EXAMPLES

1. The double pulley consists of two parts that are attached to one another. It has a weight of 50 lb and a centroidal radius of gyration of 0.6 ft and is turning with an angular velocity of 20 rad/s clockwise. Determine the kinetic energy of the system. Assume that neither cable slips on the pulley.

**SOLUTION**

\[
T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} I_0 \omega_0^2
\]

\[v_1 = r_1 \omega_0 = 1 \times 20 = 20 \text{ ft/s}
\]

\[v_2 = r_2 \omega_0 = 0.5 \times 20 = 10 \text{ ft/s}
\]

\[
T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} I_0 \omega_0^2
\]

\[v_1 = \frac{1}{5} \times 0.62112 \times 20^2 = 124 \text{ lb-ft}
\]

\[v_2 = \frac{1}{2} \times 0.93168 \times 10 = 46.6 \text{ lb-ft}
\]

\[I_0 = \frac{1}{2} I_1 \omega_0^2 + \frac{1}{2} I_2 \omega_0^2
\]

\[T = 1.5528 \times 0.6^2 = 20 \text{ lb-ft}
\]

\[T = 283 \text{ lb-ft}
\]

2. The soap-box car has a weight of 110 lb, including the passenger but excluding its four wheels. Each wheel has a weight of 5 lb, radius of 0.5 ft, and a radius of gyration 0.3, computed about an axis passing through the wheel’s axle. Determine the car’s speed after it has traveled 100 ft starting from rest. The wheels roll without slipping. Neglect air resistance.

**Data:**

\[W_c = 110 \text{ lb}
\]

\[W_w = 5 \text{ lb}
\]

\[m_c = 3.4161 \text{ slug}
\]

\[m_w = 0.15528 \text{ slug}
\]

\[r_w = 0.5 \text{ ft}
\]

\[s = 100 \text{ ft}
\]

\[v_i = ?
\]

**SOLUTION**

\[T_0 = \sum_{W_h} T
\]

\[T_0 = 0
\]

\[\sum_{U_{1-2}} = W_t h = 6500 \text{ lb-ft}
\]

\[T_1 + \sum_{U_{1-2}} = T_2
\]

\[T_2 = 0.31056 v^2 + 0.1118 v^2 + 1.7081 v^2
\]

\[v = 55.2 \text{ ft/s}
\]

**NOTE:**

If we do not account for rotational energy then

\[v = 59.8 \text{ ft/s}
\]

(8% difference)

3. The 4 kg slender rod is subjected to the force and couple moment. When it is in the position shown it has an angular velocity \(\omega_0 = 6 \text{ rad/s}\). Determine its angular velocity at the instant it has rotated downwards 90°. The force is always applied perpendicular to the axis of the rod. Motion occurs in the vertical plane.

**Data:**

\[m = 4 \text{ kg}
\]

\[W = 39.24 \text{ N}
\]

\[\omega_0 = ?
\]

**SOLUTION**

\[T_1 = \sum_{U_{1-2}} = T_2
\]

\[\omega_n = \omega_0 / 2
\]

\[\omega_2 = 8.25 \text{ rad/s}
\]

CONSERVATION OF ENERGY

If all the forces acting on the system are conservative, then:

\[T_1 + V_1 = T_2 + V_2
\]

Typically \(V = V_g + V_c\)

where

\[V_g = \frac{1}{2} k \omega_n^2
\]

\[V_c = \frac{1}{2} \omega_n^2
\]

EXAMPLES

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1. The door is made from one piece whose ends move along the horizontal and vertical tracks. If the door is in the open position \( \theta = 0^\circ \), and then released, determine the speed at which its end \( A \) strikes the stop at \( C \). Assume the door is a 1 lb thin plate having a width of 10 ft.

**SOLUTION**

\[
I_0 = \frac{1}{12} m l^2 = \frac{1}{12} \times 5.59 \times 8^2 = 29.813 \text{ lb-ft}^2
\]

(From tables, note that the width does not matter)

\[y_g = -4 \text{ ft} \]

(totals drop of the mass center \( G \))

\[V_0 = 0, \quad V_t = 0\]

\[T_1 + V_1 = T_2 + V_2\]

\[T_1 = 0\]

\[V_1 = 0\]

\[\omega = 2.5803 \omega^2\]

\[h_G = 3.5 \text{ ft}\]

\[V_G = r_G \omega = 3.5 \omega\]

\[T_s = 0.4659 \omega^2 + 0.19668 \omega^2 + 1.9177 \omega^2\]

\[T_s = 2.5803 \omega^2\]

Potential energy at 2:

\[V_2 = V_{spring} + V_{rod} + V_{sphere}\]

\[V_{spring} = \frac{1}{2} k (r \theta)^2 = \frac{1}{2} \times 4 \times \left(1 + \frac{\pi}{2}\right)^2 = 3.4306 \text{ lb-ft}\]

Note that the forces \( F_s \) and \( F_r \) do not perform any work.

Also, \( |\mathbf{x}_y - \mathbf{x}_b| = 1 \text{ ft} \)

\[T_2 = \frac{1}{2} I_x \omega^2 + \frac{1}{2} m v_G^2 = \frac{1}{2} \times 29.81 \times \omega^2 + \frac{1}{2} \times 5.59 \times (1 \times \omega^2)\]

\[T_2 = 17.7 \omega^2\]

\[V_2 = -Wh = -180 \times 4 = -720 \text{ lb-ft}\]

\[T_2 + V_2 = 0 = 17.7 \omega^2 - 720 \Rightarrow \omega = 6.3779 \text{ rad/s}\]

\[v_A = v_B + \omega \times r_{d/B} = (6.3779 \text{ k}) \times (5 \text{ j})\]

\[v_y = -31.91\]

2. The disk \( A \) is pinned at \( Q \) and weighs 15 lb. A 1 ft rod weighting 2 lb and a 1 ft diameter sphere weighting 10 lb are welded to the disk, as shown. If ft spring is originally stretched 1 ft and the sphere is released from the position shown, determine the angular velocity of the disk when it has rotated 90°.

**SOLUTION**

\[I_0 = 0.9318 \text{ slug-ft}^2\]

\[I_{gs} = 0.00518 \text{ slug-ft}^2\]

\[I_{gs} = 0.03106 \text{ slug-ft}^2\]

\[m_B = 0.46584 \text{ slug}\]

\[m_s = 0.06211 \text{slug}\]

\[m_s = 0.31065 \text{slug}\]

\[\theta = 90°\]

\[\omega = 2\]

\[T_1 + V_1 = T_2 + V_2\]

\[T_1 = 0\]

\[V_1 = \frac{1}{2} k \Delta s^2 = \frac{1}{2} \times 4 \times 1^2 = 2 \text{ lb-ft}\]

\[T_2 = T_D + T_k + T_s\]

\[T_D = \frac{1}{2} I_{gs} \omega^2 = 0.4659 \omega^2\]

\[k = 2.5 \text{ ft}\]

\[v_{r/G} = r_G \omega = 2.5 \omega\]

\[T_k = \frac{1}{2} m_s v_{r/G}^2 = \frac{1}{2} \times 0.00518 \times \omega^2 + \frac{1}{2} \times 0.06211 \times (2.5 \omega)^2\]

\[T_s = 0.19668 \omega^2\]

\[V_{rod} = -W_h k_{GR} = -2 \times 2.5 = -5 \text{ lb-ft}\]

\[V_{sphere} = -W_s h_{GS} = -10 \times 3.5 = -35 \text{ lb-ft}\]

\[V_2 = 34.306 - 5 - 35 = -5.694 \text{ lb-ft}\]

Put all together:

\[2 = 2.5803 \omega^2 - 5.694\]

\[\omega = 1.73 \text{ rad/s}\]
**IMPULSE AND MOMENTUM**

Linear momentum: \( L = m \mathbf{v}_G \)

Angular Momentum: Consider a particle \( i \) of mass \( dm \) in a body rotating with angular velocity \( \omega \) and where the point \( P \) has velocity \( \mathbf{v}_P \).

The velocity of the particle \( i \) is \( \mathbf{v}_i = \mathbf{v}_P + \omega \times \mathbf{r}_{iP} \).

The "moment" of the particle about \( P \) is \( d(H_p) = \mathbf{r}_{iP} 	imes dm \mathbf{v}_i \).

In terms of the coordinate components:

\[
(H_p)_k = (x_i + y_j) \times \int dm [v_{iP} i + v_{iP} j + (\alpha k) \times (x_i + y_j)]
\]

Since all terms are in the \( k \) direction the scalar equation is

\[
d(H_p) = -dm y v_{Pz} + dm x v_{Py} + dm \omega v_i^2
\]

Integrating over the whole body:

\[
H_p = -\int dm y v_{Pz} + \int dm x v_{Py} + \int r_i^2 dm \omega
\]

So we get:

\[
H_p = -m \bar{y} v_{Pz} + m \bar{x} v_{Py} + I_p \omega
\]

And if \( P = G \)

\[
H_G = I_G \omega
\]

The angular momentum of a body about its center of gravity \( G \) is the product of its moment of inertia \( I \) and its angular velocity \( \omega \).

To find an expression for the angular momentum about an arbitrary point \( P \) in terms of \( G \), by the parallel axis theorem:

\[
I_p = I_G + m(x^2 + y^2)
\]

Substituting in the expression for \( H_p \) derived above:

\[
H_p = \mathbf{r} m[-v_{Pz} + \bar{x} \omega] + \mathbf{x} m[v_{Py} + \bar{x} \omega] + I_p \omega
\]

Expressing \( \mathbf{v}_i \) in terms of \( \mathbf{v}_{Pz} \):

\[
\mathbf{v}_G = \mathbf{v}_P + \omega \times \mathbf{r}_{iP}
\]

or in terms of components:

\[
\mathbf{v}_G = \mathbf{v}_P + \omega \times (\mathbf{x} i + \bar{x} j)
\]

This yields:

\[
\mathbf{v}_G = \mathbf{v}_P - \bar{y} \omega
\]

\[
\mathbf{v}_G = \mathbf{v}_P + \bar{x} \omega
\]

The equation for \( H_p \) can then be written as:

\[
H_p = -\mathbf{y} m v_{Gz} + \mathbf{x} m v_{Gy} + I_p \omega
\]

If the angular momentum is computed about a point \( P \), it is equivalent to the linear momentum \( m \mathbf{v}_G \) about \( P \) plus the angular momentum \( I_p \omega \).

Let us now examine linear and angular momentum for the three different types of motion under consideration:

**TRANSLATION:**

\[
L = m \mathbf{v}_G
\]

\[
H_G = 0
\]

**ROTATION ABOUT A FIXED AXIS:**

\[
L = m \mathbf{v}_G
\]

\[
H_G = I_G \omega \quad \text{or} \quad H_G = I_0 \omega
\]

**GENERAL PLANE MOTION:**

\[
L = m \mathbf{v}_G
\]

\[
H_G = I_G \omega \quad \text{or} \quad H_A = I_0 \omega + d \mathbf{v}_G
\]

Principle of Linear Impulse and Momentum

\[
m(\mathbf{v}_G)_1 + \int_0^t \mathbf{F} dt = m (\mathbf{v}_G)_2
\]

Principle of Angular Impulse and Momentum

\[
I_0 \omega_1 + \int_0^t M_G dt = I_G \omega_2
\]

For rotation about a fixed axis \( O \) we can also write:

\[
I_0 \omega_1 + \int_0^t M_G dt = I_0 \omega_2
\]

These equations can be applied to an entire system of connected bodies rather than to each body independently to eliminate the effect of reactions at the connections.
If two bodies with smooth surfaces collide when their mass centers and velocity are aligned with the line of impact, the result is a central impact and is treated as we did before.

If the line connecting the mass centers does not coincide with the line of impact, as in the case when one of the bodies is rotating about a fixed axis, the impact is eccentric.

In this case, in general two equations need to be solved to determine all the velocities. The first will generally involve the application of conservation of angular momentum and the second is obtained from the coefficient of restitution.

\[(H_0)_1 = (H_0)_2 \quad \text{and} \quad e = \frac{(v_x)_1 - (v_x)_2}{(v_y)_1 - (v_y)_2}\]

Where the velocities are along the line of impact.

**EXAMPLES**

1. A flywheel has a mass of 60 kg and a radius of gyration of \(k_a = 150\) mm about an axis of rotation passing through its mass center. If a motor supplies a clockwise torque having a magnitude of \(M = (5t)\) N·m, where \(t\) is in seconds, determine the wheel’s angular velocity in \(3\) s. Initially the flywheel is rotating clockwise at \(\omega = 2\) rad/s.

Data:

\[m = 60\, \text{kg} \quad I_0 = 60 \times 0.15^2 = 1.35\, \text{kg} \cdot \text{m}^2\]

\[r_a = 0.15\, \text{m} \quad M = -5t \]

\[\omega = -2\, \text{rad/s} \quad \omega(t = 3s) = ?\]

SOLUTION

\[I_0 \omega + \int_0^t (-5t) dt = I_0 \omega_2\]

\[\omega(t = 3s) = \frac{1.35 \times (-2) - \frac{5 \times 3^2}{2}}{1.35} \quad \omega_2 = 18.67\, \text{rad/s}\]

2. The spool has a mass of 30 kg and a radius of gyration \(k_a = 0.25\) m. Block \(A\) has a mass of 25 kg, and block \(B\) has a mass of 10 kg. If they are released from rest, determine the time required for block \(A\) to attain speed of 2 m/s. Neglect the mass of the ropes.

Data:

\[m_A = 25\, \text{kg} \quad m_B = 10\, \text{kg}\]

\[v_{x_A} = v_{x_B} = v_A = v_B = 2\, \text{m/s}\]

\[r_{a_A} = 0.25\, \text{m}\]

SOLUTION

First calculate \(I_O\)

\[I_0 = 30 \times 0.25^2 = 1.875\, \text{kg} \cdot \text{m}^2\]

\[(H_0)_1 = 0\]

\[
\sum \int_0^t M_\omega dt = 1 / 2 \left( m_A r_{a_A} v_{x_A} + m_B r_{a_B} v_{x_B} \right) + I_0 \omega
\]

\[= 55.917\, \text{t}\]

\[(H_0)_2 = r_A m_A (v_{x_A})_2 + r_B m_B (v_{x_B})_2 + I_0 \omega
\]

\[= 0.3 \times 2 + 0.18 \times 10 \times 1.2 + 1.875 \times 6.6667 = 29.66\, \text{kg} \cdot \text{m}^2 / \text{s}
\]

\[t = 0.53\, \text{s}\]