COMPLEX ANALYSIS PROBLEMS

D. VASSILEV


1.1. Homework

Problem 1.1.1. a) Let \( f_n \) be a sequence of functions that are holomorphic on the punctured unit disc \( \mathbb{D}^\times \) and suppose that each \( f_n \) has a pole at \( z = 0 \). If the sequence \( \{f_n\} \) converges uniformly on compact subsets of \( \mathbb{D}^\times \), then does the limit function \( f \) necessarily have a pole at \( z = 0 \)?

b) Answer the same question with "pole" replaced by "removable singularity" or "essential singularity."

Problem 1.1.2. Compute the following residues \( \text{Res}_f(z_0) \) for the given function and point.

\[ z_0 = 2i, \quad f(z) = \frac{z^2}{(z-2i)(z+3)}. \]

Problem 1.1.3. a) Find the Laurent series of the function \( f(z) = \frac{9z-z^2}{(z^2-9)(z+1)} \) on the annulus \( A = \{z \in \mathbb{C} : 3 < |z| < \infty \} \).

b) Compute the integral \( \int_{|z|=4} f(z) \, dz \).

Problem 1.1.4. Determine the number of zeros of \( p(z) = z^5 + \frac{1}{3}z^3 + \frac{1}{4}z^2 + \frac{1}{3} \) inside the annulus \( A = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 1 \} \).

Problem 1.1.5. Let \( f \) be a function holomorphic on the punctured unit disk \( \mathbb{D}^\times \) and \( f' \) has a pole of order \( k \) at \( z = 0 \). Show that \( k \geq 2 \) and \( f \) has a pole of order \( k-1 \) at \( z = 0 \).

Problem 1.1.6. Let \( f \) be holomorphic function on the closed unit disk and \( f^{(k)}(0) = 0 \) for \( k = 0, 1, \ldots, n \) and \( \max_{|z|=1} |f(z)| = M \). Show that \( |f(z)| \leq M|z|^{n+1} \) for \( |z| < 1 \).

Problem 1.1.7. Let \( f(z) \) denote a function which is holomorphic in \( \mathbb{C} \setminus \{0\} \) and has the Laurent expansion \( f(z) = \sum_{j=-\infty}^{\infty} a_j z^j \). Assuming that \( f(z) \) is real for all real \( z \), does it follow that all coefficients \( a_j \) are real? Give a proof or counterexample.
1.2. Homework

Problem 1.2.1. Let \( z_1 \) and \( z_2 \) be two distinct points in the complex plane and \( l \) the line determined by them. Find the formula for the orthogonal reflection with respect to \( l \), i.e., find the function \( S(z) \), called the Schwarz function of \( l \), such that given any point \( z \in \mathbb{C} \) the reflected image is \( z^* = S(z) \). Notice that \( S(z) = \overline{z} \) for \( z \in l \). (Have a look at Problems (2.3.2) and (2.3.3) from last semester.)

Problem 1.2.2. Find the Schwarz function \( S \) of the circle of radius \( R \) centered at the point \( z_0 \in \mathbb{C} \). This will be a holomorphic function \( S \) defined in a neighborhood of the circle and satisfying \( S(z) = \overline{z} \) for points on the circle. First consider the case \( R = 1 \) and \( z_0 = 0 \).

Problem 1.2.3. a) Let \( f \) the holomorphic function \( f \) on \( \Omega \) be a sequence of points in \( \Omega \). Assume that there is a constant \( C < \infty \) such that for some (finitely many - some could be the same) \( a \)'s of modulus \( 1 < |a| < 1 \). Let \( h(z) = \frac{z-a}{1-\overline{a}z} \) for some \( (\text{finitely many} - \text{some could be the same}) a \)'s of modulus \( |a| < 1 \).

Problem 1.2.4. Show that if \( f \) is an entire function which is real valued on some open interval on the real axis and purely imaginary on some open interval on the imaginary axis, then \( f \) is an odd function, \( f(-z) = -f(z) \) for all \( z \in \mathbb{C} \).

Problem 1.2.5. Let \( h \in \mathcal{A}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}}) \) and \( |h(z)| = 1 \), when \( |z| = 1 \). Prove that \( h \) is a rational function. Show that up to a constant multiple of unit modulus \( h \) is a finite product of Blaschke factors \( B_{a}(z) = \frac{z-a}{1-\overline{a}z} \) for some \( (\text{finitely many} - \text{some could be the same}) a \)'s of modulus \( |a| < 1 \).

Problem 1.2.6. Let \( |a| < 1 \) and \( L(z) = \frac{z-a}{1-\overline{a}z} \). Let \( L_n = L \circ \cdots \circ L \ n \text{-times} \). Prove that the sequence of holomorphic functions \( \{L_n\} \) converges locally uniformly on \( \mathbb{D} \) and find its limit.

Problem 1.2.7. Let \( \Omega \) be a normal family of holomorphic functions on \( U \). Show that the family \( \Omega' \) of all complex derivatives of the functions in \( \Omega \) is also a normal family. Hint: Review Problem (2.7.2).

Problem 1.2.8. Suppose that \( \{f_n\} \) is a uniformly bounded family of holomorphic functions on a domain \( \Omega \). Let \( \{z_k\} \) be a sequence of points in \( \Omega \) converging to a point \( z_0 \in \Omega \). Show that if for every fixed \( k \) the sequence \( \{f_n(z_k)\} \) is convergent, then the \( \{f_n\} \) converges locally uniformly on compact subsets of \( \Omega \).

Problem 1.2.9. Let \( \Omega \) be a bounded domain of the complex plane \( \mathbb{C} \) and \( \{f_n\} \) a sequence of holomorphic function on \( \Omega \). Assume that there is a constant \( C < \infty \) such that

\[
\int_{\Omega} |f_n(z)|^2 \, dxdy < C
\]

for all \( n \). Prove that \( \{f_n\} \) is a normal family. Hint: Use that \( |f(z)|^2 \) can be bounded by the mean value of \( |f|^2 \) on a small disc centered at \( z \). This inequality was also used in Problem (2.10.6) b). See also Problem (1.2.10).

Problem 1.2.10. Let \( \Omega \) be an open set and \( K \subset L \subset \Omega \). Show that there exists a constant \( M \) (depending on \( K \) and \( L \)) such that for any holomorphic function \( f \) on \( \Omega \) we have

\[
\sup_{K} |f| \leq M \left( \int_{L} |f|^2 \, dxdy \right)^{1/2}.
\]

Hint: First show the inequality for \( K = D(z_0, r) \) and \( L = D(z_0, R) \) using the Cauchy formula applied to the holomorphic function \( f^2 \). See also Problem (2.6.7) and (2.7.10).
1.3. Homework

Note: Here *conformal* means injective and surjective holomorphic map, hence a biholomorphism.

**Problem 1.3.1.** Determine $\text{Aut}(\mathbb{C})$.

**Problem 1.3.2.**

a) Describe all holomorphic maps from $\mathbb{C}$ to $\mathbb{D}$?

b) Can there be a holomorphic map of $\mathbb{D}$ onto $\mathbb{C}$?

**Problem 1.3.3.**

a) Prove that if $f : D \to D$ is holomorphic and has two distinct fixed points then $f(z) = z$.

b) Is it true that if $f : D \to D$ is holomorphic then it must have a fixed point?

**Problem 1.3.4.** Let $\Omega$ be a holomorphically simply connected domain in $\mathbb{C}$ and let $P$ and $Q$ be distinct points of $\Omega$. Let $F_1$ and $F_2$ be conformal self-maps of $\Omega$, i.e., $F_1, F_2 \in \text{Aut}(\Omega)$. If $F_1(P) = F_2(P)$ and $F_1(Q) = F_2(Q)$ then prove that $F_1 \equiv F_2$. Note: Be careful to distinguish the case of $\mathbb{C}$ from that of $\Omega$ a proper subset of $\mathbb{C}$.

**Problem 1.3.5.** Let $\Omega$ be a bounded domain and let $\phi$ be a conformal mapping of $\Omega$ to itself. Let $P \in \Omega$ and suppose that $\phi(P) = P$. a) Show that if $\phi'(P) = 1$, then $\phi$ must be the identity.

b) Show that in the general case (i.e. without the assumption a)) $\phi(z) = \lambda z$ for some constant $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

**Problem 1.3.6.** Show that $f(z) = -\frac{1}{2} \left( z + \frac{1}{z} \right)$ is a conformal map of the “upper” half disc $\mathbb{D} \cap \{z : \Im z > 0\}$ to the upper half-plane.

**Problem 1.3.7.** Let $\Omega$ be a holomorphically simply connected domain in $\mathbb{C}$ and let $\phi$ be a conformal mapping of $\Omega$ to $\mathbb{D}$. Set $P = \phi^{-1}(0)$. Let $f : \Omega \to \mathbb{D}$ be any holomorphic function such that $f(P) = 0$. Prove that $|f'(P)| \leq |\phi'(P)|$.

**Problem 1.3.8.** Let $U$ be an open subset of $\mathbb{C}$. If there is a continuous function $f : U \to \mathbb{C}$ such that $e^{f(z)} = z$ for $z \in U$, show that then $f$ is necessarily holomorphic and hence a branch of $\log z$ on $U$.

**Problem 1.3.9.** Let $n \in \mathbb{N}$. Find a conformal map of the sector $0 < \arg z < \frac{n}{n}$ onto $\mathbb{D}$.
1.4. Homework

Note: Harmonic function means real-valued unless explicitly stated otherwise.

**Problem 1.4.1.** Let \( u : U \to \mathbb{R} \) be a continuous function. Show the equivalence of the next two properties.

(a) \( u(z_0) = \frac{1}{2\pi} \int_{\partial D} u(z) \, ds(z) \) for any \( D = D(z_0, r) \subset U \), where \( ds \) is arc length measure on \( \partial D \).

(b) \( u(z_0) = \frac{1}{\pi r^2} \int_D u(z) \, dA(z) \) for any \( D = D(z_0, r) \subset U \), where \( dA = dx \, dy \).

**Problem 1.4.2.** Prove that there is no nonconstant harmonic function \( u : U \to \mathbb{R} \) such that \( u(z) \leq 0 \) for all \( z \in \mathbb{C} \).

**Problem 1.4.3.** Let the function \( v(z) = \Im \exp[(1 + z)/(1 - z)] \).

(a) Calculate \( v(z) \) explicitly.

(b) What limiting value does \( v \) have as \( z \to 1, z \in \mathbb{D} \)?

**Problem 1.4.4.** Give two distinct harmonic functions on \( \mathbb{C} \) that vanish on the entire real axis. Why is this not possible for holomorphic functions?

**Problem 1.4.5.** Use the open mapping principle for holomorphic functions to prove an open mapping principle for harmonic functions.

**Problem 1.4.6.** (a) Let \( u \) be a harmonic on a connected open set \( U \subset \mathbb{C} \). Show that the following are equivalent:

i) \( u \equiv 0 \) on \( U \);

ii) \( u \) vanishes identically on some disc \( D(z_0, r) \subset U \);

iii) there is a point \( z_0 \in U \) where all partial derivatives of \( u \) vanish, \( \frac{\partial^m u}{\partial x^n \partial y^{m-n}} (z_0) = 0 \).

(b) State and prove a unique continuation property for harmonic functions defined on a connected domain \( U \subset \mathbb{C} \).

**Problem 1.4.7.** Prove that a nonconstant harmonic function on a connected domain \( U \subset \mathbb{C} \) cannot reach a local maximum (minimum).

**Problem 1.4.8.** Using Poisson’s formula find a harmonic function on \( \mathbb{D} \) which takes continuously the value \( +1 \) on the semi-circle \( \partial \mathbb{D} \cap \{\Im z > 0\} \) and the value \( 0 \) on the semi-circle \( \partial \mathbb{D} \cap \{\Im z < 0\} \).
1.5. Homework

**Problem 1.5.1.** Find a formula for the upper half-plane $\Omega = \{z \mid \Im z > 0 \}$ analogous to the Poisson integral formula by mapping $\Omega$ conformally to the unit disc.

**Problem 1.5.2.** Let $u \in C(\Omega)$, $\Omega$-open subset of $\mathbb{C}$, and $u$ harmonic in $\Omega \setminus \{z_0\}$. Prove that $u$ is harmonic on $\Omega$.

**Problem 1.5.3.** Let $\Omega$ be an open subset of $\mathbb{C}$, $u$ a harmonic function on $\Omega$ with harmonic conjugate $v$. Show that for any $D(z_0, R) \subset \Omega$ we have the expansions

$$u(z) = u(z_0 + re^{i\theta}) = u_0 + \sum_{n=1}^{\infty} (u_n \cos n\theta - v_n \sin n\theta) r^n$$

$$v(z) = v(z_0 + re^{i\theta}) = v_0 + \sum_{n=1}^{\infty} (u_n \cos n\theta + v_n \sin n\theta) r^n$$

for $z = z_0 + re^{i\theta} \in D(z_0, R)$ and some numbers $u_n, v_n$.

**Problem 1.5.4.** Let $z = re^{i\theta} \in D(0, \rho)$. Show that

$$\frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos t + r^2} = 1 + 2 \sum_{n=1}^{\infty} \left( \frac{r}{\rho} \right)^n \cos nt$$

$$\frac{2\rho r \sin t}{\rho^2 - 2\rho r \cos t + r^2} = 2 \sum_{n=1}^{\infty} \left( \frac{r}{\rho} \right)^n \sin nt$$

converge uniformly on every compact subset of $D(0, \rho)$. Hint: use the holomorphic function

$$f(z) = \frac{\rho + z}{\rho - z}$$

Do also the computation for $f(z) = \frac{pe^{i\theta} + re^{i\phi}}{pe^{i\theta} - re^{i\phi}}$

**Problem 1.5.5.** Is there a version of Harnack’s principle for a decreasing sequence of harmonic functions? If so, formulate and prove it. If not, give a counterexample.
1.6. **Homework**

**Problem 1.6.1.** Recall the Poisson formula for the unit disc $\mathbb{D}$,

$$u(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} f(e^{i\theta}) \, d\theta, \quad |z| < 1,$$

where $f$ is a piece-wise continuous function on $\partial\mathbb{D}$. Show that

$$v(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{|z|^2 - 1}{|z - e^{i\theta}|^2} f(e^{i\theta}) \, d\theta, \quad |z| > 1$$

defines a bounded harmonic function on the exterior of the unit disc, which is continuous at every point of $\partial\mathbb{D}$ where $f$ is continuous.

**Problem 1.6.2.** Show that both $u(z) = 1$ and $v(z) = \ln |z|$ are solution of the Dirichlet problem on $\mathbb{C} \setminus \mathbb{D}$ with boundary data on $\partial\mathbb{D}$ given by $f \equiv 1$. What solution does the Poisson formula of Problem 1.6.1 give?

**Problem 1.6.3.** Show that the Dirichlet problem on $\mathbb{C} \setminus \mathbb{D}$ with continuous boundary data on $\partial\mathbb{D}$ given by a function $f$ has a unique bounded solution. Hint: Problem 1.5.2 and the map $w = \frac{1}{2} \zeta$ can be helpful.

**Problem 1.6.4.** Let $\phi : \Omega \to \mathbb{D}$ be a conformal map, $\phi(z_0) = 0$ and $\phi'(z_0) > 0$ for some given point $z_0 \in \Omega$ and $u$ a harmonic function on $\Omega$. Let $\Omega_r = \phi^{-1}(D(0, r))$, $0 < r < 1$, be the pre-image of the disc $D(0, r)$.

a) Show that

$$u(z) = \frac{1}{2\pi} \int_{\partial\Omega_r} u(\zeta) \frac{|\phi'(\zeta)|^2}{|\phi'(\zeta)|^2 - |\phi(\zeta)|^2} \frac{|\phi'(\zeta)|}{|\phi(\zeta)|} \, |d\zeta|, \quad z \in \Omega_r,$$

b) Show that at $z_0$ the above formula can be written in the form

$$u(z_0) = \frac{1}{2\pi} \int_{\partial\Omega_r} u(\zeta) \frac{|\phi'(\zeta)|}{|\phi(\zeta)|} \, |d\zeta| = -\frac{1}{2\pi} \int_{\partial\Omega} u(\zeta) \frac{\partial}{\partial \nu} \ln |\phi(\zeta)| \, |d\zeta|,$$

where $\nu$ is the inner unit normal to $\Omega_r$.

**Problem 1.6.5.** Let $\Omega$ be a simply connected domain, $\Omega \neq \mathbb{C}$. For any $w \in \Omega$ let $\phi_w : \Omega \to \mathbb{D}$ be the conformal map such that $\phi_w(z) = 0$ and $\phi'_w(z) > 0$. Define the function

$$G(z, w) = -\ln |\phi_w(z)|, \quad w, z \in \Omega, \ z \neq w.$$ 

Show that for any $w \in \Omega$ the function $z \to G(z, w)$ is positive and harmonic in $\Omega \setminus \{w\}$. Furthermore, it has the properties $\lim_{z \to w} G(z, w) = +\infty$ and $\lim_{z \to \zeta} G(z, w) = 0$ for any $\zeta \in \partial\Omega$.

**Problem 1.6.6.** Show that a sub-harmonic function on a connected open set cannot reach a local maximum unless it is identically constant. Show that unlike harmonic functions, sub-harmonic functions can reach a local minimum without being constant. Hint: Think of a sub-harmonic function on the real line that is constant on $(-\infty, 0)$.

**Problem 1.6.7.** Show that if $h : \Omega \to \mathbb{R}$ is harmonic and $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function, then $\phi \circ h$ is subharmonic. Hint: Use Jensen's inequality: if $\phi$ is a convex function on $\mathbb{R}$, then $\phi\left(\frac{1}{b-a} \int_{a}^{b} f(t) \, dt\right) \leq \frac{1}{b-a} \int_{a}^{b} (\phi \circ f)(t) \, dt$ valid for any $a < b$ and $f$ a continuous function on $[a, b]$.

**Problem 1.6.8.** a) Show that if $f$ is holomorphic on an open set $\Omega$ and $p > 0$, then $|f|^p$ is subharmonic.

b) Now suppose that $f$ is merely harmonic. Prove that $|f|^p$ is subharmonic when $p \geq 1$, but that in general it fails to be subharmonic for $p < 1$. 
1.7. Homework

Problem 1.7.1. Prove that in the definition of a sub-harmonic function it is enough to consider harmonic functions that are continuous on the closed disc and harmonic inside.

Problem 1.7.2. Let $u : U \to \mathbb{R}$ be a continuous function. Show the equivalence of the next properties.
   (a) $u(z_0) \leq \frac{1}{\pi r^2} \int_D u(z) \, dA(z)$ for any disc $D = D(z_0, r) \subset U$ and $r < r_0 = r_0(z_0)$, where $dA = dx dy$.
   (b) $u$ is sub-harmonic.

Problem 1.7.3. Prove that if $f \in C^2(U)$ on an open set $U \subset \mathbb{C}$ and $f$ is subharmonic, then $\triangle f \geq 0$ on $U$. Hint: To check the inequality at $0 \in U$ (this is enough!) use Taylor’s formula to obtain
   \[
   \frac{1}{\pi r^2} \int_{D(0, r)}^{} u(z) - u(0) \, dx \, dy = \frac{r^2}{8} \Delta u(0) + o(r^2).
   \]
   For the converse see the next problem.

Problem 1.7.4. Let $f : U \to \mathbb{R}$ be a $C^2$ function on an open set in $U \subset \mathbb{C}$.
   (a) Recall that if $\Delta f > 0$ at a point $P$, then $f$ cannot have a local maximum at $P$. Use this observation to deduce that if $\Delta f > 0$ everywhere on $U$, then $f$ is subharmonic.
   (b) If $\Delta f \geq 0$ everywhere, then, for each $\epsilon > 0$, $\Delta(f + \epsilon|z|^2) > 0$ everywhere. Use a limiting argument and part (a) to deduce that if $\Delta f \geq 0$ everywhere, then $f$ is subharmonic.

Problem 1.7.5. Show that if the bounded open set $U$ has the outer segment property at the boundary point $z_0 \in \partial U$, i.e., there is a point $z_1 \in \mathbb{C}$ so that the closed segment $I$ connecting $z_0$ to $z_1$ has the property $I \cap \overline{U} = \{z_0\}$ then $z_0$ is a regular point.
1.8. HOMEWORK

Problem 1.8.1. Determine the convergence of the product
\[ \prod_{k=1}^{\infty} \left( 1 + \frac{(-1)^k}{\sqrt{k}} \right). \]

Problem 1.8.2. Let \( a_n \in \mathbb{C}, n \in \mathbb{N} \), and set \( \Pi_n = \prod_{k=1}^{n} (1 + a_k) \) and \( P_n = \prod_{k=1}^{n} (1 + |a_k|) \). Show that
\[ P_n \leq \exp(\sum_{k=1}^{n} |a_k|) \quad \text{and} \quad |\Pi_n - 1| \leq P_n - 1. \]

Problem 1.8.3. Show that if \( 0 \leq a_n < 1 \) and \( \sum_{n=1}^{\infty} a_n \) is a divergent series, then
\[ \lim_{n \to \infty} \prod_{k=1}^{n} (1 - a_k) = 0. \]

Problem 1.8.4. Let \( \Pi = \prod_{k=1}^{\infty} (1 + a_k) \) be an absolutely convergent product. Show that for any reordering function \( Q : \mathbb{N} \to \mathbb{N} \) a bijection of \( \mathbb{N} \) we have
\[ \Pi = \prod_{k=1}^{\infty} (1 + a_{Q(k)}). \]

Problem 1.8.5. Determine the regions of convergence and absolute convergence of the product
\[ \prod_{k=1}^{\infty} (1 + z^k). \]

Problem 1.8.6. Determine the regions of convergence and absolute convergence of the product
\[ \prod_{k=1}^{\infty} (1 - \frac{z^k}{k}). \]

Problem 1.8.7. Determine the region of convergence of the product
\[ \prod_{k=1}^{\infty} \left| 1 - \frac{z}{k} \right|. \]

Problem 1.8.8. Determine the region of convergence of the product
\[ \prod_{k=1}^{\infty} \left( 1 + z^{3k} \right). \]

Problem 1.8.9. Suppose \( \sum |a_n - b_n| < \infty \). Determine the largest open set \( \Omega \) for which the product
\[ \prod_{k=1}^{\infty} \frac{z - a_n}{z - b_n} \]
converges normally on \( \Omega \).
1.9. Homework

Problem 1.9.1. Find an entire function with zeros of multiplicity one at the points $a_n = n^2$.

Problem 1.9.2. Show that $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$.

Problem 1.9.3. Show that $f(z) = \prod_{n=1}^{\infty} \left[(1 - \frac{z}{n}) e^{\frac{z}{n}}\right]$ and $g(z) = \prod_{n=1}^{\infty} \left[(1 + \frac{z}{n}) e^{-\frac{z}{n}}\right]$ define entire functions.

Problem 1.9.4. Show that for some entire function $h(z)$ we have

$$\sin \pi z = zf(z) g(z) e^{h(z)},$$

where $f$ and $g$ are the functions from the Problem 1.9.3. (Actually $e^h \equiv \pi$.)

Problem 1.9.5. Show that

$$\pi \cot \pi z = h'(z) + \frac{1}{z} + \sum_{n=-\infty}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n}\right),$$

where $h$ is the function from Problem 1.9.4. Here the $\sum$ is over $n \neq 0$.

Problem 1.9.6. Let $a_n$ be a sequence of non-zero complex numbers without an accumulation point in the complex plane. Let $k$ be the largest non-negative integer for which the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^k}$$

diverges. The function

$$f(z) = \prod_{n=1}^{\infty} \left\{\left(1 - \frac{z}{a_n}\right) \exp \left(\frac{z}{a_n} + \cdots + \frac{1}{k} \frac{z^k}{a_n^k}\right)\right\}$$

is an entire function with zeros given by the sequence $\{a_n\}$. In particular, when $k = 0$ we obtain

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)$$

is an entire function with zeros precisely given by the sequence $a_n$. Note: Equivalently, $k$ is the smallest integer for which the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{k+1}}$$

converges.

Problem 1.9.7. Suppose that $a_n$ is a sequence of distinct complex numbers, $\lim |a_n| = \infty$ and that the $A_n$ are arbitrary complex numbers. Show that there exists an entire function $f$ such that $f(a_n) = A_n$. Hint: Let $g$ be an entire function with simple zeros at $a_n$. Show that

$$\sum \frac{g(z)}{z-a_n} \cdot \frac{A_n}{g'(a_n)} e^{\gamma_n(z-a_n)}$$

converges for some choice of the numbers $\gamma_n$.

Problem 1.9.8. Show that the field generated by all entire functions is the field of all meromorphic functions on $\mathbb{C}$. In other words, if $q(z)$ is a meromorphic function on $\mathbb{C}$, then $q(z) = f(z)/g(z)$ for some entire functions $f$ and $g$. 
1.10. Homework

Problem 1.10.1. a) Suppose that \( f, g \in A(\Omega) \) have no common zeros in the open set \( \Omega \). Show that there are \( u, v \in A(\Omega) \) such that

\[
fu + gv = 1.
\]

b) Prove the result of part a) for finitely many functions \( f_1, \ldots, f_n \in A(\Omega) \) without a common zero.

Problem 1.10.2. Show that every finitely generated maximal ideal \( m \) of the ring \( A(\Omega) \) is of the form \( m_{\lambda} = \{ f \in A(\Omega) : f(\lambda) = 0 \} \) for some \( \lambda \in \Omega \).

Problem 1.10.3. Let \( Z \) be an infinite discrete subset of \( \Omega \).

a) Show that the set \( I \) of all functions vanishing at all but finitely many points of \( Z \) is an ideal of \( A(\Omega) \).

b) Show that there is no point of \( Z \) at which all functions in \( I \) have a zero.

Problem 1.10.4. Show that the ring \( A(\Omega) \) has maximal ideals which are not of the form \( m_{\lambda} \). Hint: Consider the maximal ideal \( m \) containing \( I \) from Problem 1.10.3.

Problem 1.10.5. Find a meromorphic function with simple poles at every integer number and residue one.

Problem 1.10.6. Show that

\[
f(z) = \pi \cot \pi z - \frac{1}{z} - \sum_{n=1}^{\infty} \left( \frac{1}{z - n + 1/n} - \frac{1}{z + n - 1/n} \right)
\]

is an entire function which is periodic \( f(z) = f(z+1) \). Hint: Show that \( f(z+1) - f(z) \equiv 0 \) by grouping terms (carefully).

Problem 1.10.7. Show that the function \( f \) defined in Problem 1.10.6 vanishes identically. For this you can proceed by following either of the following steps (do both but turn-in one solution).

a) Let \( \Gamma_n \) be the square with vertices \((n + \frac{1}{2}) + i(n + \frac{1}{2})\) and its reflections across the coordinate axes and the origin. Show that

\[
\frac{1}{2\pi i} \int_{\Gamma_n} \frac{\cot \pi \xi}{\xi(z - \xi)} \, d\xi = \frac{\cot(\pi z)}{z} - \frac{1}{\pi z^2} + \frac{1}{\pi} \sum_{j=1}^{n} \frac{1}{j(j - z)} + \frac{1}{\pi} \sum_{j=1}^{n} \frac{1}{j(j + z)}.
\]

Hint: See Example 4.6.6. or your notes from last semester.

b.1) Show that

\[
f'(z) = -\frac{\pi^2}{\sin^2 \pi z} + \sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2}.
\]

b.2) Show that \( f' \) is bounded on \( \{ z \mid |\Re(z)| \leq \frac{1}{2} \} \).

b.3) Show that

\[
\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2}.
\]

b.4) Show that \( f \equiv 0 \) using that \( f \) is an odd function.

Problem 1.10.8. Show that

\[
\sin \pi z = \pi z \cdot f(z) \cdot g(z) = \pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right),
\]

where \( f \) and \( g \) are the functions from the Problem 1.9.3.
1.11. Homework

Problem 1.11.1. Let $f$ be holomorphic in a neighborhood of the closure of the disk $D(0,R)$, $f(0) \neq 0$, and $n(r)$ be the number of zeros (counted with their multiplicities) inside the circle $|z| = r$, $r < R$. Show that
\[
\int_0^R \frac{n(r)}{r} \, dr = \sum_{|a_k| < R} \ln \left| \frac{R}{a_k} \right|,
\]
where $a_k$ are the zeros repeated with their multiplicities.

Problem 1.11.2. Let $f$ be holomorphic in a neighborhood of the closure of the disk $D(0,R)$, $f(0) \neq 0$, and $n(r)$ be the number of zeros (counted with their multiplicities) inside the circle $|z| = r$, $r < R$. Show that
\[
\int_0^R \frac{n(r)}{r} \, dr = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(Re^{i\theta})| \, d\theta - \ln |f(0)|.
\]

Problem 1.11.3. Let $f$ be holomorphic in a neighborhood of the closure of the disk $D(0,R)$, $f(0) \neq 0$, and $n(r)$ be the number of zeros (counted with their multiplicities) inside the circle $|z| = r$, $r < R$. If
\[
M(R) = \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\theta})| \, d\theta
\]
on $|z| = R$ show that
\[
n(R/2) \leq \frac{1}{\ln 2} \ln \frac{M(R)}{|f(0)|}.
\]

Problem 1.11.4. Let $f$ be holomorphic in a neighborhood of the closure of the unit disk $\mathbb{D}$, $|f(0)| = 1$ and $|f(z)| \leq 17$ on $|z| = 1$. Give an estimate of the number of roots inside $D(0,1/2)$.

Problem 1.11.5. Suppose $a_n \in \mathbb{D}$ and $\sum (1 - |a_n|) = \infty$. Show that if $f$, $g \in H^\infty(\mathbb{D})$ and $f(a_n) = g(a_n)$ then $f \equiv g$ on $\mathbb{D}$.

Problem 1.11.6. Prove the Jensen-Poisson formula: if $f$ is holomorphic in a neighborhood of the closure of the disk $D(0,r)$, $z_0 \in D(0,r)$ and $f(z_0) \neq 0$ then
\[
\ln |f(z_0)| + \sum_{k=1}^n \ln \left| \frac{r^2 - \bar{a_k}z_0}{r(z_0 - a_k)} \right| = \frac{1}{2\pi} \int_0^{2\pi} \Re \left\{ \frac{re^{i\theta} + z_0}{re^{i\theta} - z_0} \right\} \ln |f(re^{i\theta})| \, d\theta
\]
where $a_k$ are the zeros of $f$ each repeated as many times as its multiplicity.

Problem 1.11.7. Let $f$ is holomorphic in a neighborhood of the closure of the unit disk $\mathbb{D}$ and $f(z_0) \neq 0$. Show:
\[a)\]
\[
|f(0)| \leq |a_1 \ldots a_k| \cdot \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| \, d\theta.
\]
b) if $|f(z)| \leq M$ on $|z| = 1$ then
\[
|f(0)| \leq |a_1 \ldots a_k| \cdot M.
\]
1.12. Homework

**Problem 1.12.1.** Show that $f(z) = 1/z$ cannot be approximated uniformly on the unit circle by holomorphic polynomials.

**Problem 1.12.2.** Show that if the Taylor series of an entire function $f$ converges uniformly on $\mathbb{C}$ to $f$, then $f$ is a polynomial.

**Problem 1.12.3.** Let $K = D(4,1) \cup D(-4,1)$ and $L = D(4i,1) \cup D(-4i,1)$. Construct a sequence $(f_j)$ of entire functions which converges uniformly to 1 on $K$ and uniformly to $-1$ on $L$.

**Problem 1.12.4.** Show that if $\Omega \subset \mathbb{C}$ is an open set and $f \in A(\Omega)$, then there exists a sequence $(r_n)$ of rational functions with poles in $\hat{\mathbb{C}} \setminus \Omega$ such that $r_n$ converges normally to $f$.

**Problem 1.12.5.** Let $u \in C([0,1] \times K)$ where $K$ is a compact subset of $\mathbb{C}$. Show that

$$S_n(z) = \frac{1}{n} \sum_{j=1}^{n} u(j/n, z)$$

converges uniformly and find its limit.

**Problem 1.12.6.** Let $K$ be a compact for which $\mathbb{C} \setminus K$ is not connected. Show that there is a function $f$ holomorphic in a neighborhood of $K$ which cannot be approximated uniformly on $K$ by polynomials. Hint: Take a $z_o$ which belongs to a bounded connected component of $K^c$. Take a polynomial $p$ so that

$$|(z - z_o)p(z) - 1| < 1.$$ 

Use the maximum principle to see that this inequality holds true at $z_o$ as well (in fact in the component containing $z_o$).

**Problem 1.12.7.** Using Runge’s theorem construct a function $f \in A(\mathbb{D})$ which has no radial limit at any boundary point, i.e.,

$$\lim_{r \to 1^-} f(re^{i\theta})$$

does not exist for all $\theta$.

**Problem 1.12.8.** * Construct an entire function with the following property: given any bounded open set $U$ with a connected complement $\hat{\mathbb{C}} \setminus U$ and $g \in A(U)$ there is a sequence of positive integers $n_k$ such that $F_k(z) = F(z + n_k)$ converges uniformly to $g$ on $U$.
13. Homework

Problem 1.13.1. Show that for two lattices we have \( \Lambda(\omega_1, \omega_2) = \Lambda(\omega'_1, \omega'_2) \) iff there is a matrix \( A \in GL(2, \mathbb{Z}) \) with \( \det A = \pm 1 \) such that
\[
\begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.
\]

Problem 1.13.2. a) Consider \( SL(2, \mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, \det A = 1 \right\} \). Using your knowledge of \( \text{Aut}(\mathbb{H}) \) show that the group of automorphisms of the upper half-plane \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im} z > 0 \} \) is the group
\[
\text{Aut}(\mathbb{H}) = SL(2, \mathbb{R})/\{\pm I\} \overset{def}{=} PSL(2, \mathbb{R}).
\]
Write explicitly the isomorphism.

b) Show that
\[
\text{Aut}(\hat{\mathbb{C}}) = SL(2, \mathbb{C})/\{\pm I\} \overset{def}{=} PSL(2, \mathbb{C}).
\]
Write explicitly the isomorphism.

Problem 1.13.3. Show that \( \Gamma \), the congruence group mod 2 (\( \subset \text{Aut}(\mathbb{H}) \)), is generated by
\[
\mu(z) = \frac{z}{2z + 1} \text{ and } \omega(z) = z + 2.
\]

Definition. Let \( z_2, z_3, z_4 \) be three distinct points of \( \hat{\mathbb{C}} \). Consider the function
\[
z \mapsto S_{z_2, z_3, z_4}(z) = \frac{z - z_3}{z - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}
\]
which is called the cross ratio of \( z, z_2, z_3, z_4 \). If \( z_2, z_3 \) or \( z_4 \) is \( \infty \) the formula is defined by taking the limit of the above formula when the corresponding point approaches \( \infty \). It is customary to use the notation \( (z, z_2, z_3, z_4) \) for \( S_{z_2, z_3, z_4}(z) \).

Problem 1.13.4. a) Show that the cross ratio \( (z, z_1, z_2, z_3) \) is the unique linear fractional transformation which sends \( z_1, z_2, z_3 \) to \( 1, 0, \infty \) respectively.

b) Show that the cross ratio is invariant under linear fractional transformations, i.e., if \( T \in \text{Aut}(\hat{\mathbb{C}}) \) then \( (z, z_1, z_2, z_3) = (Tz, Tz_1, Tz_2, Tz_3) \).

c) The cross ratio \( (z, z_1, z_2, z_3) \in \mathbb{R} \) iff the four points \( z, z_1, z_2, z_3 \) lie on a circle or on a straight line.

Problem 1.13.5. Show that the right half \( W' \) of the fundamental domain \( W \) of \( \Gamma \) can be mapped homeomorphically onto \( \mathbb{H} \) so that \( 0, 1, \infty \) stay fixed. Hint: Use Carathéodory’s theorem.

Problem 1.13.6. Suppose \( f \) is a doubly periodic meromorphic function on \( \mathbb{C} \) with periods 1 and \( i \) so that \( f \) is holomorphic on \( \mathbb{C} \setminus \{m + ni \mid m, n \in \mathbb{Z}\} \). Prove that the residue of \( f \) at each of the poles is zero.

Problem 1.13.7. Let \( \Lambda = \Lambda(\omega_1, \omega_2) \) with \( \omega_2/\omega_1 \notin \mathbb{R} \) be a lattice. For \( a \in \mathbb{C} \) let
\[
\Phi = \Phi_a = \{a + t_1 \omega_1 + t_2 \omega_2 \mid 0 \leq t_1, t_2 < 1\}
\]
be the so-called fundamental parallelogram at \( a \).

a) Show that if \( \delta \geq \text{diam}(\Phi) \) then
\[
n - 1 - \delta \leq |z| \leq n + \delta, z \in A_n.
\]

b) Show that there exists a constant \( M \) depending on the area and diameter of \( \Phi \) such that for \( n \in \mathbb{N} \) the annulus \( A_n = \{z \in \mathbb{C} \mid n - 1 \leq |z| < n\} \) contains at most \( Mn \) lattice points. Hint: Consider the fundamental parallelograms \( \Phi_a, a \in \Lambda \) that intersect \( A_n \).

c) Show that if \( k > 2 \) then \( \sum_{\omega \in \Lambda'} \frac{1}{|\omega|^k} \), where \( \Lambda' = \Lambda \setminus \{0\} \), converges. Hint: Either use part b) or show that \( \inf \{|m\omega_1 + n\omega_2|/(|m| + |n|)\} > 0 \) and then write the sum as a suitable iterated sum.

Problem 1.13.8. Let \( \Lambda = \Lambda(\omega_1, \omega_2) \) with \( \omega_2/\omega_1 \notin \mathbb{R} \) be a lattice. Construct a function which has simple zeros at all points of \( \Lambda \) (the Weierstrass sigma function). Hint: Use Problem 1.13.7 and Problem 1.9.6.
**Problem 1.13.9.** Let $\phi \in A(\mathbb{D})$ be one-to-one on $\mathbb{D}$. Show that $\phi$ is a proper map, i.e., for every compact $K \subset G = f(\mathbb{D})$ the pre-image $\phi^{-1}(K)$ is a compact of $\mathbb{D}$. In particular for every sequence of points $z_n \in \mathbb{D}$ which converges to a boundary point $z_0 \in \partial \mathbb{D}$ the sequence $w_n = \phi(z_n)$ converges to $\partial G$ in the sense that $w_n$ leaves every compact contained in $G$.

**Problem 1.13.10.** Let $\phi \in A(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ be one-to-one on $\mathbb{D}$. Show that if $\phi$ is injective on $\mathbb{D}$ then $\phi$ is injective on $\overline{\mathbb{D}}$. Hint: Show that we only need to check injectivity on $\partial \mathbb{D}$. For the latter argue by contradiction: if $\phi(\zeta) = \phi(\zeta')$ then $\phi$ has to be constant on one of the arcs between $\zeta$ and $\zeta'$ which can be seen by considering the image of the sector defined by $0$, $\zeta$ and $\zeta'$. 
Part 2. Homework Problems, MATH561, F2010
2.1. HOMEWORK

Problem 2.1.1. Prove the following identity for any \( \bar{z} \cdot w \neq 1\),
\[
1 - \frac{|z - w|^2}{1 - \bar{z}w} = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2}.
\]

Problem 2.1.2. Prove that the function \( \phi(z) = i\frac{1 - z}{1 + z} \) maps the open unit disc one-to-one onto the open upper half-plane.

Problem 2.1.3. Let \( |z_0| < 1 \). Prove that the function \( \psi(z) = \frac{z - z_0}{1 - \bar{z}_0z} \) has the following properties:

1. \( \psi \) is holomorphic.
2. \( \psi(0) = z_0 \) and \( \psi(z_0) = 0 \);
3. the unit circle is mapped onto the unit circle;
4. maps the open unit disc \( \mathbb{D} \) one-to-one and onto itself.

Problem 2.1.4. Show that a real-valued holomorphic polynomial must be identically constant.

Problem 2.1.5. Prove that \( \Delta(|f|^2) = 4 \frac{\partial f^2}{\partial z} \) if \( f \) is a holomorphic function.

Problem 2.1.6. Prove that if \( f \in \mathcal{C}^2 \) is holomorphic and non-vanishing, then \( \ln |f| \) is harmonic.

Problem 2.1.7. Show that the function \( f(z) = \frac{1}{z} \) is holomorphic on the annulus \( A = \{ z \in \mathbb{C} : 1 < |z| < 2 \} \), but does not have a holomorphic antiderivative on \( A \).

Problem 2.1.8. Write the Cauchy-Riemann equations in polar coordinates.

Problem 2.1.9. Define at every point \( z \in \mathbb{R}^2 \) a linear map \( J_z : T_z \mathbb{R}^2 \to T_z \mathbb{R}^2 \) given by
\[
J_z \begin{pmatrix} \frac{\partial}{\partial x} \big|_z \end{pmatrix} = \frac{\partial}{\partial y} \big|_z, \quad J_z \begin{pmatrix} \frac{\partial}{\partial y} \big|_z \end{pmatrix} = -\frac{\partial}{\partial x} \big|_z,
\]
i.e., \( J_z \) is a c.c.w. rotation by \( 90^\circ \). Suppose \( f : \mathbb{R}^2 \to \mathbb{R}^2 \), \( f(z) = (u(z), v(z)) \), is a smooth map.

We showed that \( f_* \circ J_z = J_{f(z)} \circ f_* \) at every \( z \in \mathbb{R}^2 \) iff \( f \) is a holomorphic function in the sense that it satisfies the Cauchy-Riemann equations.

(a) Derive the system satisfied by the derivatives of \( f_* \circ J_z = -J_{f(z)} \circ f_* \).
(b) What can you say if \( J_z \) was defined to be a c.c.w rotation by angle \( \phi \) instead of \( \pi/2 \) and should we have a continuum of complex analysis qualifying exams?
2.2. Homework

Problem 2.2.1. Let \( U_1 \subset U_2 \subset \ldots \) be a sequence of connected open subsets of \( \mathbb{C} \) and \( U \) be their union. Show that if \( f \in A(U) \) and for every \( j \), the restriction of \( f \) to \( U_j \) has a holomorphic antiderivative, then \( f \) has a holomorphic antiderivative on \( U \).

Problem 2.2.2. Find all solutions of \( z^{1+i} = 9 \).

Problem 2.2.3. (a) Show that the function \( \cos z = \frac{e^{iz} + e^{-iz}}{2} \) is holomorphic.

(b) Describe the holomorphic map \( z \mapsto \cos z \) by showing images of some suitable curves in the complex plane, for example, lines parallel to the coordinate axes. In particular, show that the vertical strip \( 0 < x < \pi \) is mapped one-to-one and onto the region \( \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty)) \).

(c) Is \( \cos z, z \in \mathbb{C} \), a bounded function? Show that \( |\cos(x + iy)| \leq e^y \) for \( y \geq 0 \).

Problem 2.2.4. (a) Let \( \sin z = \frac{e^{iz} - e^{-iz}}{2i} \). Find \( \frac{\partial \sin z}{\partial z} \) and \( \frac{\partial \sin z}{\partial \bar{z}} \).

(b) Show that \( (\sin z)' = \cos z \).

Problem 2.2.5. (a) Show that the function \( \cosh z = \frac{e^z + e^{-z}}{2} \) and \( \sinh z = \frac{e^z - e^{-z}}{2} \) are holomorphic. Find their complex derivatives.

(b) Describe the holomorphic map \( z \mapsto \cosh z \) by showing images of some suitable curves in the complex plane, for example, lines parallel to the coordinate axes.

Problem 2.2.6. (a) Write the complex polynomial \( P(z, \bar{z}) = z^3 - \bar{z}^2z \) in real notation, i.e., in terms of \( x \) and \( y \), \( z = x + iy \).

(b) Write the polynomial \( F(x, y) = (x - y^2) + i(x + y^2) \) as a polynomial in \( z \) and \( \bar{z} \).

Problem 2.2.7. If possible, find a holomorphic function \( f = u + iv \) with the given real part.

(a) \( u = x^2 - y^2 \).

(b) \( u = \cosh x \cos y \).

(c) \( u = x^3 \).

Problem 2.2.8. Find all holomorphic functions \( w = f(z) \) defined on \( \mathbb{C} \) whose image is contained in the coordinate axes \( uv = 0, w = u + iv \).
2.3. Homework

**Problem 2.3.1.** Show that if $h$ is a complex valued harmonic polynomial, then $h$ can be written in the form $h(z) = p(z) - \overline{q(z)}$, where $p$ and $q$ are holomorphic polynomials.

**Problem 2.3.2.** a) Let $C$ be the unit circle in $\mathbb{C}$. Find a holomorphic function $S(z)$ defined in a neighborhood of $C$ (all point in an annulus around $C$) so that $S(z) = \overline{z}$ for every $z \in C$.
   
   b) Show that $|S'(z)| = 1$ for $z \in C$.

**Problem 2.3.3.** Suppose $\gamma$ is a smooth arc in $\mathbb{C}$ for which there is a holomorphic function $S(z)$ defined near $\gamma$ and such that $S(z) = \overline{z}$ for every $z \in \gamma$. Let $z_o \in \gamma$ and $k = S'(z_o) \in \mathbb{C}$.
   
   a) Show that $|S'(z)| = 1$ for $z \in \gamma$.
   
   b) Show that the equation of the tangent line to $\gamma$ at $z_o$ can be written as
      
      $$\overline{z} = S'(z_o) \cdot (z - z_o) + \overline{z_o}.$$  
   
   c) Show that $\frac{1}{2} \arg S'(z_o)$ is the angle the tangent to $\gamma$ at $z_o$ makes with the real axis (so the slope of the tangent is determined).

**Problem 2.3.4.** Suppose $f$ and $g$ are $C^1$ complex valued functions (not necessarily holomorphic!) and $\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial z}$ for all $z \in \mathbb{C}$. How are $f$ and $g$ related?

**Problem 2.3.5.** Compute the following line integrals.
   
   a) $\int_\gamma \frac{z^2}{z-1} \, dz$, where $\gamma$ is the circle of radius 3 centered at the origin and c.c.w. orientation.
   
   b) $\int_\gamma \frac{z}{(z+4)(z-1+4i)} \, dz$, where $\gamma$ is the circle of radius 1 centered at the origin and c.c.w. orientation.
   
   c) $\int_\gamma \overline{z} + z^2 \, dz$ where $\gamma$ is the unit square of side of length two, centered at $(0,0)$ with clockwise orientation.

**Problem 2.3.6.** Show that if $F$ is holomorphic and $F \in C^2$, then $f(z) = F'(z)$ is also holomorphic.

**Problem 2.3.7.** Let $f : \mathbb{C} \to \mathbb{C}$ be a $C^1(\mathbb{C})$ function (so $f$ is not given to be holomorphic!). Suppose that for any piecewise linear curve $\gamma : [a, b] \to \mathbb{C}$ satisfying $\gamma(a) = \gamma(b)$ it holds that $\int_\gamma f(z) \, dz = 0$.
   
   a) Prove that $f$ has a $C^2$ holomorphic antiderivative $F$, i.e., there is $F \in C^2(\mathbb{C})$, $\frac{\partial F}{\partial \bar{z}} = 0$ and $F' = f$.
   
   You might want to fix a point $z_0 \in \mathbb{C}$ and consider $F(z) = \int_\gamma f(\zeta) \, d\zeta$ where $\gamma$ is a suitable curve connecting $z_0$ to $z$.
   
   b) Show that $f$ is holomorphic.

**Problem 2.3.8.** Let $\varphi$ be a real valued $C^1$ function, $\varphi : \mathbb{C} \to \mathbb{R}$, for which the integral $\int \int_{\mathbb{R}^2} |\nabla \varphi|^2 \, dA$ exists, where $dA$ is the area element in the plane. Let $z = f(w)$ be any holomorphic function on $\mathbb{C}$ and $\psi(w) = \varphi(f(w))$. Show that
   
   $$\int \int_{\mathbb{R}^2} |\nabla \varphi|^2 \, dA = \int \int_{\mathbb{R}^2} |\nabla \psi|^2 \, dA.$$  
   
   You might want to recall that $|\nabla \varphi|^2 = 4 |\frac{\partial \varphi}{\partial \bar{z}}|^2$ and $dz \wedge d\bar{z} = -2i dx \wedge dy$ in order to do the verification in one or two lines.
2.4. Homework

Problem 2.4.1. Prove that the Cauchy integral formula
\[
f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{D}
\]
holds for any function \( f \in \mathbb{C}(\mathbb{D}) \cap A(\mathbb{D}) \), where \( \mathbb{D} \) is the unit disc. The point is that \( f \) is not a \( C^1(\overline{\mathbb{D}}) \) function. Notice that you can use the proven formula on every disc \( D(0, r), 0 < r < 1 \), and its boundary.

Problem 2.4.2. Let \( D(z_0, r) \) denote the open disc centered at \( z_0 \) and radius \( r \). Compute the integrals:

a) \[
\int_{\partial D(8i, 2)} z^3 \, dz;
\]

b) \[
\int_{\partial D(6+i, 2)} (\bar{z} - i)^2 \, dz;
\]

c) \[
\frac{1}{2\pi i} \int_{\partial D(0, 1)} \frac{1}{z + 2} \, dz;
\]

d) \[
\frac{1}{2\pi i} \int_{\partial D(0, 2)} \frac{1}{z + 1} \, dz.
\]

Problem 2.4.3. Let \( f \) be a continuous function on \( \partial \mathbb{D} \)– the boundary of the unit circle oriented c.c.w. Let \( F(z) = \begin{cases} f(z), & |z| = 1; \\ \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta, & |z| < 1. \end{cases} \)

Is it true that \( F \in \mathbb{C}(\overline{\mathbb{D}}) \)? You might want to consider \( f(z) = \bar{z} \).

Problem 2.4.4. Let \( D \subset \mathbb{C} \) be an open disc with center \( 0 \). Suppose that both \( f \) and \( g \) are holomorphic functions on \( D \setminus 0 \). Show that if \( \frac{\partial f}{\partial z} = \frac{\partial g}{\partial z} \) on \( D \setminus 0 \), then \( f \) and \( g \) differ by a constant.

Problem 2.4.5. Calculate \( \int_{\gamma} \frac{d\zeta}{(\zeta - 1)(\zeta - 2)} \), where \( \gamma \) is the circle with center \( 0 \), radius \( 4 \), and counterclockwise orientation.

Problem 2.4.6. Let \( f : \mathbb{C} \to \mathbb{C} \) be \( \mathcal{C}^1 \) function. Suppose that for any piecewise linear curve \( \gamma : [a, b] \to \mathbb{C} \) satisfying \( \gamma(a) = \gamma(b) \) it holds that \( \int_{\gamma} f(z) \, dz = 0 \).

a) Prove that there is a holomorphic function \( F \) on \( \mathbb{C} \) such that \( \frac{\partial F}{\partial z} = f \).

b) Show that \( f \) is holomorphic.
2.5. Homework

Problem 2.5.1. (a) Prove that if $U \subset \mathbb{C}$ is open and connected and if $p, q \in U$, then there is a piecewise $C^1$ curve from $p$ to $q$ consisting of horizontal and vertical line segments. [Hint: Show that, with $p \in U$ fixed, the set of points $q \in U$ that are reachable from $p$ by curves of the required type is both open and closed in $U$.]

(b) Let $f \in \mathcal{O}(\mathbb{C})$ and holomorphic on the complement of the coordinate axes. Prove that $f$ is actually holomorphic on $\mathbb{C}$.

Problem 2.5.2. Show that the conclusion of Morera’s theorem still holds if it is only assumed that the integral of $f$ around the boundary of every rectangle in $U$ or around every triangle in $U$ is 0.

Problem 2.5.3. Let $\gamma : [0, 1] \to \mathbb{C}$ be any $C^1$ curve. Define

$$f(z) = \int_{\gamma} \frac{1}{\zeta - z} \, d\zeta.$$  

(a) Prove that $f$ is holomorphic on $\mathbb{C} \setminus T_\gamma$, where $T_\gamma$ is the trace (image) of $\gamma$.

(b) In case $\gamma(t) = t$, show that there is no way to extend $f$ to a continuous function on all of $\mathbb{C}$.

Problem 2.5.4. Let $\gamma$ be the unit circle $|z| = 1$ oriented c.c.w. and

$$f(z) = \int_{\gamma} \frac{1}{\zeta - z} \, d\zeta.$$  

Find $f''(0)$.

Problem 2.5.5. Explain why the following string of equalities is incorrect:

$$\frac{d^2}{dx^2} \int_{-1}^{1} \ln|x-t| \, dt = \int_{-1}^{1} \frac{d^2}{dx^2} \ln|x-t| \, dt = \int_{-1}^{1} \frac{-1}{(x-t)^2} \, dt.$$  

Problem 2.5.6. Show that if $f$ is an entire function such that for some real number $M > 0$ and positive integer $k$ we have

$$|f(z)| \leq M|z|^k$$

for all $|z| \geq 1$, then $f$ is a polynomial of degree at most $k$.

Problem 2.5.7. Evaluate the following integrals:

(a) $\int_{|z|=2} \frac{2z}{z^2-3} \, dz$;

(b) $\int_{|z|=1} \frac{z}{2(z-3)} \, dz$;

(c) $\int_{|z+2|=1} \frac{z}{4z-2z^2} \, dz$;

(d) $\int_{|z|=2} \frac{z}{2(z-3)} \, dz$. 


2.6. Homework

Problem 2.6.1. Show that if the sequence \( f_n \in \mathbb{C}(U : \mathbb{C}) \), \( U \subset \mathbb{C} \) converges locally uniformly, then the limit function \( f(z) = \lim_{n \to \infty} f_n(z) \), \( z \in U \) is a continuous function on \( U \), i.e., \( f \in \mathbb{C}(U : \mathbb{C}) \).

Problem 2.6.2. Show the mean value property for a holomorphic function, i.e., if \( f \) is holomorphic in some open set \( U \) then for any disc \( D = D(z_0, R) \subset U \) we have

\[
\frac{1}{2\pi R} \int_{\partial D} f(\zeta) |d\zeta|.
\]

Note: \( |d\zeta| \) is the arc-length parameter on the circle.

Problem 2.6.3. Let \( f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \) be the holomorphic function defined on the disc of convergence around \( z_0 \) of the power series (suppose the disc is of non-zero radius). Show that

\[
a_n = \frac{f^{(n)}(z_0)}{n!}.
\]

Problem 2.6.4. Determine the disc of convergence of each of the following series.

(a) \( \sum_{k=3}^{\infty} k^2 \cdot z^k \);
(b) \( \sum_{k=2}^{\infty} k \log(k+1) \cdot z^k \);
(c) \( \sum_{k=2}^{\infty} 2(\log(k))^k \cdot (z - 3)^k \);
(d) \( \sum_{k=0}^{\infty} p(k) \cdot z^k \) where \( p \) is some fixed polynomial;
(e) \( \sum_{k=1}^{\infty} 3^k \cdot (z + 2i)^k \);
(f) \( \sum_{k=2}^{\infty} \frac{k^2 + 4}{k^2 + 4} \cdot z^k \);
(g) \( \sum_{k=0}^{\infty} k \cdot e^{-k} \cdot z^k \);
(h) \( \sum_{k=1}^{\infty} \frac{1}{k!} \cdot (z - 5)^k \);
(i) \( \sum_{k=1}^{\infty} k \cdot (-1)^k \cdot z^k \).

Problem 2.6.5. The functions \( f_k(x) = \sin(kx) \) are \( \mathbb{C}^\infty \) and bounded by 1 on the interval \([-1, 1]\) yet their derivatives at 0 are unbounded. Contrast this situation with the functions \( f_k(z) = \sin(kz) \) on the unit disc. The Cauchy estimates provide bounds for \( |\frac{df_k}{dz}(0)| \). Why are these two examples not contradictory?

Problem 2.6.6. Suppose that \( f : D(0, 1) \to \mathbb{C} \) is holomorphic and that \( |f(z)| \leq 2 \), \( z \in D(0, 1) \). Derive an estimate for \( \left| \frac{df}{dz}(\frac{1}{2}) \right| \).

Problem 2.6.7. Show that if \( f : D(0, R) \to \mathbb{C} \) is holomorphic, then

\[
|f(0)|^2 \leq \frac{1}{\pi R^2} \int_{D(0,R)} |f(z)|^2 \, dx \, dy.
\]

Hint: The function \( f^2 \) is holomorphic too. Use the Cauchy integral formula to obtain on any circle \( |\zeta| = r < R \) to obtain a bound from above of \( |f(0)|^2 \) by a constant multiple of \( \int_{|\zeta|=r} |f(\zeta)|^2 \, |d\zeta| \). Then use that in polar coordinates \( dx \, dy = r \, d\phi \, dr \) with \( r \, d\phi = |d\zeta| - \) the arc-length parameter on the circle of radius \( r \).
2.7. Homework

Problem 2.7.1. Show that a uniformly convergent sequence of continuous functions defined on some subset $A \subset \mathbb{C}$ have a limit which is a continuous function.

Problem 2.7.2. Let $K_1 \subseteq K_2 \subseteq \Omega$, where $\Omega$ is an open subset of $\mathbb{C}$. Show that there exists a constant $C = C(K_1, K_2, \Omega)$ such that for any $f \in A(\Omega)$ we have

$$\max_{z \in K_1} |f'(z)| \leq C \max_{z \in K_2} |f(z)|.$$

Problem 2.7.3. Show that if $f \in A(\mathbb{C})$ and the Taylor series of $f$ centered at 0 converges uniformly to $f$ on $\mathbb{C}$, then $f$ is a polynomial.

Find the power series expansion for each of the following holomorphic functions about the given point. Determine the radius of convergence of each series.

Problem 2.7.4. $f(z) = 1/(1 + 2z), \ P = 0$.

Problem 2.7.5. $f(z) = z^2/(4 - z), \ P = i$.

Problem 2.7.6. $f(z) = 1/z, \ P = 2 - i$.

Problem 2.7.7. $f(z) = (z - \frac{1}{2})/(1 - \frac{z}{2}), \ P = 0$.

Problem 2.7.8. $f(z) = \frac{\sin z}{1 + z^2}, \ P = 0$.

Problem 2.7.9. Suppose $f \in A(D), \ D = D(0, 1)$. Is it possible to have $|f^{(n)}(0)| \geq e^n n!$?

Problem 2.7.10. (a) For $n, m \in \mathbb{N}$ compute $\int_{|z| = r} z^n \bar{z}^m |dz|$.

(b) Let $f \in A(D), \ D = D(0, 1)$, and $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Prove Parseval’s equality

$$\frac{1}{2\pi} \int_{|z| = r} |f(z)|^2 |dz| = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

valid for any $0 < r < 1$. 

2.8. Homework

Problem 2.8.1. For a natural number \( q \) let \( z^{1/q} \) be the \( q \)-th root function, which is real and positive for \( z \)-real and positive, i.e., if \( z = re^{i\phi}, -\pi \leq \phi < \pi \), then \( z^{1/q} = \sqrt[r]{r} e^{i\phi/2q} \).

(a) Let \( \alpha = p/q \)-rational number. Show that

\[
\frac{d^n}{dz^n} (z^\alpha) = \alpha(\alpha - 1) \ldots (\alpha - n + 1)z^{\alpha - n}, \quad z \in \mathbb{C} \setminus \{(-\infty, 0]\}.
\]

In particular \( z \mapsto z^{p/q} \) is a holomorphic function on \( \mathbb{C} \setminus \{(-\infty, 0]\} \).

(b) Let \( \alpha = 1/2 \). Show that

\[
z^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha - 1) \ldots (\alpha - n + 1)}{n!} (z - 1)^n, \quad |z - 1| < 1.
\]

This is the Taylor series of \( z^{1/2} \) centered at \( z = 1 \).

(c) Let \( \alpha = 1/2 \). Show that

\[
(1 + z)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha - 1) \ldots (\alpha - n + 1)}{n!} z^n, \quad |z| < 1.
\]

This is the Taylor series of \( (1 + z)^{1/2} \) centered at \( z = 0 \).

Problem 2.8.2. Let \( f \in A(\Omega), \Omega \)-open bounded subset of \( \mathbb{C} \). Show that \( f \) has at most finitely many zeros in \( \Omega \).

Problem 2.8.3. Using the open mapping theorem, show that if \( f \in A(\Omega), \Omega \)-open subset of \( \mathbb{C} \), then we have the following Maximum Principle.

If \( |f(z)| \) has a local maximum \( M \) at \( z_0 \in \Omega \), then \( f \equiv M \). As usual, \( z_0 \) is a point of local maximum of the modulus of \( f \) if \( |f(z)| \leq |f(z_0)| \) for all \( z \) in some neighborhood \( D \) of \( z_0 \). Hint: Note that \( w \in f(D) \) must satisfy \( |w| \leq |f(z_0)| \).

Problem 2.8.4. Show that if \( f \in A(\Omega), \Omega \)-open subset of \( \mathbb{C} \), then we have the following Minimum Principle. If \( |f(z)| \) has a local minimum \( m \) at \( z_0 \in \Omega \), then \( f \equiv m \) or \( m = 0 \). Thus, the modulus of a non-constant holomorphic function cannot achieve a strictly positive local minimum. (Minimum principle)

Problem 2.8.5. Let \( f \in A(\Omega) \cap C(\bar{\Omega}), \Omega \)-open bounded subset of \( \mathbb{C} \). Show the following versions of the maximum/minimum principles.

(a) The \( \max_{z \in \Omega} |f(z)| \) is achieved on \( \partial \Omega \), thus

\[
|f(z)| \leq \max_{\zeta \in \partial \Omega} |f(\zeta)|, \quad z \in \bar{\Omega}.
\]

(b) The \( \min_{z \in \Omega} |f(z)| \) is achieved on \( \partial \Omega \) if \( f \) has no zeros in \( \Omega \).

Problem 2.8.6. Show that the claim of Problem 2.8.5(b) is not true if we drop the condition \( f(z) \neq 0, z \in \Omega \).

Problem 2.8.7. Show that if \( f \in A(\mathbb{D}), \mathbb{D} = D(0, 1) \) with \( |f(z)| = 1, z \) when \( |z| = 1 \), then \( \mathbb{D} \subset f(\mathbb{D}) \).

Problem 2.8.8. (a) Show that there is no universal \( r > 0 \) such that for all \( f \in A(\mathbb{D}), \mathbb{D} = D(0, 1) \) satisfying \( f(0) = 0 \) we have that \( D(0, r) \subset f(D) \). Hint: \( f_n(z) = \frac{z^n}{n!} \) might be helpful.

(b) Show that \( f_n(z) = n(e^{i/n} - 1) \) is a better example since \( f'_n(0) = 1 \).
2.9. Homework

Problem 2.9.1. Find the maximum of $|f(z)|$ on $\hat{D}$ for $f(z) = \frac{z^3}{\bar{z}^5}$.

Problem 2.9.2. Show that if we have a sequence of functions $f_n \in \mathbb{C}(\Omega) \cap A(\Omega)$ which converges uniformly on $\partial \Omega$ then $f_n$ converges uniformly to a function $f$ on $\hat{\Omega}$ and $f \in \mathbb{C}(\hat{\Omega}) \cap A(\Omega)$. Hint: Use the maximum principle.

Problem 2.9.3. Show that if $f \in A(\Omega)$, then either $f$ is constant or $\Re f$ cannot have a local maximum in $\Omega$.

Problem 2.9.4. Show that if $f_n \in A(\Omega)$ is a sequence of nowhere vanishing functions, $f_n(z) \neq 0$ for $z \in \Omega$, which converges locally uniformly to a function $f$, then either $f \equiv 0$ or $f(z) \neq 0$ for $z \in \Omega$.

Problem 2.9.5. Show that if $f_n \in A(\Omega)$ is a sequence of one-to-one functions, which converge locally uniformly to a function $f$, then either $f \equiv \text{const}$ or $f$ is also one-to-one.

Problem 2.9.6. Show that the linear fractional transformations of the form

$$f(z) = \lambda \frac{z - c}{1 - \bar{c}z}$$

where $|\lambda| = 1$ and $c \in D$ form a group of biholomorphisms of the unit disc $\mathbb{D}$.

Problem 2.9.7. Show that the group defined in the last problem is $\text{Aut}(\mathbb{D})$. Hint: Recall the proof for the group $SU(1, 1)/\{\pm I\}$ or see directly that the group in this problem is $SU(1, 1)/\{\pm I\}$.

Problem 2.9.8. Show that if $f \in \text{Aut}(\mathbb{D})$, then we have equalities in the Schwarz-Pick inequalities.

Problem 2.9.9. For $z_1, z_2 \in \mathbb{D}$ let

$$\rho(z_1, z_2) = \tanh^{-1} \left( \frac{|z_1 - z_2|}{1 - \bar{z}_1 z_2} \right).$$

a) Show that if $f \in \text{Aut}(\mathbb{D})$, then $\rho(f(z_1), f(z_2)) = \rho(z_1, z_2)$ for $z_1, z_2 \in \mathbb{D}$. Hint: Use Problem 2.9.8.

b) Show that $\rho$ defines a distance function on $\mathbb{D}$. This is the Poincare distance of $\mathbb{D}$. Hint: For the triangle inequality, reduce to the case $\rho(z_1, z_2) \leq \rho(z_1, 0) + \rho(0, z_2) = \rho(|z_1|, 0) + \rho(0, |z_2|)$ using that $\text{Aut}(\mathbb{D})$ is transitive and (a). Use also that $\tanh^{-1} \left( \frac{r_1 + r_2}{1 + r_1 r_2} \right) = \tanh^{-1}(r_1) + \tanh^{-1}(r_2)$ for $r_1, r_2 \in [0, 1]$.

Problem 2.9.10. a) Show that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic then $f$ is a contraction on $\mathbb{D}$ considered with the Poincare distance. Hint: Use Schwarz-Pick.

b) Show that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic then $f \in \text{Aut}(\mathbb{D})$ iff $f$ is an isometry of $\mathbb{D}$ considered with the Poincare distance.
2.10. Homework

Problem 2.10.1. Determine the type of singularity of each of the following function at \( z = 0 \). 
\( a) \ f(z) = \frac{1}{z}; \ b) \ f(z) = \sin \frac{1}{z}; \ c) \ f(z) = \frac{1}{e^{1/z}}. \)

Problem 2.10.2. Determine the type of singularity of each of the following function at \( z = 0 \). 
\( a) \ f(z) = \frac{\sin z}{z}; \ b) \ f(z) = \frac{\cos z}{z}; \)

Problem 2.10.3. Determine the type of singularity of each of the following function at \( z = 0 \). 
\( a) \ f(z) = \frac{e^{z} - e^{-z}}{z^{2}}; \ b) \ \sum_{k=2}^{\infty} \frac{z^{k}}{k} \)

Problem 2.10.4. Let \( f \) be holomorphic on \( U \setminus \{z_{0}\} \), \( z_{0} \in U \), \( U \)- open subset of \( \mathbb{C} \). If \( f \) has an essential singularity at \( z_{0} \), then what type of singularity does \( \frac{1}{f} \) have at \( z_{0} \)? What about when \( f \) has a removable singularity or a pole at \( z_{0} \)?

Problem 2.10.5. Let \( z_{0} \in U \)-open subset of \( \mathbb{C} \). Let 
\[ A_{j} = \{ f \text{ - holomorphic on } U \setminus \{z_{0}\} : f \text{ has a singularity of type } (j) \text{ at } z_{0} \}, \]
where \( (j) \) refers to singularities of types (i) removable, (ii) pole, or (iii) essential singularity. Is \( A_{j} \) closed under addition? Multiplication? Division?

Problem 2.10.6. Let \( D^{\times} = D \setminus \{0\} \). Prove the following two refined versions of Riemann’s theorem. 
\( a) \) If \( f \) is holomorphic on \( D^{\times} \) and \( \lim_{z \to 0} f(z) = 0 \) when \( z \to 0 \) then 0 is a removable singularity. 
\( b) \) If \( f \) is holomorphic on \( D^{\times} \) and if \( \int_{D} |f(z)|^{2} \, dx \, dy < \infty \), then 0 is a removable singularity.

Problem 2.10.7. Suppose that \( z = 0 \) for some integer \( N \geq 0 \) we have that \( z^{N} f(z) \) is bounded near \( z = 0 \). Show that \( f(z) = \frac{g(z)}{z^{N}} \) for some function \( g \) holomorphic on the unit disc \( D \). If we take \( N \) to be the smallest such integer, then the corresponding \( g \) will satisfy the condition \( g(0) \neq 0 \).

Problem 2.10.8. Suppose \( z = 0 \) is not a removable singularity of the holomorphic function \( f \) on \( D^{\times} = D(0, 1) \setminus \{0\} \). Show that \( z = 0 \) is a pole singularity iff for some integer \( N \geq 1 \) we have that \( z^{N} f(z) \) is bounded near \( z = 0 \).

Problem 2.10.9. In this problem you are asked to do what we did for \( S U(1, 1)/\{\pm I\} \) starting with another subgroup of \( GL(2, \mathbb{C}) \). Consider \( SL(2, \mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, \det A = 1 \right\} \).
\( a) \) Show that \( SL(2, \mathbb{R}) \) is a subgroup of \( GL(2, \mathbb{C}) \) (even \( GL(2 : \mathbb{R}) \)). 
\( b) \) Using Problem 2.1.2 and your knowledge of \( \text{Aut}(\mathbb{D}) \) show that the group of automorphisms of the upper half-plane \( \mathbb{H} = \{ z \in \mathbb{C} : \mathfrak{R} z > 0 \} \) is the group \( SL(2, \mathbb{R})/\{\pm I\} \).
2.11. Homework

Problem 2.11.1. Find the Laurent series on the annulus $1 < |z| < 5$ for the function

$$f(z) = \frac{z + 1}{(z - 1)(z - 5)}.$$ 

Problem 2.11.2. Calculate the annulus of convergence for each of the following Laurent series: a) $\sum_{n=-\infty}^{\infty} 2^{-n}z^n$; b) $\sum_{n=-\infty}^{\infty} 3^n z^n$.

Problem 2.11.3. Calculate the annulus of convergence (including possible boundary points) for each of the following Laurent series: a) $\sum_{n=-\infty}^{\infty} \frac{z^n}{n!}$; b) $\sum_{n=-\infty}^{\infty} \frac{z^n}{n^n}$.

Problem 2.11.4. Calculate the annulus of convergence (including possible boundary points) for each of the following Laurent series: a) $\sum_{n=-\infty}^{10} \frac{z^n}{|n|!}$; b) $\sum_{n=-20}^{\infty} n^2 z^n$.

Problem 2.11.5. Give an example of a (formal) doubly infinite series such that

$$\lim_{n \to \infty} \sum_{k=-n}^{n} a_k z^k$$

exists for some $z \neq 0$ but such that

$$\sum_{k=-\infty}^{\infty} a_k z^k$$

fails to converge for that same $z$.

Problem 2.11.6. a) Let $f_n$ be a sequence of functions that are holomorphic on the punctured unit disc $D^\times$ and suppose that each $f_n$ has a pole at $z = 0$. If the sequence $\{f_n\}$ converges uniformly on compact subsets of $D^\times$, then does the limit function $f$ necessarily have a pole at $z = 0$?

b) Answer the same question with “pole” replaced by “removable singularity” or “essential singularity.”

Problem 2.11.7. Prove that $f$ has an essential singularity on the punctured unit disc $D^\times = D(0, 1) \setminus \{0\}$ if for each positive integer $N$ there is a sequence $z_n \in D^\times$ with $\lim_{n \to \infty} z_n = 0$ and $|z_n^N f(z_n)| \geq N$. Thus, $f$ “blows up” faster than any positive power of $1/|z|$ along some sequence converging to 0.

Problem 2.11.8. Let $f$ be holomorphic on the punctured disc $D^\times = D(P, r) \setminus \{z_0\}$ and suppose that $f$ has a pole of order $k$ at $z_0$. Then the Laurent series coefficients $a_n$ of $f$ expanded about the point $z_0$, for $n = -k, -k+1, -k+2, \ldots$, are given by the formula

$$a_n = \frac{1}{(k+n)!} \left( \frac{\partial}{\partial z} \right)^{k+n} (z - z_0) \cdot f(z) \bigg|_{z = z_0}.$$ 

Problem 2.11.9. Prove that if $f$ has a non-removable singularity at $z_0$, then $e^f$ has an essential singularity at $z_0$.

Problem 2.11.10. Calculate the first four terms of the Laurent expansion of the given function $f$ about the given point $z_0$. In each case, specify the annulus of convergence of the expansion and the residue of $f$ at $z_0$.

a) $f(z) = \sin(1/z)$ at $z_0 = 0$;

b) $f(z) = \frac{z}{(z+1)^3}$ at $z_0 = -1$;

c) $f(z) = e^{z^2/z^3}$ at $z_0 = 0$;
2.12. Homework

Problem 2.12.1. Suppose \( f, g \in A(D) \), \( D = D(z_0, R_0) \), are two holomorphic functions such that \( f \) has a simple zero at \( z_0 \) while \( g(z_0) \neq 0 \). Show that \( \text{Res} \left( \frac{g}{f}, z_0 \right) = \frac{g(z_0)}{f'(z_0)} \).

Problem 2.12.2. Compute each of the following residues \( \text{Res}_f(z_0) \) for the given function and point.

a) \( z_0 = 2i \), \( f(z) = \frac{z^3}{(z-2)(z+3)} \);

b) \( z_0 = -3 \), \( f(z) = \frac{z^2 + 1}{z(z+3)} \);

c) \( z_0 = 2 \), \( f(z) = \frac{z}{z(z+1)(z-2)} \).

Problem 2.12.3. Compute each of the following residues \( \text{Res}_f(z_0) \) for the given function and point.

a) \( z_0 = i + 1 \), \( f(z) = \frac{e^z}{(z-1)^2} \);

b) \( z_0 = \pi \), \( f(z) = \frac{\cot z}{z(z+1)} \);

c) \( z_0 = 0 \), \( f(z) = \frac{\sin z}{z(z-2)(z+1)} \).

Problem 2.12.4. Use the calculus of residues to compute the integral \( \frac{1}{2\pi i} \oint f(z) \, dz \), where \( f(z) = e^{iz} / [\sin z \cos z] \) and \( \gamma \) is the quadrilateral with vertices \( \pm 5i, \pm 10 \).

Problem 2.12.5. Use the calculus of residues to compute the following integrals, where the boundary is positively oriented unless said explicitly otherwise.

a) \( \frac{1}{2\pi i} \oint_{\partial D(0,5)} f(z) \, dz \), where \( f(z) = z / [(z + 1)(z + 2i)] \);

b) \( \frac{1}{2\pi i} \oint_{\partial D(0,5)} f(z) \, dz \), where \( f(z) = e^z / [(z + 1)\sin z] \).

Problem 2.12.6. Use the calculus of residues to compute the following integrals, where the boundary is positively oriented unless said explicitly otherwise.

a) \( \frac{1}{2\pi i} \oint_{\partial D(0,8)} f(z) \, dz \), where \( f(z) = \cot z / [(z - 6i)^2 + 64] \);

b) \( \frac{1}{2\pi i} \oint_{\partial D(0,8)} f(z) \, dz \), where \( f(z) = f(z) = e^z / [z(z + 1)(z + 2)] \) and \( \gamma \) is the negatively oriented triangle with vertices \( 1 \pm i \) and \(-3 \).

Problem 2.12.7. Compute the integral \( \int_{-\infty}^{\infty} \frac{x^2}{x^4 - 4x + 5} \, dx \).

Problem 2.12.8. Compute the integral \( \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 1} \, dx \).

Problem 2.12.9. Compute the integral \( \int_{0}^{2\pi} \frac{d\theta}{(2 - \sin \theta)^2} \) by using \( z = e^{i\theta} \).

Problem 2.12.10. Compute the integral \( \int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}} \, dx \) by using \( f(z) = e^{iz} / (e^z + e^{-z}) \).
2.13. Homework

Problem 2.13.1. Suppose \( f \) is a holomorphic function on a set of the form \( \mathbb{C} \setminus D(0, R) \) for some \( R > 0 \) and \( F(z) = f(1/z), |z| < 1/R \).

a) If \( f \) has a pole of order \( k \) at \( \infty \), what property does \( 1/f \) have at \( \infty \)?

b) What property does \( 1/F(z) \) have at \( 0 \)?

Problem 2.13.2. Suppose \( f : \hat{\mathbb{C}} \to \mathbb{C} \) is holomorphic. Show that \( f = \text{const} \).

Problem 2.13.3. Calculate the residue of the given function at \( \infty \).

a) \( f(z) = z^3 - 7z^2 + 8 \);

b) \( f(z) = z^2 e^z \).

Problem 2.13.4. Calculate the residue of the given function at \( \infty \).

a) \( f(z) = e^z/p(z), \ p(z) \text{ a polynomial} \);

b) \( f(z) = p(z)e^z, \ p(z) \text{ a polynomial} \).

Problem 2.13.5. Calculate the residue of the given function at \( \infty \).

a) \( f(z) = \sin z \);

b) \( f(z) = \cot z \). Note: Be careful what you answer here!

Problem 2.13.6. Determine all poles of the function \( f(z) = 1/\sin(1/(1-z)) \) in the unit disc \( \mathbb{D} \).

Problem 2.13.7. Let \( f \) be a non-constant holomorphic function on a neighborhood of the closed disc \( \overline{D} = \overline{D}(z_0, r_0) \). Show that \( f \) has at most finitely many zeros on \( \overline{D} \).

Problem 2.13.8. Classify the singularities of the following functions on \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) and find the principal (singular) part at each of them.

a) \( f(z) = 1-z^2/1-z^2 \);

b) \( f(z) = \sin z/z^2 \).

Problem 2.13.9. Show that if \( \Omega \) is an open subset of \( \mathbb{C} \) and \( S \) is a discrete subset of \( \Omega \) then \( S \) is relatively closed in \( \Omega \).

Problem 2.13.10. Let \( \Omega \) be an open connected subset of \( \mathbb{C} \) and \( S \) a discrete subset of \( \Omega \). Show that \( S \) is at most countable. Hint: If \( K \) is a compact, \( K \subset \Omega \), what can you say about \( S \cap K \)? Sets of the form \( K_n = B(0, n) \cap \{z \in \Omega : \text{dist}(z, \partial \Omega) \geq 1/n\}, \ n \in \mathbb{N} \) might be useful.
2.14. Homework

Problem 2.14.1. Let $f$ be holomorphic on the punctured unit disc $\mathbb{D}^\times$. What kind of singularity is $z = 0$ if we know that $f$ is an injective map, i.e., $z \neq z'$ implies $f(z) \neq f(z')$?

Problem 2.14.2. Show that if $f : \mathbb{C} \to \mathbb{C}$ is a one to one holomorphic map which is conformal, then $f(z) = az + b$, i.e., $f$ is a linear function.

Hint: What kind of singularity is $\infty$? You can use without a proof that a polynomial of degree $n$ has $n$ complex roots.

Problem 2.14.3. Let $f(z) = -2/(z^2 - z - 2)$.

a) Find the Laurent expansion of $f$ in the region $|z + 1| > 3$.
b) Classify the singularities of $f$ including the $\infty$ point and find the residues at the singular points.

Problem 2.14.4. Evaluate the integral

$$\int_{\gamma} \frac{e^z}{(z + 1)(z - 2i + 1)} \, dz,$$

where $\gamma$ is the ellipse $\frac{x^2}{4} + y^2 = 1$ with positive orientation (c.c.w.).

Problem 2.14.5. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and $M(r) = \max_{|z|=r} |f(z)|$.

a) Show that $M(r)$ is a continuous and non-decreasing function.
b) Show that $M(r)$ is increasing if $f \neq \text{const}$, i.e., if $M(r) \geq M(R)$ for some $r < R$, then $f = \text{const}$.

Problem 2.14.6. Let $f$ be a function holomorphic on the unit disc $\mathbb{D}$ and $f(0) = 0$. Show that

$$F(z) = \sum_{k=1}^{\infty} f_k(z)$$

defines a holomorphic function on $\mathbb{D}$, where $f_k(z) = f(z^k)$, $k = 0, 1, 2, \ldots$.

Hint: Show that $F_n = \sum_{k=1}^{n} f_k(z)$ converges locally uniformly on $\mathbb{D}$ by using Schwarz’s lemma.
2.15. Homework

Problem 2.15.1. Estimate the number of zeros of the given function in the given region $U$. a) $f(z) = z^8 + 5z^7 - 20, \ U = D(0, 6)$; b) $f(z) = z^3 - 3z^2 + 2, \ U = D(0, 1)$; c) $f(z) = z^2 e^z - z, \ U = D(0, 2)$.

Problem 2.15.2. Show that the polynomial $p(z) = z^4 + 2z^3 + 3z^2 + z + 2$ has exactly two zeros in the right half-plane.

Problem 2.15.3. Let $p$ be a polynomial on $\mathbb{C}$. 

a) Show that if the zeros are contained in a given half-plane $V \subset \mathbb{C}$, then the same is true for the zeros of $p'$. 
b) Show that the zeros of $p'$ are contained in the closed convex hull of the zeros of $p$. The closed convex hull of a set $S$ is the intersection of all closed convex sets that contain $S$.

Problem 2.15.4. Let $P(z) = \sum_{k=0}^{n} a_k(z)^k$, where $a_k \in \mathbb{C}([0, 1])$. In other words, $P$ is a one-parameter family of polynomials of the same degree depending continuously on a parameter.

a) Let $Z = \{(z, t) : P_t(z) = 0\}$. If $(z_0, t_0) \in Z$ and $\frac{\partial P_t}{\partial c} \bigg|_{z_0 = 0} \neq 0$, then show, using the argument principle, that there is an $\epsilon > 0$ such that for all $t$ sufficiently close to $t_0$ there is a unique $z \in D(z_0, \epsilon)$ with $P_t(z) = 0$.
b) Prove that if the roots of $P_t$ are distinct (no multiple roots) then the same is true for $P_t$ for all $t$ sufficiently close to $t_0$.

Problem 2.15.5. Let $g \in A(\Omega), \ \Omega$—open subset of $\mathbb{C} \setminus \{(-\infty, 0]\}$. Show that if $g$ does not have a zero in $\Omega$ then we can find $h \in A(\Omega)$ such that $h^n = g$, i.e., we can define a holomorphic $n$-th root of $g$.

Problem 2.15.6. Give a proof of Hurwitz’ preservation of zeros theorem using the Argument Principle.

Problem 2.15.7. Let $a_n$ be a strictly decreasing sequence of positive real numbers.

a) Show that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function on $\mathbb{D}$.
b) Show that every partial sum of the above series has no zeros in $\mathbb{D}$.
c) Show that $f$ has no zeros in $\mathbb{D}$.

Problem 2.15.8. Show that $\int_{0}^{2\pi} \frac{d\theta}{a + \sin \theta} = \frac{2\pi}{\sqrt{a^2 - 1}}$ for $a > 1$.

Problem 2.15.9. Show that $\int_{0}^{\infty} \frac{x^{\lambda - 1}}{x + \alpha} dx = \frac{\pi}{\sin \pi \alpha} \frac{\sin(\lambda - 1)\theta}{\sin \theta}$, where $0 < \lambda < 1$ and $-\pi < \theta < \pi$.

Problem 2.15.10. Use the calculus of residues to sum the series $\sum_{k=0}^{\infty} \frac{1}{k^{\lambda + 1}}$. Hint: Use either the tangent or cotangent function to introduce infinitely many poles that are located at the integer values that you want to study.

Problem 2.15.11. (Integral formula for the inverse function). Suppose $f$ is holomorphic on $D(0, R)$, $f(0) = 0, \ f'(z) \neq 0$ on $D^\circ(0, R)$. Let $0 < r < R$. Show that 

$$
g(w) = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{\xi f'(\xi)}{f(\xi) - w} d\xi$$

is the inverse function of $f$. More precisely, if $m = \min_{|\zeta| = r} |f(\zeta)|$, then $g$ is holomorphic on $|w| < m$ and $f \ (g(w)) = w$ there, while $g \ (f(z)) = z$ for $z \in f^{-1}(|w| < m) \cap D(0, r)$. Hint: Use Rouche and the residue theorem by noting that $g$ is the residue of a certain function.