ECE 238L
Boolean Algebra - Part I

August 29, 2008

Boolean Algebra

Objectives

- Understand basic Boolean Algebra
- Relate Boolean Algebra to Logic Networks
- Prove Laws using Truth Tables
- Understand and Use First 11 Theorems
- Apply Boolean Algebra to:
  - Simplifying Expressions
  - Multiplying Out Expressions
  - Factoring Expressions

A New Kind of Algebra

	Regular Algebra	Boolean Algebra
Values	Numbers	Zero (0)
	Integers	One (1)
	Real Numbers	
	Complex Numbers	
Operators	$+ - \times / \sqrt{}$	AND, $\bullet, \bigcap, \bigwedge$
	Log, ln, etc.	OR, $+, \bigcup, \bigvee$
		Complement

Complement Operation

Also known as *invert* or *not*.

x	$\bar{x}$
0	1
1	0

This is a *truth-table*. It gives input-output mapping by simply enumerating all possible input combinations and the associated outputs

Logical AND Operation

- $\bullet$ denotes AND

Output is true iff all inputs are true

A	B	$Q = A \bullet B$
0	0	0
0	1	0
1	0	0
1	1	1

Logical OR Operation

$+$ denotes OR

Output is true if any inputs are true

A	B	$Q = A + B$
0	0	0
0	1	1
1	0	1
1	1	1

Truth Tables

A truth table provides a *complete enumeration* of the nputs and the corresponding output for a function.

A	B	F
0	0	1
0	1	1
1	0	0
1	1	1

If there n inputs, there will be 2^n rows in the table.

Unlike with regular algebra, full enumeration is possible (and useful) in Boolean Algebra.

Boolean Expressions

Boolean expressions are made up of variables and constants combined by AND, OR and NOT.

Examples: 1, A', $A \bullet B$, $C + D$, AB, $A(B + C)$, $AB + C$

$A \bullet B$ is the same as AB ($\bullet$ is omitted when obvious) Parentheses are used like in regular algebra for grouping.

A **literal** is each instance of a variable or constant.

This expression has 4 variables and 10 literals:

$$a'bd + bcd + ac' + a'd'$$

Boolean Expressions

Each Boolean expression can be specified by a truth table which lists all possible combinations of the values of all variables in the expression.

$$F = A' + BC$$

A	B	C	A'	BC	F=A'+BC
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	1	0	1
0	1	1	1	1	1
1	0	0	0	0	0
1	0	1	0	0	0
1	1	0	0	0	0
1	1	1	0	1	1

Boolean Expressions From Truth Tables

Each 1 in the output of a truth table specifies one term in the corresponding boolean expression.

The expression can be read off by inspection...

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

F is true when:
$\leftarrow$ A is false and B is true and C is false
OR
$\leftarrow$ A is true and B is true and C is true
F = A'BC' + ABC

Another Example

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

$$F = ?$$

Another Example

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

$$\begin{aligned} \text{F} = {} & \text{A'B'C} + \\ & \text{A'BC'} + \\ & \text{AB'C'} + \\ & \text{ABC} \end{aligned}$$

Yet Another Example

A	B	F
0	0	1
0	1	1
1	0	1
1	1	1

$$\begin{aligned} F &= A'B' + A'B + AB' + AB \\ &= 1 \end{aligned}$$

F is always true.

May be multiple expressions for any given truth table.

Converting Boolean Functions to Truth Tables

$$F = AB + BC$$

	A	B	C	F	
	0	0	0	0	
	0	0	1	0	
	0	1	0	0	
BC {	0	1	1	1	
	1	0	0	0	
	1	0	1	0	
	1	1	0	1	} AB
BC {	1	1	1	1	} AB

Some Basic Boolean Theorems

A	B	F= A • B
0	0	0
0	1	0
1	0	0
1	1	1

$\longleftarrow$

From these ...

A	B	F= A • B
0	0	0
0	1	1
1	0	1
1	1	1

$\longrightarrow$

We can derive these...

A	B	F= A • 0 = 0
0	0	0
0	0	0
1	0	0
1	0	0

A	B	F= A • 1 = A
0	1	0
0	1	0
1	1	1
1	1	1

A	B	F= A + 0 = A
0	0	0
0	0	0
1	0	1
1	0	1

A	B	F= A + 1 = 1
0	1	1
0	1	1
1	1	1
1	1	1

Proof Using Truth Tables

Truth Tables can be used to prove that two Boolean expressions are equal.

If the two expressions have the same values for all possible combinations of variables, they are equal.

$$XY' + Y = X + Y$$

X	Y	Y'	XY'	XY' + Y	X + Y
0	0	1	0	0	0
0	1	0	0	1	1
1	0	1	1	1	1
1	1	0	0	1	1

These two expessions are equal.

Basic Boolean Algebra Theorems

Here are the first five Boolean Algebra theorems we will study and use:

X + 0 = X	X • 1 = X
X + 1 = 1	X • 0 = 0
X + X = X	X • X = X
(X')' = X	
X + X' = 1	X • X' = 0

Basic Boolean Algebra Theorems

While these laws don't seem very exciting, they can be very useful in simplifying Boolean expressions:

Simplify:

$$\underbrace{\underbrace{\text{(MN' + M'N) P + P'}}_{\text{X}} + 1}_{1}$$

Boolean Algebra Theorems

Commutative Laws

$$X \bullet Y = Y \bullet X \qquad\qquad X + Y = Y + X$$

Associative Laws

$$(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z) = X \bullet Y \bullet Z$$

$$(X + Y) + Z = X + (Y + Z) = X + Y + Z$$

Just like regular algebra

Distributive Law

$$X(Y+Z) = XY + XZ$$

Prove with a truth table:

X	Y	Z	Y+Z	X(Y+Z)	XY	XZ	XY + XZ
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	0	0	1
1	1	1	1	1	1	1	1

Again, like algebra

Other Distributive Law

Proof: $X + YZ = (X + Y)(X + Z)$

$$
\begin{array}{rcl}
(X+Y)(X+Z) & = & X(X+Z) + Y(X+Z) \\
 & = & XX + XZ + YX + YZ \\
 & = & X + XZ + XY + YZ \\
Factor\ Out\ X \rightarrow & = & X \bullet 1 + XZ + XY + YZ \\
 & = & X(1 + Z + Y) + YZ \\
 & = & X \bullet 1 + YZ \\
 & = & X + YZ
\end{array}
$$

NOT like regular algebra!

Simplification Theorems

$$X\,Y + X\,Y' = X \qquad (X + Y)(X + Y') = X$$
$$X + X\,Y = X \qquad X(X + Y) = X$$
$$(X + Y')\,Y = X\,Y \qquad X\,Y' + Y = X + Y$$

These are useful for simplifying Boolean Expressions.

The trick is to find X and Y.

$$(A' + B + CD)(B' + A' + CD)$$
$$(A' + CD + B)(A' + CD + B')$$
$$A' + CD$$

Using the rule at the top right.

Multiplying Out

- All terms are products of single variables only
- (no parentheses)

A B C' + D E + F G H	Yes
A B + C D + E	Yes
A B + C (D + E)	No

Multiplied out = sum-of-products form (**SOP**)

Multiplying Out - Example

$(A' + B)(A' + C)(C + D)$

Use $(X + Y)(X + Z) = X + Y Z$

$(A' + B C)(C + D)$

Multiply Out

$A' C + A' D + B C C + B C D$

Use $X \bullet X = X$

$A' C + A' D + B C + B C D$

Use $X + X Y = X$

$A' C + A' D + BC$

Using the theorems may be simpler than brute force. But brute force does work...

Factoring

- Final form is products only
- All sum terms are single variables only

(A + B + C')(D + E)	Yes
(A + B)(C + D') E'F	Yes
(A + B +C')(D+E) + H	No
(A' + BC)(D + E)	No

This is called product-of-sums form or **POS**.

Factoring - Example

$$AB + CD$$

Use $X + YZ = (X + Y)(X + Z)$

$$(AB + C)(AB + D)$$

Use $X + YZ = (X + Y)(X + Z)$ again

$$(A + C)(B + C)(AB + D)$$

And again

$$(A + C)(B + C)(A + D)(B + D)$$

POS vs SOP

- Any expression can be written either way
- Can convert from one to another using theorems
- Sometimes SOP looks simpler
  - $AB + CD = (A + C)(B + C)(A + D)(B + D)$
- Sometimes POS looks simpler
  - $(A + B)(C + D) = BD + AD + BC + AC$

SOP will be most commonly used in this class but learn both.

Duality in Boolean Algebra

- If an equality is true
  - Its dual will be true as well
- To form a dual:
  - AND $\Longleftrightarrow$ OR
  - Invert constant 0's or 1s
  - Do NOT invert variables

Because these are true	These are also true
$X + 0 = X$	$X \bullet 1 = X$
$X + 1 = 1$	$X \bullet 0 = 0$
$X + X = X$	$X \bullet X = X$
$X + X' = 1$	$X \bullet X' = 0$

This will help you to remember the rules.