Boolean Algebra - Part 2

September 4, 2008

Inversion

Inversion or Complement of a Function means all 0 outputs become 1 and all 1 outputs become 0.

A	B	F	F'
0	0	0	1
0	1	1	0
1	0	1	0
1	1	0	1

Inversion

DeMorgan's Laws

(X+Y)'=X'Y'

(XY)'=X'+Y'

Proof

X	Y	X'	Y'	X+Y	(X+Y)'	X'Y'	XY	(XY)'	X'+Y'
0	0	1	1	0	1	1	0	1	1
0	1	1	0	1	0	0	0	1	1
1	0	0	1	1	0	0	0	1	1
1	1	0	0	1	0	0	1	0	0

DeMorgan's Laws

For n variables:

$$(X_1 + X_2 + X_3 + ... + X_n)' = X_1' X_2' X_3' ... X_n'$$

$$(X_1 X_2 X_3 ... X_n)' = X_1' + X_2' + X_3' + ... + X_n'$$

DeMorgan's Laws

For complex expressions, apply DeMorgan's Laws successively. Example:

$$F = A'B + AB'$$

$$\begin{aligned}(F)' &= (A'B + AB')' \\ &= (A'B)' \bullet (AB')' \\ &= (A + B') \bullet (A' + B) \\ &= AA' + AB + A'B' + BB' \\ F' &= AB + A'B'\end{aligned}$$

A	B	F	F'
0	0	0	1
0	1	1	0
1	0	1	0
1	1	0	1

DeMorgan's Laws

Another Example:

$$((a'b + 1)(cd + e' + 0))'$$

$$(a'b +1)' + (cd + e' + 0)'$$

$$((a'b)' \bullet 1') + (cd)' \bullet (e')' \bullet 0'$$

$$(a+b') \bullet 0+(c'+d') \bullet e \bullet 1$$

Note Expression is not simplified.

DeMorgan's Laws - One Step Rule

$$(f(X_1, X_2, ...X_N, 0, 1, +, \bullet))' = f(X_1', X_2', ...X_N', 1, 0, \bullet, +)$$

1. Replace all variables with the inverse.
2. Replace $+$ with $\bullet$ and $\bullet$ with $+$.
3. Replace 0 with 1 and 1 with 0.

Be careful of hierarchy...
This is the biggest source of errors, when applying DeMorgan's Laws.
Before beginning, surround all AND terms with parentheses.

Canonical Forms

Canonical Forms

- Is this equality true?
  - AD + A'D' + CD ?=? AD + A'D' + A'C
- Hard to tell
  - Both look minimized...
  - But they look different...
- For many expressions there are multiple *minimal* forms.
- Minimizing is <u>not</u> a good way to determine equality.

Canonical Forms

- A canonical form is something written in a *standard* way.
  - Two expressions which are equal will have *identical* canonical forms.
- Is there a standard (canonical) way of representing boolean expressions?
  - Yes...

Canonical Forms

- A canonical form is something written in a *standard* way.
  - Two expressions which are equal will have *identical* canonical forms.
- Is there a standard (canonical) way of representing boolean expressions?
  - Yes... Any ideas about how?

Truth tables $\Longleftrightarrow$ Canonical?

f = AD + A'D' + CD

A	B	C	D	f
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	1
0	1	0	1	0
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	1
1	1	1	0	0
1	1	1	1	1

f = AD + A'D' + A'C

A	B	C	D	f
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	1
0	1	0	1	0
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	1
1	1	1	0	0
1	1	1	1	1

Same truth table —two different minimal expressions.
Are truth tables canonical forms?

Minterm Expansions

f = AD + A'D' + A'C

A	B	C	D	f	
0	0	0	0	1	m0
0	0	0	1	0	m1
0	0	1	0	1	m2
0	0	1	1	1	m3
0	1	0	0	1	m4
0	1	0	1	0	m5
0	1	1	0	1	m6
0	1	1	1	1	m7
1	0	0	0	0	m8
1	0	0	1	1	m9
1	0	1	0	0	m10
1	0	1	1	1	m11
1	1	0	0	0	m12
1	1	0	1	1	m13
1	1	1	0	0	m14
1	1	1	1	1	m15

- Truth table is unique <u>if</u> rows in set order
- Each row can be numbered
- Call each row's AND term a minterm

$$f(A,B,C,D) = m0 + m2 + m3 + m4 + m6$$
$$m7 + m9 + m11 + m13 + m15$$
$$f(A,B,C,D) = \sum \mathrm{m}(0,2,3,4,6,7,9,11,13,15)$$

Minterms

- Are these minterms of four variables?

abcd
ab'cd'
ab'c
a'bc'bd

Minterms

- Are these minterms of four variables?

abcd	yes
ab'cd'	yes
ab'c	no, only 3 literals
a'bc'bd	no, b more than once

Minterms have names

A	B	C	f	
0	0	0	1	m0 = A'B'C'
0	0	1	0	m1 = A'B'C
0	1	0	1	m2 = A'BC'
0	1	1	1	m3 = A'BC
1	0	0	1	m4 = AB'C'
1	0	1	0	m5 = AB'C
1	1	0	1	m6 = ABC'
1	1	1	1	m7 = ABC

Note: the order of the variables is significant.

Minterm Expansion

- A minterm expansion is *unique*.

$$f(A, B, C, D) = \sum \mathsf{m}(0, 2, 3, 7)$$

- Useful for:
  - Proving equality
  - Shorthand for representing boolean expressions

Maxterms are POS Equivalents to Minterms

A	B	C	f	Minterms	Maxterms
0	0	0	1	m0 = A'B'C'	M0 = m0' = A+B+C
0	0	1	0	m1 = A'B'C	M1 = m1' = A+B+C'
0	1	0	1	m2 = A'BC'	M2 = m2' = A+B'+C
0	1	1	1	m3 = A'BC	M3 = m3' = A+B'+C'
1	0	0	1	m4 = AB'C'	M4 = m4' = A'+B+C
1	0	1	0	m5 = AB'C	M5 = m5' = A'+B+C'
1	1	0	1	m6 = ABC'	M6 = m6' = A'+B'+C
1	1	1	1	m7 = ABC	M7 = m7' = A'+B'+C'

The maxterms are the inverse of the minterms.

Maxterm Expansion

- Are these maxterms of four variables?

$$a+b+c+d$$
$$a+b'+c+d'$$
$$a+b'+d'$$
$$a'+b+c'+a'+d$$

Maxterm Expansion

- Are these maxterms of four variables?

a+b+c+d	yes
a+b'+c+d'	yes
a+b'+d'	no, only three literals
a'+b+c'+a'+d	no, a' more than once

Maxterm Expansion

Any function can be written as a product of maxterms. This is called a:

Standard Product of Sums
(Standard POS)

Use the **Zeros** for f to write the POS:

A	B	C	f	
0	0	0	0	M0
0	0	1	1	M1
0	1	0	0	M2
0	1	1	0	M3
1	0	0	0	M4
1	0	1	1	M5
1	1	0	1	M6
1	1	1	1	M7

$$f(A,B,C) = M_0 M_2 M_3 M_4$$

$$f(A,B,C) = \prod M(0,2,3,4)$$

$$f = (A+B+C)(A+B'+C)(A+B'+C')(A'+B+C)$$

Maxterm Expansion

- Maxterm expansions are useful
  - For the same things as minterm expansions
- We will use them when we want POS instead of SOP representations

Minterm / Maxterm Relationships

For any function, if a minterm is in the Minterm Expansion, the corresponding maxterm is not in the Maxterm Expansion.

A	B	C	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$f(A,B,C) = m_1 + m_5 + m_6 + m_7$$
$$f(A,B,C) = M_0 M_2 M_3 M_4$$

Makes it easy to convert between SOP and POS.

Minterm Example

- Is this equation true?

$$AD + A'D' + CD = AD + A'D' + A'C$$

1. Write each side as a truth table.

2. Compare which minterms each contains

Minterm expansions $\iff$ shorthand truth table

Minterm Example

- Minimize the following:

$$f(A, B, C) = m_1 + m_5 + m_7$$

Minterm Example

- Minimize the following:

$$f(A,B,C) = m_1 + m_5 + m_7$$

1. Write out as SOP $\quad f = A'B'C + AB'C + ABC$

2. Minimize

$$\begin{aligned} f &= A'B'C + AB'C + ABC \\ &= A'B'C + AB'C + AB'C + ABC \\ &= (A' + A)(B'C) + AC(B' + B) \\ &= B'C + AC \end{aligned}$$

minterm expansions $\Longleftrightarrow$ shorthand expressions

Maxterm Example

- Minimize the following:

$$f(A, B, C) = M_1 M_5 M_7$$

1. Write out as SOP

$$f = (A+B+C')(A'+B+C')(A'+B'+C')$$

2. Minimize

$$f = (B+C')(A'+C')$$

maxterm expansions $\Longleftrightarrow$ shorthand expressions...

Minterm/Maxterm Example

Convert the following to POS:

$$F = AB + C$$

Minterm/Maxterm Example

Convert the following to POS:

$$F = AB + C$$

1. Write minterm expansion

   (use truth table if it helps)

2. Convert to maxterm expansion

3. Write POS from maxterm expansion

4. Simplify if desired

Minterm/Maxterm Example

Convert the following to POS:

$$F = AB + C$$

1. Write minterm expansion
   $F = m_1 + m_3 + m_5 + m_6 + m_7$ (use truth table if it helps)

2. Convert to maxterm expansion
   $F = M_0 M_2 M_4$

3. Write POS from maxterm expansion
   $F = (A + B + C)(A + B' + C)(A' + B + C)$

4. Simplify if desired
   $F = (B + C)(A + C)$

Algebraic Simplification

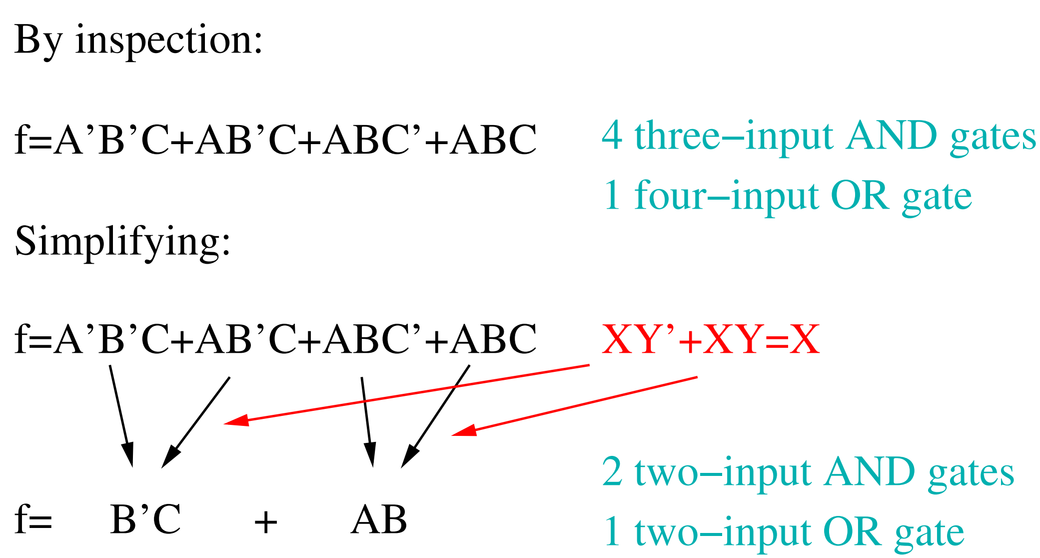
Algebraic Simplification

Objectives

- Why simplify?

  Simpler equations lead to less hardware, faster, cheaper, and less power.

A	B	C	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1



Hardware Cost

- Exact cost depends on technology
  - nMOS, CMOS, TTL, I2L, GaAs, FPGA, ...
- In general:
  - Fewer literals $\Longleftrightarrow$ smaller/faster/lower power circuit
  - Inverters don't count
    - Free in some technologies
    - Insignificant difference in others

Algebraic Simplification: Which Theorems To Use?

Essential Identities	
X + 0 = X	X • 1 = X
X + 1 = 1	X • 0 = 0
X + X = x	X • X = X
(X')' = X	
X + X' = 1	X • X' = 0
Essential Commutative, Associative, Distributive and DeMorgan's Laws	
X + Y = Y + X	X • Y = Y • X
(X + Y) + Z = X + (Y + Z) = X+Y+Z	(XY)Z = X(YZ) = XYZ
X(Y + Z) = XY + XZ	X + YZ = (X + Y) (X + Z)
$[f(X_1, X_2, ...X_N, 0, 1, +, \bullet)]' = f(X_1', X_2', ...X_N', 1, 0, \bullet, +)$	
Essential	
X Y + X Y' = X	(X + Y) (X + Y') = X
X + XY = X	X(X+Y)=X
Useful, hard to remember, easy to re-derive	
(X + Y') Y = XY	XY' + Y = X + Y

Suggestions:

1. Focus on blue ones!
2. Create duals onright as needed.
3. Be familiar with the last group.

Four Methods of Algebraic Simplification

1. Combine terms
2. Eliminate terms
3. Eliminate literals
4. Add redundant terms

Methods of Algebraic Simplification

1. Combine terms

Use XY+XY'=X

ABC'+A'B'C+A'BC'=

BC'+A'B'C

2. Eliminate terms

3. Eliminate literals

4. Add redundant terms

Methods of Algebraic Simplification

1. Combine terms

2. Eliminate terms

$$\begin{aligned} f &= A'B + B(A'D + C) \\ &= A'B + A'BD + BC \\ &= A'B + BC \end{aligned}$$

Use $X + XY = X$

3. Eliminate literals

4. Add redundant terms

Methods of Algebraic Simplification

1. Combine terms

2. Eliminate terms

3. Eliminate literals

$$
\begin{aligned}
f &= A'B + A'B'C'D + ABCD' \\
&= A'(B + B'C'D') + ABCD' \\
&= A'(B + C'D') + ABCD' \\
&= A'B + A'C'D' + ABCD' \\
&= A'C'D' + B(A' + ACD') \\
&= A'C'D' + B(A' + CD') \\
&= A'B + BCD' + A'C'D'
\end{aligned}
$$

Use X + X' Y = X + Y.

4. Add redundant terms

Methods of Algebraic Simplification

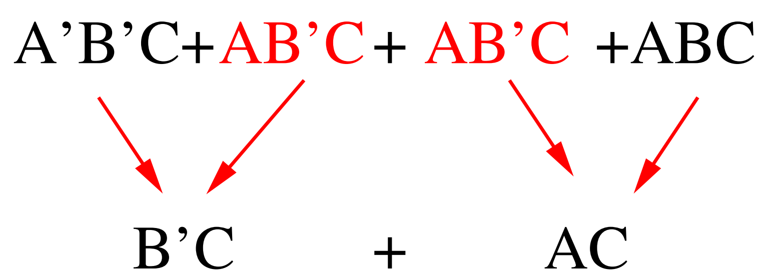
1. Combine terms

2. Eliminate terms

3. Eliminate literals

4. Add redundant terms

$$A'B'C + AB'C + ABC$$



Proving an Equation is True

1. Construct a truth table for both sides
   (a) Tedious, not very elegant
   (b) Computers love this method
2. Convert Both Sides to minterm/maxterm expansion
3. Manipulate one side algebraically to equal the other
4. Perform the same operation on both sides

Perform the same operation on both sides

- Valid Operations
  - Complement
  - Construct the duals
- Invalid Operations
  - Add a term to both sides (not reversible)
  - Multiply (AND) both sides by a term (not reversible)
  - Subtract a term from both sides (subtraction?)

Proving an Equation – Add a term to both sides

Prove $y = z$

Add 1 to both sides.

$$y + 1 = z + 1$$

Use $X + 1 = 1$

$$1 = 1$$

Therefore, $y = z$.

Proving an Equation – Add a term to both sides

Prove $y = z$

Add 1 to both sides.

$$y + 1 = z + 1$$

Use $X + 1 = 1$

$$1{=}1$$

Therefore, $y = z$.

No, NO, **NO**! You *cannot* add an arbitrary term to both sides. However, this does not prevent you from duplicating an existing term:

$$Y = Z \iff Y{+}Y = Z$$

Proving an Equation – Multiply (AND) both sides by a term

Prove $y = z$

Multiply both sides by 0

$$y \bullet 0 = z \bullet 0$$

Use $X \bullet 0 = 0$

$$0 = 0$$

Therefore, $y = z$.

Proving an Equation – Multiply (AND) both sides by a term

Prove y = z

Multiply both sides by 0

$$y \bullet 0 = z \bullet 0$$

Use $X \bullet 0 = 0$

$$0{=}0$$

Therefore, y = z.

No, NO, **NO**! You *cannot* multiply both size by an arbitrary term. However, this does not prevent you from duplicating an existing term:

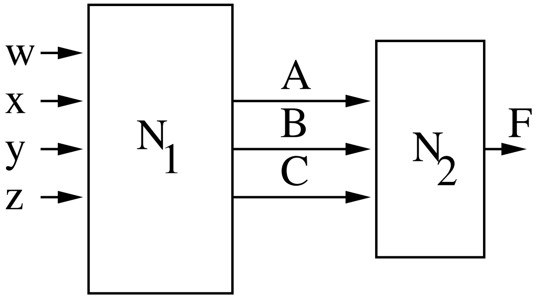
$$Y \bullet Z = Y \bullet Y \bullet Z$$

Proving an Equation is False

- Show one combination of values which is false (e.g., a counter example).
- Use Boolean algebra manipulation
  - Reduce to sum of products form
  - Minimize
  - Compare the 2 sides
  - Be careful—some expressions have more than one minimal form ...

Incompletely Specified Functions

Incompletely Specified Functions



Known: N_1 does not generate outputs A B C with values of 101 or 111.

A	B	C	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	X
1	1	0	1
1	1	1	X

↙ The X's represent don't care since those input combos never occur, you don't care what the output is.

Incompletely Specified Functions

A	B	C	f	f_1	f_2	f_3	f_4
0	0	0	0	0	0	0	0
0	0	1	1	1	1	1	1
0	1	0	0	0	0	0	0
0	1	1	0	0	0	0	0
1	0	0	1	1	1	1	1
1	0	1	X	0	0	1	1
1	1	0	1	1	1	1	1
1	1	1	X	0	1	0	1

Multiple ways to choose the X's when minimizing. The cost of the *minimal* solution will depend on which combination of values are chosen for the X's.

$$f_1 = A'B'C + AB'C + ABC'$$
$$f_1 = A'B'C + AC'$$

$$f_2 = A'B'C + AB'C' + ABC' + ABC$$
$$f_2 = A'B'C + AC'(B' + B) + AB(C' + C)$$
$$f_2 = A'B'C + AC' + AB$$

Incompletely Specified Functions

A	B	C	f	f_1	f_2	f_3	f_4
0	0	0	0	0	0	0	0
0	0	1	1	1	1	1	1
0	1	0	0	0	0	0	0
0	1	1	0	0	0	0	0
1	0	0	1	1	1	1	1
1	0	1	X	0	0	1	1
1	1	0	1	1	1	1	1
1	1	1	X	0	1	0	1

$$
\begin{aligned}
f_1 &= A'B'C + AC' \\
f_2 &= A'B'C + AC' + AB \\
f_3 &= A'B'C + AB'C' + AB'C + ABC' \\
&= A'C(B + B') + AB'(C' + C) \\
f_3 &= A'C + AB'
\end{aligned}
$$

$$
\begin{aligned}
f_4 &= A'B'C + AB'C' + AB'C + ABC' + ABC \\
&= (A' + A)B'C + AB'(C' +' C) + AB(C' + C) \\
&= B'C + AB' + AB \\
f_4 &= B'C + A
\end{aligned}
$$

f_4 is the best.

Incompletely Specified Functions

- If the function has n don't cares
  - Must solve 2n truth tables
  - Ouch!
- A better way will be shown in a later chapter...

Don't Cares - Minterm Representation

A	B	C	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	**X**
1	1	0	1
1	1	1	**X**

Use $d_0..d_7$ to represent don't care minterms

$$F = m_1 + m_4 + m_6 + d_5 + d_7$$

$$F = \sum m(1,4,6) + \sum d(5,7)$$

Don't Cares - Maxterm Representation

A	B	C	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	**X**
1	1	0	1
1	1	1	**X**

Use $D_0..D_7$ to represent don't care maxterms

$$F = M_0 M_2 M_3 D_5 D_7$$

$$F = \prod M(0,2,3) \prod D(5,7)$$