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Strong Unique Continuation Properties of Generalized Baouendi–Grushin Operators

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This paper gives a quantitative control of the order of zero of a weak solution to perturbations of the Baouendi–Grushin operator, which generalizes the result due to Aronszaijn, Krzywicki, and Szarski valid for elliptic operators in divergence form with Lipschitz continuous coefficients.

Keywords Baouendi–Grushin operator; Unique continuation.

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1. Introduction and Statement of the Results

The uniqueness in the Cauchy problem and the closely connected unique continuation property (ucp) for subelliptic operators is a subject which is far from being understood and to a large extent unexplored. On the negative side there exists a general counterexample of Bahouri (1986) to the ucp for zero order perturbations of sub-Laplacians $\mathscr{L} - V = \sum_{j=1}^{m} X_j X_j - V$, when, besides the finite rank condition on the Lie algebra, some additional geometric conditions are fulfilled by the vector fields $X_1, \ldots, X_m$ (such additional assumptions are not necessary in dimension three or four). What happens, however, if one considers the unperturbed operator corresponding to the case $V = 0$? In this situation Bony (1969) has proved uniqueness in the Cauchy problem if the vector fields are real analytic. A general satisfactory answer to this question in the C^∞ or less regular case does not seem to be presently available. In this paper we study the strong unique continuation property (sucp) for a class of variable coefficient operators whose "constant coefficient" model at one point is the so called Baouendi–Grushin operator (Baouendi, 1967; Grushin, 1970, 1971). We recall that the latter is the

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following operator on $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$, $N = n + m$,

$$\mathscr{L}_o = \sum_{i=1}^{N} X_i X_i u, \tag{1.1}$$

where the vector fields (which are not in fact constant coefficient) are given by

$$X_k = \frac{\partial}{\partial x_k}, \quad k = 1, \dots, n, \quad X_{n+j} = |x|^\alpha \frac{\partial}{\partial y_j}, \quad j = 1, \dots, m. \tag{1.2}$$

Here $\alpha > 0$ is a fixed parameter, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_m) \in \mathbb{R}^m$. When $\alpha = 0$, $\mathscr{L}_o$ is just the standard Laplacian in $\mathbb{R}^N$. For $\alpha > 0$ the ellipticity of the operator $\mathscr{L}_o$ becomes degenerate on the *characteristic submanifold* $M = \mathbb{R}^n \times \{0\}$ of $\mathbb{R}^N$. When $\alpha = 2k$, with $k \in \mathbb{N}$, then $\mathscr{L}_o$ is a sum of squares of C^∞ vector fields satisfying Hörmander finite rank condition on the Lie algebra *rank* $Lie[X_1, \dots, X_m] \equiv N$. We note that there exists a family of anisotropic dilations

$$\delta_t(\xi) = \delta_t(x, y) = (tx, t^{(\alpha+1)}y), \quad t > 0 \tag{1.3}$$

naturally associated with the vector fields in (1.2). Consequently, in the analysis of $\mathscr{L}_o$ the number

$$Q = n + (\alpha + 1)m \quad (> N = n + m), \tag{1.4}$$

plays the role of a dimension. We refer to Q as the *homogeneous dimension* relative to the vector fields (1.2). Operators modeled on (1.1) have been intensively studied after the pioneering works of Franchi and Lanconelli (1983), see Franchi and Serapioni (1987), and the references therein.

The analysis of the operator $\mathscr{L}_o$ is subtle and, at least in the case $\alpha = 1$, it is closely connected to that of the real part of the Kohn sub-Laplacian on the Heisenberg group $\mathbb{H}^n$ (Franchi and Lanconelli, 1983; Garofalo, 1993; Garofalo and Shen, 1994, 1996; Rothschild and Stein, 1976). Since the latter operator is real-analytic hypoelliptic, harmonic functions in $\mathbb{H}^n$ cannot vanish to infinite order at one point unless they are identically zero. However, to present date there exists no quantitative proof of such sucp in $\mathbb{H}^n$ (by this we mean a proof based on estimates and which does not directly hinge on the real-analyticity of solutions). In particular, it would be important to know whether the generalized frequency in $\mathbb{H}^n$ introduced in Garofalo and Lanconelli (1990) is increasing, but this remains at the moment a challenging open question. Such and related questions constitute some of the motivations of the present paper. Returning to the operator $\mathscr{L}_o$, we mention that it was proved in Garofalo (1993) that the frequency attached to the horizontal energy is indeed increasing at points of the degeneracy manifold M, thus the sucp holds for $\mathscr{L}_o$. In the same paper this is also proved for the operator $\mathscr{L}_o - \langle \vec{b}, Du \rangle - V$ with suitable assumptions on $\vec{b}$ and V. To give an idea, for example

$$|V| \le \frac{C}{\rho}\psi \quad \text{and} \quad |\langle \vec{b}, Du \rangle| \le C|Xu|\psi^{1/2}$$

is enough. Here Du is the gradient of u, $|Xu|$ is the horizontal gradient (1.10) of u, and ρ and ψ are defined correspondingly in (1.8) and (1.9). With a completely different method, based on a subtle two-weighted Carleman estimate, the sucp was established in Garofalo and Shen (1994) for zero order perturbations $\mathscr{L}_o - V$, where the potential V is allowed to belong to some appropriate L^p spaces.

In this paper we consider equations of the type

$$\mathscr{L}u = \sum_{i,j=1}^{N} X_j(a_{ij}(x,y)X_i u) = 0. \tag{1.5}$$

We assume that $A = \big(a_{ij}(x,y)\big)$, $i, j = 1, \ldots, N$, is a $N \times N$ matrix-valued function on $\mathbb{R}^N$ which, for simplicity, we take such that

$$A(0) = Id. \tag{1.6}$$

Furthermore, we assume A is symmetric and uniformly elliptic matrix. Thus $a_{ij}(g) = a_{ji}(g)$ and there exists $\lambda > 0$ such that for any $\eta \in \mathbb{R}^N$

$$\lambda|\eta|^2 \leq \langle A\eta, \eta\rangle \leq \lambda^{-1}|\eta|^2. \tag{1.7}$$

Our main concern is whether, under suitable assumptions on the matrix A, the sucp continues to hold for the operator $\mathscr{L}$. To put our result in perspective we mention that when $\alpha = 0$ in (1.2), so that $\mathscr{L}_o$ is the standard Laplacian, a famous result due to Aronszajn et al. (1962) states that if the matrix A has Lipschitz continuous coefficients, then the operator $\mathscr{L}$ possesses the sucp. Furthermore, it was shown in Miller (1974) that such assumption is optimal. Our results, Theorems 1.2 and 1.3 can be seen as a generalization of that in Aronszajn et al. (1962), in the sense that, in the limit as $\alpha \to 0$ we recapture both the assumptions and the conclusion of the elliptic case, see Remark 1.3. The approach, however, is different from that in Aronszajn et al. (1962), which is based on Carleman inequalities along with results from Riemannian geometry that do not seem to be adaptable to our context due to the lack of ellipticity. Instead, we have borrowed the ideas developed in Garofalo and Lin (1986, 1987) and Garofalo (1991, 1993), see also the subsequent simplification in Kukavica (1998). Our main result is Theorem 1.2, which gives a quantitative control of the order of zero of a weak solution to (1.5). Such result is proved under some hypothesis on the matrix A which are listed as assumptions (H) below. The latter are tailored on the geometry of the operator $\mathscr{L}_o$ and should be interpreted as a sort of Lipschitz continuity with respect to a suitable pseudo-distance associated to the system of vector fields (1.2).

In order to state the main result we recall the definition of the *gauge* ρ associated to $\mathscr{L}_o$ (Garofalo, 1993). With $\xi = (x, y) \in \mathbb{R}^N$ we let

$$\rho = \rho(\xi) \stackrel{def}{=} (|x|^{2(\alpha+1)} + (\alpha+1)^2|y|^2)^{\frac{1}{2(\alpha+1)}}. \tag{1.8}$$

We stress that ρ is homogeneous of degree one with respect to the anisotropic dilations (1.3). In the sequel we indicate with $B_r = \{\rho < r\}$ the pseudo-balls centered at the origin in $\mathbb{R}^N$ with radius r with respect to the gauge ρ. Since $\rho \in C^\infty(\mathbb{R}^N\backslash\{0\})$, the outer unit normal on ∂B_r is given by $\nu = |D\rho|^{-1}D\rho$. As we mentioned, if $\alpha = 2k$, with $k \in \mathbb{N}$, then the system (1.2) satisfies Hörmander's condition, and the ensuing

Carnot-Carathéodory distance of ξ from the origin can be shown to be comparable to $\rho(\xi)$. We will also need the angle function ψ defined as follows (Garofalo, 1993)

$$\psi = \psi(\xi) \overset{def}{=} |X\rho|^2(\xi) = \frac{|x|^{2\alpha}}{\rho^{2\alpha}}, \quad \xi \neq 0. \tag{1.9}$$

Hereafter, given a function f, we denote by

$$Xf = (X_1 f, \ldots, X_N f) \tag{1.10}$$

the gradient along the system of vector fields in (1.2) (called also *horizontal gradient of* f), and let $|Xf|^2 = \sum_{j=1}^{N}(X_j f)^2$. The function ψ vanishes at every point of the characteristic manifold M, and clearly satisfies $0 \leq \psi \leq 1$.

Definition 1.1. A weak solution to $\mathscr{L}u = 0$ in an open set Ω is a function $u \in L^2_{loc}(\Omega)$ such that the (distributional) horizontal gradient $Xu \in L^2_{loc}(\Omega)$, and for which the equation $\mathscr{L}u = 0$ is satisfied in the variational sense in Ω, i.e.,

$$\int_\Omega \langle AXu, X\phi \rangle dV = 0$$

for every $\phi \in C_o^\infty(\Omega)$.

We note that, under the hypothesis in the present paper, thanks to the basic results in Franchi and Lanconelli (1983) and Franchi and Serapioni (1987) a weak solution u is (after modification on a set of measure zero) Hölder continuous with respect to the Euclidean distance. We are ready to state our main result.

Theorem 1.2. *Let A be a symmetric matrix satisfying* (1.7) *and the hypothesis* (H) *below with relative constant Λ. Suppose u is a weak solution of* (1.5) *in a neighborhood of the origin Ω. Under these assumptions, there exist positive constants $C = C(u, \alpha, \lambda, \Lambda, N)$ and $r_o = r_o(u, \alpha, \lambda, \Lambda, N)$, such that, for any $2r \leq r_o$, we have*

$$\int_{B_{2r}} u^2\psi \, dV \leq C \int_{B_r} u^2 \psi \, dV.$$

The dependence of the constant C on u is quite explicit. It involves the L^2 norm of $|Xu|$ on B_1, and the L^2 norm of u on ∂B_1 with respect to the weighted measure $\psi \, dH_{N-1}$. We remark that, although we have stated Theorem 1.2 when the point of consideration is the origin, this result continues to be true for any other point with the appropriate modification of the hypothesis (H).

We say that $u \in L^2_{loc}(\mathbb{R}^N)$ vanishes to infinite order at some $z_o \in \mathbb{R}^N$ if for every $k > 0$ one has

$$\lim_{r \to 0} \frac{1}{r^k} \int_{B_r(z_o)} |u|^2 dV = 0.$$

A given partial differential operator $\mathscr{L}$ in $\mathbb{R}^N$ is said to possess the strong unique continuation property (SUCP) if for every $z_o \in \mathbb{R}^N$, and any weak solution u of $\mathscr{L}u = 0$, the assumption that u vanishes to infinite order at z_o implies that $u \equiv 0$ in some neighborhood of z_o. In other words non-trivial solutions can have at most finite order of vanishing. As it is well known (Garofalo and Lin, 1986, Theorem 1.2; see also Giaquinta, 1983), implies the following SUCP.

Theorem 1.3. *With the assumptions of Theorem* 1.2, *the operator* $\mathscr{L}$ *has the SUCP.*

In order to state our main assumptions (H) on the matrix A it will be useful to represent the latter in the following block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

Here, the entries are respectively $n \times n$, $n \times m$, $m \times n$ and $m \times m$ matrices, and we assume that $A_{12}^t = A_{21}$. We shall denote by B the matrix

$$B = A - \boldsymbol{I}_{N \times N}$$

and thus

$$B(0) = \boldsymbol{O}_{N \times N}, \tag{1.11}$$

thanks to (1.6). The proof of Theorem 1.2 relies crucially on the following assumptions on the matrix A. These will be our main hypothesis and, without further mention, will be assumed to hold throughout the paper.

Hypothesis. There exists a positive constant Λ such that, for some $\epsilon > 0$, one has in B_ϵ the following estimates

$$\begin{aligned} |b_{ij}| = |a_{ij} - \delta_{ij}| &\leq \begin{cases} \Lambda\rho, & \text{for } 1 \leq i, j \leq n \\ \Lambda\psi^{\frac{1}{2}+\frac{1}{2\alpha}}\rho = \Lambda\dfrac{|x|^{\alpha+1}}{\rho^\alpha}, & \text{else} \end{cases} \\ |X_k b_{ij}| = |X_k a_{ij}| &\leq \begin{cases} \Lambda, & \text{for } 1 \leq k \leq n \quad \text{and} \quad 1 \leq i, j \leq n \\ \Lambda\psi^{\frac{1}{2}} = \Lambda\dfrac{|x|^\alpha}{\rho^\alpha}, & \text{else} \end{cases} \end{aligned} \tag{H}$$

An interesting, typical example of a matrix satisfying the conditions (H) is

$$A = \begin{pmatrix} 1 + \rho f(x, y) & |x|^{\alpha+1} g(x, y) \\ |x|^{\alpha+1} g(x, y) & 1 + |x|^{\alpha+1} h(x, y) \end{pmatrix},$$

where f, g and h are functions which are Lipschitz continuous at the origin of $\mathbb{R}^2$ with respect to the Euclidean metric. In this example $m = n = 1$.

Remark 1.4. It is important to observe that, thanks to (1.9), if we take formally $\alpha = 0$ in (H) we obtain a Lipschitz condition at the origin for the matrix A. Our results thus encompass those in the cited paper (Aronszajn et al., 1962), see also Garofalo and Lin (1986).

For a vector field F we denote by FA the matrix with elements (Fa_{ij}). We will apply the same notation to all matrices under consideration. Throughout the paper we will tacitly assume that all vectors are column vectors. Also, we will use the same notation for first order partial differential operators and for the corresponding tangent vectors, with meaning determined by the context.

The plan of the paper is as follows. In section two we prove Theorem 1.2. The proof involves various technical estimates. For the reader's convenience and ease of exposition we have collected all the auxiliary material in section three.

2. The Frequency Function

The purpose of this section is to prove Theorem 1.2. The main step is to show the monotonicity of the frequency Theorem 2.2. We begin by introducing the relevant quantities that will appear in the proof. Since our results are local in nature, from now on, we focus our attention on the pseudo-ball B_2. The notation dH_{N-1} will indicate $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$. Let u be a weak solution u of (1.5) in B_2.

Definition 2.1. For every $0 < r < 2$ we let

$$H(r) = \int_{\partial B_r} u^2 \frac{\langle AX\rho, X\rho\rangle}{|D\rho|} dH_{N-1},$$

$$D(r) = \int_{B_r} \langle AXu, Xu\rangle dV.$$

The generalized frequency of u on B_r is defined by

$$N(r) \overset{def}{=} \begin{cases} \dfrac{rD(r)}{H(r)}, & \text{if } H \neq 0 \\ 0, & \text{if } H = 0. \end{cases}$$

We shall denote by S the matrix relating the gradient along the vector fields in (1.2) and the standard gradient in $\mathbb{R}^N$, i.e., $X = SD$, where

$$S = \begin{pmatrix} \boldsymbol{I}_{n\times n} & \boldsymbol{0} \\ \boldsymbol{0} & |x|^\alpha \boldsymbol{I}_{m\times m} \end{pmatrix}. \tag{2.1}$$

Trivially, we have

$$S = S^t \quad \text{and} \quad \mathscr{L}u = \operatorname{div}(SASDu). \tag{2.2}$$

The following theorem constitutes the main result of this section.

Theorem 2.2. *Let u be a nontrivial weak solution of $\mathscr{L}u = 0$ in the pseudo-ball B_2, then there exist positive constants $r_o = r_o(\alpha, \lambda, \Lambda, N)$ and $M = M(u, \alpha, \lambda, \Lambda, N)$ such that*

$$\widetilde{N}(r) = \exp(Mr)N(r)$$

is a continuous monotonically nondecreasing function for $r \in (0, r_o)$.

Proof. The proof of Theorem 2.2 rests on Lemmas 2.5 and 2.12 below. Let $M = \max\{C_1, C_2\}$, where C_1 and C_2 are the constants from Lemmas 2.5 and 2.12. Let Q be the homogeneous dimension in (2.3) associated with the non-isotropic dilations (2.4). With r_o as defined in Lemma 2.5 we have that, either $u \equiv 0$ in B_{r_o}, or $H(r) > 0$ for $0 < r < r_o$. In the former case the frequency is identically zero on $(0, r_o)$, so let us consider the latter case, in which $H(r) > 0$. The continuity of $\widetilde{N}(r)$ follows from the continuity of each of the functions involved in its definition. Furthermore, for a.e. $r \in (0, r_o)$ we have

$$\begin{aligned}\left(\ln \frac{rD(r)}{H(r)} e^{2Mr}\right)' &= \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} + 2M \\ &\geq \frac{1}{r} + \frac{Q-2}{r} + \frac{2}{D(r)} \int_{\partial B_r} \frac{\langle AXu, X\rho\rangle^2}{\langle AX\rho, X\rho\rangle} \frac{dH_{N-1}}{|D\rho|} \\ &\quad - \frac{Q-1}{r} - 2\frac{D(r)}{H(r)} \geq 0,\end{aligned}$$

where we have applied first Lemmas 2.5 and 2.12, and then Proposition 2.4 and the Cauchy-Schwarz inequality. □

With the help of the monotonicity it is easy to prove Theorem 1.2, see Section 3 of Garofalo and Lin (1986). We include the proof in the current setting for completeness.

Proof of Theorem 1.2. If the solution vanishes in some neighborhood of the origin then the doubling for all sufficiently small balls is trivially satisfied. Let us consider next the case of a non-trivial solution. Let r_o be the number defined in Lemma 2.5 and $2r \leq r_o$. By the co-area formula

$$\int_0^R \int_{\partial B_r} u^2 \psi \frac{dH_{N-1}}{|D\rho|} dr = \int_{B_R} u^2 \psi \, dV.$$

From the ellipticity of A in (1.7), we have

$$\int_0^R H(r) dr \approx \int_{B_R} u^2 \psi \, dV,$$

with constant of proportionality depending only on $\lambda > 0$. This shows it is enough to prove the doubling property for the height function H. Now, we obtain from Lemma 2.5

$$\begin{aligned}\ln \frac{H(2r)}{2^{Q-1} H(r)} &= \ln \frac{H(2r)}{2^{Q-1} r^{Q-1}} - \ln \frac{H(r)}{r^{Q-1}} = \int_r^{2r} \left\{\frac{H'(t)}{H(t)} - \frac{Q-1}{t}\right\} dt \\ &\leq \int_r^{2r} \left\{2\frac{D(t)}{H(t)} + C_1\right\} dt \leq \int_r^{2r} 2\widetilde{N}(t) \frac{e^{-2Mt}}{t} dt + Mr \\ &\leq 2\widetilde{N}(r_o) \int_r^{2r} \frac{1}{t} dt + M = 2\widetilde{N}(r_o) \ln 2 + M,\end{aligned}$$

where in the last inequality we have used the monotonicity of the modified frequency expressed by Theorem 2.2. We thus conclude

$$H(2r) \leq 2^{Q-1} e^{\{2\widetilde{N}(r_o) \ln 2 + M\}} H(r).$$

Integrating the latter inequality we obtain the doubling property in the conclusion of Theorem 1.2. □

Remark 2.3. We observe that for non-trivial solution we have the doubling property for all balls $B_{2r} \subset \Omega$ and $2r \leq 1$, since for "big" balls, i.e., $2r \geq r_o$ we have

$$\frac{\int_{B_{2r}} u^2\psi\, dV}{\int_{B_r} u^2\psi\, dV} \leq \frac{\int_{B_1} u^2\psi\, dV}{\int_{B_{r_o/2}} u^2\psi\, dV}.$$

Of course, in this case the constant C in the doubling property depends on $\widetilde{N}(1)$.

Finally, we establish Theorem 1.3.

Proof of Theorem 1.3. Suppose u is a solution which vanishes to infinite order at the origin. Let $|B_r| = \omega_o r^Q$. Fix a number $\kappa > 0$ such that $C_o 2^{-Q\kappa} = 1$. For any r sufficiently small and $p \in \mathbb{N}$ the doubling property applied p times gives

$$\begin{aligned}
\int_{B_r} u^2\psi\, dV &\leq C_o^p \int_{B_{r/2^p}} u^2\psi\, dV \\
&\leq \omega_o^\kappa C_o^p \frac{r^{Q\kappa}}{2^{Qp\kappa}} \frac{1}{|B_{r/2^p}|^\kappa} \int_{B_{r/2^p}} u^2\psi\, dV \\
&\leq \omega_o^\kappa r^{Q\kappa} \frac{1}{|B_{r/2^p}|^\kappa} \int_{B_{r/2^p}} u^2\psi\, dV \to 0
\end{aligned}$$

when $p \to \infty$ since $0 \leq \psi \leq 1$. This ends the proof. □

The remainder of this section is devoted to establishing Lemmas 2.5 and 2.12.

Proposition 2.4. *For a.e.* $r \in (0, 2)$ *the horizontal energy of* u *on* B_r *can be expressed by the surface integral*

$$D(r) = \int_{\partial B_r} u \frac{\langle AXu, X\rho \rangle}{|D\rho|} dH_{N-1}.$$

Proof. By the definition of weak solution we have u is continuous and $Xu \in L^2(B_2)$, thus for a.e. $r \in (0, 2)$ one has $Xu \in L^2(\partial B_r)$. The outer unit normal on ∂B_r is given by $\nu = |D\rho|^{-1} D\rho$ and thus

$$u\frac{\langle AXu, X\rho \rangle}{|D\rho|} = u\frac{\langle AXu, SD\rho \rangle}{|D\rho|} = \langle uSAXu, \nu \rangle.$$

The divergence theorem, (2.2) and the fact that $\mathscr{L}u = 0$ imply

$$\begin{aligned}
\int_{\partial B_r} u\frac{\langle AXu, X\rho \rangle}{|D\rho|} dH_{N-1} &= \int_{B_r} \operatorname{div}(uSAXu) dV \\
&= \int_{B_r} \langle AXu, Xu \rangle dV + \int_{B_r} u\mathscr{L}u\, dV \\
&= \int_{B_r} \langle AXu, Xu \rangle dV,
\end{aligned}$$

as claimed in the proposition. □

We proceed with proving the main estimate for the generalized height function $H(r)$. This is the first place where the assumptions (H) on the matrix A play a decisive role. We observe that $r \to H(r)$ is absolutely continuous, thus differentiable a.e. on $(0, 2)$. In the subsequent analysis the number

$$Q = n + (\alpha + 1)m \quad (>N = n + m), \tag{2.3}$$

will play an important role. We note that Q is the *homogeneos dimension* relative to the anisotropic dilations

$$\delta_t(\xi) = \delta_t(x, y) = (tx, t^{(\alpha+1)}y), \quad t > 0 \tag{2.4}$$

naturally associated with the vector fields in (1.2). The infinitesimal generator of (2.4) is

$$Z = \sum_{1 \le i \le n} x_i \frac{\partial}{\partial x_i} + (\alpha + 1) \sum_{1 \le j \le m} y_i \frac{\partial}{\partial y_i}, \tag{2.5}$$

so that a function u is δ_t*-homogeneous* of degree $k \in \mathbb{R}$ if and only if $Zu = ku$. At this point it is worth observing that if u is homogeneous of degree k, and solves the "constant coefficient" equation $\mathcal{L}_o u = 0$ (i.e., u is a fundamental $\mathcal{L}_o$-harmonic of degree k), then the corresponding frequency is constant and equal to k. This justifies the name generalized frequency. To prove this fact one uses Proposition 2.4 with $A \equiv I$ which gives

$$D(r) = \int_{B_r} \langle Xu, Xu \rangle dV = \int_{\partial B_r} u \frac{\langle Xu, X\rho \rangle}{|D\rho|} dH_{N-1}.$$

A calculation, see (2.12) in Garofalo (1993) or Proposition 3.1, shows ($X = SD$!)

$$X\rho = \frac{\psi}{\rho} S^{-1} Z, \tag{2.6}$$

for any function u. When u is $\mathcal{L}_o$-harmonic of degree k we have $Zu = ku$, and one infers from (2.6)

$$\langle Xu, X\rho \rangle = \frac{\psi}{\rho} Zu.$$

Substitution of the latter identity in (2.6) gives

$$D(r) = \frac{k}{r} \int_{\partial B_r} u^2 \frac{\psi}{|D\rho|} dH_{N-1} = \frac{k}{r} H(r),$$

which proves $N(r) \equiv k$.

Lemma 2.5.

a) *There exists a positive constant* $C_1 = C_1(\alpha, \lambda, \Lambda, N)$ *such that for a.e.* $r \in (0, 2)$ *one has*

$$\left| H'(r) - \frac{Q-1}{r} H(r) - 2D(r) \right| \le C_1 H(r).$$

b) *There exists a positive number* $r_o = r_o(\alpha, \lambda, \Lambda, N) \leq 1$ *such that, either* $H(r) = 0$ *on* $(0, r_o)$, *or* $H(r) > 0$ *on* $(0, r_o)$.

Proof. a) Using the definition (2.1) of S we have

$$\frac{\langle AX\rho, X\rho\rangle}{|D\rho|} = \langle SAX\rho, \nu\rangle.$$

The divergence theorem gives

$$\begin{aligned} H(r) &= \int_{\partial B_r} u^2\langle SAX\rho, \nu\rangle dH_{N-1} = \int_{B_r} \operatorname{div}(u^2 SAX\rho)dV \\ &= \int_{B_r} \langle AX\rho, Xu^2\rangle dV + \int_{B_r} u^2 \mathscr{L}\rho\, dV \\ &= \int_{B_r} 2u\langle AX\rho, Xu\rangle dV + \int_{B_r} u^2\mathscr{L}\rho\, dV. \end{aligned} \tag{2.7}$$

Since the gauge ρ is not smooth at the origin, to make rigorous the previous calculation one must integrate on the set $B_r\backslash\overline{B}_\epsilon$ and then let $\epsilon \to 0$. We note that the last integral on the second line of the above chain of equalities is convergent since $\mathscr{L}\rho \in L^1_{loc}(\mathbb{R}^N)$. This can be seen from the remarkable formula

$$\mathscr{L}_o\rho = \frac{Q-1}{\rho}|X\rho|^2, \quad \text{in } \mathbb{R}^N\backslash\{0\}, \tag{2.8}$$

which is (2.18) in Garofalo (1993). Once 2.8 is available one easily obtains by a rescaling, using (2.4), that $\rho^{-p} \in L^1_{loc}(\mathbb{R}^N)$ if and only if $p < Q$. This shows, in particular, that $\mathscr{L}_o\rho \in L^1_{loc}(\mathbb{R}^N)$. We note explicitly that (2.8) expresses, in disguise, the fact that for a suitable constant $C > 0$ the function

$$\Gamma = C\rho^{2-Q} \tag{2.9}$$

is a fundamental solution of $\mathscr{L}_o$ with pole at 0.

Returning to (2.7), after an application of the Federer's co-area formula we differentiate at a.e. $r > 0$, and use Proposition 2.4, obtaining

$$H'(r) = 2D(r) + \int_{\partial B_r} \frac{u^2\mathscr{L}\rho}{|D\rho|} dH_{N-1}.$$

This implies

$$\begin{aligned} H'(r) - \frac{Q-1}{r}H(r) - 2D(r) &= \int_{\partial B_r} \frac{u^2\mathscr{L}\rho}{|D\rho|} dH_{N-1} - \frac{Q-1}{r}H(r) \\ &= \int_{\partial B_r} u^2\frac{\operatorname{div}(SBX\rho)}{|D\rho|} dH_{N-1} + \int_{\partial B_r} u^2\frac{\mathscr{L}_0\rho}{|D\rho|} dH_{N-1} \\ &\quad - \frac{Q-1}{r}\int_{\partial B_r} u^2\frac{|X\rho|^2}{|D\rho|} dH_{N-1} \\ &\quad - \frac{Q-1}{r}\int_{\partial B_r} u^2\frac{\langle BX\rho, X\rho\rangle}{|D\rho|} dH_{N-1}. \end{aligned}$$

We recall that $(b_{ij}) = B = A - Id$. Now, thanks to (2.8) the two middle terms in the last equality above are equal. The last term is easily estimated as follows on ∂B_r

$$\frac{\langle BX\rho, X\rho\rangle}{|D\rho|} \le Cr\frac{\langle AX\rho, X\rho\rangle}{|D\rho|},$$

for some positive constant $C = C(\alpha, \lambda, \Lambda, N)$. This is recognized observing that by (H) we have $||B||_{L^\infty(\partial B_r)} \le Cr$, and using also (1.7). Finally, we estimate the first term in the right-hand side. Writing the divergence term as

$$\operatorname{div}(SBX\rho) = \sum_{i,j=1}^{N} X_i(b_{ij}X_j\rho) = \sum_{i,j=1}^{N} X_i b_{ij} X_j\rho + b_{ij}X_iX_j\rho,$$

and taking into account the assumptions (H), Propositions 3.1 and 3.3 we find, by splitting the terms into the four groups that appear in the block form of A (and hence of B), the following inequalities

$$\sum_{i,j=1}^{N} |X_i b_{ij} X_j\rho| \le C\big(\psi^{1+\frac{1}{2\alpha}} + \psi^{\frac{1}{2}}\psi^{1+\frac{1}{2\alpha}} + \psi^{\frac{1}{2}}\psi^{\frac{1}{2}} + \psi^{\frac{1}{2}}\psi^{\frac{1}{2}}\big) \le C\psi,$$

$$\begin{aligned}\sum_{i,j=1}^{N} |b_{ij}X_iX_j\rho| &\le C\left(\rho\frac{\psi}{\rho} + \rho\psi^{\frac{1}{2}+\frac{1}{2\alpha}}\rho\psi^{\frac{1}{2}-\frac{1}{2\alpha}} + \rho\psi^{\frac{1}{2}+\frac{1}{2\alpha}}\frac{\psi^{\frac{3}{2}+\frac{1}{2\alpha}}}{\rho} + \rho\psi^{\frac{1}{2}+\frac{1}{2\alpha}}\frac{\psi}{\rho}\right)\\ &\le C\psi.\end{aligned}$$

This completes the proof of part a).

b) From part a) we have

$$H'(r) \ge \left(\frac{Q-1}{r} - C_1\right)H(r) + 2D(r).$$

Let $r_1 = \min\left\{1, \frac{Q-1}{2C_1}\right\}$ so that $H'(r) \ge C_1H(r) + 2D(r) \ge 0$ on the interval $(0, r_1)$. Therefore there exists an $0 < r_o \le r_1$ with the required properties. $\square$

Our next objective is to obtain estimates of the first variation $D'(r)$ of the horizontal energy. Let

$$\mu \stackrel{def}{=} \langle AX\rho, X\rho\rangle. \tag{2.10}$$

Consider the vector field F defined as follows

$$F = \rho\sum_{i,j=1}^{N}\frac{a_{ij}X_j\rho}{\mu}X_i, \quad x \ne 0, \tag{2.11}$$

i.e.,

$$Fu = \frac{\rho}{\mu}\langle AX\rho, Xu\rangle = \frac{\rho}{\mu}\langle SAX\rho, Du\rangle,$$

for any smooth function u. We now see that the assumptions on the matrix A guarantee that F can be continuously extended to all of $\mathbb{R}^N$. Furthermore, near the characteristic manifold, such extension gives a small perturbation of the Euler vector field Z in (2.5). To prove this latter claim, we recall (2.6), and let

$$\sigma \stackrel{def}{=} \langle BX\rho, X\rho \rangle = \mu - \psi. \tag{2.12}$$

Thus, F can be re-written as

$$F = \frac{\psi}{\mu} Z + \frac{\rho}{\mu} SBX\rho = Z - \frac{\sigma}{\mu} Z + \rho \sum_{i,j=1}^{N} \frac{b_{ij} X_j \rho}{\mu} X_i. \tag{2.13}$$

From (H), the coercivity of A, and from Lemma 3.1 we find easily

$$\left| \frac{\sigma}{\mu} Z \right| \leq C \frac{\rho \psi^{1+\frac{1}{2\alpha}} \psi^{1+\frac{1}{2\alpha}} + \rho \psi^{\frac{1}{2}+\frac{1}{2\alpha}} \psi^{\frac{1}{2}} \psi^{1+\frac{1}{2\alpha}}}{\psi} |Z| \leq \Lambda \rho \psi^{1+\frac{1}{\alpha}} |Z| \leq \Lambda |x| |Z|, \tag{2.14}$$

and

$$\left| \frac{b_{ij} X_j \rho}{\mu} X_i \right| \leq C \leq C \rho \psi^{\frac{1}{2\alpha}} \leq C|x|.$$

Substituting the two estimates (2.14) in (2.13), we obtain the above claim.

Our next goal is establishing a basic Rellich-type identity involving the vector field F, Lemma 2.11, which we shall use to prove the main estimate on the derivative of the horizontal energy, see Lemma 2.12. The proof of such Rellich-type identity relies on some basic estimates on the divergence and the commutators of F which are collected in the subsequent Lemmas 2.6–2.10. We mention that, in turn, the proofs of these five lemmas rely on some auxiliary technical estimates which, in order to keep the flow of this section, we have collected separately in the next section. Hereafter, the summation convention over repeated indices will be adopted.

Lemma 2.6. *There exists a constant* $C = C(\alpha, \lambda, \Lambda, N) > 0$ *such that for* $1 \leq i \leq N$ *we have*:

$$\left| \left[X_i, \frac{\rho}{\mu} SBX\rho \right] u \right| \leq C\rho |Xu|.$$

Proof. By a direct calculation

$$\begin{aligned}\left[X_i, \frac{\rho}{\mu} SBX\rho \right] u &= X_i \left\langle \frac{\rho}{\mu} BX\rho, Xu \right\rangle - \left\langle \frac{\rho}{\mu} BX\rho, XX_i u \right\rangle \\ &= X_i \left(\frac{\rho}{\mu} \right) b_{kj} X_j \rho X_k u + \frac{\rho}{\mu} X_i (b_{kj} X_j \rho) X_k u + \frac{\rho}{\mu} b_{kj} X_j \rho [X_i, X_k] u.\end{aligned}$$

Now, Lemmas 3.8 and 3.9 and Remark 3.6 give the desired bound for the first and the second sum in the last line. To estimate the last sum we use that

$$|[X_i, X_k]u| \leq \frac{\alpha}{|x|} |Xu|$$

and Lemma 3.9. □

Lemma 2.7. *There exists a constant* $C = C(\alpha, \lambda, \Lambda, N) > 0$ *such that for* $1 \le i \le N$ *we have*:

$$\left|\left[X_i, -\frac{\sigma}{\mu}Z\right]u\right| \le C\rho|Xu|.$$

Proof. From Proposition 3.1 we have

$$Zu = \frac{\rho}{\psi}\langle X\rho, Xu\rangle.$$

Thus

$$\begin{aligned}\left[X_i, \frac{\sigma}{\mu}Z\right]u &= X_i\left(\frac{\sigma}{\mu}\frac{\rho}{\psi}\langle X\rho, Xu\rangle\right) - \frac{\sigma}{\mu}\frac{\rho}{\psi}\langle X\rho, XX_iu\rangle \\ &= X_i\left(\frac{\sigma}{\psi}\frac{\rho}{\mu}X_k\rho\right)X_ku + \frac{\sigma\rho}{\mu\psi}X_k\rho[X_i, X_k]u \\ &= \frac{\rho}{\mu}X_i\left(\frac{\sigma}{\psi}\right)X_k\rho X_ku + \frac{\sigma}{\psi}X_i\left(\frac{\rho}{\mu}\right)X_k\rho X_ku + \frac{\sigma\rho}{\mu\psi}X_iX_k\rho X_ku + \frac{\sigma\rho}{\mu\psi}X_k\rho[X_i, X_k]u.\end{aligned}$$

Using Lemmas 3.5, 3.7, 3.8, and Proposition 3.3 together with

$$|[X_i, X_k]u| \le \frac{\alpha}{|x|}|Xu|$$

we can bound each of the terms above and finish the proof. □

Lemma 2.8. *There exists a constant* $C = C(\alpha, \lambda, \Lambda, N) > 0$ *such that*

$$\left|\operatorname{div}\left(\frac{\rho}{\mu}SBX\rho\right)\right| \le C\rho.$$

Proof. We have

$$\begin{aligned}\operatorname{div}\left(\frac{\rho}{\mu}SBX\rho\right) &= \left\langle BX\rho, X\left(\frac{\rho}{\mu}\right)\right\rangle + \frac{\rho}{\mu}\operatorname{div}(SBX\rho) \\ &= \frac{\sigma}{\mu} - \frac{\rho}{\mu^2}\big(\langle BX\rho, X\sigma\rangle + \langle BX\rho, X\psi\rangle\big) + \frac{\rho}{\mu}X_k(b_{kj}X_j).\end{aligned}$$

Invoking Lemmas 3.5, 3.9, Proposition 3.2, and Remark 3.6, we end the proof. □

Lemma 2.9. *There exists a constant* $C = C(\alpha, \lambda, \Lambda, N) > 0$ *such that*

$$\left|\operatorname{div}\left(\frac{\sigma}{\mu}Z\right)\right| \le C\rho.$$

Proof. The proof is straightforward after we make use of the fact that ψ is homogeneous of order 0, i.e., $Z\psi = 0$. Recall also that $divZ = Q$, and that

$\mu = \psi + \sigma$.

$$\mathrm{div}\left(\frac{\sigma}{\mu}Z\right) = Z\left(\frac{\sigma}{\mu}\right) + Q\frac{\sigma}{\mu} = Z\left(\frac{\mu - \psi}{\mu}\right) + Q\frac{\sigma}{\mu}$$
$$= -Z\left(\frac{\psi}{\mu}\right) + Q\frac{\sigma}{\mu} = -\psi Z\left(\frac{1}{\mu}\right) + Q\frac{\sigma}{\mu} = \frac{\psi}{\mu^2}Z\sigma + Q\frac{\sigma}{\mu}.$$

Clearly

$$\left|\frac{\sigma}{\mu}\right| \le C\rho\psi$$

while

$$|Z\sigma| \le \frac{\rho}{\psi}|X\rho||X\sigma| \le C\rho\psi$$

by Lemma 3.5. □

Lemma 2.10. $|\langle FAXu, Xu\rangle| \le C\rho|Xu|^2$.

Proof. It is enough to show that

$$|Fa_{rs}| \le C\rho,$$

i.e.,

$$\frac{\rho}{\psi}|\langle AX\rho, Xa_{rs}\rangle| \le C\rho,$$

which is the same as

$$|\langle AX\rho, Xa_{rs}\rangle| \le C\psi \quad \text{for all } (r, s).$$

The assumption (H) implies

$$|a_{ij}X_i\rho X_j a_{rs}| \le C\psi^{\frac{1}{2}}\psi^{\frac{1}{2}} \le C\psi, \quad n+1 \le j \le N,$$
$$|a_{ij}X_i\rho X_j a_{rs}| \le C\left(\psi^{1+\frac{1}{2\alpha}} + \psi^{\frac{1}{2}+\frac{1}{2\alpha}}\rho\psi^{\frac{1}{2}}\right) \le C\psi^{1+\frac{1}{2\alpha}} \le C\psi, \quad 1 \le j \le n.$$ □

We can now prove the above mentioned Rellich-type indentity.

Lemma 2.11. *Let $X_1, \ldots, X_N$ and F be the above considered vector fields in $\mathbb{R}^N$. We have the following identity*

$$\int_{\partial B_r} \langle AXu, Xu\rangle\langle F, \nu\rangle dH_{N-1}$$
$$= 2\int_{\partial B_r} a_{jk}X_j u\langle X_k, \nu\rangle Fu\, dH_{N-1}$$
$$- 2\int_{B_r} (\mathrm{div}X_k)a_{jk}X_j uFu\, dV$$

$$- 2\int_{B_r} a_{jk}X_j u[X_k, F]u\, dV + \int_{B_r} (\text{div}F)\langle AXu, Xu\rangle dV$$
$$+ \int_{B_r} \langle (FA)Xu, Xu\rangle dV - 2\int_{B_r} Fu\mathcal{L}u\, dV,$$

where FA is the matrix with elements Fa_{ij}. Here, ν denotes the outer unit normal to B_r.

Proof. The proof of the above integral identity is based on the divergence theorem and can be carried similarly to its classical counterpart, see Ch. 5 in Necǎs (1967). Since the vector fields and the matrix A are not smooth, one has to justify the use of such result by a standard approximation argument which can be carried using the following key estimates from Lemmas 2.6–2.10. Specifically, Lemmas 2.8 and 2.9 give

$$|Q - \text{div}\, F| \le C\rho,$$

whereas Lemmas 2.6, 2.7 imply

$$|[X, F]u - Xu| \le C\rho|Xu|.$$

Finally, Lemma 2.10 gives

$$\|FA\|_\infty \le C\rho. \qquad \square$$

Lemma 2.12. *There exists a constant $C_2 = C_2(\alpha, \lambda, \Lambda, N) > 0$ such that*

$$D'(r) \ge 2\int_{\partial B_r} \frac{1}{\mu}\frac{\langle AXu, X\rho\rangle^2}{|D\rho|} dV + \frac{Q-2}{r} D(r) - C_2 D(r),$$

where μ is defined in (2.10).

Proof. By the co-area formula $D(r) = \int_0^r \int_{\partial B_s} \frac{\langle AXu, Xu\rangle}{|D\rho|} dH_{N-1} ds$. Hence,

$$D'(r) = \int_{\partial B_r} \frac{\langle AXu, Xu\rangle}{|D\rho|} dH_{N-1} = \frac{1}{r}\int_{\partial B_r} \langle AXu, Xu\rangle\langle F, \nu\rangle dH_{N-1},$$

taking into account that on ∂B_r one has $\langle F, \nu\rangle = \frac{r}{|D\rho|}$. The latter follows from the following calculation

$$\langle F, \nu\rangle = \frac{F\rho}{|D\rho|} = \rho\frac{\langle SAX\rho, D\rho\rangle}{\mu|D\rho|} = \rho\frac{\langle AX\rho, X\rho\rangle}{\mu|D\rho|} = \frac{r}{|D\rho|}.$$

From Lemma 2.11 we obtain

$$D'(r) = 2\int_{\partial B_r} \frac{1}{\mu}\frac{\langle AXu, X\rho\rangle^2}{|D\rho|} dV + \frac{1}{r}\int_{B_r} (\text{div}\, F)\langle AXu, Xu\rangle dV$$
$$- \frac{2}{r}\int_{B_r} a_{jk}X_j u[X_k, F]u\, dV + \frac{1}{r}\int_{B_r} \langle (FA)Xu, Xu\rangle dV.$$

In view of (2.13), the fact that $\operatorname{div} Z = Q$, and of the identities $[X_i, Z] = X_i$, $i = 1, \ldots, N$, we can rewrite the above formula in the following form

$$
\begin{aligned}
D'(r) = {} & 2\int_{\partial B_r} \frac{1}{\mu}\frac{\langle AXu, X\rho\rangle^2}{|D\rho|} dH_{N-1} + \frac{Q-2}{r} D(r) \\
& + \frac{1}{r}\int_{B_r} \operatorname{div}\left(-\frac{\sigma}{\mu} Z + \frac{\rho}{\mu} SBX\rho\right)\langle AXu, Xu\rangle dV \\
& - \frac{2}{r}\int_{B_r} a_{jk} X_j u\left[X_k, -\frac{\sigma}{\mu} Z + \frac{\rho}{\mu} SBX\rho\right] u\, dV \\
& + \frac{1}{r}\int_{B_r} \langle (FA)Xu, Xu\rangle dV.
\end{aligned}
$$

At this point we are left with showing that the assumption (H) implies the correct estimates for the last three integrals. The absolute value of the integral involving the divergence is estimated by Lemmas 2.8 and 2.9. The integral involving the commutators is estimated by Lemmas 2.6 and 2.7, using also the ellipticity of A, cf. (1.7). Finally, the absolute value of the last integral is estimated by Lemma 2.10 and by (1.7). This finishes the proof of Lemma 2.12. □

3. Auxiliary Results

In this section we collect some basic estimates that have been used in Section 2. Recall that the matrix S was defined in (2.1).

Proposition 3.1.

i) *The following formula holds true*

$$
Z = \frac{\rho}{\psi} SX\rho
$$

ii) *The horizontal gradient of the gauge satisfies*

$$
\begin{aligned}
|X_k\rho| &\le \psi^{1+\frac{1}{2\alpha}} \quad \textit{for } 1 \le k \le n, \\
|X_{n+k}\rho| &\le (\alpha+1)\psi^{\frac{1}{2}} \quad \textit{for } 1 \le k \le m.
\end{aligned}
$$

Proof. By definition

$$
\begin{aligned}
X_k\rho &= \psi\frac{x_k}{\rho} \quad \text{for } 1 \le k \le n, \\
X_{n+k}\rho &= (\alpha+1)\psi^{\frac{1}{2}}\frac{y_k}{\rho^{\alpha+1}} \quad \text{for } 1 \le k \le m.
\end{aligned}
$$

In other words, we have

$$
X\rho = \left(\frac{\psi}{\rho}x, (\alpha+1)\frac{\psi^{1/2}}{\rho^{\alpha+1}}y\right) = \frac{\psi}{\rho}\left(x, (\alpha+1)|x|^{-\alpha}y\right) = \frac{\psi}{\rho}S^{-1}Z,
$$

having in mind the definition of the radial vector field Z, see 2.5. From $\frac{|x|}{\rho} = \psi^{\frac{1}{2\alpha}}$ and $|y| \leq \rho^{\alpha+1}$ we obtain that the estimates in ii). □

In the next proposition we compute the horizontal gradient of the angle function.

Proposition 3.2. *The angle function ψ satisfies the estimates*

$$|X_k\psi| \leq C\alpha\frac{\psi}{|x|}, \quad \text{if } 1 \leq k \leq n$$
$$|X_{n+k}\psi| \leq C\alpha\frac{\psi}{\rho}, \quad \text{if } 1 \leq k \leq m.$$

Proof. Since $\psi = \frac{|x|^{2\alpha}}{\rho^{2\alpha}}$ we have

$$\begin{aligned} X\psi &= \frac{2\alpha|x|^{2\alpha-1}X|x|}{\rho^{2\alpha}} - \frac{2\alpha|x|^{2\alpha}}{\rho^{2\alpha+1}}X\rho \\ &= \frac{2\alpha|x|^{2\alpha-1}}{\rho^{2\alpha}}\begin{pmatrix}\frac{x}{|x|}\\ 0\end{pmatrix} - \frac{2\alpha|x|^{3\alpha}}{\rho^{2(2\alpha+1)}}\begin{pmatrix}|x|^{\alpha}x\\ (\alpha+1)y\end{pmatrix} \\ &= 2\alpha\psi\begin{pmatrix}\frac{x}{|x|^2}\\ 0\end{pmatrix} - 2\alpha\frac{\psi^2}{\rho^2}\begin{pmatrix}x\\ (\alpha+1)|x|^{-\alpha}y\end{pmatrix}. \end{aligned}$$

This shows that

$$X_i\psi = \begin{cases} 2\alpha\psi\frac{x_i}{|x|^2} - 2\alpha\psi^2\frac{x_i}{\rho^2} & \text{if } 1 \leq i \leq n, \\ -2\alpha(\alpha+1)\psi\frac{y_{i-n}|x|^{\alpha}}{\rho^{2\alpha+2}} & \text{if } n+1 \leq i \leq N. \end{cases}$$

Now, $|x| \leq \rho$ and $|y| \leq \rho^{\alpha+1}$ lead to the desired estimates. □

In the proof of Theorem 1.2 the following estimates on the horizontal Hessian of ρ play an important role.

Proposition 3.3.

$$|X_iX_j\rho| \leq C\frac{\psi}{\rho} \quad \text{for } 1 \leq i, j \leq n \text{ or } n+1 \leq i, j \leq N,$$
$$|X_iX_{n+j}\rho| \leq C\frac{\psi^{\frac{1}{2}}}{|x|} = C\rho\psi^{\frac{1}{2}-\frac{1}{2\alpha}} \quad \text{for } 1 \leq i \leq n,\ 1 \leq j \leq m,$$
$$|X_{n+j}X_i\rho| \leq C\frac{\psi^{\frac{3}{2}}|x|}{\rho^2} = C\frac{\psi^{\frac{3}{2}+\frac{1}{2\alpha}}}{\rho} \quad \text{for } 1 \leq i \leq n,\ 1 \leq j \leq m.$$

Proof. We need to compute the second derivatives of ρ and this is done easily for example by using the product rule and the formulas from Propositions 3.1 and 3.2. We shall write only the expressions for the second derivatives.

If $1 \le i, j \le n$, we have:

$$X_iX_j\rho = -(2\alpha+1)\frac{\psi^2}{\rho^3}x_ix_j + 2\alpha\frac{\psi}{\rho|x|^2}x_ix_j + \frac{\psi}{\rho}\delta_{ij}.$$

If $1 \le i \le n$ and $1 \le j \le m$ we have:

$$X_iX_{n+j}\rho = -(2\alpha+1)\frac{\psi^2}{\rho^3}(\alpha+1)|x|^{-\alpha}x_iy_j \\ + 2\alpha\frac{\psi}{\rho|x|^2}(\alpha+1)|x|^{-\alpha}x_iy_j - \frac{\psi}{\rho}\alpha(\alpha+1)|x|^{-\alpha-2}x_iy_j.$$

If $1 \le i \le n$ and $1 \le j \le m$ we have:

$$X_{n+j}X_i\rho = -(2\alpha+1)\frac{\psi^2}{\rho^3}(\alpha+1)|x|^{-\alpha-2}x_iy_j.$$

If $1 \le i, j \le m$, we have:

$$X_{n+i}X_{n+j}\rho = -(2\alpha+1)\frac{\psi^2}{\rho^3}(\alpha+1)|x|^{-\alpha-2}x_jy_i + (\alpha+1)\frac{\psi}{\rho}\delta_{ij}.$$

At this point the estimates follow in an obvious way using $|x| \le \rho$ and $|y| \le \rho^{\alpha+1}$. □

Definition 3.4. Let:

$$\mu \overset{def}{=} \langle AX\rho, X\rho\rangle,$$

and also

$$B \overset{def}{=} A - Id, \quad \sigma \overset{def}{=} \langle BX\rho, X\rho\rangle.$$

One more notation we will use is: $(b_{ij}) = B$.

Lemma 3.5. *If* (H) *holds then*:

$$|\sigma| \le C\rho\psi^{\frac{3}{2}+\frac{1}{2\alpha}},$$
$$|X_k\sigma| \le C\psi^{\frac{3}{2}} 1 \le k \le N.$$

Proof. We have $\sigma = b_{ij}X_i\rho X_j\rho$. Thus Proposition 3.3 and (H) give:

$$|\sigma| \le C\big(\rho\psi^{1+\frac{1}{2\alpha}}\psi^{1+\frac{1}{2\alpha}} + \rho\psi^{\frac{1}{2}+\frac{1}{2\alpha}}\psi^{1+\frac{1}{2\alpha}}\psi^{\frac{1}{2}} + \rho\psi^{\frac{1}{2}+\frac{1}{2\alpha}}\psi^{\frac{1}{2}}\psi^{\frac{1}{2}}\big) \\ \le C\big(\rho\psi^{2+\frac{1}{\alpha}} + \rho\psi^{2+\frac{1}{\alpha}} + \rho\psi^{\frac{3}{2}+\frac{1}{2\alpha}}\big) \le C\rho\psi^{\frac{3}{2}+\frac{1}{2\alpha}}.$$

The derivatives are given by $X_k\sigma = b_{ij}X_kX_i\rho X_j\rho + X_kb_{ij}X_i\rho X_j\rho$ and we can use Propositions 3.1 and 3.3 to obtain the desired estimates.

For $1 \leq k \leq n$ we have

$$
\begin{aligned}
|X_k\sigma| &\leq C\left(\rho\frac{\psi}{\rho}\psi^{1+\frac{1}{2\alpha}} + \rho\psi^{\frac{1}{2}+\frac{1}{2\alpha}}\frac{\psi}{\rho}\psi^{\frac{1}{2}} + \rho\psi^{\frac{1}{2}+\frac{1}{2\alpha}}\rho\psi^{\frac{1}{2}-\frac{1}{2\alpha}}\psi^{1+\frac{1}{2\alpha}} + \rho\psi^{\frac{1}{2}+\frac{1}{2\alpha}}\rho\psi^{\frac{1}{2}-\frac{1}{2\alpha}}\psi^{\frac{1}{2}}\right) \\
&\quad + C\big(\psi^{1+\frac{1}{2\alpha}}\psi^{1+\frac{1}{2\alpha}} + \psi^{\frac{1}{2}}\psi^{1+\frac{1}{2\alpha}}\psi^{\frac{1}{2}} + \psi^{\frac{1}{2}}\psi^{\frac{1}{2}}\psi^{\frac{1}{2}}\big) \\
&\leq C\big(\psi^{2+\frac{1}{2\alpha}} + \psi^{2+\frac{1}{\alpha}} + \psi^{\frac{3}{2}}\big) \leq C\psi^{\frac{3}{2}}.
\end{aligned}
$$

For $n+1 \leq k \leq N$ we find

$$
\begin{aligned}
|X_k\sigma| &\leq C\left(\rho\frac{\psi^{\frac{3}{2}+\frac{1}{2\alpha}}}{\rho}\psi^{1+\frac{1}{2\alpha}} + \rho\psi^{\frac{1}{2}+\frac{1}{2\alpha}}\frac{\psi^{\frac{3}{2}+\frac{1}{2\alpha}}}{\rho}\psi^{\frac{1}{2}} + \rho\psi^{\frac{1}{2}+\frac{1}{2\alpha}}\frac{\psi}{\rho}\psi^{1+\frac{1}{2\alpha}} + \rho\psi^{\frac{1}{2}+\frac{1}{2\alpha}}\frac{\psi}{\rho}\psi^{\frac{1}{2}}\right) \\
&\quad + C\psi^{\frac{1}{2}}\big(\psi^{1+\frac{1}{2\alpha}}\psi^{1+\frac{1}{2\alpha}} + \psi^{1+\frac{1}{2\alpha}}\psi^{\frac{1}{2}} + \psi^{\frac{1}{2}}\psi^{\frac{1}{2}}\big) \\
&\leq C\big(\psi^{\frac{5}{2}+\frac{1}{\alpha}} + \psi^{2+\frac{1}{2\alpha}} + \psi^{\frac{3}{2}}\big) \leq C\psi^{\frac{3}{2}}.
\end{aligned}
$$

□

Remark 3.6. Notice that a careful examination of the second part of the above proof shows that we also proved:

$$\left|X_k b_{ij} X_i \rho\right| \leq C\psi.$$

Lemma 3.7. *If* (H) *holds then*:

$$\left|X_k\left(\frac{\psi}{\mu}\right)\right| \leq C\psi^{\frac{1}{2}} \quad \textit{for } 1 \leq k \leq N.$$

Proof. It is enough to estimate the reciprocal $\frac{\mu}{\psi}$ since

$$X_k\left(\frac{\psi}{\mu}\right) = -\frac{\psi^2}{\mu^2}X_k\left(\frac{\mu}{\psi}\right) \quad \text{and} \quad 0 < \lambda \leq \frac{\mu}{\psi} \leq \lambda^{-1}.$$

From $X_k\left(\frac{\mu}{\psi}\right) = X_k\left(\frac{\sigma}{\psi}\right)$, using Lemma 3.5 and Proposition 3.2 we obtain:

$$\left|X_k\left(\frac{\sigma}{\psi}\right)\right| = \left|\frac{X_k\sigma}{\psi} - \frac{\sigma}{\psi^2}X_k\psi\right| \leq C\left(\psi^{\frac{1}{2}} + \frac{\rho\psi^{\frac{3}{2}+\frac{1}{2\alpha}}}{\psi^2}\rho\psi^{1-\frac{1}{2\alpha}}\right) = C\psi^{\frac{1}{2}}.$$

The proof is complete. □

Lemma 3.8. *If* (H) *holds then*:

$$\left|X_k\left(\frac{\rho}{\mu}\right)\right| \leq C\psi^{-1-\frac{1}{2\alpha}} \quad \textit{for } 1 \leq k \leq N.$$

Proof.

$$X_k\left(\frac{\rho}{\mu}\right) = X_k\left(\frac{\psi}{\mu}\frac{\rho}{\psi}\right) = X_k\left(\frac{\psi}{\mu}\right)\frac{\rho}{\psi} + \frac{X_k\rho}{\mu} - \frac{\rho X_k\psi}{\psi\mu}.$$

Now Lemma 3.7 and Propositions 3.1 and 3.2 give:

$$\left|X_k\left(\frac{\rho}{\mu}\right)\right| \le C\left(\frac{\psi^{\frac{1}{2}}\rho}{\psi} + \frac{\psi^{\frac{1}{2}}}{\psi} + \frac{\rho}{\psi^2}\frac{\psi}{|x|}\right) \le C\psi^{-1-\frac{1}{2\alpha}},$$

recalling also that $0 < \lambda \le \frac{\mu}{\psi} \le \lambda^{-1}$. □

Lemma 3.9. *If* (H) *holds then*:

$$|b_{kj}X_j\rho| \le C\rho\psi^{1+\frac{1}{2\alpha}}$$

Proof. If $1 \le j \le n$ we have $|X_j\rho| \le C\psi^{1+\frac{1}{2\alpha}}$ and $b_{kj} \le C\rho$. If $n+1 \le j \le N$ we have $|X_j\rho| \le C\psi^{\frac{1}{2}}$ and $b_{kj} \le C\rho\psi^{\frac{1}{2}+\frac{1}{2\alpha}}$. □

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