1,1}$ which is the total space of a $\mathbb{T}^3$-bundle over M^4 and has a G_2-structure $\Theta=\omega_1\wedge\eta_1+\omega_2\wedge\eta_2-\omega_3\wedge\eta_3+\eta_1\wedge\eta_2\wedge\eta_3$,solving the first two Killing spinor equations in (9.2) with constant dilaton in dimension 7, where η_i, $1\leq i\leq 3$, is a 1-form on $M^{1,1,1}$ such that $d\eta_i=\Gamma_i$, $1\leq i\leq 3$.

For any smooth function f on M^4, the G_2-structure on $M^{1,1,1}$ given by

$$\Theta_f=e^{2f}\Big[\omega_1\wedge\eta_1+\omega_2\wedge\eta_2-\omega_3\wedge\eta_3\Big]+\eta_1\wedge\eta_2\wedge\eta_3$$

solves the first two Killing spinor equations in (9.2) with non-constant dilaton $\phi=-2f$.

To achieve a smooth solution to the Strominger system *we still have to determine* an auxiliary vector bundle with an G_2-instanton in order to satisfy the anomaly cancellation condition (9.1).

9.1. **The quaternionic Heisenberg group.** The Lie algebra $\mathfrak{g}(\mathbb{H})$ of the seven dimensional group $\boldsymbol{G}\,(\mathbb{H})$ has structure equations

$$d\gamma^1 = d\gamma^2 = d\gamma^3 = d\gamma^4 = 0, \quad d\gamma^5 = \gamma^{12} - \gamma^{34}, \quad d\gamma^6 = \gamma^{13} + \gamma^{24}, \quad d\gamma^7 = \gamma^{14} - \gamma^{23}. \tag{9.6}$$

where $\gamma_1, \dots, \gamma_7$ is a basis of left invariant 1-forms on $\boldsymbol{G}\,(\mathbb{H})$. In particular, the quaternionic Heisenebrg group $\boldsymbol{G}\,(\mathbb{H})$ in dimension seven is an R^3-bundle over the flat Calabi-Yau space R^4 and therefore fits the geometric model described above.

In order to obtain results in dimensions less than seven through contractions of $\boldsymbol{G}\,(\mathbb{H})$ it will be convenient to consider the orbit of $G(\mathbb{H})$ under the natural action of $GL(3,\mathbb{R})$ on the *span* $\{\gamma^5, \gamma^6, \gamma^7\}$. Accordingly let K_A be a seven-dimensional real Lie group with Lie bracket $[x, x']_A = A[A^{-1}x, A^{-1}x']$ for $A \in GL(3,\mathbb{R})$ defined by a basis of left-invariant 1-forms $\{e^1, \dots, e^7\}$ such that $e^i = \gamma^i$ for $1 \le i \le 4$ and $(e^5\ e^6\ e^7) = A\,(\gamma^5\ \gamma^6\ \gamma^7)^T$. Hence, the structure equations of the Lie algebra $\mathfrak{K}_A$ of the group K_A are

$$de^1 = de^2 = de^3 = de^4 = 0, \qquad de^{4+i} = \sum_{j=1}^{3} a_{ij}\,\sigma_j, \quad i = 1,2,3, \tag{9.7}$$

where $\sigma_1 = e^{12} - e^{34}$, $\sigma_2 = e^{13} + e^{24}$, $\sigma_3 = e^{14} - e^{23}$ are the three anti-self-dual 2-forms on $\mathbb{R}^4$ and $A = \{a_{ij}\}$ is a 3 by 3 matrix. We will denote the norm of A by $|A|$, $|A|^2 = \sum_{i,j=1}^3 a_{ij}^2$.

Since $\mathfrak{K}_A$ is isomorphic to $\mathfrak{g}(\mathbb{H})$, if K_A is connected and simply connected it is isomorphic to $G(\mathbb{H})$. Furthermore, any lattice Γ_A gives rise to a (compact) nilmanifold $M_A = K_A/\Gamma_A$, which is a $\mathbb{T}^3$-bundle over a $\mathbb{T}^4$ with connection 1-forms of anti-self-dual curvature on the four torus.

The three closed hyperKähler 2-forms on $\mathbb{R}^4$ are given by $\omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} - e^{24}, \quad \omega_3 = e^{14} + e^{23}$. Following [85], for a smooth function f on $\mathbb{R}^4$, we consider the G_2 structure on K_A defined by the 3-form

$$\bar\Theta = e^{2f}\Big[\omega_1 \wedge e^7 + \omega_2 \wedge e^5 - \omega_3 \wedge e^6\Big] + e^{567}, \tag{9.8}$$

The corresponding metric $\bar g$ on K_A has an orthonormal basis of 1-forms given by

$$\bar e^1 = e^f\,e^1, \quad \bar e^2 = e^f\,e^2, \quad \bar e^3 = e^f\,e^3, \quad \bar e^4 = e^f\,e^4, \quad \bar e^5 = e^5, \quad \bar e^6 = e^6, \quad \bar e^7 = e^7 \tag{9.9}$$

with self-dual 2-forms $\bar\omega_i = e^{2f}\omega_i$ and anti-self-dual 2-forms $\bar\sigma_i = e^{2f}\sigma_i$, i=1,2,3.

It is easy to check using (9.7) and the property $\sigma_i \wedge \omega_j = 0$ for $1 \le i,j \le 3$ that the (9.4) is satisfied, i.e., the G_2 structure $\bar\Theta$ solves the gravitino and dilatino equations with non-constant dilaton $\phi = -2f$ [87]. Furthermore, with $f_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}$, $1 \le i,j \le 4$, we obtain the next formula for the torsion $\bar T$ of $\bar\Theta$, see [87] for details,

$$d\bar T = -e^{-4f}\left[\triangle e^{2f} + 2|A|^2\right]\bar e^{1234} = -\left[\triangle e^{2f} + 2|A|^2\right] e^{1234}, \tag{9.10}$$

where $\triangle e^{2f} = (e^{2f})_{11} + (e^{2f})_{22} + (e^{2f})_{33} + (e^{2f})_{44}$ is the Laplacian on $\mathbb{R}^4$.

9.2. **The first Pontrjagin form of the $(-)$-connection.** The connection 1-forms of a connection ∇ are determined by $\nabla_X e_j = \sum_{s=1}^7 \omega_j^s(X)e_s$. From Koszul's formula, we have that the Levi-Civita connection 1-forms $(\omega^{\bar g})_{\bar j}^{\bar\imath}$ of the metric $\bar g$ are given by

$$\begin{aligned}(\omega^{\bar g})_{\bar j}^{\bar\imath}(\bar e_k) &= -\tfrac12\Big(\bar g(\bar e_i, [\bar e_j, \bar e_k]) - \bar g(\bar e_k, [\bar e_i, \bar e_j]) + \bar g(\bar e_j, [\bar e_k, \bar e_i])\Big)\\ &= \tfrac12\Big(d\bar e^i(\bar e_j, \bar e_k) - d\bar e^k(\bar e_i, \bar e_j) + d\bar e^j(\bar e_k, \bar e_i)\Big)\end{aligned} \tag{9.11}$$

taking into account $\bar g(\bar e_i, [\bar e_j, \bar e_k]) = -d\bar e^i(\bar e_j, \bar e_k)$. With the help of (9.11) we compute the expressions for the connection 1-forms $(\omega^-)_{\bar j}^{\bar\imath}$ of the connection ∇^-,

$$(\omega^-)_{\bar j}^{\bar\imath} = (\omega^{\bar g})_{\bar j}^{\bar\imath} - \frac12(\bar T)_{\bar j}^{\bar\imath}, \qquad \text{where} \qquad (\bar T)_{\bar j}^{\bar\imath}(\bar e_k) = \bar T(\bar e_i, \bar e_j, \bar e_k). \tag{9.12}$$

A long straightforward calculation based on (9.11), (9.12) yields that the first Pontrjagin form of ∇^- is a scalar multiple of e^{1234} given by [87]

$$\pi^2 p_1(\nabla^-) = \left[\mathcal{F}_2[f] + \triangle_4 f - \frac{3}{8}|A|^2 \triangle e^{-2f}\right] e^{1234}, \tag{9.13}$$

where $\mathcal{F}_2[f]$ is the 2-Hessian of f, i.e., the sum of all principle 2×2-minors of the Hessian, and $\triangle_4 f = div(|\nabla f|^2 \nabla f)$ is the 4-Laplacian of f. This formula shows, in particular, that even though the curvature 2-forms of ∇^- are quadratic in the gradient of the dilaton, the Pontrjagin form of ∇^- is also quadratic in these terms. Furthermore, if f depends on two of the variables then $\mathcal{F}_2[f] = det(Hess f)$ while if f is a function of one variable $\mathcal{F}_2[f]$ vanishes.

What remains is to solve the anomaly cancellation condition. We use the G_2-instanton D_Λ defined in [87], which depends on a 3 by 3 matrix $\Lambda = (\lambda_{ij}) \in \mathfrak{gl}_3(\mathbb{R})$.

It is shown in [87] that the connection D_Λ is a G_2-instanton with respect to the G_2 structure defined by (9.8) which preserves the metric if and only if $\mathrm{rank}(\Lambda) \leq 1$. In this case, the first Pontrjagin form $p_1(\mathrm{D}_\Lambda)$ of the G_2-instanton D_Λ is given by

$$8\pi^2 p_1(\mathrm{D}_\Lambda) = -4\lambda^2\, e^{1234}, \tag{9.14}$$

where $\lambda = |\Lambda\, A|$ is the norm of the product matrix $\Lambda\, A$.

After this preparation, we are left with solving the anomaly cancellation condition $d\bar{T} = \frac{\alpha'}{4} 8\pi^2 \Big(p_1(\nabla^-) - p_1(D_\Lambda)\Big)$, which in general is a highly overdetermined system for the dilaton function f. Remarkably, in our case taking into account (9.10), (9.13) and (9.14) the anomaly becomes *the single* non-linear equation

$$\triangle e^{2f} + 2|A|^2 + \frac{\alpha'}{4}\left[8\mathcal{F}_2[f] + 8\triangle_4 f - 3|A|^2 \triangle e^{-2f} + 4\lambda^2\right] = 0. \tag{9.15}$$

We remind that this is an equation on $\mathbb{R}^4$ for the dilaton function f.

Remark 9.1. *An important question interesting for both string theory and nonlinear analysis is whether the non-linear PDE* (9.15) *admits a periodic solution.*

In [87] was found a one dimensional (non-smooth) solution, which we describe briefly. If we assume that the function f depends on one variable, $f = f(x^1)$, and for a *negative* α' we choose $2|A|^2 + \alpha'\lambda^2 = 0$, i.e., we let $\alpha' = -\alpha^2$ so that $2|A|^2 = \alpha^2\lambda^2$. This simplifies (9.15) to the ordinary differential equation

$$\left(e^{2f}\right)' + \frac{3}{4}\alpha^2 |A|^2 \left(e^{-2f}\right)' - 2\alpha^2 f'^3 = C_0 = const. \tag{9.16}$$

A solution of the last equation for $C_0 = 0$ was found in [86, Section 4.2]. The substitution $u = \alpha^{-2} e^{2f}$ allows us to write (9.16) in the form

$$\left(e^{2f}\right)' + \frac{3}{4}\alpha^2 |A|^2 \left(e^{-2f}\right)' - 2\alpha^2 f'^3 = \frac{\alpha^2 u'}{4u^3}\left(4u^3 - 3\frac{|A|^2}{\alpha^2} u - u'^2\right).$$

For $C_0 = 0$ we solve the following ordinary differential equation for the function $u = u(x^1) > 0$

$$u'^2 = 4u^3 - 3\frac{|A|^2}{\alpha^2} u = 4u\,(u-d)\,(u+d)\,, \qquad d = \sqrt{3|A|^2}/\alpha. \tag{9.17}$$

Replacing the real derivative with the complex derivative leads to the Weierstrass' equation $\left(\frac{d\mathcal{P}}{dz}\right)^2 = 4\mathcal{P}\,(\mathcal{P}-d)\,(\mathcal{P}+d)$for the doubly periodic Weierstrass $\mathcal{P}$ function with a pole at the origin. Letting $\tau_\pm$ be the basic half-periods such that τ_+ is real and τ_- is purely imaginary we have that $\mathcal{P}$ is real valued on the lines $\mathfrak{Re}\, z = m\tau_+$ or $\mathfrak{Im}\, z = im\tau_-$, $m \in \mathbb{Z}$. Thus, $u(x^1) = \mathcal{P}(x^1)$ defines a non-negative $2\tau_+$-periodic function with singularities at the points $2n\tau_+$, $n \in \mathbb{Z}$, which solves the real equation (9.17). By construction, $f = \frac{1}{2}\ln(\alpha^2 u)$ is a periodic function with singularities on the real line which is a solution to equation (9.15). Therefore the G_2 structure defined by $\bar{\Theta}$ descends to the 7-dimensional nilmanifold $M^7 = \Gamma\backslash K_A$ with singularity, determined by the singularity of u, where K_A is the 2-step nilpotent Lie group with Lie algebra $\mathfrak{K}_A$, defined by (9.7), and Γ is a lattice with the same period as f, i.e., $2\tau_+$ in all variables.

In fact, M^7 is the total space of a $\mathbb{T}^3$ bundle over the asymptotically hyperbolic manifold M^4 which is a conformally compact 4-torus with conformal boundary at infinity a flat 3-torus. Thus, we obtain the complete solution to the

Strominger system in dimension seven with non-constant dilaton, non-trivial instanton and flux and with a negative α' parameter found in [87].

9.3. **Solutions through contractions.** A contraction of the quaternionic heisenberg algebra can be obtained considering the matrix

$$A_\varepsilon \overset{def}{=} \begin{pmatrix} 0 & b & 0 \\ a & 0 & -b \\ 0 & 0 & \varepsilon \end{pmatrix}.$$

Letting $\varepsilon \to 0$ into A_ε we get in the limit, using (9.7), the structure equations of a six dimensional two step nilpotent Lie algebra known as $\mathfrak{h}_5$. On the corresponding simply connected two-step nilponent Lie group H_5 non-trivial solutions to the Strominger system in dimension 6 were prsented in[86]. It is a remarkable fact [87] that the geometric structures, the partial differential equations and their solutions found in dimension seven starting with the quaternionic Heisenbeg group as above converge trough contraction to the heterotic solutions on 6-dimensional non-Kähler space on H_5 found in [86]. Moreover, using suiatable contractraction it is possible to obtain non-trivial solutions to the Strominger system in dimension 5 as well (see [87] for details).

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REFERENCES

[1] Agrachev, A.,& Lee, Paul W. Y., *Generalized Ricci curvature bounds for three dimensional contact subriemannian manifolds*. Math. Ann. 360 (2014), no. 1-2, 209–253. 8

[2] Agrachev, A., & Lee, P., *Bishop and Laplacian Comparison Theorems on Three Dimensional Contact Subriemannian Manifolds with Symmetry*, to appear in J. Geom. Anal., DOI 10.1007/s12220-013-9437-2. 8

[3] Agrachev, A., Barilari, D., & Rizzi, L., *The curvature: a variational approach.* arXiv:1306.5318 8

[4] Alexandrov, A.D., *A characteristic property of the spheres*, Ann. Mat. Pura Appl. **58** (1962), 303–354. 14

[5] Alexander, H., *Holomorphic mappings from the ball and polydisc.*, Math. Ann., 209, (1974), 249–257. 43

[6] de Andres, L., Fernández, M., Ivanov, St., Joseba, S., Ugarte, L., & Vassilev, D.,*Quaternionic Kähler and Spin(7) metrics arising from quaternionic contact Einstein structures*, Ann. Mat. Pura Appl. (4) 193 (2014), no. 1, 261–290. 18, 25

[7] Armstrong, S., & Biquard, O., *Einstein metrics with anisotropic boundary behavior.* Int. Math. Res. Not. IMRN 2010, no. 11, 2094–2118. 18

[8] Astengo, F., Cowling, M. & Di Blasio, B., *The Cayley transform and uniformly bounded representations*, J. Funct. Anal., **213** (2004), 241–269. 13, 20

[9] Aubin, Th., *Some nonlinear problems in Riemannian geometry*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998 5, 6

[10] Aubin, Th., *Problèms isopérimétriques et espaces de Sobolev*, J. Diff. Geometry, **11** (1976), 573–598. 5

[11] ———, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. (9) **55** (1976), no. 3, 269–296. 5, 6

[12] ———, *Espaces de Sobolev sur les variétés Riemanniennes*, Bull. Sc. Math., **100** (1976), 149–173. 5

[13] Bahri, A., *Another proof of the Yamabe conjecture for locally conformally flat manifolds*. Nonlinear Anal. 20 (1993), no. 10, 1261–1278. 5, 6

[14] Banner, A. D., *Some properties of boundaries of symmetric spaces of rank one.* Geom. Dedicata 88 (2001), no. 1-3, 113-133. 13

[15] Barletta, E., *The Lichnerowicz theorem on CR manifolds.* Tsukuba J. Math. 31 (2007), no. 1, 77–97. 8

[16] Baudoin, F., & Garofalo, N., *Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries*, J. of the European Math. Soc. (JEMS), to appear. 8

[17] Baudoin, F,; Bonnefont, M.; Garofalo, N.; Munive, I. H., *Volume and distance comparison theorems for sub-Riemannian manifolds*. J. Funct. Anal. 267 (2014), no. 7, 2005–2027. 8

[18] Baudoin, F., Kim, B., & Wang, J,. *Transverse Weitzenböck formulas and curvature dimension inequalities on Riemannian foliations with totally geodesic leaves.* arXiv:1408.0548 8

[19] Baudoin, F.; Wang, J., *Curvature dimension inequalities and subelliptic heat kernel gradient bounds on contact manifolds*. Potential Anal. 40 (2014), no. 2, 163–193. 8

[20] Bedford, E., $(\partial\bar\partial)_b$ *and the real parts of CR functions.* Indiana Univ. Math. J. 29 (1980), no. 3, 333–340. 36

[21] Bedford, E., & Federbush, P., *Pluriharmonic boundary values.* Tohoku Math. J. (2) 26 (1974), 505–511. 36

[22] Bergshoeff, E.A., de Roo,M., *The quartic effective action of the heterotic string and supersymmetry*, Nuclear Phys. **B 328** (1989), 439–468. 45

[23] Sub-Riemannian geometry. Edited by André Bellaïche and Jean-Jacques Risler. Progress in Mathematics, 144. Birkhäuser Verlag, Basel, 1996. 11

[24] Berndt, J., Tricerri, F., & Vanhecke, L., Generalized Heisenberg groups and Damek-Ricci harmonic spaces. Lecture Notes in Mathematics, 1598. Springer-Verlag, Berlin, 1995. 12

[25] Biquard, O., *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **265** (2000). 9, 18, 19, 23, 24

[26] Biquard, O., *Quaternionic contact structures*, Quaternionic structures in mathematics and physics (Rome, 1999), 23–30 (electronic), Univ. Studi Roma "La Sapienza", Roma, 1999. 9, 18, 19, 23

[27] Blair, D. E., *On the characterization of complex projective space by differential equations*. J. Math. Soc. Japan 27 (1975), 919. 4, 22

[28] Bojarski, B., & Iwaniec, T., *Another approach to Liouville theorem.* Math. Nachr. 107 (1982), 253–262. 43

[29] Bourguignon, J.-P., $ Ezin, J.-P., *Scalar curvature functions in a conformal class of metrics and conformal transformations*. Trans. Amer. Math. Soc. 301 (1987), no. 2, 723–736. 7

[30] Boyer, C. & Galicki, K., *3-Sasakian manifolds*, Surveys in differential geometry: essays on Einstein manifolds, 123–184, Surv. Differ. Geom., **VI**, Int. Press, Boston, MA, 1999. 25

[31] Boyer, C., Galicki, K. & Mann, B., *The geometry and topology of* 3*-Sasakian manifolds* J. Reine Angew. Math., **455** (1994), 183–220. 22, 25

[32] Branson, T. P., Fontana, L., & Morpurgo, C., *Moser-Trudinger and Beckner-Onofri's inequalities on the CR sphere*, Ann. of Math. (2) 177 (2013), no. 1, 1–52. 8, 9, 15, 20

[33] Brendle, S., *Blow-up phenomena for the Yamabe equation*. J. Amer. Math. Soc. 21 (2008), no. 4, 951–979. 6

[34] Brendle, S., & Marques, F. C., *Recent progress on the Yamabe problem*. Surveys in geometric analysis and relativity, 29–47, Adv. Lect. Math. (ALM), 20, Int. Press, Somerville, MA, 2011. 6

[35] ———, *Blow-up phenomena for the Yamabe equation*. II. J. Differential Geom. 81 (2009), no. 2, 225–250. 6

[36] Brinkmann,H.W., *Einstein spaces which are mapped conformally on each other.* Math. Ann. 94 (1925) 119–145. 7

[37] Burns, D., Jr. & Shnider, S., *Spherical hypersurfaces in complex manifolds*. Invent. Math. 33 (1976), no. 3, 223–246. 44

[38] Čap, A., & Slovák, J., Parabolic geometries. I. Background and general theory. Mathematical Surveys and Monographs, 154. American Mathematical Society, Providence, RI, 2009. x+628 pp. 18, 43

[39] Capogna, L., *Regularity of quasi-linear equations in the Heisenberg group*. Comm. Pure Appl. Math. 50 (1997), no. 9, 867–889. 44

[40] Capogna, L., & Cowling, M., *Conformality and Q-harmonicity in Carnot groups.* Duke Math. J. 135 (2006), no. 3, 455–479. 13, 44

[41] Capogna, L., Danielli, D., & Garofalo, N., *The geometric Sobolev embedding for vector fields and the isoperimetric inequality*. Comm. Anal. Geom. 2 (1994), no. 2, 203–215. 9

[42] ———, *An imbedding theorem and the Harnack inequality for nonlinear subelliptic equations*, Comm. Part. Diff. Eq., **18** (1993), 1765–1794. 14

[43] Capogna, L., Danielli, D., Pauls, S. D., & Tyson, J. T., An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem. *Progress in Mathematics*, 259. Birkhuser Verlag, Basel, 2007. 11

[CSal] Capria, M. & Salamon, S., *Yang-Mills fields on quaternionic spaces* Nonlinearity **1** (1988), no. 4, 517–530.

[44] Cartan, E., *Sur la geometrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes*,
I. Ann. di Mat., **11** (1932), 17–90,
II. Ann. Sci. Norm. Sup. Pisa, **1** (1932), 333–354. 23, 43

[45] Chang, S.-C., Cheng, J.-H., & Chiu, H.-L., *A fourth order curvature flow on a CR 3-manifold.* Indiana Univ. Math. J. 56 (2007), no. 4, 1793–1826. 36

[46] Chang, S.-C., & Chiu, H.-L., *Nonnegativity of CR Paneitz operator and its application to the CR Obata's theorem.* J. Geom. Anal. 19 (2009), 261–287. 8, 31, 38, 39, 40

[47] ———, *On the CR analogue of Obata's theorem in a pseudohermitian 3-manifold.* Math. Ann. 345(2009), 33–51. 8, 31, 38, 39, 40

[48] Chang, S.-C., & Chiu, H.-L., *On the estimate of the first eigenvalue of a subLaplacian on a pseudohermitian 3-manifold.* Pacific J. Math. 232 (2007), no. 2, 269–282. 8

[49] Chang, S.-C., & Wu, C.-T., *The entropy formulas for the CR heat equation and their applications on pseudohermitian $(2n+1)$-manifolds.* Pacific J. Math., 246 (2010), no. 1, 1–29. 8

[50] ———, *The diameter estimate and its application to CR Obata's Theorem on closed pseudohermitian $(2n+1)$-manifolds*, Trans. Amer. Math. Soc. 364 (2012), no. 7, 3349–3363. 38

[51] Chanillo, S., Chiu, H.-L., & Yang, P., *Embedded three-dimensional CR manifolds and the non-negativity of Paneitz operators.* Geometric analysis, mathematical relativity, and nonlinear partial differential equations, 65–82, Contemp. Math., 599, Amer. Math. Soc., Providence, RI, 2013. 36

[52] Chanillo, S., & Yang, P. C., *Isoperimetric inequalities & volume comparison theorems on CR manifolds.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 8 (2009), no. 2, 279–307. 8

[53] Chavel, I., Eigenvalues in Riemannian geometry. Including a chapter by Burton Randol. With an appendix by Jozef Dodziuk. Pure and Applied Mathematics, 115. Academic Press, Inc., Orlando, FL, 1984. 3

[54] Chen, W.,& Li, C., *Classification of solutions of some nonlinear elliptic equations*, Duke Math. Journ., **63** (1991), 615–622. 14

[55] Cheng, J.-H., Chiu, H.-L., & Paul Yang, P., *Uniformization of spherical CR manifolds.* arXiv:1301.1133.

[56] Cheng, S.-Y., *Eigenvalue comparison theorems and its geometric applications*. Math. Z. 143 (1975), no. 3, 289–297. 3

[57] Chłng, J.-H., & Lee, John M., *A local slice theorem for 3-dimensional CR structures.* Amer. J. Math. 117 (1995), no. 5, 12491298. 31

[58] Cheng, J.-H., Malchiodi, A., & Yang, P., *A positive mass theorem in three dimensional Cauchy-Riemann geometry*. arXiv:1312.7764. 8

[59] Cheng, S. Y., & Yau, S. T., *On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation*. Comm. Pure Appl. Math. 33 (1980), no. 4, 507–544. 18

[60] Chern, S.S. & Moser, J., *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271. 23, 43

[61] Chiu, H.-L., *The sharp lower bound for the first positive eigenvalue of the subLaplacian on a pseudohermitian 3-manifold.* Ann. Global Anal. Geom. 30 (2006), no. 1, 81–96. 8, 35, 36

[62] Christ, M., Liu, H., & Zhang, A., *Sharp Hardy-Littlewood-Sobolev Inequalities on Octonionic Heisenberg Group,* arXiv:1407.3419 8, 10, 20

[63] ——— Christ, M., Liu, H., & Zhang, A., *Sharp Hardy-Littlewood-Sobolev Inequalities on Quaternionic Heisenberg Groups,* arXiv:1407.3417 8

[64] Ciatti, P., *A new proof of the J^2-condition for real rank one simple Lie algebras and their classification.* Proc. Amer. Math. Soc. 133 (2005), no. 6, 1611-1616. 11

[65] Cowling, M., Dooley, A. H., Korányi, A. & Ricci, F., *H-type groups and Iwasawa decompositions*, Adv. Math., **87** (1991), 1–41. 11, 12, 14, 44

[66] Cowling, M., Dooley, A. H., Korányi, A. & Ricci, F., *An approach to symmetric spaces of rank one via groups of Heisenberg type.*, J. of Geom. Anal., **8** (1998), no. 2, 199–237. 11, 12, 13

[67] Cowling, M., De Mari, F., Korányi, A., & Reimann, H.M., *Contact and conformal maps in parabolic geometry. I*. Geom. Dedicata 111 (2005), 65–86. 44

[68] Cowling, M. & Korányi, A., *Harmonic analysis on Heisenberg type groups from a geometric viewpoint*, in "Lie Group Representation III", pp.60-100, Lec. Notes in Math., **1077** (1984), Springer-Verlag. 14, 44

[69] Cowling, M., De Mari, F., Korányi, A., & H.M. Reimann, H.M, *Contact and conformal maps on Iwasawa N groups.* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 13 (2002), 219–232. 13

[70] Cowling, M., De Mari, F., Korányi, A., & H.M. Reimann, H.M, *Contact and conformal maps in parabolic geometry.* I, Geom. Dedicata 111 (2005), 65–86. 13

[71] Cowling, M., & Otazi, A., *Conformal maps of Carnot groups*, arXiv:1312.6423. 43

[72] Damek, E., *The geometry of a semidirect extension of a Heisenberg type nilpotent group.* Colloq. Math. 53 (1987), no. 2, 255–268. 11, 12

[73] Damek, E., & Ricci, F., *A class of nonsymmetric harmonic Riemannian spaces.* Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 1, 139–142. 12

[74] Damek, E., & Ricci, F., *Harmonic analysis on solvable extensions of H-type groups.* J. Geom. Anal. 2 (1992), no. 3, 213–248. 11, 12, 13

[75] de Lima, L. L.; Piccione, P.; Zedda, M., *A note on the uniqueness of solutions for the Yamabe problem*. Proc. Amer. Math. Soc. 140 (2012), no. 12, 4351–4357. 7

[76] De Mari, F., & Ottazzi, A., *Rigidity of Carnot groups relative to multicontact structures.* Proc. Amer. Math. Soc. 138 (2010), no. 5, 1889–1895. 13, 44

[77] Dragomir, S. & Tomassini, G. Differential geometry and analisys on CR manifolds, Progress in Math. vol. 246, Birkhäuser Boston, Inc., Boston, MA, 2006. 22, 27, 39

[78] Duchemin, D., *Quaternionic contact structures in dimension 7*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 4, 851–885. 19, 24

[79] Eisenhart, L.P., Riemannian geometry, Princeton University Press, 1966. 23

[80] Erkekoğlu, F.; García-Río, E.; Kupeli, D. N.; nal, B., *Characterizing specific Riemannian manifolds by differential equations.* Acta Appl. Math. 76 (2003), no. 2, 195–219. 4

[81] Escobales, R., *Riemannian submersions with totally geodesic fibers*. J. Differential Geom. 10 (1975), 253–276.

[82] Federer, Herbert; Fleming, Wendell H., *Normal and integral currents*. Ann. of Math. (2) 72 1960 458–520. 9

[83] Faraut, J., & Korányi, A., Analysis on symmetric cones. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1994. 13

[84] Fefferman, C., & Graham, C. R., *Conformal invariants*. The mathematical heritage of lie Cartan (Lyon, 1984). Astérisque 1985, Numéro Hors Série, 95–116. 18

[85] Fernández, M., Ivanov, S., Ugarte, L., Villacampa, R., *Compact supersymmetric solutions of the heterotic equations of motion in dimensions 7 and 8*, Adv. Theor. Math. Phys. **15** (2011), 245–284. 45, 46

[86] Fernández, M., Ivanov, S., Ugarte, L., & Vassilev, D., *Non-Kähler heterotic string solutions with non-zero fluxes and non-constant dilaton*, J. High Energy Phys **1406** (2014), 073. 47, 48

[87] ——— *The quaternionic Heisenberg group and Heterotic String Solutions with non-constant dilaton in dimensions 7 and 5.* arXiv:1410.4130 46, 47, 48

[88] Folland, G. B., *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Math., **13** (1975), 161–207. 14

[89] Folland, G. B. *Applications of analysis on nilpotent groups to partial differential equations.* Bull. Amer. Math. Soc. 83 (1977), no. 5, 912–930. 14

[90] Folland, G. B. & Stein, E. M., *Estimates for the $\bar\partial_b$ Complex and Analysis on the Heisenberg Group*, Comm. Pure Appl. Math., **27** (1974), 429–522. 8, 14, 19, 20

[91] Frances, C., *Une preuve du thèorème de Liouville en gèomtrie conforme dans le cas analytique*. (French) [Proof of the Liouville theorem of conformal geometry in the analytic case] Enseign. Math. (2) 49 (2003), no. 1-2, 95–100. 43

[92] Frank, R. L., & Lieb, E.H., *Sharp constants in several inequalities on the Heisenberg group*, Ann. of Math. (2) 176 (2012), no. 1, 349–381. 8, 9, 10, 15, 20

[93] Friedrich, Th., Ivanov, S. *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. **6** (2002), 303–336. 45

[94] Friedrich, Th., Ivanov, S., *Killing spinor equations in dimension 7 and geometry of integrable G_2 manifolds*, J. Geom. Phys., **48** (2003), 1–11. 45

[95] Gallot, S., *Équations différentielles caract ¨eristiques de la sphére.* Ann. Sci. École Norm. Sup. (4) 12 (1979), no. 2, 235–267. 2

[96] Gamara, N., *The CR Yamabe conjecture the case $n = 1$*, J. Eur. Math. Soc. (JEMS) **3** (2001), no. 2, 105–137. 8, 27

[97] Gamara, N. & Yacoub, R., *CR Yamabe conjecture – the conformally flat case*, Pacific J. Math. **201** (2001), no. 1, 121–175. 8, 27

[98] García-Ro, E.; Kupeli, D. N.; Ünal, B., *On a differential equation characterizing Euclidean spheres*. J. Differential Equations 194 (2003), no. 2, 287–299. 4

[99] Garofalo, N. & Vassilev, D., *Symmetry properties of positive entire solutions of Yamabe type equations on groups of Heisenberg type*, Duke Math J, **106** (2001), no. 3, 411–449. 10, 14, 32

[100] ———, *Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups*, Math Ann., **318** (2000), no. 3, 453–516. 10, 14

[101] Gauntlett, J.P., Kim, N., Martelli, D., Waldram,D., *Fivebranes wrapped on SLAG three-cycles and related geometry*, JHEP **0111** (2001), 018. 45

[102] Gauntlett, J.P., Martelli, D., Waldram,D., *Superstrings with intrinsic torsion*, Phys. Rev. **D 69** (2004), 086002, 27 pp. 45

[103] Gehring, F. W., *Rings and quasiconformal mappings in space.* Trans. Amer. Math. Soc. 103 1962 353–393. 43

[104] Gidas, B., Ni, W. M. & Nirenberg, L., *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., **68** (1979), 209–243. 14

[105] Gidas, B., Ni, W. M. & Nirenberg, L., *Symmetry of positive solutions of nonlinear elliptic equations*, Math. Anal. Appl., Part A, Adv. in Math. Suppl. Studies, **7 A** (1981), 369–402. 14

[106] Graham, C. R., *The Dirichlet problem for the Bergman Laplacian.* I. Comm. Partial Differential Equations 8 (1983), no. 5, 433–476. II. Ibid. 8 (1983), no. 6, 563–641. 36

[107] Graham, C. R., & Lee, John M., *Einstein metrics with prescribed conformal infinity on the ball.* Adv. Math. 87 (1991), no. 2, 186–225. 18

[108] ——— *Smooth solutions of degenerate Laplacians on strictly pseudoconvex domains.* Duke Math. J., 57(1988), 697–720. 36

[109] Greenleaf, A., *The first eigenvalue of a subLaplacian on a pseudohermitian manifold.* Commun. Partial Diff. Equations, 10 (1985), no. 2, 191–217. 8, 35, 37

[110] Grong, E., & Thalmaier, A., *Curvature-dimension inequalities on sub-Riemannian manifolds obtained from Riemannian foliations, Part II*. math.DG.arXiv:1408.6873 8

[111] ———, *Curvature-dimension inequalities on sub-Riemannian manifolds obtained from Riemannian foliations, Part I.* math.DG.arXiv:1408.6872. 8

[112] Hartman, P., *On isometries and on a theorem of Liouville.* Math. Z. 69 1958 202–210. 43

[113] Heber, J., *Noncompact homogeneous Einstein spaces.* Invent. Math. 133 (1998), no. 2, 279–352. 19

[114] Helgason, S., Differential geometry, Lie groups, and symmetric spaces. Pure and Applied Mathematics, 80. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. 11

[115] Hersch, J., *Quatre propriétés isopérimétriques de membranes sphériques homogènes*. C. R. Acad. Sci. Paris Sr. A-B **270** (1970). 15

[116] Hirachi, K., *Scalar pseudo-hermitian invariants and the Szegö kernel on three-dimensional CR manifolds*, in: Complex Geometry, 1990 Osaka Conf. Proc. Marcel Dekker Lect. Notes Pure Appl. Math. 143 (1993), 67–76. 36

[117] Hladky, R. K., *Bounds for the first eigenvalue of the horizontal Laplacian in positively curved sub-Riemannian manifolds*. Geom. Dedicata 164 (2013), 155–177. 8

[118] Hughen, W. K., The sub-Riemannian geometry of three-manifolds. Thesis (Ph.D.), Duke University. 1995. 8

[119] Ishihara, Shigeru; Tashiro, Yoshihiro, *On Riemannian manifolds admitting a concircular transformation*. Math. J. Okayama Univ. 9 1959/1960 1947. 3, 4

[120] Ivanov, St., Minchev, I., & Vassilev, D., *Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem*, Mem. of AMS, Volume 231, Number 1086 (2014). 7, 8, 20, 24, 25, 26, 31, 32, 33, 34, 41, 43

[121] ———, *Extremals for the Sobolev inequality on the seven dimensional quaternionic Heisenberg group and the quaternionic contact Yamabe problem,* J. Eur. Math. Soc. (JEMS) 12 (2010), no. 4, 1041–1067. 8, 10, 26, 31, 32, 33, 34, 35, 44

[122] ———, *The optimal constant in the L^2 Folland-Stein inequality on the quaternionic Heisenberg group*, Ann. Sc. Norm. Super. Pisa Cl. Sci.(5) Vol. XI (2012), 1–18. 8, 10, 15, 20

[123] ———, *Quaternionic contact Einstein manifolds*, arXiv:1306.0474. 24, 25, 41

[124] ———, *Quaternionic contact hypersurfaces in hyper-Kähler manifolds*, arXiv:1406.4256 27

[125] ———, *Solution of the qc Yamabe problem on a qc-conformally qc-Einstein manifold*, arxiv:15?? 31, 35

[126] Ivanov, S., Petkov, A., & Vassilev, D., *The sharp lower bound of the first eigenvalue of the sub-Laplacian on a quaternionic contact manifold in dimension 7*, Nonlinear Anal-Theory 93 (2013) 51-61. 8, 40, 41

[127] ———, *The sharp lower bound of the first eigenvalue of the sub-Laplacian on a quaternionic contact manifold*, J. Geom. Anal. 24 (2014), no. 2, 756–778.8, 40, 41

[128] ———, *The Obata sphere theorems on a quaternionic contact manifold of dimension bigger than seven*, arXiv:1303.0409.8, 40, 41, 42

[129] Ivanov, St. & Vassilev, D., *Conformal quaternionic contact curvature and the local sphere theorem*, J. Math. Pures Appl. **93** (2010), 277–307. 8, 25, 26, 27, 32

[130] ———, *Extremals for the Sobolev Inequality and the Quaternionic Contact Yamabe Problem*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.14, 17, 22, 23, 25, 28, 33, 35, 37, 44

[131] ———, *An Obata type result for the first eigenvalue of the sub-Laplacian on a CR manifold with a divergence free torsion*, J. Geom. 103 (2012), 475–504. 8, 31, 37, 38, 39, 40

[132] ———, *An Obata-type theorem on a three dimensional CR manifold*, Glasg. Math. J., 56 (2014), no. 2, 283–294. 8, 25, 37, 38, 40

[133] Ivanov, St., Vassilev, D., & Zamkovoy, S., *Conformal paracontact curvature and the local flatness theorem,* Geom. Ded. **144** (2010), no.1, 79–100. 23, 28, 37

[134] Iwaniec, T., & Martin, G., *The Liouville theorem*. (English summary) Analysis and topology, 339–361, World Sci. Publ., River Edge, NJ, 1998. 43

[135] Jacobowitz, H., *Two notes on conformal geometry.* Hokkaido Math. J. 20 (1991), no. 2, 313–329. 43

[136] Jelonek, W., *Positive and negative 3-K-contact structures*, Proc. Amer. Math. Soc. **129** (2000), 247–256. 25

[137] Jerison, D., & Lee, J., *A subelliptic, nonlinear eigenvalue problem and scalar curvature on CR manifolds*, Contemporary Math., **27** (1984), 57–63. 8, 14, 27

[138] Jerison, D., & Lee, J., *The Yamabe problem on CR manifolds*, J. Diff. Geom., **25** (1987), 167–197. 14, 15, 27

[139] Jerison, D., & Lee, J., *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc., **1** (1988), no. 1, 1–13. 8, 10, 27, 28, 29, 31, 34

[140] Jerison, D., & Lee, J., *Intrinsic CR normal coordinates and the CR Yamabe problem*, J. Diff. Geom., **29** (1989), no. 2, 303–343. 8, 27

[141] Jung, S. D., Lee, K. R., & Richardson, K., *Generalized Obata theorem and its applications on foliations.* J. Math. Anal. Appl. **376** (2011), no. 1, 129–135. 8

[142] Kaplan, A., *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Amer. Math. Soc., **258** (1980), 147–153. 10

[143] Kaplan, A., & Putz, R., *Harmonic forms and Riesz transforms for rank one symmetric spaces.* Bull. Amer. Math. Soc. 81 (1975), 128-132. 10

[144] Kaplan, A., & Putz, R., *Boundary behavior of harmonic forms on a rank one symmetric space.* Trans. Amer. Math. Soc. 231 (1977), no. 2, 369-384. 11

[145] Kaplan, A., & Ricci, F., *Harmonic analysis on groups of Heisenberg type.* Harmonic analysis (Cortona, 1982), 416435, Lecture Notes in Math., 992, Springer, Berlin, 1983. 11

[146] Kashiwada, T., *A note on Riemannian space with Sasakian 3-structure*, Nat. Sci. Reps. Ochanomizu Univ., **22** (1971), 1–2. 25

[147] Khuri, M. A.; Marques, F. C.; Schoen, R. M., *A compactness theorem for the Yamabe problem*. J. Differential Geom. 81 (2009), no. 1, 143–196. 6

[148] Kiosak, Volodymyr; Matveev, Vladimir S., *There are no conformal Einstein rescalings of complete pseudo-Riemannian Einstein metrics.* C. R. Math. Acad. Sci. Paris 347 (2009), no. 17-18, 1067–1069. 7

[149] Konishi, M., *On manifolds with Sasakian 3-structure over quaternion Kähler manifolds*, Kodai Math. Sem. Rep. **26** (1975), 194–200. 25

[150] Koiso, N., *A decomposition of the space $\mathcal{M}$ of Riemannian metrics on a manifold*. Osaka J. Math. 16 (1979), no. 2, 423–429. 7

[151] Korányi, A., *Kelvin transform and harmonic polynomials on the Heisenberg group*, Adv.Math. **56** (1985), 28–38. 14, 44

[152] Korányi, A., *Geometric properties of Heisenberg-type groups.* Adv. in Math. 56 (1985), no. 1, 28-38. 11

[153] Korányi, A., *Multicontact maps: Results and conjectures.* Lect. Notes Semin. Interdiscip. Mat., IV, S.I.M. Dep. Mat. Univ. Basilicata, Potenza, 2005, 57–63. 13

[154] Korányi, A., & Reimann, H. M., *Quasiconformal mappings on the Heisenberg group*. Invent. Math. 80 (1985), no. 2, 309–338. 44

[155] Kühnel, W., *Conformal transformations between Einstein spaces*. Conformal Geometry (R. S. Kulkarni and U. Pinkall, eds.). aspects of math. E 12 Vieweg, Braunschweig 1988, pp. 105–146. 3, 7

[156] Kühnel, W., & Rademacher, H.-B., *Einstein spaces with a conformal group*. Results Math. 56 (2009), no. 1-4, 421–444. 7

[157] Kuiper, N. H., *On conformally-flat spaces in the large.* Ann. of Math. (2) 50, (1949), 916–924. 44

[158] Kunkel, C.S., *Quaternionic contact normal coordinates* , arXiv:0807.0465 32

[159] Lanconelli, E., & Uguzzoni, F., *Asymptotic behavior and non-existence theorems for semilinear Dirichlet problems involving critical exponent on unbounded domains of the Heisenberg group*, Boll. Un. Mat. Ital., (8) **1-B** (1998), 139–168. 15

[160] LeBrun, C., *On complete quaternionic-Kähler manifolds.* Duke Math. J. 63 (1991), no. 3, 723–743. 18

[161] ———, *Fano manifolds, contact structures, and quaternionic geometry.* Int. J. Math. 6 (1995), 419–437.

[162] Le Donne, E., & Ottazzi, A., *Isometries of Carnot groups and subFinsler homogeneous manifolds.*, arXiv:1210.5097. 44

[163] Lee, Jeffrey M., & Richardson, K., *Lichnerowicz and Obata theorems for foliations.* Pacific J. Math. **206** (2002), no. 2, 339–357. 8

[164] ———, *Riemannian foliations and eigenvalue comparison.* Ann. Global Anal. Geom. **16** (1998), no. 6, 497–525. 8

[165] Lee, J., *Pseudo-einstein structures on CR manifolds*, Amer. J. Math., **110** (1988), 157–178. 36

[166] Lee, J., *The Fefferman metric and pseudohermitian invariants*, Trans. AMS, **296** (1986), 411–429. 28

[167] Lee, J. M. & Parker, T., *The Yamabe Problem*, Bull Am. Math. Soc. **17** (1987), no. 1, 37–91. 5, 6, 7

[168] Lempert, L., *On three-dimensional Cauchy-Riemann manifolds.* J. Amer. Math. Soc. 5 (1992), no. 4, 923–969. 36

[169] Li, S.-Y., & Luk, H.-S., *The sharp lower bound for the first positive eigenvalue of a sub-Laplacian on a pseudo-Hermitian manifold.* Proc. Amer. Math. Soc. 132 (2004), no. 3, 789–798. 8, 35

[170] Li, S.-Y., Wang, X., *An Obata type theorem in CR geometry*, arXiv:1207.4033, version 1. 8, 37, 38

[171] ———, *An Obata type theorem in CR geometry*, J. Differential Geom. 95 (2013), no. 3, 483-502, arXiv:1207.4033, version 2. 8, 37, 38, 40

[172] Li, Y.-Y., & Zhang, L. *Compactness of solutions to the Yamabe problem. III*. J. Funct. Anal. 245 (2007), no. 2, 438–474. 6

[173] ———, *Compactness of solutions to the Yamabe problem. II.* Calc. Var. Partial Differential Equations 24 (2005), no. 2, 185237. 6

[174] ———, *Compactness of solutions to the Yamabe problem*. C. R. Math. Acad. Sci. Paris 338 (2004), no. 9, 693695.

[175] Lichnerowicz, A., *Géométrie des groupes de transformations.* Travaux et Recherches Mathematiques, III. Dunod, Paris 1958. 2, 5

[176] Liouville, J., *Théorème sur l'équation $dx^2 + dy^2 + dz^2 = \lambda(d\alpha^2 + d\beta^2 + d\gamma^2)$*. J. Math. Pures Appl. 15 (1850), 103. 43

[177] ———, *Extension au cas des trois dimensions da la question du tracégéograhpique.* Note VI. - In: Monge, G. (ed.), Applications de la analysis à la géométrie, Bachelier, Paris, 1850, 609–617. 43

[178] Maldacena, J. M., *The Large N limit of superconformal field theories and supergravity.* Adv.Theor.Math.Phys., 2:231–252. 18

[179] Mancini, G., Fabbri, I., & Sandeep, K., *Classification of solutions of a critical Hardy-Sobolev operator.* J. Differential Equations 224 (2006), no. 2, 258–276. 15

[180] Marques, F. C., *A priori estimates for the Yamabe problem in the non-locally conformally flat case*. J. Differential Geom. 71 (2005), no. 2, 315346. 6

[181] Matveev, V. S.; Rademacher, H.-B.; Troyanov, M.; Zeghib, A., *Finsler conformal Lichnerowicz-Obata conjecture*. Ann. Inst. Fourier (Grenoble) 59 (2009), no. 3, 937–949. 7

[182] Maz'ya, V. G., *Classes of domains and imbedding theorems for function spaces.* Dokl. Akad. Nauk SSSR 133 527–530 (Russian); translated as Soviet Math. Dokl. 1 1960 882–885. 9

[183] Mok, N., & Yau, S.-T., *Completeness of the Khler-Einstein metric on bounded domains and the characterization of domains of holomorphy by curvature conditions*. The mathematical heritage of Henri Poincar, Part 1 (Bloomington, Ind., 1980), 4159, Proc. Sympos. Pure Math., 39, Amer. Math. Soc., Providence, RI, 1983. 18

[184] Morbidelli, D., *Liouville theorem, conformally invariant cones and umbilical surfaces for Grushin-type metrics*. Israel J. Math. 173 (2009), 379–402. 44

[185] Mostow, G. D., *Strong rigidity of locally symmetric spaces*, Annals of Mathematics Studies, No. 78. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973. v+195 pp. 18, 43

[186] Nevanlinna, R., *On differentiable mappings*. 1960 Analytic functions pp. 3–9 Princeton Univ. Press, Princeton, N.J. 43

[187] Obata, M., *Conformal changes of Riemannian metrics on a Euclidean sphere*. Differential geometry (in honor of Kentaro Yano), pp. 347–353. Kinokuniya, Tokyo, 1972. 7

[188] ———, *The conjecture of conformal transformations in Riemannian manifolds*, J. Diff. Geom., **6** (1971), 247–258. 2, 7

[189] ———, *Certain conditions for a Riemannian manifold to be iosometric with a sphere.* J. Math. Soc. Japan **14**, no.3, (1962), 333–340. 2, 4

[190] ———, *Riemannian manifolds admitting a solution of a certain system of differential equations.* 1966 Proc. U.S.-Japan Seminar in Differential Geometry (Kyoto, 1965) pp. 101114 Nippon Hyoronsha, Tokyo. 4

[191] Ottazzi, A., *Multicontact vector fields on Hessenberg manifolds*. J. Lie Theory 15 (2005), no. 2, 357–377. 13, 44

[192] ———, *A sufficient condition for nonrigidity of Carnot groups*. Math. Z. 259 (2008), no. 3, 617–629. 13, 44

[193] Ottazzi, Alessandro; Warhurst, Ben, *Contact and 1-quasiconformal maps on Carnot groups*. J. Lie Theory 21 (2011), no. 4, 787811. 13

[194] Pak, H., & Park, J., *A note on generalized Lichnerowicz-Obata theorems for Riemannian foliations.* Bull. Korean Math. Soc. **48** (2011), no. 4, 769–777. 8

[195] Pansu, P., *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. (2) 129 (1989), no. 1, 1–60. 43

[196] Pansu, P., *Quasiisométries des variétés á courbure négative.* Thése, Paris 1987. 13

[197] Poincaré, H., *Les fonctions analytiques de deus variables et la représentation conforme*. Rend. Circ. Mat. Palermo, 23 (1907), 185–220; Oeuvres, t. IV, 244–289. 43

[198] Reimann, H.M., *Rigidity of H-type groups.* Math. Z. 237 (2001), no. 4, 697–725. 13, 44

[199] Rešetnjak, Ju. G., *Liouville's conformal mapping theorem under minimal regularity hypotheses.* (Russian) Sibirsk. Mat. Ž. 8 1967 835–840. 43

[200] Rifford, L., *Ricci curvatures in Carnot groups*. Math. Control Relat. Fields 3 (2013), no. 4, 467–487.

[201] Rothschild, Linda Preiss; Stein, E. M., *Hypoelliptic differential operators and nilpotent groups.* Acta Math. 137 (1976), no. 3-4, 247–320. 19

[202] Rumin, M., *Formes différentielles sur les variétés de contact.* (French) [Differential forms on contact manifolds] J. Differential Geom. 39 (1994), no. 2, 281–330. 8

[203] Saloff-Coste, Laurent Aspects of Sobolev-type inequalities. London Mathematical Society Lecture Note Series, 289. Cambridge University Press, Cambridge, 2002. 9

[204] Schoen, R., *On the number of constant scalar curvature metrics in a conformal class*. Differential geometry, 311–320, Pitman Monogr. Surveys Pure Appl. Math., 52, Longman Sci. Tech., Harlow, 1991. 6

[205] ———, *A report on some recent progress on nonlinear problems in geometry*, Surveys in differential geometry (Cambridge, MA, 1990), 201–241, Lehigh Univ., Bethlehem, PA, 1991. 6

[206] ———, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom., **20** (1984), no. 2, 479–495. 5, 6

[207] ———, *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation*, Comm. Pure Appl. Math. **41** (1988), no. 3, 317–392. 5

[208] ———, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*. Topics in calculus of variations (Montecatini Terme, 1987), 120154, Lecture Notes in Math., 1365, Springer, Berlin, 1989. 5, 6

[209] Schoen, R., & Yau, S.-T., *Lectures on differential geometry.* Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994. 44

[210] ———, *On the proof of the positive mass conjecture in general relativity*. Comm. Math. Phys. 65 (1979), no. 1, 45–76. *Proof of the positive mass theorem. II*. ibid. 79 (1981), no. 2, 231–260. 5

[211] Serrin, J., *A symmetry problem in potential theory*, Arch. Ration. Mech., **43** (1971), 304–318. 14

[212] Strominger, A., *Superstrings with torsion*, Nuclear Phys. **B 274** (1986), 253–284. 45

[213] Szegö, G., *Inequalities for certain eigenvalues of a membrane of given area* J. Rational Mech. Anal. 3, (1954). 343–356. 15

[214] Talenti, G., *Best constant in the Sobolev inequality*, Ann. Mat. Pura Appl., **110** (1976), 353–372. 5, 9

[215] Tanaka, N., *On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables.* J. Math. Soc. Japan 14 1962, 397–429. 9, 19, 20, 21, 22, 43

[216] Tang, P., *Regularity and extremality of quasiconformal homeomorphisms on CR 3 -manifolds.* Ann. Acad. Sci. Fenn. Math. 21 (1996), no. 2, 289–308. 44

[217] Tanno, S., *Remarks on a triple of K-contact structures*, Tohoku Math. J. **48** (1996), 519–531. 25

[218] Tashiro, Yoshihiro, *Complete Riemannian manifolds and some vector fields*. Trans. Amer. Math. Soc. 117 1965 251275. 4

[219] Trudinger, N., *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa, **22** (1968), 265–274. 5, 6

[220] Vassilev, D., *Lp estimates and asymptotic behavior for finite energy solutions of extremals to Hardy-Sobolev inequalities*. Trans. Amer. Math. Soc. 363 (2011), no. 1, 37–62. 15

[221] ______, *Regularity near the characteristic boundary for sub-laplacian operators*, Pacific J Math, **227** (2006), no. 2, 361–397. 9, 14, 35

[222] Webster, S. M., *Real hypersurfaces in complex space*, Thesis, University of California, 1975. 14, 15, 21, 23

[223] ______, *Pseudo-Hermitian structures on a real hypersurface*, J. Differential Geom. **13** (1978), no. 1, 25–41. 19, 21, 22

[224] ______, *On the pseudo-conformal geometry of a Kähler manifold*, Math. Z. **157** (1977), no. 3, 265–270.

[225] Wang, W., *The Yamabe problem on quaternionic contact manifolds*, Ann. Mat. Pura Appl., **186** (2007), no. 2, 359–380. 8, 14, 20, 31

[226] X. Wang, *On a remarkable formula of Jerison and Lee in CR geometry*, arXiv:1308.5182. 31

[227] Wolf, J.A., Spaces of constant curvature. Second edition. Department of Mathematics, University of California, Berkeley, Calif., 1972. 11

[228] Yamabe, H. , *On a deformation of Riemannian structures on compact manifolds*, Osaka Math J., (12) (1960), 21–37. 5, 6

[229] Yamaguchi, K., *Differential systems associated with simple graded Lie algebras*, In: Progress in Differential Geometry, Adv. Stud. Pure Math. 22, Math. Soc. Japan, Tokyo, 1993, pp. 413–494. 44

[230] Yano, K., *The theory of Lie derivatives and its applications*. North-Holland Publishing Co., Amsterdam; P. Noordhoff Ltd., Groningen; Interscience Publishers Inc., New York, 1957. 5

[231] Yano K., & T. Nagano, Einstein spaces admitting a oneparameter group of conformal transformations. Ann. Math. (2) 69 (1959) 451–461. 3, 4

[232] Yano, K., & Obata, M., *Conformal changes of Riemannian metrics*. J. Differential Geometry 4 1970 53–72. 5

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