From Algebra to Astrophysics

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“The miracle of the appropriateness of the language of mathematics to the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve”.

Solving Algebraic Equations

Recall quadratic equations:

\[ ax^2 + bx + c = 0, \]

where \( a, b, c \) are given real numbers (coefficients). Completing the square we can easily derive

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a},
\]

where \( \Delta = b^2 - 4ac \), the discriminant. Then, the number of \textbf{real} solutions of the equation depends on \( \Delta \). Yet, the number of \textbf{complex} solutions counting multiplicities is always 2!
A Brief History

Quadratic Equations - 1800-1600 B.C., Babylonians

Cubic Equations - 16th century A. D. (S. del Ferro, N. Tartaglia, G. Cardano)

Quartic Equations - 16th century A.D., (L. Ferrari)

All the above solutions are expressed as explicit formulas.

19th century - N.-H. Abel, E. Galois proved that general equations of degree 5 and higher CANNOT be solved by explicit formulas.

Question: How many complex solutions does an equation of degree $n \geq 1$ have?
Fundamental Theorem of Algebra

Theorem 1. Every complex polynomial \( p(z) := a_nz^n + \ldots + a_0, \ a_n \neq 0 \) of degree \( n \) has precisely \( n \) complex roots (counted with multiplicities).

First proved in 1799 by C. F. Gauss (1777-1855).
In the 1990s T. Sheil-Small, A. Wilmshurst proposed to extend FTA to a larger class of polynomials, harmonic polynomials.

\[ h(z) := p(z) - \overline{q(z)}, n := \text{deg } p > m := \text{deg } q. \]

(For a complex number \( a+ib, a, b \in \mathbb{R}, a + ib = a - ib. \))

**Theorem 2.** (A. Wilmshurst, '92)

\[ \#\{z : h(z) = 0\} \leq n^2. \]

Moreover, there exist \( p, q : \text{deg } q = n - 1 \) such that the upper bound \( n^2 \) is attained.
Wilmshurst’s example for $n = 2$.

\[
h(z) := \text{Im}(e^{-\frac{i\pi}{4}z^n}) + i\text{Im}(e^{\frac{i\pi}{4}(z - 1)^n}).
\]

A little bit of algebra gives a more elegant example:

\[
h(z) := z^n + (z - 1)^n + i\bar{z}^n - i(\bar{z} - 1)^n.
\]

Note: $m = n - 1$. 

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**Question:** If $m << n$, what is the precise upper bound for the number of zeros of the polynomial $p(z) - q(z)$?

**Conjecture 1.** (A. Wilmshurst, '92)

$$\#\{z : p(z) - \overline{q(z)} = 0\} \leq m(m - 1) + 3n - 2.$$ 

For $m = n - 1$, the above example shows that Conjecture 1 holds and is sharp. For $m = 1$, it becomes

**Conjecture 2.** (T. Sheil-Small - A. Wilmshurst, '92)

$$\#\{z : p(z) - \bar{z} = 0, n > 1\} \leq 3n - 2.$$
History of Conjecture 2

• In the 1990s D. Sarason and B. Crofoot and, independently, D. Bshouty, A. Lyzzaik and W. Hengartner verified it for \( n = 2, 3 \).

• In 2001, using elementary complex dynamics and the argument principle for harmonic mappings, G. Swiatek and DK proved Conjecture 2 for all \( n > 1 \).

• In 2003-2005 L. Geyer showed, using dynamics, that \( 3n - 2 \) bound is sharp for all \( n \).
Theorem 3. (G. Swiatek -DK, ’01)

\[ \# \{ z : p(z) - \bar{z} = 0, n > 1 \} \leq 3n - 2. \]

The bound $3n - 2$ is sharp for all $n$ (L. Geyer, ’03 -’05).

Example. Consider

\[ h(z) = z - \frac{1}{2}(3z - z^3), \quad n = 3. \]

It has $3 \times 3 - 2 = 7$ zeros $0, \pm 1, \frac{1}{2}(\pm \sqrt{7} \pm i)$. 
Let \( r(z) := \frac{p(z)}{q(z)} \) be a rational function, \( p(z), q(z) \) are polynomials. \( \text{deg} r := \max\{\text{deg} p, \text{deg} q\} \). For example,

\[
r(z) = \sum_{j=1}^{n} \frac{a_j}{z - z_j}.
\]

**Theorem 4.** (G. Neumann -DK, ’05)

\[
\#\{z : r(z) - \bar{z} = 0, n := \text{deg} r > 1\} \leq 5n - 5.
\]

The bound \( 5n - 5 \) is sharp for all \( n \) (S. Rhie, ’03).

It turns out that this result opens a door to another world.
Geometric Optics in the Perfect World
Optics in Less Than Perfect World

[Diagram of Optics with labels: Coma, Spherical aberration, Chromatic aberration]
Multiple Images by a System of Mirrors
Gravitational Microlensing

- $n$ co-planar point-masses (e.g. condensed galaxies, black holes, etc.) in lens plane or deflector plane.

- Consider a light source in the plane parallel to the lens plane (source plane) and perpendicular to the line of sight from the observer.

- Due to deflection of light by masses multiple images of the source are formed. This phenomenon is known as gravitational microlensing.
Basics of gravitational lensing.
Gravitational lensing: the gravitational field of a massive object(s) acts as a lens for background sources

Exciting fact: the map from the distorted picture to the original is a planar harmonic map.
Lensing by Multiple Massive Objects
Lens Equation

Light source is located in the position $w$ in the source plane. The lensed image is located at the position $z$ in the lens plane while the masses are located at the positions $z_j$ in the lens plane.

\[ w = z - \sum_{1}^{n} \frac{\sigma_j}{(z - z_j)}, \]

where $\sigma_j \neq 0$ are real constants.

Letting $r(z) = \sum_{1}^{n} \frac{\sigma_j}{(z - z_j)} + w$, the lens equation becomes

\[ z - r(z) = 0, \quad \text{deg } r = n. \]

The number of solutions = the number of “lensed” images.
5 images of a quasar=quasi-stellar-radio object
History

• $n = 1$ (one mass) A. Einstein (1912 - 1933), either two images or the whole circle ("Einstein ring").

• H. Witt ('90) For $n > 1$ the maximum number of observed images is $\leq n^2 + 1$. S. Mao, A. Petters and H. Witt ('97) showed that the maximum is $\geq 3n + 1$.

• S.H. Rhie ('01) conjectured the upper bound for the number of lensed images for an $n$-lens is $5n - 5$.

Corollary 1. (G. Neumann-DK, '05). The number of lensed images by an $n$-mass lens cannot exceed $5n - 5$ and this bound is sharp (Rhie, '03). Moreover, it follows from the proof that the number of images is even when $n$ is odd and vice versa.
Rhie’s Construction

13 images for the non-perturbed lens and 20 images after adding a small mass at the origin.
Questions

1. How many zeros can a polynomial

\[ h := z^m - p(z), \text{deg} \ p = n > m \]

have?

Wilmshurst’s conjecture for \( m = 2 \) suggests the upper bound \( 3n \). Is it true?

2. **Lensing.** G. Neumann-DK’s theorem applies to \( n \) “spherically symmetric” mass distributions in the lens plane and gives at most \( 5n - 5 \)-lensed images outside the support of the mass distribution.

**Question.** How many lensed images can a uniform elliptic mass distribution produce?
Theorem 5. (C. Fassnacht - C. Keeton - DK, '07.)

An elliptic galaxy $\Omega$ with a uniform mass density may produce at most 4 “bright” lensing images of a point light source outside $\Omega$, and at most one “dim” image inside $\Omega$, i.e., at most 5 lensing images altogether.

Moreover, an elliptic galaxy $\Omega$ with mass density that is constant on ellipses confocal with $\Omega$, may produce at most 4 “bright” lensing images of a point light source outside $\Omega$. 
An “Astronomical” Proof
Einstein Rings are Ellipses

**Theorem 6.** (Fassnacht - Keeton - DK,'07.)
For any lens $\mu$, if the lensing produces an image “curve” surrounding the lens, it is either a circle in the case when the shear, i.e., a gravitational “pull” by a galaxy “far, far away”, $= 0$, or an ellipse.
Einstein Ring Gravitational Lenses

*Hubble Space Telescope* • Advanced Camera for Surveys

NASA, ESA, A. Bolton (Harvard-Smithsonian CfA), and the SLACS Team

STScI-PRC05-32
“Isothermal” Elliptical Lenses

- The density, important from the physical viewpoint, is a so-called “isothermal density” obtained by projecting onto the lens plane the “realistic” three-dimensional density \( \sim 1/\rho^2 \), where \( \rho \) is the (three-dimensional) distance from the origin. It could be included into the whole class of densities that are constant on all ellipses homothetic rather than confocal with the given one.

- Lens equation becomes transcendental.

\[
z - const \int_0^1 \frac{dt}{\sqrt{z^2 - c^2 t^2}} - \gamma \bar{z} = w.
\]
Final Remarks

- An isothermal sphere with a shear is covered by ’06 DK - G. Neumann theorem and may produce at most 4 images (observed).

- DK and E. Lundberg (’09) have proved that an isothermal elliptical lens without a shear may produce up to 8 bright images. Instantly, Bergweiler and Eremenko improved the estimate to 6 images, and showed that 6 is sharp. No more than 5 images (4 bright +1 dim) have been observed up to now.

- In 2000 Ch. Keeton, S. Mao and H. J. Witt constructed models with a tidal gravitational perturbation (shear) having 9, (8 bright + 1 dim), images.
Three-Dimensional Lensing

• The 3-dimensional lens equation with mass-distribution $dm(y)$ with source at $\vec{w}$ becomes

$$\vec{x} - \nabla_x \left( \int \frac{dm(y)}{|x-y|} \right) = \vec{w}.$$ 

• If the mass-distribution $dm(y)$ consists of $n$ point-masses, there are some estimates for the maximal number of images (A. Petters, ’90s) based on geometric topology (Morse theory). No sharp estimates are known.

• A difficult Maxwell’s problem concerns a number of stationary points of the Newtonian potential of $n$ point-masses (conjectured $\leq (n - 1)^2$). Most recent progress due to Eremenko, Gabrielov, D. Novikov, B. Shapiro. But this is the beginning of a new tale.
THANK YOU!