Exploring LU Decomposition

4.0 LU Decomposition

Last time, we discussed using Gaussian elimination to solve systems of equations. We divided the process into four steps:

- Row normalization
- Partial pivoting
- Forward sweep – putting the matrix into upper triangular form
- Backward sweep – back substitution of the values to solve the problem

The first two steps condition the matrix to improve the quality of the solution and the last two steps actually compute the solution. In looking at these last two steps, it is easy to see that the back substitution is by far the easier process. It is easier when you do the work by hand and it is faster when the equations are solved on the computer. We are going to look at a method for solving systems of equations that uses two back substitutions instead of the forward and back sweep method in ordinary Gaussian elimination.

In a previous lecture we discussed computing the element stresses in a truss by solving a system of linear equations. Problems of this type can be written as shown in equation 4.1. The matrix \([A]\) symbolizes the geometry of the truss and the vector \(\{C\}\) contains the external forces on the truss. We can examine many different external loading cases by changing \(\{C\}\). \([A]\) which is based upon the geometry would not change.

\[
[A] \times \{X\} = \{C\} \quad (4.1)
\]

Equation 4.1 can be rewritten as:

\[
[A] \times \{X\} - \{C\} = 0 \quad (4.2)
\]

We know from our work with Gaussian elimination that we can take the matrix \([A]\) and convert it to upper triangular form. We can illustrate this in the following 4x4 matrix.
In matrix notation

\[
[U]{X} - \{D\} = 0
\]  
(4.4)

Now assume there is a lower diagonal matrix \( L \)

\[
[L] = \begin{bmatrix}
l_{11} & 0 & 0 & 0 \\
l_{21} & l_{22} & 0 & 0 \\
l_{31} & l_{32} & l_{33} & 0 \\
l_{41} & l_{42} & l_{43} & l_{44}
\end{bmatrix}
\]  
(4.5)

That has the property:

\[
[L][U]{X} - [L]{D} = [A]{X} - \{C\}
\]  
(4.6)

Multiplying through

\[
[L][U]{X} - [L]{D} = [A]{X} - \{C\}
\]  
(4.7)

or

\[
[L][U]{X} = [A]{X}
\]  
(4.8)

removing \({X}\) from both sides

\[
[L][U] = [A]
\]  
(4.9)

and

\[
[L]{D} = \{C\}
\]  
(4.10)

### 4.1 Using it to Solve Problems

We can use this technique to solve problems of the form

\[
[A]{X} = \{C\}
\]  
(4.11)
by first computing the D vector from the equation

\[ [L][D] = [C] \]  \hspace{1cm} (4.10)

We can do this with back substitution because \([L]\) is a lower triangular matrix.

\[
\begin{bmatrix}
  l_{11} & 0 & 0 & 0 \\
  l_{21} & l_{22} & 0 & 0 \\
  l_{31} & l_{32} & l_{33} & 0 \\
  l_{41} & l_{42} & l_{43} & l_{44}
\end{bmatrix}
\begin{bmatrix}
  d_1 \\
  d_2 \\
  d_3 \\
  d_4
\end{bmatrix} =
\begin{bmatrix}
  c_1 \\
  c_2 \\
  c_3 \\
  c_4
\end{bmatrix}
\]  \hspace{1cm} (4.2)

Solving we have

\[
d_1 = c_1 / l_{11}
\]
\[
d_2 = (c_2 - l_{21} d_1) / l_{22}
\]
\[
d_3 = (c_3 - l_{31} d_1 - l_{32} d_2) / l_{33}
\]
\[
d_4 = (c_4 - l_{41} d_1 - l_{42} d_2 - l_{43} d_3) / l_{44}
\]  \hspace{1cm} (4.12)

We can express this generally with

\[
d_i = \frac{c_i - \sum_{j=i}^{i=1} l_{ij} d_j}{l_{ii}}, i = 2, 3, 4, \ldots, n
\]  \hspace{1cm} (4.13)

Once we have the Ds, we can compute the Xs with the equation

\[ [U][X] = [D] \]  \hspace{1cm} (4.4)

or

\[
\begin{bmatrix}
  u_{11} & u_{12} & u_{13} & u_{14} \\
  0 & u_{22} & u_{23} & u_{24} \\
  0 & 0 & u_{33} & u_{34} \\
  0 & 0 & 0 & u_{44}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix} =
\begin{bmatrix}
  d_1 \\
  d_2 \\
  d_3 \\
  d_4
\end{bmatrix}
\]  \hspace{1cm} (4.14)

or

\[
x_4 = d_4 / u_{44}
\]
\[
x_3 = (d_3 - u_{34} x_4) / u_{33}
\]
\[
x_2 = (d_2 - u_{23} x_3 - u_{24} x_4) / u_{22}
\]
\[
x_1 = (d_1 - u_{12} x_2 - u_{13} x_3 - u_{14} x_4) / u_{11}
\]  \hspace{1cm} (4.15)
We can generalize this as

\[ x_n = d_n / u_{nn} \]  
\[ (4.16) \]

\[ x_i = (d_i - \sum_{j=i+1}^{n} u_{ij} x_j) / u_{ii}, i = n-1, n-2, n-3,... \]  
\[ (4.17) \]

### 4.2 Computing L and U (Doolittle Technique)

Looking back at the Gaussian elimination we developed in the last lecture we started with:

\[
\begin{bmatrix}
 a_{11} & a_{12} & a_{13} & | & c_1 \\
 a_{21} & a_{22} & a_{23} & | & c_2 \\
 a_{31} & a_{32} & a_{33} & | & c_3 \\
\end{bmatrix}
\]  
\[ (4.18) \]

We multiplied the top row by \( a_{21} / a_{11} \) and subtracted it from the second row then multiplied the first row by \( a_{31} / a_{11} \) and subtracted it from the third row. This gave us:

\[
\begin{bmatrix}
 a_{11} & a_{12} & a_{13} & | & c_1 \\
 0 & b_{22} & b_{23} & | & d_2 \\
 0 & b_{32} & b_{33} & | & d_3 \\
\end{bmatrix}
\]  
\[ (4.19) \]

Next we multiplied the third row by \( b_{32} / b_{22} \) and subtracted it from the second row. This resulted in:

\[
\begin{bmatrix}
 a_{11} & a_{12} & a_{13} & | & c_1 \\
 0 & b_{22} & b_{23} & | & d_2 \\
 0 & 0 & e_{33} & | & f_3 \\
\end{bmatrix}
\]  
\[ (4.20) \]

The U matrix is:

\[
U = \begin{bmatrix}
 a_{11} & a_{12} & a_{13} \\
 0 & b_{22} & b_{23} \\
 0 & 0 & e_{33} \\
\end{bmatrix}
\]  
\[ (4.21) \]

In creating these we used the factors

\[ g_{21} = a_{21} / a_{11} \]  
\[ (4.22) \]

\[ g_{31} = a_{31} / a_{11} \]  
\[ (4.22) \]
\[ g_{32} = \frac{b_{32}}{b_{22}} \] to eliminate the second term in the third equation \hspace{1cm} (4.22)

We can use these terms to create \( L \). It is:

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
\frac{g_{21}}{g_{32}} & 1 & 0 \\
\frac{g_{31}}{g_{32}} & \frac{g_{32}}{g_{32}} & 1
\end{bmatrix}
\hspace{1cm} (4.23)

### 4.3 Example

We will solve the following system of equations using LU decomposition.

\[
\begin{align*}
-12x_1 + x_2 - 8x_3 &= -80 \\
x_1 - 6x_2 + 4x_3 &= 13 \\
-2x_1 - x_2 + 10x_3 &= 90
\end{align*}
\hspace{1cm} (4.24)
\]

First we solve for \( U \) using the basic Gaussian elimination approach. Multiply row 1 by \((1/-12)\) and subtract from the second row then multiply the first row by \((-2/-12)\) and subtract from the third row.

\[
U = \begin{bmatrix}
-12 & 1 & -8 & \text{\textbar} & -80 \\
0 & -5.917 & 3.333 & 6.333 \\
0 & -1.167 & 11.333 & 03.333
\end{bmatrix}
\hspace{1cm} (4.25)
\]

Multiply the second row in (4.25) by \((-1.167/-5.917)\) and subtract from the third row.

\[
U = \begin{bmatrix}
-12 & 1 & 8 & \text{\textbar} & -80 \\
0 & -5.917 & 3.333 & 6.333 \\
0 & 0 & 10.676 & 102.084
\end{bmatrix}
\hspace{1cm} (4.26)
\]

We can now construct the \( L \) matrix using the multipliers we used to create \( U \).

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
-0.0833 & 1 & 0 \\
0.1667 & 0.1973 & 1
\end{bmatrix}
\hspace{1cm} (4.26)
\]

We have inserted the terms \( g_{21} = 1/-12, \ g_{31} = -2/-12, \) and \( g_{32} = -1.167/-5.915 \).
Now we can check to see if these are correct before proceeding to the next step. We know from our development that:

\[ [L] \times [U] = [A] \]  \hspace{1cm} (4.9)

We will multiply L and U to perform this check.

\[
\begin{bmatrix}
1 & 0 & 0 \\
-0.0833 & 1 & 0 \\
0.1667 & 0.1972 & 1
\end{bmatrix}
\begin{bmatrix}
-12 & 1 & -8 \\
-5.917 & 3.333 \\
0 & 0 & 10.676
\end{bmatrix}
= \begin{bmatrix}
-12 & 1 & -8 \\
1 & -6 & 4 \\
-2 & -1 & 10
\end{bmatrix} \hspace{1cm} (4.27)
\]

It appears that both L and U are correct so we will continue to solve the system of equations. The step is to solve:

\[ [L] [D] = \{ C \} \]  \hspace{1cm} (4.10)

Substituting in the values we have:

\[
\begin{bmatrix}
1 & 0 & 0 \\
-0.0833 & 1 & 0 \\
0.1667 & 0.1972 & 1
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2 \\
d_3
\end{bmatrix}
= \begin{bmatrix}
-80 \\
13 \\
90
\end{bmatrix} \hspace{1cm} (4.28)
\]

\[ d_1 = -80 \]
\[ d_2 = 13 - (-80)(-0.0833) = 6.3336 \]
\[ d_3 = 90 - (-80)(0.1667) - (6.3336)(0.1972) = 102.08 \hspace{1cm} (4.29) \]

We know that \[ [U] [X] = \{ D \} \] so we use U to solve for X.

\[
\begin{bmatrix}
-12 & 1 & -8 \\
0 & -5.917 & 3.333 \\
0 & 0 & 10.676
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
-80 \\
6.3336 \\
102.08
\end{bmatrix} \hspace{1cm} (4.30)
\]

\[ X_3 = 102.08 / 10.676 = 9.562 \]
\[ X_2 = (6.3336 - (3.333)(9.562))/ -5.917 = 4.315 \hspace{1cm} (4.31) \]
\[ X_1 = (-80 - 4.315 + (8)(9.562))/ -12 = 0.652 \hspace{1cm} (4.31) \]

We can check these by substituting into the original equations.

\[ (-12)(0.652) + 4.315 - (8)(9.562) = -80 \]
\[ 0.652 - (6)(4.315) + (4)(9.562) = 13 \hspace{1cm} (4.32) \]
\[ (-2)(0.652) - 4.315 + (10)(9.562) = 90 \]
It checks so we have correctly computed the solution.