SYMMETRY PROPERTIES OF POSITIVE ENTIRE SOLUTIONS OF YAMABE TYPE EQUATIONS ON GROUPS OF HEISENBERG TYPE

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1. Introduction

This paper is a continuation of the project initiated in [22], where we studied the following non-linear Dirichlet problem

$$\begin{cases} \mathcal{L}u \ = \ - \ u^{\frac{Q+2}{Q-2}} \\ u \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega), \quad u \geq 0. \end{cases} \tag{1.1}$$

Here, $\boldsymbol{G}$ is a stratified, nilpotent Lie group, in short a Carnot group, of arbitrary step, and $\Omega \subset \boldsymbol{G}$ is a domain which can be bounded or unbounded. The second order differential operator $\mathcal{L}$ represents a given sub-Laplacian on $\boldsymbol{G}$. If $\mathfrak{g} = \overset{r}{\underset{j=1}{\oplus}} V_j$ is a stratification of the Lie algebra $\mathfrak{g}$ of $\boldsymbol{G}$, with $[V_1, V_j] \subset V_{j+1}$ for $1 \leq j < r$, $[V_1, V_r] = \{0\}$, we assume that a scalar product $< \cdot, \cdot >$ is given on $\mathfrak{g}$ for which the $V_j's$ are mutually orthogonal. The stratification allows to define a natural family of non-isotropic dilations $\Delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ as follows

$$\Delta_\lambda(X_1 + ... + X_r) \ = \ \lambda X_1 + ... + \lambda^r X_r.$$

The exponential map $exp : \mathfrak{g} \to \boldsymbol{G}$ is an analytic diffeomorphism. It induces a group of dilations on $\boldsymbol{G}$ via the formula

$$\delta_\lambda(g) \ = \ exp \circ \Delta_\lambda \circ exp^{-1}(g), \quad g \in \boldsymbol{G}.$$

We denote by $dH = dH(g)$ a fixed Haar measure on $\boldsymbol{G}$. One has $dH(\delta_\lambda(g)) = \lambda^Q dH(g)$, where $Q \ = \ \sum_{j=1}^{r} j \ dim \, V_j$ is the *homogeneous dimension* of $\boldsymbol{G}$ attached to the non-isotropic dilations $\{\delta_\lambda\}_{\lambda>0}$. This number plays the role of a dimension in the analysis of Carnot groups. Let $X = \{X_1, \ldots ,X_m\}$ be a basis of V_1 and continue to denote by X the corresponding system of sections on $\boldsymbol{G}$. The *sub-Laplacian* associated with X is the second-order partial differential operator on $\boldsymbol{G}$ given by

$$\mathcal{L} \ = \ - \sum_{j=1}^{m} X_j^* X_j \ = \ \sum_{j=1}^{m} X_j^2$$

(we recall that in a Carnot group one has $X_j^* = -X_j$, see [19]). By the assumption on the Lie algebra one immediately sees that the system X satisfies the well-known finite rank condition, therefore thanks to Hörmander's theorem [27] the operator $\mathcal{L}$ is hypoelliptic. However, it fails

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to be elliptic, and the loss of regularity is measured by the step r of the stratification of $\mathfrak{g}$. For a function u on $\boldsymbol{G}$ we let $|Xu| = (\sum_{j=1}^{m}(X_j u)^2)^{1/2}$. For $1 \le p < Q$ we set

$$\overset{o}{\mathcal{D}}{}^{1,p}(\Omega) = \overline{C_o^\infty(\Omega)}^{||\cdot||_{\mathcal{D}^{1,p}(\Omega)}},$$

where $\mathcal{D}^{1,p}(\Omega)$ indicates the space of functions $u \in L^{p^*}(\Omega)$ having distributional horizontal gradient $Xu = (X_1 u, ..., X_m u) \in L^p(\Omega)$. The space $\mathcal{D}^{1,p}(\Omega)$ is endowed with the obvious norm

$$||u||_{\mathcal{D}^{1,p}(\Omega)} \ = \ ||u||_{L^{p^*}(\Omega)} + ||Xu||_{L^p(\Omega)}.$$

Here, $p^* = \frac{pQ}{Q-p}$ is the Sobolev exponent relative to p. The relevance of such number is emphasized by the following basic result due to Folland and Stein [19], [20].

Theorem (Folland and Stein). *Let $\Omega \subset \boldsymbol{G}$ be an open set. For any $1 < p < Q$ there exists $S_p = S_p(\boldsymbol{G}) > 0$ such that for $u \in C_o^\infty(\Omega)$*

$$\left(\int_\Omega |u|^{p^*}\, dH(g)\right)^{1/p^*} \ \le \ S_p \ \left(\int_\Omega |Xu|^p\, dH(g)\right)^{1/p}. \tag{1.2}$$

Returning to (1.1) we see that the exponent $\frac{Q+2}{Q-2} = 2^* - 1$ is critical for the case $p = 2$ of the embedding (1.2). In [22] we studied the regularity of a weak solution of (1.1) near the characteristic set $\Sigma = \Sigma_{\Omega,X} = \{g \in \partial\Omega \mid X_j(g) \in T_g(\partial\Omega), j = 1, ..., m\}$ of a bounded domain Ω. We proved that under certain geometric hypothesis on the ground domain near Σ, such as starlikeness and convexity (both notions are to be suitably interpreted), a weak solution possesses bounded horizontal gradient Xu up to the characteristic set of Ω. When $\boldsymbol{G}$ is a group of step two, we also established the boundedness of the derivative of u along the generator Z of the non-isotropic group dilations $\{\delta_\lambda\}_{\lambda>0}$. These results were then applied to prove that, under the given geometric assumptions on Ω, there exist no solutions to (1.1), other than the trivial one. Once the main results for bounded domains were obtained, we used the conformal invariance of the problem (1.1), and the CR Kelvin transform to obtain non-existence results for an important class of unbounded domains in groups of Heisenberg, or Iwasawa type. Such domains were called *characteristic convex cones* and *half-spaces* in [22].

When u is a solution of (1.1) for $\Omega = \boldsymbol{G}$, then we say that u is an *entire solution*. It is important to observe that a suitable adaptation of the method of concentration of compactness due to P. L. Lions [38], [39] allows to prove that in any Carnot group (1.1) always admits at least one positive entire solution, see [44]. In this regard an elementary, yet crucial observation, is that if u is an entire solution to (1.1), then such are also the two functions

$$\tau_h u \ \overset{def}{=} \ u \circ \tau_h, \qquad\qquad h \in \boldsymbol{G}, \tag{1.3}$$

where $\tau_h : \boldsymbol{G} \to \boldsymbol{G}$ is the operator of left-translation $\tau_h(g) = hg$, and

$$u_\lambda \ \overset{def}{=} \ \lambda^{(Q-2)/2}\, u \circ \delta_\lambda, \qquad\qquad \lambda > 0. \tag{1.4}$$

In this paper we are only concerned with positive entire solutions of (1.1). In this context the partial differential equation in (1.1) arises in the study of the CR Yamabe problem: *Given a compact, strictly pseudo-convex CR manifold, find a choice of contact form for which the Webster-Tanaka pseudo-hermitian scalar curvature is constant.* Such problem was solved in most cases by Jerison and Lee in a series of important papers, see [28]-[31]. A crucial step in their analysis was the explicit computation of the extremal functions in (1.2) in the special situation when $p = 2$ and $\boldsymbol{G}$ is the Heisenberg group $\mathbb{H}^n$. Jerison and Lee made the deep discovery that, up to group translations and dilations, a suitable multiple of the function

$$u(z,t) \;=\; ((1+|z|^2)^2 \;+\; t^2)^{-(Q-2)/4}, \tag{1.5}$$

is the only positive entire solution of (1.1) in $\mathbb{H}^n$. Here, we have denoted with (z,t), $z \in \mathbb{C}^n, t \in \mathbb{R}$, the variable point in $\mathbb{H}^n$.

Our final goal is to establish a similar uniqueness result for the positive entire solutions of (1.1), when $\boldsymbol{G}$ is a group of Heisenberg type. Such problem is considerably harder than its already difficult Heisenberg group predecessor. Groups of Heisenberg type were introduced by Kaplan [32] in connection with hypoellipticity questions. They constitute a direct and important generalization of the Heisenberg group, as they include, in particular, Iwasawa groups, i.e., the nilpotent component N in the Iwasawa decomposition KAN of simple groups of rank one. Since their introduction there has been a considerable amount of work in the study of groups of Heisenberg type and of their geometry, we refer the reader to the papers [32], [33], [14], [34], [35], [13], [37], [15], [16], [17], [12], [9], [22] and to the references therein.

Some years ago we discovered that for such groups problem (1.1) possesses a remarkable one-parameter family of explicit entire solutions.

Theorem 1.1. *Let $\boldsymbol{G}$ be a group of Heisenberg type. For every $\epsilon > 0$ the function*

$$K_\epsilon(g) \;=\; \left(\frac{m(Q-2)\epsilon^2}{(\epsilon^2+|x(g)|^2)^2+16|y(g)|^2}\right)^{\frac{Q-2}{4}}, \qquad g \in \boldsymbol{G}, \tag{1.6}$$

is a positive, entire solution of the Yamabe equation (1.1).

Theorem 1.1 appeared in [22]. The symbols $x(g), y(g)$ in (1.6) respectively denote the projection of the exponential coordinates of the point $g \in \boldsymbol{G}$ onto the first and second layer of the Lie algebra $\mathfrak{g}$ (see (2.1), (2.2) in the next section for the relevant definitions), whereas m indicates the dimension of the first layer. The existence of such one parameter family of solutions is due to the dilation invariance of the equation, see (1.4). After discovering the special solutions K_ϵ in Theorem 1.1 we formulated the following.

Conjecture: *In a group of Heisenberg type the functions K_ϵ in* (1.6) *are the only non-trivial entire solutions to (1.1). All other non-trivial solutions are obtained from* (1.6) *by* (1.3).

If true, the conjecture would provide a generalization of the results of Jerison and Lee to the setting of groups of Heisenberg type. It would also allow to compute, in this setting, the extremals and the best constant in the Folland-Stein embedding (1.2), thus obtaining an analogue to the famous results of Aubin [1], [2] and Talenti [43]. In this paper we provide a partial answer to this conjecture.

Definition 1.2. *Let $\boldsymbol{G}$ be a Carnot group of step two with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$. We say that a function $U : \boldsymbol{G} \to \mathbb{R}$ has* partial symmetry *(with respect to a point $g_o \in \boldsymbol{G}$) if there exists a function $u : [0,\infty) \times V_2 \to \mathbb{R}$ such that for every $g = exp(x(g) + y(g)) \in \boldsymbol{G}$ one has*

$$\tau_{g_o}\, U(g) \;=\; u(|x(g)|, y(g)). \tag{1.7}$$

A function U is said to have cylindrical symmetry *(with respect to $g_o \in \boldsymbol{G}$) if there exists $\phi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ for which*

$$\tau_{g_o}\, U(g) \;=\; \phi(|x(g)|, |y(g)|), \tag{1.8}$$

for every $g \in \boldsymbol{G}$.

Our main result is the following.

Theorem 1.3. *Let* $\boldsymbol{G}$ *be a group of Iwasawa type. If* $U \not\equiv 0$ *is an entire solution to (1.1) having partial symmetry, then up to group translations we must have* $u = K_\epsilon$*, for some* $\epsilon > 0$*, where* K_ϵ *is the function in Theorem 1.1 and* u *is as in* (1.7).

As we mentioned above, every Iwasawa group is a group of Heisenberg type. We refer the reader to [13] and [12] for an extensive study of the geometric and analytic properties of such groups. Theorem 1.3 is a direct consequence of the following two results.

Theorem 1.4. *Let* $\boldsymbol{G}$ *be an Iwasawa group. Suppose* $U \not\equiv 0$ *is an entire solution of (1.1). If* U *has partial symmetry, then* U *has cylindrical symmetry.*

Unlike the Euclidean case, in the Folland-Stein embedding there exists no spherical symmetrization, and therefore the search of minimizers cannot be reduced to an ordinary differential equation, as in the famous results of Aubin [1], [2] and Talenti [43]. Therefore, after Theorem 1.4 is in force one still needs to confront the non-trivial problem of the uniqueness of positive solutions of a certain non-linear pde in the Poincaré half-plane. This question is resolved by the following theorem.

Theorem 1.5. *Let* $U \not\equiv 0$ *be an entire solution to (1.1) in a group of Iwasawa type* $\boldsymbol{G}$ *and suppose that* U *has cylindrical symmetry. There exists* $\epsilon > 0$ *such that up to a left-translation (1.3) one has*

$$U(g) \;=\; \left(\frac{m(Q-2)\epsilon^2}{(\epsilon^2 + |x(g)|^2)^2 + 16|y(g)|^2} \right)^{\frac{Q-2}{4}}.$$

As a consequence of Theorem 1.3 we obtain the following result. In the sequel we denote by $\overset{o}{X}_{ps}(\boldsymbol{G})$ the subset of $\overset{o}{\mathcal{D}}{}^{1,2}(\boldsymbol{G})$ of the functions having partial symmetry.

Theorem 1.6. *Let* $\boldsymbol{G}$ *be a group of Iwasawa type. Consider the restriction to* $\overset{o}{X}_{ps}(\boldsymbol{G})$ *of the embedding of* $\overset{o}{\mathcal{D}}{}^{1,2}(\boldsymbol{G})$ *into* $L^{2Q/(Q-2)}(\boldsymbol{G})$*. For every* $u \in \overset{o}{X}_{ps}(\boldsymbol{G})$ *one has*

$$\left(\int_{\boldsymbol{G}} |u|^{2^*} dH(g) \right)^{1/2^*} \leq S_2 \left(\int_{\boldsymbol{G}} |Xu|^2 \; dH(g) \right)^{1/2},$$

with

$$S_2 \;=\; \frac{1}{\sqrt{m(m+2(k-1))}}\; 4^{k/(m+2k)}\; \pi^{-(m+k)/2(m+2k)} \; \left(\frac{\Gamma(m+k)}{\Gamma((m+k)/2)} \right)^{1/(m+2k)}. \tag{1.9}$$

An extremal is given by the function

$$f(g) \;=\; \gamma(m,k) \;\left[(1+|x(g)|^2)^2 \;+\; 16|y(g)|^2) \right]^{-(Q-2)/4},$$

where

$$\gamma(m,k) \;=\; \left[4^k\ \pi^{-(m+k)/2(m+2k)}\ \frac{\Gamma(m+k)}{\Gamma((m+k)/2)}\right]^{(m+2(k-1))/2(m+2k)}.$$

Any other non-negative extremal is obtained from f by (1.3) *and* (1.4)*.*

Concerning Theorem 1.6 we mention that we have recently received a preprint from W. Beckner [4] in which the author proves the following result, among others. He considers in $\mathbb{R}^2$ the Baouendi-Grushin operator

$$\mathcal{L}_o \;=\; \frac{\partial^2}{\partial x^2} \;+\; 4x^2\,\frac{\partial^2}{\partial y^2}.$$

It is clear that $\mathcal{L}_o = X_1^2 + X_2^2$, where $X_1 = \partial/\partial x$, $X_2 = 2x\ \partial/\partial y$. Since $[X_1, X_2] = 2\ \partial/\partial y$, $\mathcal{L}_o$ fulfills Hörmander finite rank condition. Moreover, as it is witnessed by Proposition 3.1 and the ensuing Propositions 3.2, 3.3, there is a deep connection between the the sub-Laplacian on a group of Heisenberg type $\boldsymbol{G}$ and the corresponding Baouendi-Grushin operator on its Lie algebra $\mathfrak{g}$, via the natural action of the k-dimensional torus $\mathbb{T}^k$ on $\boldsymbol{G}$. The sharp Sobolev embedding proved in [4] is.

Theorem (Beckner). *For $f \in C^1(\mathbb{R})$ one has*

$$||f||_{L^6(\mathbb{R}^2)} \;\le \pi^{-1/3}\left(\int_{\mathbb{R}^2}\left[\left(\frac{\partial f}{\partial x}\right)^2 \;+\; 4x^2\left(\frac{\partial f}{\partial y}\right)^2\right]\,dxdy\right)^{1/2}. \tag{1.10}$$

This inequality is sharp, and an extremal is given by $[(1+x^2)^2+y^2]^{-1/4}$.

The ideas employed in [4] are quite different from those in this paper. Inequality (1.10) is obtained as a corollary of a beautiful sharp Sobolev inequality in the hyperbolic plane $SL(2,\mathbb{R})/SO(2)$. We note that the transformation $(x,y) \to (x,4y)$ easily allows to recognize that, if we denote by S_2^* the sharp constant in (1.10), then $S_2^* = 4^{-1/3}S_2$, where S_2 is the sharp constant obtained from Theorem 1.6 in the case $m = k = 1$ (we note that in such case $Q = m + 2k = 3$). Since (1.9) gives $S_2 = 4^{1/3}\pi^{-1/3}$, our Theorem 1.6 is in perfect accordance with Beckner's result.

We now describe the plan of the paper. In section two we introduce the relevant definitions and collect various results which are needed subsequently. In section three we prove Theorem 1.4. To establish the latter we suitably adapt the method of moving hyper-planes due to Alexandrov [3] and Serrin [41]. Such method was later perfected in the two celebrated papers [23], [24] by Gidas, Ni and Nirenberg to obtain symmetry for semi-linear equations with critical growth in $\mathbb{R}^n$ or in a ball. In our proof we incorporate some important simplification of the proof in [24] due to Chen and Li [10]. We mention that a crucial role is played by Theorem 1.1 and also by the inversion and the related Kelvin transform introduced by Korányi for the Heisenberg group [36], and subsequently generalized to groups of Heisenberg type in [13], [12]. In section four we prove Theorem 1.5. The proof of this result has been strongly influenced by the approach of Jerison and Lee for the Heisenberg group, see Theorem 7.8 in [29]. After a change in the dependent variable, which relates the Yamabe equation to a new non-linear pde in a quadrant of the Poincaré half-plane, one is led to prove that the only positive solutions of the latter are quadratic polynomials of a certain type. In the case of the Heisenberg group (in which, thanks to the fact that the center is one dimensional, one has the whole half-plane at one's disposal) Jerison and Lee solve this problem by a beautiful, yet misterious, identity which involves the basic Cauchy-Riemann operators $\partial_z, \partial_{\overline{z}}$. When we tried to extend Theorem 7.8 in [29] to the setting of groups of Heisenberg type, after several unsuccessful attempts we discovered that our initial approach to the problem, based on the employement of a suitable P-function, was in essence the same as Jerison and Lee's. Once this aspect was recognized we were able to successfully combine their ideas with ours and complete the proof of Theorem 1.5. We have chosen to present the details avoiding the use of complex variables. We have also tried to emphasize the connection between the more general version of Jerison and Lee's identity in Theorem 4.1 and the method of the so-called *P-functions* introduced by Weinberger in [45] (and subsequently developed by several authors). Given a solution u of a certain pde, such method is based on the construction of a suitable non-linear function of u and *grad* u, a P-function, which is itself solution (or sub-solution) to a related pde, and therefore satisfies a maximum principle. Although more restricted in its applicability than the method of moving hyper-planes, the method of P-functions provides a remarkable alternative approach to symmetry. In this respect we mention the papers [8] and [21] which contain closely related ideas.

In closing we mention that some interesting existence and non-existence results for positive entire solutions of the equation $\mathcal{L}u = -K(x)u^p$ in Carnot groups were announced by G. Lu and J. Wei in [40]. These authors also study the asymptotic behavior at infinity of the relevant solutions. We also mention that we have recently received a preprint by I. Birindelli and J. Prajapat in which the authors, using the method of moving hyper-planes, prove in the context of the Heisenberg group $\mathbb{H}^n$ an interesting non-existence theorem for positive entire solutions having partial symmetry of the equation $\mathcal{L}u = -u^p$, with sub-critical exponent $p < (Q+2)/(Q-2)$. After the present paper was accepted this result has appeared in [7].

2. **Preliminaries**

In this section we introduce the relevant definitions and recall some known results which will be used in sections three and four. Consider a Carnot group of step r, $\boldsymbol{G}$, with Lie algebra $\mathfrak{g} = \oplus_{j=1}^r V_j$. We assume that $\mathfrak{g}$ is equipped with a scalar product with respect to which the V_j's are mutually orthogonal. We use the exponential mapping $exp : \mathfrak{g} \to \boldsymbol{G}$ to define analytic maps $\xi_i : \boldsymbol{G} \to V_i, i = 1, ..., r$, through the equation $g = exp(\xi_1(g) + \xi_2(g) + ... + \xi_r(g))$. Here, $\xi(g) = \xi_1(g) + ... + \xi_r(g)$ is such that $g = exp(\xi(g))$. With $m = dim(V_1)$, the coordinates of the projection ξ_1 in the basis $X_1, \dots ,X_m$ will be denoted by $x_1 = x_1(g), \dots, x_m = x_m(g)$, i.e.,

$$x_j(g) \; = < \xi(g), X_j > \qquad j = 1, ..., m, \tag{2.1}$$

and we set $x = x(g) = (x_1(g), \dots, x_m(g)) \in \mathbb{R}^m$. We also need to exploit the properties of the exponential coordinates in the second layer of the stratification of $\mathfrak{g}$. We thus fix an orthonormal basis $Y_1, \dots, Y_k$ of V_2 and, similarly to (2.1), we define the exponential coordinates in the second layer V_2 of a point $g \in \boldsymbol{G}$ by letting

$$y_i(g) \; = < \xi(g), Y_i >, \qquad i = 1 \dots k, \tag{2.2}$$

and $y = (y_1, \dots, y_k) \in \mathbb{R}^k$. We next recall the Baker-Campbell-Hausdorff formula, see, e.g., [27]

$$exp\, \xi \exp \eta = exp\, (\xi + \eta + 1/2[\xi, \eta] + ...), \qquad \xi, \eta \in \mathfrak{g}, \tag{2.3}$$

where the dots indicate a linear combination of terms of order three and higher which is finite due to the nilpotency of $\boldsymbol{G}$. When the group is of step two such terms do not appear. By definition the order of an element in V_j is j. We will need the following two lemmas from [22].

Lemma 2.1. *Let $\boldsymbol{G}$ be a Carnot group, then*

$$\mathcal{L}x_j \ = \ 0, \quad j = 1, \dots, m.$$

The function $\psi(g) \overset{def}{=} |x(g)|^2$ enjoys the following properties:

$$\mathcal{L}\psi = 2m, \tag{2.4}$$

$$|X\psi|^2 = 4\psi. \tag{2.5}$$

Lemma 2.2. *Let $\boldsymbol{G}$ be a Carnot group, then*

$$\mathcal{L}y_i \ = \ 0, \quad i = 1, \dots, k.$$

Furthermore, one has

$$X_l(|y|^2) \ = \ \sum_{i=1}^{k} < [\xi_1, X_l], Y_i > y_i, \qquad l = 1, ..., m, \tag{2.6}$$

$$X_l^2(|y|^2) \ = \ \frac{1}{2} \sum_{i=1}^{k} \ < [\xi_1, X_l], Y_i >^2, \qquad l = 1, ..., m. \tag{2.7}$$

From the $\mathcal{L}$-harmonicity of y_i we infer, in particular, that the function $g \to |y(g)|^2$ is $\mathcal{L}$-subharmonic and we obtain from (2.7)

$$\mathcal{L}(|y|^2) \ = \ \frac{1}{2} \sum_{l=1}^{m} \sum_{i=1}^{k} \ < [\xi_1, X_l], Y_i >^2 \ \geq \ 0. \tag{2.8}$$

We next consider for a Carnot group of step two with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ the map $J : V_2 \to End(V_1)$ defined by

$$< J(\xi_2)\xi_1', \xi_1'' > = < \xi_2, [\xi_1', \xi_1''] >, \qquad \text{for} \quad \xi_2 \in V_2 \quad \text{and} \quad \xi_1', \xi_1'' \in V_1. \tag{2.9}$$

From the definition (2.9) it is clear that

$$< J(\xi_2)\xi_1, \xi_1 > = \ 0 \qquad \text{for every} \quad \xi_1 \in V_1, \xi_2 \in V_2. \tag{2.10}$$

It was A. Kaplan who first recognized, in his work on groups of Heisenberg type [32], the important connection between the algebraic properties of the map J and the analytical properties of the relevant sub-Laplacian. The next definition was introduced in [32].

Definition 2.3. *A Carnot group of step two, $\boldsymbol{G}$, is called of* Heisenberg type *if for every vector $\xi_2 \in V_2$, with $|\xi_2| = 1$, the map $J(\xi_2) : V_1 \to V_1$ defined by (2.9) is orthogonal. This implies*

$$|J(\xi_2)\xi_1| \ = \ |\xi_2| \ |\xi_1|, \qquad \xi_1 \in V_1, \xi_2 \in V_2. \tag{2.11}$$

It is easy to see that

$$< J(\xi_2)\xi_1, \ J(\xi_2')\xi_1 > = < \xi_2, \ \xi_2' > |\xi_1|^2, \ \text{for any } \xi_1 \in V_1 \text{ and } \xi_2, \xi_2' \in V_2. \tag{2.12}$$

The following properties of groups of Heisenberg type can be found in [32], [9]. For the reader's convenience we have collected them in one theorem.

Theorem 2.4. *Let $\boldsymbol{G}$ be a group of Heisenberg type, then*

$$\begin{aligned}\mathcal{L}(|y|^2)(g) &= \frac{k}{2}\ |x(g)|^2\\ |X(|y|^2)|^2(g) &= |x(g)|^2\ |y(g)|^2\\ < X(|x|^2)(g), X(|y|^2)(g) > &= 0.\end{aligned}$$

In the rest of the paper we will use the gauge

$$N(g) = \left(|x(g)|^4 + 16|y(g)|^2\right)^{1/4}. \tag{2.13}$$

This choice is justified by the following fact, discovered by Kaplan [32], which generalizes an analogous formula due to Folland [18] for the Heisenberg group $\mathbb{H}^n$. In a group of Heisenberg type the fundamental solution Γ of the sub-Laplacian $\mathcal{L}$ is given by the formula

$$\Gamma(g,h) \ = \ C(\boldsymbol{G})\ N(h^{-1}g)^{-(Q-2)}, \qquad g,h\in\boldsymbol{G}, g\neq h, \tag{2.14}$$

where $C(\boldsymbol{G}) > 0$ is a suitable constant. We recall that in a group of Heisenberg type the gauge (2.13) gives rise to an actual distance $\rho(g,h) = N(g^{-1}h)$, see [14]. For every $g \in \boldsymbol{G}$ and $r > 0$ we denote by $B(g,r) = \{h \in \boldsymbol{G} \mid \rho(g,h) < r\}$ the relative ball centered at g with radius r.

In [32] Korányi introduced an inversion on the Heisenberg group and used it to define an analogue of the Kelvin transform in such setting. Subsequently, the inversion, as well as the Kelvin transform, were generalized in [13] and [12] to all groups of Heisenberg type. The properties of the CR Kelvin transform are particularly far reaching in the context of Iwasawa groups. In [22] we proved that for such groups the Kelvin transform is an isometry between the spaces $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega)$ and $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega^*)$, where Ω^* denotes the image of Ω under the CR inversion, see Theorem 2.8 below. Henceforth, for a Carnot group $\boldsymbol{G}$ we let $\boldsymbol{G}^* = \boldsymbol{G} \setminus \{e\}$.

Definition 2.5. *Let $\boldsymbol{G}$ be a group of Heisenberg type with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$. For $g = exp(\xi) \in \boldsymbol{G}$, with $\xi = \xi_1 + \xi_2$, the* inversion *$\sigma : \boldsymbol{G}^* \to \boldsymbol{G}^*$ is defined by*

$$\sigma(g) \ = \ \Big(-\ \big(|x(g)|^2\ I\ +\ 4J(\xi_2)\big)^{-1}\xi_1, -\ \frac{\xi_2}{|x(g)|^4 + 16|y(g)|^2}\Big),$$

where the map J is as in (2.9), and I denotes the identity map on V_1. One easily verifies that

$$\sigma^2(g) \ = \ g, \qquad g\in\boldsymbol{G}^*.$$

Writing $\sigma(g) = \exp(\eta)$, with $\eta = \eta_1 + \eta_2$, for the image of g we obtain from Definition 2.5 and (2.11) that

$$|\eta_1| \ = \ \frac{|\xi_1|}{N(g)^2}, \qquad \text{and} \qquad |\eta_2| \ = \ \frac{|\xi_2|}{N(g)^4}. \tag{2.15}$$

An immediate consequence of (2.15) is that

$$N(\sigma(g)) \ = \ N(g)^{-1}, \qquad g\in\boldsymbol{G}^*. \tag{2.16}$$

Definition 2.6. *Let $\boldsymbol{G}$ be a group of Heisenberg type, and consider a function u on $\boldsymbol{G}$. The* CR Kelvin transform *of u is defined by the equation*

$$u^*(g) \;=\; N(g)^{-(Q-2)}\, u(\sigma(g)), \qquad\qquad g \in \boldsymbol{G}^*.$$

When $\boldsymbol{G}$ is a group of Iwasawa type, then it was proved in [12] that the inversion and the Kelvin transform possess some remarkable properties. The next theorem collects those which are most important in the sequel.

Theorem 2.7 (see [12])**.** *Let $\boldsymbol{G}$ be a group of Iwasawa type. The Jacobian of the inversion is given by*

$$d(H\circ\sigma)(g) \;=\; N(g)^{-2Q}\, dH(g), \qquad\qquad g \in \boldsymbol{G}^*.$$

The Kelvin transform u^* of a function u satisfies the equation

$$\mathcal{L}u^*(g) \;=\; N(g)^{-(Q+2)}(\mathcal{L}u)(\sigma(g)), \qquad\qquad g \in \boldsymbol{G}^*.$$

The next four results were established in [22].

Theorem 2.8. *In a group of Iwasawa type the Kelvin transform is an isometry of $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega)$ onto $\overset{o}{\mathcal{D}}{}^{1,2}(\Omega^*)$, where Ω^* denotes the image of Ω through the inversion σ.*

Another basic property of the CR Kelvin transform is expressed by the following simple, yet crucial, lemma.

Lemma 2.9. *Given a group of Iwasawa type, let u be a solution of*

$$\begin{cases} \mathcal{L}u \;=\; -\, u^p, & p \geq 1, \\ u \in \overset{o}{\mathcal{D}}{}^{1,2}(\Omega). \end{cases} \tag{2.17}$$

The Kelvin transform of u, u^*, satisfies the equation

$$\mathcal{L}u^*(g) \;=\; -\, N(g)^{p(Q-2)-(Q+2)}\, u^*(g)^p \qquad\qquad g \in \Omega^*. \tag{2.18}$$

In particular, when $p = \frac{Q+2}{Q-2}$ we conclude that if u satisfies problem (1.1), then u^* is a solution to the same problem in Ω^*.

We will also need the following theorem on the removability of the singularity. We say that a domain Ω^* is a neighborhood of infinity if its complement is contained in a gauge ball centered at the identity.

Theorem 2.10. *Let $\boldsymbol{G}$ be a group of Iwasawa type. Suppose that u^* is a solution of* (1.1) *in Ω^*, with Ω^* a neighborhood of infinity. Let u be the Kelvin transform of u^* defined in Ω, then the group identity e is a removable singularity, i.e., u can be extended as a smooth function in a neighborhood of e where the equation is satisfied.*

Finally, we have a regularity result for the problem (1.1).

Theorem 2.11. *Let $\boldsymbol{G}$ be a Carnot group and consider an open set $\Omega \subset \boldsymbol{G}$. Suppose that u be a weak solution to the problem (1.1), then $u \in L^\infty(\Omega)$.*

We emphasize that in the above statement the open set Ω need not be bounded, and that in fact one can take $\Omega = \boldsymbol{G}$.

We next recall a basic result of Bony [5], the strong maximum principle. For the sake of simplicity, we specialize its statement to the context of Carnot groups. The reader should keep in mind, however, that Theorem 2.12 holds more in general for Hörmander type operators.

Theorem 2.12. *Let Ω be a connected open set in a Carnot group $\boldsymbol{G}$. Assume that $c \leq 0$ in Ω and that $c \in C(\overline{\Omega})$. If $u \in C^2(\Omega)$ satisfies*

$$\mathcal{L}u \;+\; Yu \;+ c\, u \;\leq\; 0 \qquad\qquad in \quad \Omega,$$

then u cannot achieve a non-positive infimum at an interior point, unless $u \equiv const$ in Ω. Here, Y denotes a smooth section of $\boldsymbol{G}$.

The following result constitutes a generalization of the Hopf boundary point lemma, see section 3.2 in [25]. A version for the Heisenberg group first appeared in [6].

Theorem 2.13. *In a group of Heisenberg type $\boldsymbol{G}$ let $\Omega \subset \boldsymbol{G}$ be a connected open set possessing an interior gauge ball $B(g_1, R)$ tangent at $g_o \in \partial\Omega$ (by this we mean that $B(g_1, R) \subset \Omega$ and that moreover $g_o \in \partial\Omega \cap \partial B(g_1, R)$). Let $u \in C^2(\Omega)$ be a non-negative solution of*

$$\mathcal{L}u \;+\; c\, u \;\leq\; 0, \tag{2.19}$$

which is continuous at g_o, and such that

$$u(g_o) \;=\; 0, \tag{2.20}$$

$$u(g) \;>\; 0, \qquad\quad g \in B(g_1, R) \cap \Omega. \tag{2.21}$$

Assume in addition that $c \in L^\infty(\Omega)$. Let η be any exterior direction at g_o such that $\frac{\partial u}{\partial \eta}(g_o)$ exists, then one has

$$\frac{\partial u}{\partial \eta}(g_o) \;<\; 0. \tag{2.22}$$

Proof. As in [22] we consider $\psi(g) = |x(g)|^2$ and introduce the function $\zeta = e^{-\alpha\psi}u$. A computation based on Lemma 2.1 gives

$$\mathcal{L}u \;=\; \mathcal{L}(e^{\alpha\psi})\, \zeta \;+\; e^{\alpha\psi}\, \mathcal{L}\zeta \;+\; 2\, \alpha\, e^{\alpha\psi} \;\; < X\psi, X\zeta > .$$

Using (2.19) we obtain from the latter equation

$$\begin{aligned} &(4\alpha^2\psi \;+\; 2m\alpha \;+\; c)\, \zeta \;+\; \mathcal{L}\zeta \;+\; 2\alpha < X\psi, X\zeta > \\ &=\; e^{-\alpha\psi}\; (\mathcal{L}u \;+\; cu) \;\leq\; 0. \end{aligned}$$

This inequality and (2.21) imply in $B(g_o, R) \cap \Omega$

$$\mathcal{L}\zeta \;+\; 2\alpha \;\; < X\psi, X\zeta > \leq \; - \; [2m\alpha + c]\; \zeta.$$

At this point we choose $\alpha > 0$ such that

$$\alpha \;\geq\; \frac{||c||_{L^\infty(\Omega)}}{2m},$$

to conclude

$$\mathcal{L}\zeta \ + \ 2\alpha \ < X\psi, X\zeta > \ \leq \ 0 \qquad \text{in} \quad B(g_o, R) \cap \Omega. \tag{2.23}$$

We next use the hypothesis that Ω possesses an interior gauge ball $B(g_1, R)$ tangent at $g_o \in \partial\Omega$. By left-translation we assume without restriction that $g_1 = e$, where e is the identity in $\boldsymbol{G}$. Recalling (2.13), which we now rewrite $N(g) = (\psi(g)^2 + 16|y(g)|^2)^{1/4}$, we introduce the auxiliary function

$$h(g) \ = \ e^{-MN^2(g)} \ - \ e^{-MR^2}$$

on the ring $A = A(R, r) = B(e, R) \setminus \overline{B}(e, r)$, where $0 < r < R$ has been fixed. The constant $M > 0$ will be chosen shortly. An elementary computation, using the fact that $\mathcal{L}(N^{2-Q})(g) = 0$ for every $g \neq e$, see (2.14), gives

$$\mathcal{L}(N^2)(g) \ = \ \frac{Q}{2N^2(g)} \ |X(N^2)(g)|^2, \qquad \text{for} \quad g \neq e. \tag{2.24}$$

Formula (2.24) allows to find

$$\begin{aligned} &\mathcal{L}h \ + \ 2\alpha \ < X\psi, Xh > \\ &= \ M \ e^{-MN^2} \left[(M \ - \ \frac{Q}{2N^2}) \ |X(N^2)|^2 \ - \ 2\alpha \ < X\psi, X(N^2) > \right]. \end{aligned} \tag{2.25}$$

Using Theorem 2.4 we find

(2.26)

$$< X\psi, X(N^2) > = \ \frac{1}{2} \ N^{-2} \ [2\psi|X\psi|^2 + \ < X\psi, X(|y|^2) >] \ = \ 4N^{-2} \ \psi^2 \ = \ 4N^2 \ |XN|^4,$$

since in a group of Heisenberg type one has $|XN|^2 = N^{-2}\psi$, see [9]. The identity (2.26) allows to conclude that choosing $M > 0$ sufficiently large in (2.25) one obtains

$$\mathcal{L}h \ + \ 2\alpha \ < X\psi, Xh > \ \geq \ 0 \qquad \text{in} \quad A. \tag{2.27}$$

The continuity of u in Ω and the compactness of $\partial B(e, r)$ implies the existence of $\epsilon > 0$ such that the function $\zeta - \epsilon h \geq 0$ on $\partial B(e, r)$. This inequality continues to hold on $\partial B(e, R)$ since $h = 0$ on that set. By (2.23), (2.27) and Theorem 2.12 we conclude $\zeta - \epsilon h \geq 0$ in A. Since u, ζ and h vanish in g_o we conclude

$$\frac{\partial u}{\partial \eta}(g_o) \ = \ e^{\alpha\psi(g_o)} \ \frac{\partial \zeta}{\partial \eta}(g_o) \ \leq \ \epsilon \ e^{\alpha\psi(g_o)} \ \frac{\partial h}{\partial \eta}(g_o),$$

where η is any direction such that $< \eta, \mathbf{N} > (g_o) > 0$, with $\mathbf{N}$ being the exterior unit normal to $\partial B(e, R)$. At this point the conclusion follows by observing that the function $N(g)$ is homogeneous of degree one and therefore denoting by Z the infinitesimal generator of group dilations we have $ZN(g) = N(g)$ for every $g \neq e$. This identity implies in particular that the Riemannian gradient of $N(g)$, ∇N, never vanishes in $\boldsymbol{G} \setminus \{e\}$. Since $\partial B(e, R)$ is a level set of N and ∇N is directed outward, we infer

$$\frac{\partial h}{\partial \eta}(g_o) \ = \ -2MRe^{-MR^2} \ \frac{\partial N}{\partial \eta}(g_o) \ < \ 0.$$

This completes the proof of the theorem.

$\square$

Corollary 2.14. *Let $u \in C^2(\Omega)$ be a non-negative solution of* (2.19) *in $\Omega \subset \boldsymbol{G}$, where $\boldsymbol{G}$ is a group of Heisenberg type, then u cannot become equal to zero at an interior point without being identically zero in Ω.*

Proof. The proof follows the lines of its elliptic counterpart, see Theorem 3.5 in [25]. Assume by contradiction that u vanishes at a point inside Ω without being identically zero. Define $\Omega^+ \overset{def}{=} \{g \in \Omega \mid u(g) > 0\}$, which is non-empty according to the assumption, and satisfies $\Omega^+ \subset \Omega$, $\partial\Omega^+ \cap \Omega \neq \varnothing$. Let $g_o \in \Omega^+$ be closer to $\partial\Omega^+$ than to $\partial\Omega$, with respect to the gauge distance. Consider the largest gauge ball $B \subset \Omega^+$ having g_o as its center. Then $u(g) = 0$ for some point $g \in \partial B$, while $u > 0$ in B. By left-translation we can assume that $g = e$, the group identity. Since g is a point of an interior minimum on Ω, the Riemannian gradient at g must vanish. This is a contradiction with Theorem 2.13, by considering for example the derivative along the generator Z of the group dilations.

□

We end this section with a simple geometric result which is used in the application of the method of moving hyper-planes, see Lemma 2.2 in [26].

Proposition 2.15. *If a connected compact surface in $\mathbb{R}^k$ has the property that for every direction $\xi \in \mathbb{R}^k$ there exists a hyper-plane Π_ξ perpendicular to ξ, such that S is symmetric with respect to Π_ξ, then S is a Euclidean sphere.*

3. **Partial symmetry implies cylindrical symmetry**

In this section we use the method of moving hyper-planes to prove Theorem 1.4. We will use the letters α, α' to index coordinates in the first layer, and β, β' for indexing the coordinates in the center, so that we have $1 \leq \alpha,\ \alpha' \leq m$ and $1 \leq \beta,\ \beta' \leq k$. Unless explicitely said otherwise, we shall use the same letter for a function f defined on $\boldsymbol{G}$ and for the corresponding function $f \circ exp$ defined on $\mathfrak{g} \approx \mathbb{R}^m \times \mathbb{R}^k$. In the next proposition we express the sub-Laplacian in the exponential coordinates, see also [15].

Proposition 3.1. *For every $\beta = 1, ..., k$, let T_β denote the vector field*

$$T_\beta \;=\; \sum_{\alpha,\alpha'=1}^{m} x_{\alpha'} < [X_{\alpha'}, X_\alpha], Y_\beta > \frac{\partial}{\partial x_\alpha}.$$

Using the exponential coordinates we have the following formula for the sub-Laplacian of a function $u : \boldsymbol{G} \to \mathbb{R}$

$$\mathcal{L}u(g) \;=\; \triangle_x u(g) \;+\; \sum_{\beta=1}^{k} T_\beta \frac{\partial u}{\partial y_\beta}(g) \;+\; \frac{1}{4}|x(g)|^2\, \triangle_y u(g). \tag{3.1}$$

In (3.1) we have respectively denoted with $\triangle_x$ and $\triangle_y$ the standard Laplacian in $\mathbb{R}^m$ and $\mathbb{R}^k$.

Proof. To avoid confusion, in the course of the proof we will keep the distinct notation $v(x,y)$ for the function $u \circ exp$ on the Lie algebra. Here, we note explicitly that $g = exp(\xi(g)) = exp(\xi_1(g) + \xi_2(g)) = exp\left(\sum_{\alpha=1}^m x_\alpha(g)X_\alpha + \sum_{\beta=1}^k y_\beta(g)Y_\beta\right)$. By the Baker-Campbell-Hausdorff formula we have for every $\alpha = 1, ..., m$

$$
\begin{aligned}
(3.2) \qquad X_\alpha u(g) = & \ \frac{d}{dt}u(exp(\xi + tX_\alpha \ + \ \frac{t}{2}\ [\xi, X_\alpha]))\ |_{t=0} = \ \frac{d}{dt}v(\xi \ + \ tX_\alpha \ + \ \frac{t}{2}\ [\xi, X_\alpha])\ |_{t=0} \\
= & \ \sum_{\alpha'} < X_\alpha \ + \ \frac{1}{2}[\xi, X_\alpha], X_{\alpha'} > \ \frac{\partial v}{\partial x_{\alpha'}} \ + \ \sum_{\beta} < X_\alpha \ + \ \frac{1}{2}[\xi, X_\alpha], Y_\beta > \frac{\partial v}{\partial y_\beta} \\
= & \ \frac{\partial v}{\partial x_\alpha} \ + \ \frac{1}{2}\sum_{\beta} < [\xi, X_\alpha], Y_\beta > \ \frac{\partial v}{\partial y_\beta},
\end{aligned}
$$

where we have used the orthonormality of the involved vectors and the stratification of $\mathfrak{g}$. We also note that

$$< [\xi, X_\alpha], Y_\beta > = < [\xi_1, X_\alpha], Y_\beta > .$$

We thus obtain from (3.2)

$$X_\alpha u \ = \ \frac{\partial v}{\partial x_\alpha} \ + \ \frac{1}{2}\sum_{\beta} \ < [\xi_1, X_\alpha], Y_\beta > \ \frac{\partial v}{\partial y_\beta}. \tag{3.3}$$

Notice that $[X_\alpha, X_\alpha] = 0$ gives

$$\frac{\partial}{\partial x_\alpha} < [\xi_1, X_\alpha], Y_\beta > = \ \sum_{\alpha'} \frac{\partial}{\partial x_\alpha} x_{\alpha'} \ \ < [X_{\alpha'}, X_\alpha], Y_\beta > = \ 0. \tag{3.4}$$

Obviously, we have also

$$\frac{\partial}{\partial y_{\beta'}} < [\xi_1, X_\alpha], Y_\beta > = \ 0. \tag{3.5}$$

Applying (3.3) twice and using (3.4) and (3.5) we find

$$
\begin{aligned}
X_\alpha^2 u(g) = & \ \left(\frac{\partial}{\partial x_\alpha} \ + \ \frac{1}{2}\sum_{\beta'} < [\xi_1, X_\alpha], Y_{\beta'} > \frac{\partial}{\partial y_{\beta'}}\right) \left(\frac{\partial v}{\partial x_\alpha} \ + \ \frac{1}{2}\sum_{\beta} < [\xi_1, X_\alpha], Y_\beta > \frac{\partial v}{\partial y_\beta}\right) \\
= & \ \frac{\partial^2 v}{\partial {x_\alpha}^2} \ + \ \sum_{\beta} < [\xi_1, X_\alpha], Y_\beta > \frac{\partial^2 v}{\partial x_\alpha \partial y_\beta} + \frac{1}{4}\sum_{\beta,\beta'} < [\xi_1, X_\alpha], Y_\beta >< [\xi_1, X_\alpha], Y_{\beta'} > \frac{\partial^2 v}{\partial y_\beta \partial y_{\beta'}}.
\end{aligned}
$$

Summing in α we obtain

$$\mathcal{L}u = \ \triangle_x v \ + \ \sum_{\beta} T_\beta \frac{\partial v}{\partial y_\beta} \ + \ \frac{1}{4}\sum_{\alpha,\beta,\beta'} < [\xi_1, X_\alpha], Y_\beta >< [\xi_1, X_\alpha], Y_{\beta'} > \frac{\partial^2 v}{\partial y_\beta \partial y_{\beta'}}.$$

Now we use (2.12) and orthonormality to further reduce the last term in the right hand-side of the latter expression

$$\sum_{\alpha,\beta,\beta'} < [\xi_1, X_\alpha], Y_\beta > < [\xi_1, X_\alpha], Y_{\beta'} > \ \frac{\partial^2 v}{\partial y_\beta \partial y_{\beta'}}$$
$$= \sum_{\beta\beta'} \Big(\sum_\alpha < J(Y_\beta)\xi_1, X_\alpha > < J(Y_{\beta'})\xi_1, X_\alpha > \Big) \ \frac{\partial^2 v}{\partial y_\beta \partial y_{\beta'}}$$
$$= \sum_{\beta,\beta'} < J(Y_\beta)\xi_1, J(Y_{\beta'})\xi_1 > \ \frac{\partial^2 v}{\partial y_\beta \partial y_{\beta'}} \ = \ \sum_{\beta,\beta'} < Y_\beta, Y_{\beta'} > |x|^2 \frac{\partial^2 v}{\partial y_\beta \partial y_{\beta'}} \ = \ |x|^2 \triangle_y v.$$

This completes the proof. □

The next two results are direct consequences of Proposition 3.1.

Proposition 3.2. *Suppose that U has the form $U(g) = u(|x(g)|, y(g))$. For every $\beta = 1, ..., k$ one has $T_\beta u \equiv 0$ and therefore* (3.1) *gives*

$$\mathcal{L}U(g) \ = \ \triangle_x u(g) \ + \ \frac{|x(g)|^2}{4} \triangle_y u(g). \tag{3.6}$$

In particular, the vector fields T_β, $\beta = 1, ..., k$, are tangential to domains with cylindrical symmetry.

Proof. One has

$$T_\beta u \ = \ \Big(\sum_{\alpha,\alpha'} x_{\alpha'} < [X_{\alpha'}, X_\alpha], Y_\beta > \ \frac{\partial}{\partial x_\alpha} \Big) \ u$$
$$= \sum_{\alpha,\alpha'} \frac{x_{\alpha'} x_\alpha}{r} \ < J(Y_\beta) X_{\alpha'}, X_\alpha > \ \frac{\partial u}{\partial r}$$
$$= \ \frac{1}{r} \ < J(Y_\beta)\xi_1, \xi_1 > \ \frac{\partial u}{\partial r} \ = \ 0,$$

where the last equality is justified by (2.10). The proof is completed. □

Proposition 3.3. *Suppose that U has the form $U(g) = u(|x(g)|, y(g))$. One has the following formula for the horizontal gradient of U*

$$|XU(g)|^2 \ = \ |\nabla_x u(g)|^2 \ + \ \frac{|x(g)|^2}{4} \ |\nabla_y u(g)|^2. \tag{3.7}$$

Proof. Taking the squares and summing in α equation (3.3) we obtain

$$|Xu|^2 \ = \ |\nabla_x u|^2 \ + \ \frac{1}{4} \sum_\alpha \Big(\sum_\beta < [\xi_1, X_\alpha], Y_\beta > \ \frac{\partial u}{\partial y_\beta} \Big)^2 \ + \ \sum_\beta T_\beta u \ \frac{\partial u}{\partial y_\beta}.$$

Since u has partial symmetry $T_\beta u \ = \ 0$ and the last term in the above equality is zero. The second term is computed by using the orthogonality of the map J and the orthonormality of the vector fields X_α and Y_β,

$$\begin{aligned}\sum_\alpha\Big(\sum_\beta <[\xi_1, X_\alpha], Y_\beta> \frac{\partial u}{\partial y_\beta}\Big)^2 \;&=\; \sum_\alpha <[\xi_1, X_\alpha], \sum_\beta \frac{\partial u}{\partial y_\beta} Y_\beta >^2 \\ =\; \sum_\alpha < J\Big(\sum_\beta \frac{\partial u}{\partial y_\beta} Y_\beta\Big)\xi_1,\; X_\alpha >^2 \;&=\; |J\Big(\sum_\beta \frac{\partial u}{\partial y_\beta} Y_\beta\Big)\xi_1|^2 \\ =\; |x(g)|^2 |\sum_\beta \frac{\partial u}{\partial y_\beta} Y_\beta|^2 \;&=\; |x(g)|^2 |\nabla_y u|^2.\end{aligned}$$

The proof is complete.

□

Remark 3.4. *Proposition 3.2 underlines the important connection between the sub-Laplacian on a group of Heisenberg type and the Baouendi-Grushin operator*

$$\triangle_x \;+\; \frac{|x(g)|^2}{4}\,\triangle_y$$

acting on functions which possess partial symmetry.

After the above preliminaries our next goal is to prove Theorem 1.4. Before starting the proof however we introduce the relevant notation and develop some preparatory results. In the remainder of this section we will always identify a point $g = exp(\xi_1 + \xi_2) \in \boldsymbol{G}$ with its exponential coordinates $(x, y) = (x(g), (y(g)) \in \mathbb{R}^m \times \mathbb{R}^k$, where $x = x(g) = (x_1(g), ..., x_m(g))$ and $y = y(g) = (y_1(g), ..., y_k(g))$ are defined by (2.1) and (2.2). For any $\lambda \in \mathbb{R}$ we consider the *characteristic half-spaces* in $\boldsymbol{G}$ introduced in [22]

$$\Sigma_\lambda \;=\; \{g = (x, y) \in \boldsymbol{G} \mid\; y_1 < \lambda\}, \qquad \lambda < 0, \tag{3.8}$$

and

$$\Sigma_\lambda \;=\; \{g = (x, y) \in \boldsymbol{G} \mid\; y_1 > \lambda\}, \qquad \lambda > 0. \tag{3.9}$$

We denote by T_λ the *characteristic hyper-planes* $\partial\Sigma_\lambda = \{g = (x, y) \in \boldsymbol{G} \mid\; y_1 = \lambda\}$. For any $g \in \Sigma_\lambda$ we let g^λ be the *symmetric* point with respect to the hyperplane T_λ, i.e., $g^\lambda = (x(g),\ 2\lambda - y_1(g),\ y_2(g), \ldots,\ y_k(g))$. Finally, we let $g_\lambda = (0, 2\lambda,\ 0, \ldots,\ 0) \in \Sigma_\lambda$ be the reflexion of $(0,\ 0) \in \mathbb{R}^m \times \mathbb{R}^k$ with respect to T_λ.

Next, we assume that $u \not\equiv 0$ be an entire solution to the problem (1.1). From Theorem 2.11 we know that $u \in L^\infty(\boldsymbol{G})$ and applying the local regularity theory of Folland and Stein [20] as it was done in [22] we conclude $u \in C^\infty(\boldsymbol{G})$. From the strong maximum principle, Theorem 2.12, we have also $u > 0$ on $\boldsymbol{G}$. Consider in $\boldsymbol{G}^*$ the Kelvin transform of u, $v(g) = N(g)^{2-Q} u(\sigma(g))$, as in Definition 2.6. We notice explicitly that

$$\lim_{N(g)\to\infty} N(g)^{Q-2}\, v(g) \;=\; u(e) \;>\; 0. \tag{3.10}$$

Setting $v^\lambda(g) = v(g^\lambda)$, we define

$$\bar{w}_\lambda(g) \;=\; \frac{v^\lambda(g) \;-\; v(g)}{K_\epsilon(g)} \;\overset{def}{=}\; \frac{w_\lambda(g)}{K_\epsilon(g)}, \qquad g \in \bar{\Sigma}_\lambda, \tag{3.11}$$

where $K_\epsilon(g)$ is the function in Theorem 1.1. We observe that $\bar{w}_\lambda \equiv 0$ on T_λ. It is clear that v^λ is singular in $g = g_\lambda \in \Sigma_\lambda$ and that v is singular in $g = e$. However, thanks to Theorem 2.8, Lemma 2.9 and Theorem 2.10 we can remove the singularities so that v^λ and v become entire solutions to (1.1). This guarantees that $\bar{w}_\lambda$ is now globally defined on $\boldsymbol{G}$. At this point we note explicitly the following simple, yet important, fact. For every fixed λ one has

$$\lim_{N(g)\to\infty} \bar{w}_\lambda(g) \;=\; 0. \tag{3.12}$$

To prove (3.12) we first observe that

$$N(g^\lambda)^{2-Q} \;-\; N(g)^{2-Q} \;=\; N(g)^{2-Q}\, \Omega_\lambda(g), \tag{3.13}$$

where $|\Omega_\lambda(g)| \to 0$ as $N(g) \to \infty$. From (3.13) one easily infers that

$$\lim_{N(g)\to\infty} N(g)^{Q-2}\, v^\lambda(g) \;=\; u(e) \;>\; 0. \tag{3.14}$$

We now write

$$v^\lambda(g) - v(g) \;=\; N(g^\lambda)^{2-Q}\, [u(\sigma(g^\lambda)) - u(\sigma(g))] \;+\; [N(g^\lambda)^{2-Q} \;-\; N(g)^{2-Q}]\, u(\sigma(g)). \tag{3.15}$$

Using (3.13) in (3.15) we obtain (3.12).

To apply the method of moving hyper-planes we establish next a result analogous to Lemma 2.1 in [10].

Lemma 3.5. (i) *If* $\inf_{\Sigma_\lambda} \bar{w}_\lambda < 0$*, then the infimum is achieved.*

(ii) *There exists* $R_o > 0$ *independent of* λ *such that, if* $g_o \in \Sigma_\lambda$ *is a point at which a strictly negative* $\inf_{\Sigma_\lambda} \bar{w}_\lambda$ *is attained, then* $N(g_o) < R_o$*. Furthermore, for all* $|\lambda| \;\geq\; R_0^2$ *we have* $\bar{w}_\lambda \;\geq\; 0$ *on* Σ_λ*.*

Proof. The proof of (i) is easy. Suppose that for a certain λ one has

$$\inf_{\Sigma_\lambda} \bar{w}_\lambda \;=\; m_\lambda \;<\; 0.$$

Consider the set $A_\lambda = \{g \in \bar{\Sigma}_\lambda \mid \bar{w}_\lambda(g) \leq m_\lambda/2\}$. The equation (3.12) and $w_\lambda \equiv 0$ on T_λ imply that A_λ is a compact set. By the continuity of $\bar{w}_\lambda$ on $\boldsymbol{G}$ we conclude the validity of (i).

To prove (ii) we begin by observing that thanks to Proposition 3.2 we have

$$\mathcal{L} v^\lambda(g) \;=\; \mathcal{L} v(g^\lambda). \tag{3.16}$$

Using (3.16) and the mean value theorem we find in Σ_λ

$$\mathcal{L} w_\lambda(g) \;=\; \mathcal{L} v(g^\lambda) \;-\; \mathcal{L} v(g) \;=\; v^{2^*-1}(g) \;-\; v^{2^*-1}(g^\lambda) \;=\; -\, c_\lambda(g)\, w_\lambda(g), \tag{3.17}$$

where

$$c_\lambda(g) \;=\; (2^* - 1)\, \psi_\lambda^{2^*-2}(g), \tag{3.18}$$

with $\psi_\lambda(g)$ a real number between $v(g^\lambda)$ and $v(g)$. The equation satisfied by $\bar{w}_\lambda$ can be obtained from (3.17) and from $w_\lambda = K_\epsilon \bar{w}_\lambda$,

$$\mathcal{L}\bar{w}_\lambda \;+\; \frac{2}{K_\epsilon} < XK_\epsilon, X\bar{w}_\lambda > \;+\; \big(c_\lambda + \frac{\mathcal{L}K_\epsilon}{K_\epsilon}\big)\bar{w}_\lambda \;=\; 0. \tag{3.19}$$

From (3.10) we infer the existence of $C_o > 0$ such that

$$v(g) \ \le \frac{C_o}{(1+N(g))^{Q-2}}, \qquad g\in \boldsymbol{G}. \tag{3.20}$$

We now show that there exist $\epsilon = \epsilon(C_o) > 0$ and $R_o \ = \ R_o(C_o) > 0$ such that

$$c_\lambda(g) \ + \ \frac{\mathcal{L}K_\epsilon}{K_\epsilon}(g) \ < \ 0, \tag{3.21}$$

for $N(g) \ge R_o$ whenever $\bar{w}_\lambda(g) < 0$. For λ such that

$$\Omega_\lambda \ \overset{def}{=} \ \{g \in \Sigma_\lambda \mid \ \bar{w}_\lambda(g) \ < \ 0\} \ \ne \ \varnothing$$

consider (3.19) on Ω_λ. If $g \in \Omega_\lambda$, we have $v(g^\lambda) < v(g)$ and thus (3.20) gives

$$v(g^\lambda) \ < \ v(g) \ \le \ \frac{C_o}{N(g)^{Q-2}}. \tag{3.22}$$

Since ψ_λ is between $v(g)$ and $v(g^\lambda)$ we conclude from (3.18) and (3.22) that

$$c_\lambda(g) \ \le \ \frac{(2^*-1)C_o^{2^*-2}}{N(g)^{(Q-2)(2^*-2)}} \ = \ \frac{C_1}{N(g)^4}. \tag{3.23}$$

Thanks to Theorem 1.1 we have

$$\frac{\mathcal{L}K_\epsilon(g)}{K_\epsilon(g)} \ = \ - \ K_\epsilon(g)^{2^*-2} \ = \ - \ \frac{m(Q-2)\epsilon^2}{(\epsilon^2 \ + \ |x(g)|^2)^2 \ + \ 16|y(g)|^2}. \tag{3.24}$$

From (3.23) and (3.24) we see that

$$\begin{aligned} c_\lambda(g) \ + \ \frac{\mathcal{L}K_\epsilon(g)}{K_\epsilon(g)} \ &\le \ \frac{C_1}{N(g)^4} \ - \ \frac{m(Q-2)\epsilon^2}{(\epsilon^2 \ + \ |x(g)|^2)^2 \ + \ 16|y(g)|^2} \\ &= \ \frac{C_1 N(g)^4 \ + \ C_1(\epsilon^4 \ + \ 2\epsilon^2|x(g)|^2) \ - \ m(Q-2)\epsilon^2 N(g)^4}{N(g)^4\big((\epsilon^2 \ + \ |x(g)|^2)^2 \ + \ 16|y(g)|^2\big)} \\ &\le \ \frac{(C_1 - m(Q-2)\epsilon^2)N(g)^4 \ + \ 2C_1\epsilon^2 N(g)^2 \ + \ C_1\epsilon^4}{N(g)^4\big((\epsilon^2 \ + \ |x(g)|^2)^2 \ + \ 16|y(g)|^2\big)}. \end{aligned}$$

If we choose $\epsilon^2 = \frac{2C_1}{m(Q-2)}$ in the latter inequality, the coefficient of $N(g)^4$ is negative, and it is then clear that we can fulfill (3.21) for $N(g) \ge R_o$ for some $R_o = R_o(C_1) > 0$. From Theorem 2.12 we conclude that $\bar{w}_\lambda$ cannot achieve a negative infimum for $N(g) \ge R_o$ in Ω_λ. This proves the first part of (ii). At this point we observe that

$$N(g) \ \ge \sqrt{|\lambda|}, \qquad \text{for every} \quad g \in \Sigma_\lambda.$$

It is therefore clear from the above argument that if we take $|\lambda| \ge R_o^2$, then $\bar{w}_\lambda$ cannot achieve a negative infimum in Σ_λ. This completes the proof of Lemma 3.5. □

We are now ready to present the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $u(x,y) \ \overset{def}{=} \ \tau_{g_0}U(x(g),y(g)) \ = \ u(|x(g)|, y(g))$ and v be the Kelvin transform of u. Since $\mathcal{L}$ is a translation invariant operator, u is an entire solution to (1.1). For ease of notation we are using the same letter to denote functions on $\boldsymbol{G}$, which have partial symmetry with respect to the identity element of $\boldsymbol{G}$, and the corresponding symmetric part defined on $[0,\infty) \times \mathbb{R}^k$, see Definition 1.2. As already mentioned in the paragraph after

(3.11), v is also an entire solution on $\boldsymbol{G}$. Furthemore, from (2.15) it is easy to see that the Kelvin transform of a function that has partial symmetry with respect to the identity element of $\boldsymbol{G}$ is a function with partial symmetry with respect to the identity element as well. The first step of the proof is to show that v has cylindrical symmetry.

Let $\lambda_o = \sup\{\lambda \le 0 \mid \bar{w}_\lambda \ \ge \ 0 \text{ in } \Sigma_\lambda\}$. Clearly $\lambda_o \le 0$. Assume first that $\lambda_o < 0$. We want to show that $\bar{w}_{\lambda_o} \equiv 0$. Suppose the contrary holds. Since

$$\mathcal{L} w_{\lambda_o} \ + \ c_{\lambda_o}\, w_{\lambda_o} \ = \ 0 \qquad \text{in} \quad \Sigma_{\lambda_o}, \tag{3.25}$$

with c_{λ_o} bounded and $w_{\lambda_o} \ \ge \ 0$, Theorem 2.13 implies that either $w_{\lambda_o} \ > \ 0$, or $w_{\lambda_o} \ \equiv \ 0$. Since we are assuming $w_{\lambda_o} \ \not\equiv \ 0$, we conclude $w_{\lambda_o} \ > \ 0$. This implies $\bar{w}_{\lambda_o} \ > \ 0$ in Σ_{λ_o}. The maximality of λ_o allows to find a sequence $\lambda_k \searrow \lambda_o$ and points $g_k \in \Sigma_{\lambda_k}$ such that

$$\bar{w}_{\lambda_k}(g_k) \ < \ 0. \tag{3.26}$$

Without restriction we can suppose $\bar{w}_{\lambda_k}(g_k) \ = \ \inf_{\Sigma_{\lambda_k}} \bar{w}_{\lambda_k}$, since by (i) of Lemma 3.5 the infimum is achieved when it is strictly negative. We thus have

$$\nabla \bar{w}_{\lambda_k}(g_k) \ = \ 0. \tag{3.27}$$

In the proof of Lemma 3.5 we saw that the sequence $\{g_k\}$ is uniformly bounded, in fact $N(g_k) \ \le \ R_o$. Possibly passing to a subsequence, we can thus assume that $g_k \to g_o \in \bar{\Sigma}_{\lambda_o}$. By continuity from (3.26) and (3.27) we have $\bar{w}_{\lambda_o}(g_o) \ \le \ 0$ and $\nabla \bar{w}_{\lambda_o}(g_o) \ = \ 0$. Since $\bar{w}_{\lambda_o} \ > \ 0$ in Σ_{λ_o}, it must be $g_o \in T_{\lambda_o}$. Finally, $\bar{w}_{\lambda_o} \ > \ 0$ and $\nabla \bar{w}_{\lambda_o}(g_o) \ = \ 0$ contradict Theorem 2.13 by considering the derivative along any direction non-tangential to the boundary. This shows that when $\lambda_o < 0$ we have $w_{\lambda_o} \equiv 0$, i.e., v is symmetric with respect to the hyperplane T_{λ_o}.

If $\lambda_o \ = \ 0$ we can repeat the above reasoning starting from $\lambda = +\infty$ and then either stop at some $\lambda_1 \ > \ 0$, or at $\lambda_1 \ = \ 0$. In the former case we can finish as above. In the latter we combine the conclusions of the two cases to see that $v(g) \ > \ v(g^\lambda)$ and $v(g) \ < \ v(g^\lambda)$, i.e., $v(g) \ = \ v(g^\lambda)$ for any g and g^λ symmetric with respect to the hyperplane $y_1 \ = \ 0$. In either case, we conclude that v is symmetric with respect to T_λ for some λ.

We note also that the restriction of v to *lines perpendicular* to T_λ is a monotonically decreasing function of the Euclidean distance to T_λ. In order to see this, suppose $\lambda_o \ < \ 0$ so that $T_\lambda \ = \ T_{\lambda_o}$. Consider an arbitrary line l perpendicular to T_{λ_o} and let $P_1, P_2 \in \Sigma_{\lambda_o} \cap l$, with P_2 between P_1 and the intersection of T_{λ_o} and l. By considering the plane $T_{\bar{\lambda}}$ with respect to which P_1 and P_2 are symmetric, using also the definition of λ_o, we see that $v(P_1) \ < \ v(P_2)$. Arguing similarly in the case of $\lambda_o \ \ge \ 0$ we see that v has the described monotonicity, when restricted to any line perpendicular to T_{λ_o}.

From Proposition 3.2, $\mathcal{L}$ is an operator invariant with respect to rotations in the center, when restricted to partially symmetric functions. Since v has partial symmetry, v is invariant under such rotations. The previous arguments show that for every direction in the center, $\mathbb{R}^k$, there exists a hyperplane $T \ = \ T_{\lambda_o} \cap \mathbb{R}^k$ perpendicular to it, such that for every $r \ > \ 0$, $v(r,\cdot)$ is symmetric with respect to T. We note explicitely that T is independent of r. In addition, v has the above monotonicity on lines perpendicular to T. Since v is a continuous function and $v(g) \to 0$ when $N(g) \to \infty$, every level set is compact. Therefore, using also the monotonicity of v, for every $r \ \ge \ 0$ and every regular value a, the level set $v(r,\cdot) \ = \ a$ is a connected closed hypersurface of $\mathbb{R}^k$, when it is non-empty. Furthemore, from the symmetry of v, every level set of the function $v(r,\dot{)}$ defined on $\mathbb{R}^k$, is symmetric with respect to the hyperplane T. In view of Proposition 2.15 we infer that every level set is a sphere. Spheres corresponding to different regular values a are concentric, for otherwise we can argue as follows. Let $O_1 \ \ne \ O_2$ be the centers of two such non-concentric spheres. Let us consider the plane of symmetry, which is perpendicular to the direction of O_1O_2. Using again the monotonicity of v, we have on one hand that it should pass through O_1, while on the other it should pass through O_2, which is impossible. Finally, for any $b > 0$ consider the level set $\Lambda_b \overset{def}{=} \{v > b\}$. Clearly $\Lambda_b = \underset{a>b}{\cup} \Lambda_a$ and from Sard's theorem there exists a sequence $\{a_k\}$ of regular values such that $a_k \searrow b$. Since the level sets corresponding to regular values are Euclidean balls in $\mathbb{R}^k$, their union is a ball as well. This shows that $v(r, \cdot)$ is a radial function of its argument, after choosing suitably the origin of $\mathbb{R}^k$. Since the planes of symmetry are independent of r, the above choice of the origin of $\mathbb{R}^k$ is independent of r as well. In other words v is a cylindrical function.

The final step is to reverse the roles of u and v, using the properties of the Kelvin transform. In the beginning of the proof we noted that v is an entire solution that has partial symmetry with respect to the identity element. Since the Kelvin transform is an involution, from the first step of the proof we see that u has cylindrical symmetry, i.e., there exists an $h_o \in \boldsymbol{G}$ (in fact h_o belongs to the center of $\boldsymbol{G}$) such that $\tau_{h_o} u = \phi(|x(g)|, |y(g)|)$. Therefore,

$$\tau_{h_o g_o} U = \tau_{h_o} \tau_{g_o} U = \phi(|x(g)|, |y(g)|)$$

and the proof of Theorem 1.4 is complete.

□

4. **Uniqueness of cylindrically symmetric solutions**

In this section we establish the uniqueness, modulo group translations and dilations, of the positive solutions with cylindrical symmetry to the equation

$$\mathcal{L}u = - u^{(Q+2)/(Q-2)}. \tag{4.1}$$

Our main objective is to prove Theorem 1.5. As we explained in the introduction, the proof of Theorem 1.5 is based on a beautiful idea of Jerison and Lee. The latter uses a variation on the theme of the so-called method of P-functions, introduced by H. Weinberger in [45], see also the book [42]. We divide the proof into several steps. The main trust of the proof will be the establishment of Theorem 4.1, which generalizes Theorem 7.8 in [29]. We begin with some preliminary reductions. The first observation is that if we let $v = \lambda u$, then by choosing

$$\lambda = \left(\frac{Q-2}{4}\right)^{-(Q-2)/2}, \tag{4.2}$$

we are reduced to consider the equation

$$\mathcal{L}v = - (\frac{Q-2}{4})^2 \, v^{(Q+2)/(Q-2)}. \tag{4.3}$$

Next, we introduce the function $\Phi = v^{-4/(Q-2)} = h(v)$. Since

$$\mathcal{L}\Phi = h''(v)|Xv|^2 + h'(v)\mathcal{L}v,$$

we easily find that Φ must satisfy the equation

$$\mathcal{L}\Phi \ = \ (\frac{Q-2}{4}+1)\ \frac{|X\Phi|^2}{\Phi}\ +\ \frac{Q-2}{4}. \tag{4.4}$$

At this point we assume that u, and therefore Φ, have cylindrical symmetry with respect to the identity, i.e., there exists a function $\phi : [0,\infty) \times [0,\infty) \to \mathbb{R}^+$ such that we can write with $g = exp(\xi_1 + \xi_2) \in \boldsymbol{G}$

$$\Phi(g) \ = \ \phi(|\xi_1|, |\xi_2|). \tag{4.5}$$

By Proposition 3.2 we see that the equation (4.4) now becomes

$$\Delta_{\xi_1}\ \phi\ +\ \frac{|\xi_1|^2}{4}\ \Delta_{\xi_2}\ \phi \ = \ (\frac{Q-2}{4}+1)\ \frac{1}{\phi}\ (|\nabla_{\xi_1}\phi|^2 + \frac{|\xi_1|^2}{4}\ |\nabla_{\xi_2}\phi|^2)\ +\ \frac{Q-2}{4}. \tag{4.6}$$

Passing to the spherical coordinates $r = |\xi_1|, s = |\xi_2|$, we obtain from (4.6)

$$\begin{aligned} &\phi_{rr}\ +\ \frac{m-1}{r}\ \phi_r\ +\ \frac{r^2}{4}\ (\phi_{ss}\ +\ \frac{k-1}{s}\ \phi_s) \\ &=\ (\frac{Q-2}{4}+1)\ \frac{1}{\phi}\ (\phi_r^2\ +\ \frac{r^2}{4}\ \phi_s^2)\ +\ \frac{Q-2}{4}. \end{aligned} \tag{4.7}$$

We now let

$$y \ = \ \frac{r^2}{4}, \qquad x \ = \ s, \tag{4.8}$$

obtaining from (4.7)

$$\begin{aligned} &\phi_{xx}\ +\ \phi_{yy}\ +\ \frac{m}{2y}\ \phi_y\ +\ \frac{k-1}{s}\ \phi_s \\ &=\ (\frac{Q-2}{4}+1)\ \frac{\phi_x^2\ +\ \phi_y^2}{\phi}\ +\ \frac{Q-2}{4}\ \frac{1}{y}. \end{aligned} \tag{4.9}$$

Defining the integers

$$a \ = \ k-1\ \geq\ 0, \qquad b \ = \ \frac{m}{2}\ \geq\ 1, \qquad n \ = \ a+b\ \geq 1, \tag{4.10}$$

and recalling that $Q = m + 2k$, we finally re-write equation (4.9) as follows

$$\Delta\phi \ = \ \frac{n+2}{2}\ \frac{|\nabla\phi|^2}{\phi}\ -\ \frac{a}{x}\ \phi_x\ -\ \frac{b}{y}\ \phi_y\ +\ \frac{n}{2y}, \tag{4.11}$$

in $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$. We remark explicitly at this point that, without loss of generality, we can assume that $k \geq 2$, and therefore $a \geq 1$. In fact, the case $k = 1$ corresponds to the Heisenberg group $\mathbb{H}^n$, and it has already been treated by Jerison and Lee in [29].

We now introduce the quantities

$$F \ = \ f\ -\ f^*, \qquad G \ = \ g\ +\ g^*, \tag{4.12}$$

where

$$f \ = \ 2 < \nabla\phi, \nabla\phi_x >\ -\ 2\delta\ \phi_{xy}, \qquad f^* \ = \ \phi_x\ \frac{|\nabla\phi|^2}{\phi}, \tag{4.13}$$

$$g \;=\; -\,2 < \nabla\phi, \nabla\phi_y > \;+ 2\delta\ \phi_{yy}, \qquad g^* \;=\; (\phi_y - \delta)\ \frac{|\nabla\phi|^2}{\phi}, \tag{4.14}$$

and $\delta \in \mathbb{R}$ will be suitably chosen subsequently. We notice that

$$f_y \;+\; g_x \;=\; 0$$

and therefore there exists a function $P = P(x, y)$ such that

$$f \;=\; P_x, \qquad -\,g \;=\; P_y.$$

This gives in particular

$$\Delta P \;=\; f_x \;-\; g_y. \tag{4.15}$$

An easy calculation shows that

$$P \;=\; |\nabla\phi|^2 \;-\; 2\delta\ \phi_y. \tag{4.16}$$

We obtain from (4.16)

$$\Delta P \;=\; 2||\nabla^2\phi||^2 \;+\; 2 < \nabla\phi, \nabla(\Delta\phi) > \;-\, 2\delta\ (\Delta\phi)_y, \tag{4.17}$$

where we have denoted with $\nabla^2\phi$ the Hessian matrix of ϕ. We now use (4.11) to compute ΔP. First, we see that

$$\begin{aligned} 2 < \nabla\phi, \nabla(\Delta\phi) > =& \;-\,(n+2)\ \frac{|\nabla\phi|^4}{\phi^2} \;+\; 2(n+2) < \nabla^2(\nabla\phi), \nabla\phi > \\ &-\ \frac{2a}{x}\ < \nabla\phi, \nabla\phi_x > \;-\; \frac{2b}{y}\ < \nabla\phi, \nabla\phi_y > \;+\; \frac{2a}{x^2}\ \phi_x^2 \;+\; \frac{2b}{y^2}\ \phi_y^2 \;-\; \frac{n}{y^2}\ \phi_y. \end{aligned} \tag{4.18}$$

We also find

$$\begin{aligned} (\Delta\phi)_y \;=&\; -\ \frac{n+2}{2}\ \frac{\phi_y|\nabla\phi|^2}{\phi^2} \;+\; (n+2)\ \frac{< \nabla\phi, \nabla\phi_y >}{\phi} \\ &-\ \frac{a}{x}\ \phi_{xy} \;-\; \frac{b}{y}\ \phi_{yy} \;+\; \frac{b}{y^2}\ \phi_y \;-\; \frac{n}{2y^2}. \end{aligned} \tag{4.19}$$

At this point we introduce the function

$$h \;=\; \gamma\ \phi^{-(n+1)}, \tag{4.20}$$

where $\gamma = \gamma(x, y)$ is a strictly positive function on Ω which will be determined subsequently. With F and G as in (4.12) we consider the differential expression

$$\begin{aligned} (hF)_x \;-\; (hG)_y \;&=\; h\ (f_x - g_y) \;-\; h\ (f_x^* + g_y^*) \;+\; h_x F \;-\; h_y G \\ &=\; h\ [\Delta P \;-\; (f_x^* + g_y^*) \;+\; \frac{n+1}{\phi}(\phi_y G - \phi_x F)] \\ &+\; \phi^{-(n+1)}(\gamma_x F \;-\; \gamma_y G). \end{aligned} \tag{4.21}$$

A computation gives

$$f_x^* \ + \ g_y^* \ = \ \Delta\phi \ \frac{|\nabla\phi|^2}{\phi} \ - \ \frac{|\nabla\phi|^4}{\phi^2} \ + \ 2\frac{< \nabla^2\phi(\nabla\phi), \nabla\phi >}{\phi} \\ + \ \delta \ \frac{\phi_y|\nabla\phi|^2}{\phi^2} \ - \ 2\delta\frac{< \nabla\phi, \nabla\phi_y >}{\phi}, \tag{4.22}$$

$$\phi_y G \ - \ \phi_x F \ = \ \frac{|\nabla\phi|^4}{\phi} \ - \ 2 < \nabla^2\phi(\nabla\phi), \nabla\phi > \\ + \ 2\delta \ < \nabla\phi, \nabla\phi_y > \ - \delta \ \frac{\phi_y|\nabla\phi|^2}{\phi}. \tag{4.23}$$

Using (4.22) and (4.23) we obtain from (4.21)

$$(hF)_x \ - \ (hG)_y \ = \ h \ \Big[\Delta P \ - \ \Delta\phi\frac{|\nabla\phi|^2}{\phi} \ + \ (n+2) \ \frac{|\nabla\phi|^4}{\phi^2} \\ - \ 2(n+2) \ \frac{< \nabla^2\phi(\nabla\phi), \nabla\phi >}{\phi} \ + \ 2\delta(n+2) \ \frac{< \nabla\phi, \nabla\phi_y >}{\phi} \\ - \ \delta(n+2) \ \frac{\phi_y|\nabla\phi|^2}{\phi^2}\Big] + \ \phi^{-(n+1)}(\gamma_x F \ - \ \gamma_y G). \tag{4.24}$$

At this point we use (4.17), (4.18) and (4.19) in (4.24) to obtain

(4.25)

$$(hF)_x \ - \ (hG)_y \\ = \ h \ \Big\{ \Big(2 \ ||\nabla^2\phi||^2 \ - \ (\Delta\phi)^2\Big) \ + \ (\Delta\phi)^2 \ + \ - \ (n+2) \ \frac{|\nabla\phi|^4}{\phi^2} \ + \ 2(n+2) < \nabla^2(\nabla\phi), \nabla\phi > \\ - \ \frac{2a}{x} \ < \nabla\phi, \nabla\phi_x > \ - \ \frac{2b}{y} \ < \nabla\phi, \nabla\phi_y > \ + \ \frac{2a}{x^2} \ \phi_x^2 \ + \ \frac{2b}{y^2} \ \phi_y^2 \ - \ \frac{n}{y^2} \ \phi_y \\ - \ 2\delta \ \Big(- \ \frac{n+2}{2} \ \frac{\phi_y|\nabla\phi|^2}{\phi^2} \ + \ (n+2) \ \frac{< \nabla\phi, \nabla\phi_y >}{\phi} \\ - \ \frac{a}{x} \ \phi_{xy} \ - \ \frac{b}{y} \ \phi_{yy} \ + \ \frac{b}{y^2} \ \phi_y \ - \ \frac{n}{2y^2}\Big) \ - \ \Delta\phi\frac{|\nabla\phi|^2}{\phi} \ + \ (n+2) \ \frac{|\nabla\phi|^4}{\phi^2} \\ - \ 2(n+2) \ \frac{< \nabla^2\phi(\nabla\phi), \nabla\phi >}{\phi} \ + \ 2\delta(n+2) \ \frac{< \nabla\phi, \nabla\phi_y >}{\phi} \\ - \delta(n+2) \ \frac{\phi_y|\nabla\phi|^2}{\phi^2}\Big\} + \ \phi^{-(n+1)}(\gamma_x F \ - \ \gamma_y G).$$

The expression in (4.25) can be simplified as follows

$$
\begin{aligned}
(4.26)\qquad & (hF)_x \ - \ (hG)_y \\
& = \ h \left\{ \left[2 \ ||\nabla^2 \phi||^2 \ - \ (\Delta\phi)^2 \right] \ + \ \Delta\phi \left[\Delta\phi - \frac{|\nabla\phi|^2}{\phi} \right] \right. \\
& - \ \frac{2a}{x} \ < \nabla\phi, \nabla\phi_x > \ - \ \frac{2b}{y} \ < \nabla\phi, \nabla\phi_y > \ + \ \frac{2a}{x^2} \ \phi_x^2 \ + \ \frac{2b}{y^2} \ \phi_y^2 \ - \ \frac{n}{y^2} \ \phi_y \\
& \left. + \ \frac{2\delta a}{x} \ \phi_{xy} \ + \ \frac{2\delta b}{y} \ \phi_{yy} \ - \ \frac{2\delta b}{y^2} \ \phi_y \ + \ \frac{\delta n}{y^2} \right\} \\
& + \ \phi^{-(n+1)} (\gamma_x F \ - \ \gamma_y G).
\end{aligned}
$$

Next we evaluate the expression

$$
\begin{aligned}
(4.27)\qquad & \Delta\phi \left[\Delta\phi - \frac{|\nabla\phi|^2}{\phi} \right] \ - \ \frac{2a}{x} \ < \nabla\phi, \nabla\phi_x > \ - \ \frac{2b}{y} \ < \nabla\phi, \nabla\phi_y > \\
& + \ \frac{2a}{x^2} \ \phi_x^2 \ + \ \frac{2b}{y^2} \ \phi_y^2 \ - \ \frac{n}{y^2} \ \phi_y \ + \ \frac{2\delta a}{x} \ \phi_{xy} \ + \ \frac{2\delta b}{y} \ \phi_{yy} \ - \ \frac{2\delta b}{y^2} \ \phi_y \ + \ \frac{\delta n}{y^2} \\
& = \\
& \frac{n(n+2)}{4} \ \frac{|\nabla\phi|^4}{\phi^2} \ - \ \frac{a(n+1)}{x} \ \frac{\phi_x |\nabla\phi|^2}{\phi} \ - \ \frac{b(n+1)}{y} \ \frac{\phi_y |\nabla\phi|^2}{\phi} \\
& + \ \frac{n(n+1)}{2y} \ \frac{|\nabla\phi|^2}{\phi} \ + \ \frac{2ab}{xy} \ \phi_x \ \phi_y \ + \ \frac{a(a+2)}{x^2} \ \phi_x^2 \ + \ \frac{b(b+2)}{y^2} \ \phi_y^2 \\
& - \ \frac{an}{xy} \ \phi_x \ - \ \frac{bn+n+2\delta b}{y^2} \ \phi_y \ + \ \frac{n(n+4\delta)}{4y^2} \\
& - \ \frac{2a}{x} \ < \nabla\phi, \nabla\phi_x > \ - \ \frac{2b}{y} \ < \nabla\phi, \nabla\phi_y > \\
& + \ \frac{2\delta a}{x} \ \phi_{xy} \ + \ \frac{2\delta b}{y} \ \phi_{yy}.
\end{aligned}
$$

We now calculate

$$
\begin{aligned}
(4.28)\qquad \gamma_x F \ - \ \gamma_y G \ = \ & 2 \Big[\gamma_x < \nabla\phi, \nabla\phi_x > \ + \gamma_y < \nabla\phi, \nabla\phi_y > \Big] \\
& - \ 2\delta \ < \nabla\gamma, \nabla\phi_y > \ - \ < \nabla\gamma, \nabla\phi > \ \frac{|\nabla\phi|^2}{\phi} \ + \ \delta\gamma_y \ \frac{|\nabla\phi|^2}{\phi}.
\end{aligned}
$$

The next step is to compute

$$
\begin{aligned}
(4.29)\qquad & \frac{n+2}{n} \left(\Delta\phi \ - \ \frac{|\nabla\phi|^2}{\phi} \right)^2 \ = \ \frac{n(n+2)}{4} \ \frac{|\nabla\phi|^4}{\phi^2} \ + \ a^2 (1 + \frac{2}{n}) \ \frac{\phi_x^2}{x^2} \ + \ b^2 (1 + \frac{2}{n}) \ \frac{\phi_y^2}{y^2} \\
& + \ \frac{n(n+2)}{4} \ \frac{1}{y^2} \ - \ \frac{a(n+2)}{x} \ \frac{\phi_x |\nabla\phi|^2}{\phi} \ - \ \frac{b(n+2)}{y} \ \frac{\phi_y |\nabla\phi|^2}{\phi} \ + \ \frac{n(n+2)}{2y} \ \frac{|\nabla\phi|^2}{\phi} \\
& \frac{2ab(n+2)}{n} \ \frac{\phi_x \phi_y}{xy} \ - \ \frac{a(n+2)}{xy} \ \phi_x \ - \ \frac{b(n+2)}{y^2} \ \phi_y.
\end{aligned}
$$

Subtracting (4.29) from (4.27) we find

$$
\begin{aligned}
(4.30)\qquad & \Delta\phi\left[\Delta\phi-\frac{|\nabla\phi|^2}{\phi}\right]\ -\ \frac{2a}{x}\ <\nabla\phi,\nabla\phi_x>\ -\ \frac{2b}{y}\ <\nabla\phi,\nabla\phi_y> \\
& +\ \frac{2a}{x^2}\,\phi_x^2\ +\ \frac{2b}{y^2}\,\phi_y^2\ -\ \frac{n}{y^2}\,\phi_y\ +\ \frac{2\delta a}{x}\,\phi_{xy}\ +\ \frac{2\delta b}{y}\,\phi_{yy}\ -\ \frac{2\delta b}{y^2}\,\phi_y\ +\ \frac{\delta n}{y^2} \\
& -\ \frac{n+2}{n}\left(\Delta\phi\ -\ \frac{|\nabla\phi|^2}{\phi}\right)^2 \\
& = \\
& \frac{a}{x}\,\frac{\phi_x|\nabla\phi|^2}{\phi}\ +\ \frac{b}{y}\,\frac{\phi_y|\nabla\phi|^2}{\phi}\ -\ \frac{n}{2y}\,\frac{|\nabla\phi|^2}{\phi} \\
& -\ \frac{4ab}{n}\,\frac{\phi_x\phi_y}{xy}\ +\ \frac{2ab}{nx^2}\,\phi_x^2\ +\ \frac{2ab}{ny^2}\,\phi_y^2\ +\ \frac{2a}{xy}\,\phi_x \\
& +\ \frac{(2\delta-1)n}{2y^2}\ -\ \frac{2a}{x}\ <\nabla\phi,\nabla\phi_x>\ -\ \frac{2b}{y}\ <\nabla\phi,\nabla\phi_y> \\
& +\ \frac{2\delta a}{x}\,\phi_{xy}\ +\ \frac{2\delta b}{y}\,\phi_{yy}\ +\ \frac{b(1-2\delta)-a}{y^2}\,\phi_y.
\end{aligned}
$$

We now multiply equation (4.28) by γ^{-1} and add it to (4.30) obtaining

$$
\begin{aligned}
(4.31)\qquad E\ & \stackrel{\text{def}}{=}\Delta\phi\left[\Delta\phi-\frac{|\nabla\phi|^2}{\phi}\right]\ -\ \frac{2a}{x}\ <\nabla\phi,\nabla\phi_x>\ -\ \frac{2b}{y}\ <\nabla\phi,\nabla\phi_y> \\
& +\ \frac{2a}{x^2}\,\phi_x^2\ +\ \frac{2b}{y^2}\,\phi_y^2\ -\ \frac{n}{y^2}\,\phi_y\ +\ \frac{2\delta a}{x}\,\phi_{xy}\ +\ \frac{2\delta b}{y}\,\phi_{yy}\ -\ \frac{2\delta b}{y^2}\,\phi_y\ +\ \frac{\delta n}{y^2} \\
& -\ \frac{n+2}{n}\left(\Delta\phi\ -\ \frac{|\nabla\phi|^2}{\phi}\right)^2\ +\ \gamma^{-1}\,(\gamma_x F\ -\ \gamma_y G) \\
& = \\
& \frac{a}{x}\,\frac{\phi_x|\nabla\phi|^2}{\phi}\ +\ \frac{b}{y}\,\frac{\phi_y|\nabla\phi|^2}{\phi}\ -\ \frac{n}{2y}\,\frac{|\nabla\phi|^2}{\phi} \\
& -\ \frac{4ab}{n}\,\frac{\phi_x\phi_y}{xy}\ +\ \frac{2ab}{nx^2}\,\phi_x^2\ +\ \frac{2ab}{ny^2}\,\phi_y^2\ +\ \frac{2a}{xy}\,\phi_x \\
& +\ \frac{(2\delta-1)n}{2y^2}\ -\ \frac{2a}{x}\ <\nabla\phi,\nabla\phi_x>\ -\ \frac{2b}{y}\ <\nabla\phi,\nabla\phi_y> \\
& +\ \frac{2\delta a}{x}\,\phi_{xy}\ +\ \frac{2\delta b}{y}\,\phi_{yy}\ +\ \frac{b(1-2\delta)-a}{y^2}\,\phi_y \\
& +\ \gamma^{-1}\Big\{2\Big[\gamma_x<\nabla\phi,\nabla\phi_x>\ +\gamma_y<\nabla\phi,\nabla\phi_y>\Big] \\
& -\ 2\delta\ <\nabla\gamma,\nabla\phi_y>\ -\ <\nabla\gamma,\nabla\phi>\ \frac{|\nabla\phi|^2}{\phi}\ +\ \delta\gamma_y\,\frac{|\nabla\phi|^2}{\phi}\Big\}.
\end{aligned}
$$

At this point we make a suitable choice of the function γ. We let

$$
(4.32)\qquad \gamma(x,y)\ =\ x^a y^b.
$$

With this choice we find

$$
\begin{aligned}
(4.33) \qquad & \gamma^{-1}\Big\{2\Big[\gamma_x < \nabla\phi, \nabla\phi_x > \ + \gamma_y < \nabla\phi, \nabla\phi_y > \Big] \\
& - \ < \nabla\gamma, \nabla\phi_y > \ - \ < \nabla\gamma, \nabla\phi > \ \frac{|\nabla\phi|^2}{\phi} \ + \ \frac{\gamma_y}{2} \ \frac{|\nabla\phi|^2}{\phi}\Big\} \\
& = \\
& \frac{2a}{x} \ < \nabla\phi, \nabla\phi_x > \ + \ \frac{2b}{y} \ < \nabla\phi, \nabla\phi_y > \ - \ \frac{2\delta a}{x} \ \phi_{xy} \ - \ \frac{2\delta b}{y} \ \phi_{yy} \\
& - \ \frac{a}{x} \ \frac{\phi_x|\nabla\phi|^2}{\phi} \ - \ \frac{b}{y} \ \frac{\phi_y|\nabla\phi|^2}{\phi} \ + \ \frac{\delta b}{y} \ \frac{|\nabla\phi|^2}{\phi}.
\end{aligned}
$$

Substituting (4.33) in (4.31) gives

$$
\begin{aligned}
(4.34) \qquad E \ = \ & \frac{2\delta b - n}{2y} \ \frac{|\nabla\phi|^2}{\phi} \ + \ \frac{2ab}{nx^2} \ \phi_x^2 \ + \ \frac{2ab}{ny^2} \ \phi_y^2 \ + \ \frac{2a}{xy} \ \phi_x \\
& - \ \frac{4ab}{n} \ \frac{\phi_x\phi_y}{xy} \ + \ \frac{(2\delta - 1)n}{2y^2} \ + \ \frac{b(1 - 2\delta) - a}{y^2} \ \phi_y.
\end{aligned}
$$

We finally choose δ in (4.34) as follows

$$
(4.35) \qquad \delta \ = \ \frac{n}{2b}.
$$

With this choice we obtain from (4.34)

$$
\begin{aligned}
(4.36) \qquad E \ & = \frac{2ab}{nx^2} \ \phi_x^2 \ + \ \frac{2ab}{ny^2} \ \phi_y^2 \ - \ \frac{4ab}{n} \ \frac{\phi_x\phi_y}{xy} \\
& \quad + \ \frac{2a}{xy} \ \phi_x \ - \ \frac{2a}{y^2} \ \phi_y \ + \ \frac{an}{2by^2} \\
& = \ \frac{2ab}{n} \ \left(\frac{\phi_x}{x} - (\frac{\phi_y}{y} \ - \ \frac{n}{2by})\right)^2.
\end{aligned}
$$

Summarizing, we have proved the following identity.

Theorem 4.1. *Let ϕ be a positive solution to the equation (4.11) in $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$. With $h = x^a y^b \phi^{-(n+1)}$, and F and G as in (4.12), the following identity holds*

$$
\begin{aligned}
& (hF)_x \ - \ (hG)_y \\
& = \ h \ \Big\{\Big[2 \ ||\nabla^2\phi||^2 \ - \ (\Delta\phi)^2\Big] \\
& + \ \frac{n+2}{n} \ \left(\Delta\phi \ - \ \frac{|\nabla\phi|^2}{\phi}\right)^2 \\
& + \frac{2ab}{n} \ \left(\frac{\phi_x}{x} - (\frac{\phi_y}{y} \ - \ \frac{n}{2by})\right)^2\Big\}.
\end{aligned}
$$

Before proceeding we note explicitly that thanks to Schwarz inequality the term within square brackets in the right-hand side of the above identity is non-negative, thus the right-hand side is the sum of three non-negative terms. Our next step is to use the Kelvin transform to obtain the asymptotic behavior of the function ϕ.

Lemma 4.2. *Let $u \not\equiv 0$ be an entire solution to (1.1) in a group of Iwasawa type $\boldsymbol{G}$. One has $u > 0$ in $\boldsymbol{G}$ and $u \in C^\infty(\boldsymbol{G})$. Suppose in addition that u is cylindrically symmetric, let $\Phi = u^{-4/(Q-2)}$ and denote by ϕ the symmetric part of Φ as in (4.5). One has for some constant $C = C(\boldsymbol{G}) > 0$ and large enough $z = (x, y) \in \Omega$*

$$C^{-1}|z|^2 \;\le\; \phi(z) \;\le\; C|z|^2, \qquad |\nabla\phi(z)| \;\le\; C|z|, \qquad |\nabla^2\phi(z)| \;\le\; C. \tag{4.37}$$

Proof. The proof is a simple consequence of the properties of the Kelvin transform in a group of Iwasawa type. Let u^* be the Kelvin transform of u. An easy computation, using (2.13) and (4.8), gives $N(g) \;=\; 2|z|^{1/2}$. From (4.8) and (2.15) we find also

$$|\eta_1| \;=\; \frac{y^{1/2}}{2|z|}, \quad |\eta_2| \;=\; \frac{x}{16|z|^2}. \tag{4.38}$$

Notice that when $z \to \infty$ we have $|\eta_1|$, $|\eta_2| \to 0$. Since the Kelvin transform is an involution, an argument very similar to that in the end of the proof of Theorem 1.4 gives the asymptotic for u in (4.37). In fact, we see that both u and u^* are entire solutions to (1.1) and they have the decay (3.10). Using (4.8) again we obtain (4.37) for both u and u^*. The bounds for the derivatives follow from the homogeneity of the arguments of

$$\phi(x, y) \;=\; |z|^2\phi^*\Big(\frac{y^{1/2}}{2|z|}, \frac{x}{16|z|^2}\Big), \tag{4.39}$$

where $\phi^* \;=\; \big(u^*\big)^{-\frac{4}{Q-2}}$ and from differentiation. $\square$

We are now ready to prove the main result of this section.

Proof of Theorem 1.5. We recall that we are assuming $dimV_2 = k \ge 2$, so that $a \ge 1$, and therefore $h \equiv 0$ on $\partial\Omega$. We consider the functions Φ and ϕ as in Theorem 4.1 and Lemma 4.2. For every $R > 0$ set $\Omega_R = \Omega \cap B(0, R)$, $\Gamma_R = \Omega \cap \partial B(0, R)$. Integrating the left-hand side of the identity in Theorem 4.1 we find

$$\int_{\Omega_R} [(hF)_x \;-\; (hG)_y]\; dxdy \;=\; \frac{1}{R}\int_{\Gamma_R} h\;[xF - yG]\; ds. \tag{4.40}$$

We now use (4.12), (4.13), (4.14) and Lemma 4.2 to infer

$$\left|\frac{1}{R}\int_{\Gamma_R} h\;[xF - yG]\; ds\right| \;\le\; C\; R^{-n} \;\to 0 \qquad \text{as} \quad R \to \infty. \tag{4.41}$$

Combining (4.40) with (4.41) and with Theorem 4.1, we finally obtain

$$\int_\Omega h\left\{\Big[2\;||\nabla^2\phi||^2 \;-\; (\Delta\phi)^2\Big] \;+\; \frac{n+2}{n}\left(\Delta\phi \;-\; \frac{|\nabla\phi|^2}{\phi}\right)^2 \right.$$
$$\left. +\; \frac{2ab}{n}\left(\frac{\phi_x}{x} \;-\; \big(\frac{\phi_y}{y} \;-\; \frac{n}{2by}\big)\right)^2\right\} dxdy \;=\; 0.$$

The latter equation implies

$$2\;||\nabla^2\phi||^2 \;=\; (\Delta\phi)^2, \quad \Delta\phi \;-\; \frac{|\nabla\phi|^2}{\phi} \;=\; 0, \quad \frac{\phi_x}{x} \;=\; \frac{\phi_y}{y} \;-\; \frac{n}{2by}. \tag{4.42}$$

From the first two equations in (4.42) and from Lemma 4.2 we conclude in a classical fashion (see, e.g., [45] or also [29]) that ϕ must be of the type

$$\phi(x,y) \;=\; A^2\,(x^2\;+\;y^2)\;+\;2A\alpha x\;+\;2B\beta y\;+\;\alpha^2\;+\;\beta^2 \tag{4.43}$$

for some numbers A, B, α and β, with $A^2 = B^2$. On the other hand, the third equation in (4.42) implies that must be

$$\alpha \;=\; 0 \qquad\qquad \text{and} \qquad \beta \;=\; \frac{n}{4bB}.$$

Recalling that $x = |\xi_2|, y = |\xi_1|^2/4$ one easily concludes from the above that

$$\phi(|\xi_1|,|\xi_2|) \;=\; \frac{A^2}{16}\;\left[(\frac{a+b}{bA^2}+|\xi_1|^2)^2\;+\;16|\xi_2|^2\right] \tag{4.44}$$

for some $A \neq 0$. Using (4.10) we can rewrite (4.44) as follows

$$\phi(|\xi_1|,|\xi_2|) \;=\; \frac{Q-2}{16m\epsilon^2}\;[(\epsilon^2+|\xi_1|^2)^2\;+\;16|\xi_2|^2] \tag{4.45}$$

where $\epsilon^2 = \frac{Q-2}{mA}$. Finally, keeping in mind that $\phi = v^{-4/(Q-2)}$, and that $u = (1/\lambda)v$, with λ given by (4.2), we obtain

$$u(g) \;=\; C_\epsilon\;((\epsilon^2+|x(g)|^2)^2\;+\;16|y(g)|^2)^{-(Q-2)/4},$$

with $C_\epsilon = [m(Q-2)\epsilon^2]^{(Q-2)/4}$. All other cylindrically symmetric solutions are obtained from this one by left-translation. This completes the proof.

□

We are now ready to present the proof of Theorem 1.6.

Proof of Theorem 1.6. Let $\overset{o}{\mathcal{D}}{}^{1,2}_{ps}(\boldsymbol{G})$ denote the subspace of $\overset{o}{\mathcal{D}}{}^{1,2}(\boldsymbol{G})$ of the functions U such that

$$U(g) \;=\; u(|x(g)|, y(g)),$$

for some function $u : [0,\infty) \times \mathbb{R}^k \to \mathbb{R}$. We start with the observation that we can restrict our considerations to the non-negative functions in $\overset{o}{\mathcal{D}}{}^{1,2}_{ps}(\boldsymbol{G})$, i.e.,

$$\Lambda \;\overset{def}{=}\; S_2^{-2} \;=\; inf\;\left\{\int_{\boldsymbol{G}}|Xu|^2 dH(g)\;\mid\; u\in\overset{o}{\mathcal{D}}{}^{1,2}_{ps}(\boldsymbol{G}),\; u\geq 0,\; \int_{\boldsymbol{G}}|u|^{2^*}dH(g)\;=\;1\right\}. \tag{4.46}$$

This follows from the invariance of the integrals under left translation, and the fact that if $U \in \overset{o}{\mathcal{D}}{}^{1,2}(\boldsymbol{G})$, then also $|U| \in \overset{o}{\mathcal{D}}{}^{1,2}(\boldsymbol{G})$ and $|XU| = |X|U||$ for a.e. $g \in \boldsymbol{G}$. A suitable adaptation of the method of concentration of compactness of P. L. Lions shows that the inf in (4.46) is achieved, see [44]. Let $v \in \overset{o}{\mathcal{D}}{}^{1,2}_{ps}(\boldsymbol{G})$ be a function for which the inf is attained, thus

$$\Lambda \;=\; \int_{\boldsymbol{G}}|Xv|^2\;dH(g), \qquad\qquad \int_{\boldsymbol{G}} v^{2^*}\;dH(g)\;=\;1.$$

Writing the Euler-Lagrange equation of the constrained problem (4.46) we see that v is a positive entire solution of $\mathcal{L}v = -\Lambda\; v^{(Q+2)/(Q-2)}$. Let $u \;\overset{def}{=}\; \Lambda^{\frac{1}{2^*-2}}\,v$, then u is a positive entire solution of (1.1). Since $u \in \overset{o}{\mathcal{D}}{}^{1,2}_{ps}(\boldsymbol{G})$, Theorem 1.3 shows that u, modulo translations in the center, belongs to the one-parameter family of positive entire solutions, namely the functions K_ϵ in Theorem 1.1. From the definition of u, it is easy to see that

$$\Lambda = \Big(\int_{\boldsymbol{G}} |Xu|^2 dH(g)\Big)^{\frac{2}{Q}}.$$

Since u is a positive entire solution of (1.1) we have $\int_{\boldsymbol{G}} |Xu|^2\, dH(g) = \int_{\boldsymbol{G}} u^{2^*}\, dH(g)$, which shows that $\Lambda = \|u\|^{2^*-2}_{L^{2^*}(G)}$. Note that $K_\epsilon = \delta_{1/\epsilon}K$, where we have let $K = K_1$, and an easy computation gives $\|\delta_{1/\epsilon}K\|_{L^{2^*}(G)} = \|K\|_{L^{2^*}(G)}$. As already remarked, all considered integrals are invariant under the translations (1.3) as well. From the above considerations we infer

$$\Lambda = \left[(m(Q-2))^{Q/2} \int_{\boldsymbol{G}} \frac{1}{[(1+|x(g)|^2)^2 + 16|y(g)|^2)]^{Q/2}}\, dH(g)\right]^{2/Q}. \tag{4.47}$$

To obtain the best constant S_2 at this point we are left with the computation of the integral in the right-hand side of (4.47). We begin with the following observation which is justified by Proposition 1.2.9 and Theorem 1.2.10 in [11]. Suppose that $U(g) = u(|x(g)|, |y(g)|) \in L^1(\boldsymbol{G})$ is a function with cylindrical symmetry on $\boldsymbol{G}$. One has

$$\int_{\boldsymbol{G}} U(g) dH(g) = \int_{\mathbb{R}^m\times\mathbb{R}^k} u(|x|, |y|) dx dy. \tag{4.48}$$

Using (4.48) we find

$$\int_{\boldsymbol{G}} \frac{1}{[(1+|x(g)|^2)^2 + 16|y(g)|^2)]^{Q/2}}\, dH(g) = \tag{4.49}$$
$$= \int_{\mathbb{R}^m\times\mathbb{R}^k} \frac{dx dy}{[(1+|x|^2)^2 + 16|y|^2)]^{Q/2}} = 4^{-k} \int_{\mathbb{R}^m} \frac{dx}{(1+|x|^2)^{Q-k}} \int_{\mathbb{R}^k} \frac{dy}{(1+|y|^2)^{Q/2}}.$$

Consider now the integral

$$\int_{\mathbb{R}^n} \frac{dt}{(1+|t|^2)^a}, \qquad a > \frac{n}{2}.$$

One easily recognizes that

$$\int_{\mathbb{R}^n} \frac{dt}{(1+|t|^2)^a} = \frac{\sigma_n}{2}\, B\left(\frac{n}{2}, a - \frac{n}{2}\right), \tag{4.50}$$

where σ_n denotes the $(n-1)$-dimensional measure of the Euclidean unit sphere in $\mathbb{R}^n$, and $B(x, y)$ is the Beta function. Recalling the two formulas

$$\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

and

$$B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

where Γ indicates Euler's Gamma function, we conclude from (4.50)

$$\int_{\mathbb{R}^n} \frac{dt}{(1+|t|^2)^a} \;=\; \pi^{n/2}\,\frac{\Gamma(a-\frac{n}{2})}{\Gamma(a)}. \tag{4.51}$$

Using (4.51) in (4.49), and recalling that $Q = m + 2k$, we finally obtain

$$\int_{\boldsymbol{G}} \frac{1}{[(1+|x(g)|^2)^2 \;+\; 16|y(g)|^2)]^{Q/2}}\, dH(g) \;=\; 4^{-k}\,\pi^{(m+k)/2}\frac{\Gamma(\frac{m+k}{2})}{\Gamma(m+k)}. \tag{4.52}$$

Substitution of (4.52) into (4.47) gives

$$\Lambda \;=\; m(Q-2)\Big(4^{-k}\,\pi^{(m+k)/2}\frac{\Gamma(\frac{m+k}{2})}{\Gamma(m+k)}\Big)^{2/Q}.$$

Therefore,

$$S_2^2 \;=\; \frac{1}{m(m+2(k-1))}\;4^{2k/(m+2k)}\;\pi^{-(m+k)/(m+2k)}\;\left(\frac{\Gamma(m+k)}{\Gamma(\frac{m+k}{2})}\right)^{2/(m+2k)}.$$

This completes the proof of Theorem 1.6. $\square$

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