1
Examples for Chapter 1, Linear Algebra

1.1 Numbers

If $z = 1 + i$, then

$$\frac{1}{z} = \frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} = \frac{1}{2} - \frac{i}{2}. \tag{1.1}$$

In other symbols,

$$\frac{1}{z} = \frac{z^*}{z\, z^*} = \frac{z^*}{|z|^2}. \tag{1.2}$$

Don't worry about Grassmann numbers. Suppose a, b are complex numbers and θ is a Grassmann number or equivalently a Grassmann variable. Then because $\theta^2 = 0$, the inverse of $\alpha = a + b\theta$ is

$$\frac{1}{\alpha} = \frac{1}{a + b\theta} = \frac{a - b\theta}{(a + b\theta)(a - b\theta)} = \frac{a - b\theta}{a^2} = \frac{1}{a} - \frac{b\theta}{a^2} \tag{1.3}$$

and $e^{a\theta}$ is

$$e^{a\theta} = 1 + a\theta. \tag{1.4}$$

1.2 Arrays

You know about inner products like

$$\boldsymbol{k} \cdot \boldsymbol{x} \equiv \vec{k} \cdot \vec{x} = k_1 x_1 + k_2 x_2 + k_3 x_3. \tag{1.5}$$

Here's a Lorentz-invariant inner product of two 4-vectors

$$p \cdot x = \vec{p} \cdot \vec{x} - p^0 x^0. \tag{1.6}$$

Particle physicists often add a minus sign and write

$$p \cdot x = p^0 x^0 - \vec{p} \cdot \vec{x}. \tag{1.7}$$

They do that to have

$$p^2 = p \cdot p = p^0 p^0 - \vec{p} \cdot \vec{p} = \left(\frac{E}{c}\right)^2 - (\vec{p})^2 = m^2 c^2. \tag{1.8}$$

1.3 Matrices

Vectors must have the same number of components to have an inner product, but any two vectors can have two kinds of outer product. For instance,

$$\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}4 & 5 & 6 & 7\end{pmatrix} = \begin{pmatrix}4 & 5 & 6 & 7\\ 8 & 10 & 12 & 14\\ 12 & 15 & 18 & 21\end{pmatrix} = \begin{pmatrix}4 & 5 & 6 & 7\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} \tag{1.9}$$

and

$$\begin{pmatrix}4\\5\\6\\7\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix} = \begin{pmatrix}4 & 8 & 12\\ 5 & 10 & 15\\ 6 & 12 & 18\\ 7 & 14 & 21\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}4\\5\\6\\7\end{pmatrix}. \tag{1.10}$$

The second of these two equations is the transpose (denoted by a T) of the first:

$$\begin{aligned}\begin{pmatrix}4\\5\\6\\7\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix} &= \begin{pmatrix}4 & 8 & 12\\ 5 & 10 & 15\\ 6 & 12 & 18\\ 7 & 14 & 21\end{pmatrix} = \begin{pmatrix}4 & 5 & 6 & 7\\ 8 & 10 & 12 & 14\\ 12 & 15 & 18 & 21\end{pmatrix}^{\mathsf{T}}\\ &= \left(\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}4 & 5 & 6 & 7\end{pmatrix}\right)^{\mathsf{T}} = \begin{pmatrix}4 & 5 & 6 & 7\end{pmatrix}^{\mathsf{T}}\begin{pmatrix}1\\2\\3\end{pmatrix}^{\mathsf{T}} \\ &= \begin{pmatrix}4\\5\\6\\7\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix}.\end{aligned} \tag{1.11}$$

Example 1.1 (Cross-products) Three-dimensional space is special in that it has a cross-product. The cross-product $\boldsymbol{A} \times \boldsymbol{B}$ of two 3-vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is the 3-vector whose ith component is the sum

$$\left(\boldsymbol{A} \times \boldsymbol{B}\right)_i = \sum_{j,k=1}^{3} \epsilon_{ijk} A_j \, B_k \tag{1.12}$$

in which ϵ_{ijk} is totally antisymmetric with $\epsilon_{123} = 1$. So $\epsilon_{213} = -1$, $\epsilon_{113} = 0$, $\epsilon_{231} = 1$, etc. $\square$

The trace of a matrix is the sum of its diagonal elements. If

$$A = \begin{pmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{pmatrix}, \tag{1.13}$$

then its trace is

$$\mathrm{Tr} A = 32 = \mathrm{Tr}\big(A^{\mathsf{T}}\big). \tag{1.14}$$

If V is a vector with complex components

$$V = \begin{pmatrix} 1+i \\ -i \\ 3+i \end{pmatrix}, \tag{1.15}$$

then its complex conjugate and adjoint (or hermitian adjoint) are

$$V^* = \begin{pmatrix} 1-i \\ i \\ 3-i \end{pmatrix} \quad \text{and} \quad V^\dagger = \begin{pmatrix} 1-i & i & 3-i \end{pmatrix}. \tag{1.16}$$

The complex conjugate of the matrix

$$B = \begin{pmatrix} i & 2 & i \\ 3 & i & 4 \\ 1 & -i & 6 \end{pmatrix}, \tag{1.17}$$

is

$$B^* = \begin{pmatrix} -i & 2 & -i \\ 3 & -i & 4 \\ 1 & i & 6 \end{pmatrix}. \tag{1.18}$$

The adjoint or hermitian adjoint of the matrix

$$B = \begin{pmatrix} i & 2 & i \\ 3 & i & 4 \\ 1 & -i & 6 \end{pmatrix} \tag{1.19}$$

is the complex conjugate of its transpose

$$B^\dagger = \begin{pmatrix} -i & 3 & 1 \\ 2 & -i & i \\ -i & 4 & 6 \end{pmatrix}. \tag{1.20}$$

The matrix

$$C = \begin{pmatrix} i & 2 & i \\ 2 & i & 4 \\ i & 4 & 6 \end{pmatrix} \tag{1.21}$$

is symmetric.

A matrix is hermitian if it is equal to its adjoint. The matrix

$$D = \begin{pmatrix} 1 & i & -i \\ -i & 2 & 2i \\ i & -2i & -7 \end{pmatrix} \tag{1.22}$$

is hemitian. Its diagonal elements are 1, 2, and -7, and they are real. All the diagonal elements of every hermitian matrix are real.

The inverse A^{-1} of a matrix A is a matrix that does this

$$A\,A^{-1} = A^{-1}\,A = I \tag{1.23}$$

in which the identity matrix I is a diagonal matrix with 1 on its main diagonal and 0's elsewhere

$$I = 1 \quad \text{or} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{etc.} \tag{1.24}$$

The Matlab command

```
A =[  1 2 3 ;   4 5 6 ;   -1 2 -3 ]
```

generates the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 2 & -3 \end{pmatrix}, \tag{1.25}$$

and the Matlab command

```
inv(A)
```

generates its inverse

$$A^{-1} = \begin{pmatrix} -1.1250 & 0.5000 & -0.1250 \\ 0.2500 & 0 & 0.2500 \\ 0.5417 & -0.1667 & -0.1250 \end{pmatrix}. \tag{1.26}$$

A matrix is unitary if its adjoint is its inverse

$$U\,U^\dagger = U^\dagger\,U = I. \tag{1.27}$$

A real unitary matrix is orthogonal, and

$$O\,O^{\dagger} = O\,O^{\mathsf{T}} = O^{\dagger}\,O = O^{\mathsf{T}}\,O = I. \tag{1.28}$$

1.4 Matrix Multiplication and Commutation Relations

```
>> X = [ 0 -1 0 0 ; -1 0 0 1; 0 0 0 0; 0 -1 0 0 ]

X =

     0    -1     0     0
    -1     0     0     1
     0     0     0     0
     0    -1     0     0

>> Y = [0 0 -1 0; 0 0 0 0; -1 0 0 1; 0 0 -1 0]

Y =

     0     0    -1     0
     0     0     0     0
    -1     0     0     1
     0     0    -1     0

>> J = [0 0 0 0; 0 0 -1 0; 0 1 0 0; 0 0 0 0]

J =

     0     0     0     0
     0     0    -1     0
     0     1     0     0
     0     0     0     0

>> J*X -X*J - Y

ans =

     0     0     0     0
     0     0     0     0
     0     0     0     0
```

```
     0     0     0     0
```

So $[J, X] = Y$.

```
>> J*Y - Y*J + X

ans =

     0     0     0     0
     0     0     0     0
     0     0     0     0
     0     0     0     0
```

So $[J, Y] = -X$. Thanks to Yu Chia Lin for pointing out a typo here.

The polarization vectors for a massless particle moving in the 3-direction are $\epsilon_{\pm}$

```
>> epp =  [ 0; 1; -1i; 0]

epp =

   0.0000 + 0.0000i
   1.0000 + 0.0000i
   0.0000 - 1.0000i
   0.0000 + 0.0000i

>> epm = [ 0; 1; 1i; 0]

epm =

   0.0000 + 0.0000i
   1.0000 + 0.0000i
   0.0000 + 1.0000i
   0.0000 + 0.0000i

>> syms a b

>> (a*X + b*Y)*epp

ans =
```

```
 - a + b*1i
          0
          0
 - a + b*1i

>> (a*X + b*Y)*epm

ans =

 - a - b*1i
          0
          0
 - a - b*1i
```

So

$$(aX + bY)^i{}_k \epsilon^k_\pm = (-a \pm ib)\, p^i/p^0. \tag{1.29}$$

1.5 Vector Spaces

You can multiply vectors by real and by complex numbers

$$2\begin{pmatrix} 3 \\ i \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 2i \\ -8 \end{pmatrix} \quad \text{and} \quad (1+i)\begin{pmatrix} 3 \\ i \\ -4 \end{pmatrix} = \begin{pmatrix} 3+3i \\ -1+i \\ -4-4i \end{pmatrix}. \tag{1.30}$$

And you can add vectors that are multiplied by (real or) complex numbers z and w

$$z\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + w\begin{pmatrix} 4i \\ 5 \\ -6i \end{pmatrix} = \begin{pmatrix} z+4iw \\ 2z+5w \\ 3z-6wi \end{pmatrix}. \tag{1.31}$$

1.6 Vector Spaces and Dimension

In quantum mechanics, we often represent systems by states, e.g., $|\ell, m\rangle$ represents a state that has angular momentum $\ell\hbar$ and angular momentum $m\hbar$ in the z direction. Here m can be $-\ell$, $-\ell+1$, $\ldots \ell-1$, ℓ. So there are $2\ell+1$ states $|\ell, m\rangle$ for a given ℓ. We can add states in quantum mechanics, so the possible states of angular momentum ℓ are a sum from $m = -\ell$ to

$m = \ell$ of the states $|\ell, m\rangle$, each multiplied by an arbitrary complex number z_m

$$|\ell, z\rangle = \sum_{m=-\ell}^{\ell} z_m |\ell, m\rangle. \tag{1.32}$$

The states $|\ell, m\rangle$ span a vector space of dimension $2\ell+1$. They form a basis for this space because every state in it can be written as a linear combination of the $|\ell, m\rangle$ as in equation (1.32). They are the orthonormal

$$\langle \ell, m | \ell', m' \rangle = \delta_{mm'} \tag{1.33}$$

eigenstates of the z component of the angular momentum $\boldsymbol{L}$

$$L_z |\ell, m\rangle = m\hbar\, |\ell, m\rangle. \tag{1.34}$$

If $\ell = 1$, then in a basis in which L_z is diagonal

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{1.35}$$

its eigenstates are

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}. \tag{1.36}$$

These states span a space of dimension 3. The state $|1, m\rangle$ is an eigenstate of L_z with eigenvalue $m\hbar$

$$L_z |1, m\rangle = m\hbar |1, m\rangle \tag{1.37}$$

for $m = -1, 0,$ and 1.

If $\ell = 2$, the diagonal form of L_z is

$$L_z = \hbar \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}. \tag{1.38}$$

In this basis, the states $|2,m\rangle$ are

$$|2,2\rangle = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \quad |2,1\rangle = \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \quad |2,0\rangle = \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},$$
$$|2,-1\rangle = \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}, \quad \text{and} \quad |2,-2\rangle = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}. \tag{1.39}$$

The state $|2,m\rangle$ is an eigenstate of L_z with eigenvalue $m\hbar$

$$L_z|2,m\rangle = m\hbar|2,m\rangle \tag{1.40}$$

for $m = -2, -1, 0, 1$, and 2. The states (1.39) are orthonormal and complete, i.e., they span the space of their dimension, 5.

If $\ell = 1/2$, the angular-momentum matrices $\boldsymbol{S}$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \tag{1.41}$$

multiplied by $\hbar/2$

$$\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}. \tag{1.42}$$

So the states $|\frac{1}{2},m\rangle$

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{1.43}$$

are eigenstates of S_z with eigenvalues $\pm\hbar/2$

$$S_z|\tfrac{1}{2},m\rangle = m\hbar|\tfrac{1}{2},m\rangle. \tag{1.44}$$

1.7 Eigenstates and Eigenvalues

What about the eigenstates and eigenvalues of $S_x = (\hbar/2)\,\sigma_1$? We call these states $|\frac{1}{2},m,x\rangle$, set

$$|\tfrac{1}{2},m,x\rangle = \begin{pmatrix}a\\b\end{pmatrix}, \tag{1.45}$$

and require that

$$\sigma_1|\tfrac{1}{2}, m, x\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2m \begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}. \tag{1.46}$$

So if $m = 1/2$, then $a = b$, while if $m = -1/2$, then $a = -b$. Normalizing these states and setting their arbitrary phases equal to unity, we find that

$$|\tfrac{1}{2}, \tfrac{1}{2}, x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |\tfrac{1}{2}, -\tfrac{1}{2}, x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{1.47}$$

Similarly, we call $|\tfrac{1}{2}, m, y\rangle$ the eigenstates and eigenvalues of $S_y = (\hbar/2)\,\sigma_2$, set

$$|\tfrac{1}{2}, m, y\rangle = \begin{pmatrix} c \\ d \end{pmatrix}, \tag{1.48}$$

and require that

$$\sigma_2|\tfrac{1}{2}, m, y\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = 2m \begin{pmatrix} c \\ d \end{pmatrix} = \pm \begin{pmatrix} c \\ d \end{pmatrix}. \tag{1.49}$$

So if $m = 1/2$, then $c = -id$, while if $m = -1/2$, then $c = id$. Normalizing these states and setting their arbitrary phases equal to unity, we find that

$$|\tfrac{1}{2}, \tfrac{1}{2}, y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad |\tfrac{1}{2}, -\tfrac{1}{2}, y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{1.50}$$

Matlab knows how to do this faster than we can do it with our fingers. The Matlab command

```
s1 = [ 0 1; 1 0]
```

makes σ_1, and the command

```
[V,D] = eig(s1)
```

returns the eigenvectors of σ_1 as

```
V =
   -0.7071    0.7071
    0.7071    0.7071
```

and its eigenvalues $2m$ as

```
D =
    -1      0
     0      1 .
```

These answers are equivalent to those (1.47) we just computed because Matlab reports eigenvalues in increasing order and because eigenvectors are defined only up to an overall factor or up to an overall phase if they are normalized.

The Matlab command

```
s2 = [ 0 -i; i 0]
```

makes σ_2, and the command

```
[V,D] = eig(s2)
```

returns the eigenvectors of σ_2 as

```
V =
   0.0000 - 0.7071i   0.0000 - 0.7071i
  -0.7071 + 0.0000i   0.7071 + 0.0000i
```

and its eigenvalues $2m$ as

```
D =
    -1      0
     0      1 .
```

As the size of the matrix increases, so does the utility of Matlab. The Matlab command

```
A =[  1 2 3 ;   4 5 6 ;  -1 2 -3 ]
```

generates the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 2 & -3 \end{pmatrix}, \tag{1.51}$$

and the Matlab command

```
[V,D] = eig(A)
```

returns the eigenvectors as

```
V =
   -0.3530   -0.8654   -0.3682
   -0.9246    0.0842   -0.4126
   -0.1431    0.4940    0.8332
```

and their eigenvalues as

```
D =
    7.4556         0         0
         0   -0.9072         0
         0         0   -3.5484 .
```

To have Matlab check this we enter

```
A*V
V*D
```

and get

```
ans =
   -2.6315    0.7851    1.3064
   -6.8936   -0.0764    1.4639
   -1.0670   -0.4481   -2.9566

ans =
   -2.6315    0.7851    1.3064
   -6.8936   -0.0764    1.4639
   -1.0670   -0.4481   -2.9566
```

which verifies Matlab's computation.

1.8 Linear Independence

The usual unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ of 3-space are linearly independent. We can write every point in 3-space uniquely as

$$\boldsymbol{r} = r_x \hat{\boldsymbol{x}} + r_y \hat{\boldsymbol{y}} + r_z \hat{\boldsymbol{z}}. \tag{1.52}$$

In terms of $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$, the only way to write the origin is

$$\boldsymbol{0} = 0\,\hat{\boldsymbol{x}} + 0\,\hat{\boldsymbol{y}} + 0\,\hat{\boldsymbol{z}}. \tag{1.53}$$

This silly equation is nearly the definition of linear independence: The n vectors $V_1, \ldots, V_n$ (all with the same number m of components) are linearly independent if (and only if) the only way to write the zero m-vector is

$$0 = 0\,V_1 + 0\,V_2 + \cdots + 0\,V_{n-1} + 0\,V_n. \tag{1.54}$$

The vectors V_i might be linearly independent if

$$m \geq n \tag{1.55}$$

but can't be linearly independent if

$$m < n. \tag{1.56}$$

The n real (complex) vectors $V_1, \ldots, V_n$ (all with the same number m of components) are linearly dependent if (and only if) one can find n real (complex) numbers r_i (z_i), not all of which are zero, with which to write the zero m-vector as

$$\begin{aligned} 0 &= r_1\, V_1 + r_2\, V_2 + \cdots + r_{n-1}\, V_{n-1} + r_n\, V_n \\ &\text{or as} \\ 0 &= z_1\, V_1 + z_2\, V_2 + \cdots + z_{n-1}\, V_{n-1} + z_n\, V_n \end{aligned} \tag{1.57}$$

So any three 2-vectors are linearly dependent.

1.9 Determinants

Levi-Civita invented various totally antisymmetric symbols such as the 2-index symbol

$$1 = \epsilon_{12} = -\epsilon_{21} \quad 0 = \epsilon_{11} = \epsilon_{22} \tag{1.58}$$

and the 3-index symbol

$$\begin{aligned} 1 &= \epsilon_{123} = \epsilon_{231} = \epsilon_{312} \\ -1 &= \epsilon_{213} = \epsilon_{132} = \epsilon_{321}. \end{aligned} \tag{1.59}$$

The Levi-Civita symbols are zero whenever any index occurs more than once

$$\begin{aligned} 0 &= \epsilon_{111} = \epsilon_{112} = \epsilon_{113} \\ 0 &= \epsilon_{221} = \epsilon_{222} = \epsilon_{223} \\ &\vdots \end{aligned} \tag{1.60}$$

In terms of the L-C symbols, the **determinant** of a 2×2 matrix A is

$$\det A = |A| = A_{11}A_{22} - A_{21}A_{12} = \sum_{i,j=1}^{2} \epsilon_{ij} A_{i1} A_{j2}, \tag{1.61}$$

that of a 3×3 matrix is

$$\det A = \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, A_{i1} A_{j2} A_{k3} = \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, A_{1i} A_{2j} A_{3k}, \tag{1.62}$$

that of a 4×4 matrix is

$$\det A = \sum_{i,j,k,\ell=1}^{4} \epsilon_{ijk\ell}\, A_{i1}A_{j2}A_{k3}A_{\ell 4} = \sum_{i,j,k=1}^{3} \epsilon_{ijk\ell}\, A_{1i}A_{2j}A_{3k}A_{4\ell} \qquad (1.63)$$

and so forth.

If the columns of a 2×2 matrix A are two 2-vectors V^1 and V^2, then its determinant is

$$\det A = \sum_{i,j=1}^{2} \epsilon_{ij}\, V_i^1 V_j^2 = V_1^1 V_2^2 - V_2^1 V_1^2. \qquad (1.64)$$

Note that because the L-C symbol is antisymmetric

$$\sum_{i,j=1}^{2} \epsilon_{ij}\, V_i^2 V_j^2 = V_1^2 V_2^2 - V_2^2 V_1^2 = 0. \qquad (1.65)$$

So the antisymmetry of the L-C symbol means that $\det A$ does not change if we add to V^1 a multiple of V^2:

$$\sum_{i,j=1}^{2} \epsilon_{ij}\, (V_i^1 + wV_i^2)V_j^2 = \sum_{i,j=1}^{2} \epsilon_{ij}\, V_i^1 V_j^2 = \det A. \qquad (1.66)$$

If the 2-vectors V^1 and V^2 are linearly dependent, then we can find numbers z_1 and z_2 such that

$$0 = z_1 V^1 + z_2 V^2 \quad \text{and so too} \quad 0 = V^1 + (z_2/z_1)V^2. \qquad (1.67)$$

So setting $V^1 = -z_1/z_2 V^2 \equiv -wV^2$, we see that the determinant of a matrix made of two linearly dependent 2-vectors vanishes

$$\det A = \sum_{i,j=1}^{2} \epsilon_{ij}\, V_i^1 V_j^2 = \sum_{i,j=1}^{2} \epsilon_{ij}\, (-wV_i^2)V_j^2 = 0. \qquad (1.68)$$

This result generalizes to every number of dimensions: The determinant of an $n \times n$ matrix whose columns are linearly dependent vanishes.

1.10 Eigenstates and Eigenvalues

Now let's consider eigenvector problem

$$AZ = \lambda Z \qquad (1.69)$$

in which A is a 2×2 matrix, Z is a 2-vector, and λ is an eigenvalue. Subtracting λ times the 2×2 identity matrix from both sides of this equation, we have

$$(A - \lambda I)Z = 0. \tag{1.70}$$

Now defining V^1 and V^2 to be the two columns of the 2×2 matrix $A - \lambda I$

$$V^1 = \begin{pmatrix} A_{11} - \lambda \\ A_{21} \end{pmatrix} \quad \text{and} \quad V^2 = \begin{pmatrix} A_{12} \\ A_{22} - \lambda \end{pmatrix}, \tag{1.71}$$

we see that the eigenvector equation

$$0 = (A - \lambda I)Z = z_1 V^1 + z_2 V^2 \tag{1.72}$$

is the statement that the two vectors V^1 and V^2 are linearly dependent. But if V^1 and V^2 are linearly dependent, then the determinant of the matrix $A - \lambda I$ must vanish

$$0 = \det(A - \lambda I). \tag{1.73}$$

This is a quadratic equation for λ. It has two solutions, λ_1 and λ_2. Sometimes the solutions are the same, and $\lambda_1 = \lambda_2$.

In terms of the eigenvalues λ_1 and λ_2, one has two equations for the components z_1 and z_2 of the eigenvectors:

$$\begin{aligned} 0 &= z_1(A_{11} - \lambda_i) + z_2 A_{12} \\ 0 &= z_1 A_{21} + z_2(A_{22} - \lambda_i). \end{aligned} \tag{1.74}$$

These two equations say that

$$z_2 = -\frac{A_{11} - \lambda_i}{A_{12}}\, z_1 \quad \text{and} \quad z_2 = -\frac{A_{21}}{A_{22} - \lambda_i}\, z_1, \tag{1.75}$$

and so are consistent only if

$$(A_{11} - \lambda_i)(A_{22} - \lambda_i) = A_{12} A_{21} \tag{1.76}$$

which is to say, only if $\det(A - \lambda I) = 0$. You might think that we have 6 unknowns here — 2 λ's, 2 z_1's, and 2 z_2's. But eigenvectors are defined only up to a complex factor. So the two equations (1.75) don't determine z_1 or z_2 but only their ratio $w = z_1/z_2$. So we have 4 unknowns: w_1, w_2, λ_1, and λ_2 and three equations, the two of (1.75) and (1.76).

We can illustrate this by dividing the two equations (1.74) by z_2 so as to get 2 equations for 2 unknowns $w = z_1/z_2$ and λ

$$\begin{aligned} 0 &= w(A_{11} - \lambda) + A_{12} \\ 0 &= w A_{21} + A_{22} - \lambda. \end{aligned} \tag{1.77}$$

The second equation gives $\lambda = wA_{21}+A_{22}$. When we substitute this formula for w into the first equation, we get the quadratic equation

$$A_{21}w^2 + (A_{22} - A_{11})w - A_{12} = 0 \tag{1.78}$$

which we can solve for its two roots $w_i = (z_1/z_2)_i$. The eigenvalues then are $\lambda_i = w_i A_{21} + A_{22}$.

Such results hold for $n \times n$ eigenvector problems: The $n \times n$ determinant $\det(A - \lambda I)$ vanishes if and only if λ is an eigenvalue of the $n \times n$ matrix A

$$AZ = \lambda Z \iff \det(A - \lambda I) = 0. \tag{1.79}$$

This is an nth-order polynomial equation

$$0 = z_n\lambda^n + z_{n-1}\lambda^{n-1} + \cdots + z_1\lambda + z_0 \tag{1.80}$$

also called an nth-degree polynomial equation. We can easily solve quadratic equations. Cubic ones are more complicated; quartic ones are even more complicated; and equations of higher order in general have only numerical solutions.

So the computer programs that find the eigenvalues and eigenvectors of $n \times n$ matrices don't compute the determinant and set it equal to zero. Indeed, the determinant of an $n \times n$ matrix has $n!$ terms, which becomes unmanageable for $n \gtrsim 10$ and hopeless for $n \gtrsim 20$ since

```
factorial(10) = 3628800 and factorial(20) = 2.4329e+18.
```

So the computer programs that solve for eigenvalues and eigenvectors use efficient algorithms such as the LU decomposition, $A = LU$ in which the matrix L is lower triangular with 1 on its main diagonal, and U is upper triangular (Tadeusz Banachiewicz 1882–1954). For example, the LU decomposition of the matrix A is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ z\alpha & z\beta + \gamma \end{pmatrix} \tag{1.81}$$

in which $\alpha = a$, $\beta = b$, $z = c/\alpha = c/a$, and $\gamma = d - z\beta = d - c\beta/a = d - bc/a$. So the LU decomposition of A is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d - bc/a \end{pmatrix}. \tag{1.82}$$

Matlab can do this for you:

```
>> syms a b c d
>> A = [ a b ; c d]
```

```
A =

[ a, b]
[ c, d]

>> [L,U] = lu(A)

L =

[   1, 0]
[ c/a, 1]


U =

[ a,            b]
[ 0, d - (b*c)/a].
>> L*U

ans =

[ a, b]
[ c, d] .
```

Matlab's webpages describe this and other ways to decompose a matrix and to find its eigenvectors and eigenvalues `https://www.mathworks.com/help/matlab/linear-algebra.html`.

The LU decomposition lets one compute determinants much more easily than with Levi-Civita's symbols. The reason is that

$$\det A = \det(LU) = \det L \det U, \tag{1.83}$$

and the determinant of a triangular matrix is just the product of its diagonal elements. So

$$\det L = 1, \tag{1.84}$$

and

$$\det U = U_{11} U_{22} \cdots U_{nn}. \tag{1.85}$$

So in the 2×2 example (1.82)

$$\det A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det a(d - bc/a) = ad - bc. \tag{1.86}$$

1.11 Eigenvectors and Eigenvalues from Matlab

First, we make a real 3×3 random matrix

```
>> M = rand(3)

M =

    0.3171    0.4387    0.7952
    0.9502    0.3816    0.1869
    0.0344    0.7655    0.4898. .
```

Matlab's eig gives its eigenvalues

```
>> e=eig(M)

e =

  -0.1327 + 0.4941i
  -0.1327 - 0.4941i
   1.4539 + 0.0000i  .
```

They are complex even though M is real because M is not symmetric, $M \neq M^{\mathsf{T}}$. Matlab gives M's eigenvectors as

```
>> [V, D] = eig(M)

V =

  -0.2541 + 0.4076i  -0.2541 - 0.4076i   0.5967 + 0.0000i
   0.6370 + 0.0000i   0.6370 + 0.0000i   0.6180 + 0.0000i
  -0.4610 - 0.3885i  -0.4610 + 0.3885i   0.5120 + 0.0000i


D =

  -0.1327 + 0.4941i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.1327 - 0.4941i   0.0000 + 0.0000i
```

```
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.4539 + 0.0000i .
```

Now we generate two 4×4 random matrices, multiply one by i, and add them to get a 4×4 complex matrix C. Then we get its eigenvalues as eig(C)

```
>> R = rand(4)

R =

    0.0975    0.9649    0.4854    0.9157
    0.2785    0.1576    0.8003    0.7922
    0.5469    0.9706    0.1419    0.9595
    0.9575    0.9572    0.4218    0.6557

>> C = R + i*rand(4)

C =

   0.0975 + 0.0357i   0.9649 + 0.7577i   0.4854 + 0.1712i   0.9157 + 0.0462i
   0.2785 + 0.8491i   0.1576 + 0.7431i   0.8003 + 0.7060i   0.7922 + 0.0971i
   0.5469 + 0.9340i   0.9706 + 0.3922i   0.1419 + 0.0318i   0.9595 + 0.8235i
   0.9575 + 0.6787i   0.9572 + 0.6555i   0.4218 + 0.2769i   0.6557 + 0.6948i

>> e = eig(C)

e =

   2.5716 + 1.9648i
  -0.0536 - 0.4794i
  -0.7540 + 0.3692i
  -0.7112 - 0.3491i  .
```

To also get the eigenvectors, we use [V,D] = eig(A):

```
>> [V,D] = eig(C)

V =

   0.4030 - 0.0924i  -0.4470 - 0.0128i  -0.2896 + 0.2366i   0.6414 + 0.0000i
   0.4741 + 0.0716i   0.0277 + 0.3895i  -0.5716 + 0.0382i  -0.2405 + 0.2058i
   0.5317 + 0.0069i   0.7387 + 0.0000i   0.4358 - 0.0069i   0.0443 - 0.6354i
   0.5625 + 0.0000i  -0.0478 - 0.3155i   0.5847 + 0.0000i  -0.2823 + 0.0554i
```

```
D =

   2.5716 + 1.9648i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.0536 - 0.4794i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.7540 + 0.3692i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.7112 - 0.3491i

>> C*V - V*D

ans =

   1.0e-14 *

   0.2442 + 0.1443i  -0.0507 - 0.0111i  -0.0389 + 0.0056i   0.0444 + 0.0222i
  -0.0888 + 0.0000i   0.0167 - 0.0860i   0.0000 + 0.0194i   0.0111 - 0.0125i
   0.1110 + 0.1332i   0.0139 - 0.0167i   0.0000 - 0.0194i   0.0000 - 0.0222i
   0.1998 + 0.1332i   0.0083 + 0.0014i  -0.0500 + 0.0250i   0.0611 - 0.0160i .
```

This last result says that apart from roundoff errors of order 1e-14

```
C V = V D
```

which says that the columns of V are the eigenvectors of C with eigenvalues that are the nonzero elements of the diagonal matrix D. In detail, this is

$$\sum_{k=1}^{4} C_{ik} V_{kj} = D_{jj} V_{ij}. \tag{1.87}$$

1.12 Linear Least Squares

Matlab says:

1.13 States and Density Operators

In quantum mechanics, we often represent systems by states, e.g., $|1,0,0\rangle$ for the ground state of hydrogen with energy E_1 and angular momentum $\ell = 0$ (here spin is neglected) and $|2,1,1\rangle$ for the $n = 2$ state with energy E_2 and angular momentum $\hbar$. We can add states

$$|\psi\rangle = z|1,0,0\rangle + w|2,1,1\rangle. \tag{1.88}$$

This state $|\psi\rangle$ is entangled because if we measure its energy to be E_1 or E_2 then we know that its angular momentum is $\ell = 0$ or $\ell = \hbar$.

We use density operators to describe systems about which we know less. For instance, the density operator

$$\rho = \tfrac{1}{2}|1,0,0\rangle\langle 1,0,0| + \tfrac{1}{2}|2,1,1\rangle\langle 2,1,1| \tag{1.89}$$

represents a system whose energy is equally likely to be E_1 or E_2.

The operators we use most often in quantum mechanics are linear operators. For example, the hamiltonian H for a hydrogen atom maps the state $|\psi\rangle$ into

$$H|\psi\rangle = zH|1,0,0\rangle + wH|2,1,1\rangle = zE_1|1,0,0\rangle + wE_2|2,1,1\rangle. \tag{1.90}$$

Here we used the fact that the state $|1,0,0\rangle$ is an eigenstate of H with eigenvalue E_1 and that $|2,0,0\rangle$ is an eigenstate of H with eigenvalue E_2

$$H|1,0,0\rangle = E_1|1,0,0\rangle \quad \text{and} \quad H|2,1,1\rangle = E_2|2,1,1\rangle. \tag{1.91}$$

These eigenstate equations are extensions to vector spaces of infinite dimension of the concepts we learned in Sections (1.6–1.10).

1.14 Triangle Inequality

The triangle inequality is

$$\|f+g\| \le \|f\| + \|g\|. \tag{1.92}$$

The square of $\|f+g\|$ is

$$\begin{aligned} \|f+g\|^2 &= (f+g, f+g) = (f,f) + (f,g) + (g,f) + (g,g) \\ &= \|f\|^2 + (f,g) + (g,f) + \|g\|^2 \\ &\le \|f\|^2 + 2|(f,g)| + \|g\|^2. \end{aligned} \tag{1.93}$$

The Schwarz inequality (1.97) says that

$$|(f,g)| \le \|f\|\|g\|. \tag{1.94}$$

So we have

$$\|f+g\|^2 \le \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 = \big(\|f\| + \|g\|\big)^2 \tag{1.95}$$

whose square root is the triangle inequality

$$\|f+g\| \le \|f\| + \|g\|. \tag{1.96}$$

Equivalently, with $f = a - c$ and $g = c - b$, this is

$$\|a-b\| \le \|a-c\| + \|c-b\|. \tag{1.97}$$

1.15 Notes on Some Problems

Problem 1.40 corrected: The coherent state $|\{\alpha(\boldsymbol{k},\ell)\}\rangle$ is an eigenstate of the annihilation operator $a(\boldsymbol{k},\ell)$ with eigenvalue $\alpha(\boldsymbol{k},\ell)$ for each mode of the electromagnetic field with wavenumber $\boldsymbol{k}$ and polarization ℓ

$$a(\boldsymbol{k},\ell)|\{\alpha(\boldsymbol{k},\ell)\}\rangle = \alpha(\boldsymbol{k},\ell)|\{\alpha(\boldsymbol{k},\ell)\}\rangle. \tag{1.98}$$

The positive-frequency part $E_i^{(+)}(t,\boldsymbol{x})$ of the electric field is a linear combination of the annihilation operators

$$E_i^{(+)}(t,\boldsymbol{x}) = \sum_{\boldsymbol{k}} \sum_{\ell=1}^{2} a(\boldsymbol{k},\ell)\, e_i(\boldsymbol{k},\ell)\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} \tag{1.99}$$

in which the wavenumber $\boldsymbol{k}$ is summed over $\boldsymbol{k} = 2\pi\boldsymbol{n}/L$ where L is an appropriate length such as the length of a laser cavity or the width of the universe, and $\boldsymbol{n}$ is a vector of integers. Show that $|\{\alpha_k\}\rangle$ is an eigenstate of $E_i^{(+)}(t,\boldsymbol{x})$

$$E_i^{(+)}(t,\boldsymbol{x})|\{\alpha(\boldsymbol{k},\ell)\}\rangle = \mathcal{E}_i^{(+)}(t,\boldsymbol{x})|\{\alpha(\boldsymbol{k},\ell)\}\rangle \tag{1.100}$$

and find its eigenvalue $\mathcal{E}_i^{(+)}(t,\boldsymbol{x})$.

2
Examples for Chapter 2, Vector Calculus

2.1 Helmholtz Decomposition

We can use the delta-function formula (2.34) to write any suitably smooth 3-dimensional vector field $\boldsymbol{V}(\boldsymbol{x})$ as

$$\boldsymbol{V}(\boldsymbol{x}) = -\int \boldsymbol{V}(\boldsymbol{r}) \, \triangle \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{x}|} \right) d^3 r \tag{2.1}$$

in which the derivatives $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \boldsymbol{\nabla}^2 = \triangle$ can be both with respect to $\boldsymbol{x}$ or both with respect to $\boldsymbol{r}$. Taking them to be with respect to $\boldsymbol{x}$, we have

$$\boldsymbol{V}(\boldsymbol{x}) = -\boldsymbol{\nabla}^2 \int \frac{\boldsymbol{V}(\boldsymbol{r})}{|\boldsymbol{r} - \boldsymbol{x}|} \, d^3 r. \tag{2.2}$$

We now use our formula (2.49) for the curl of a curl

$$\boldsymbol{\nabla}^2 \boldsymbol{V} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{V}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{V}) \tag{2.3}$$

to write $\boldsymbol{V}(\boldsymbol{x})$ as

$$\boldsymbol{V}(\boldsymbol{x}) = -\boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \int \frac{\boldsymbol{V}(\boldsymbol{r})}{|\boldsymbol{r} - \boldsymbol{x}|} \, d^3 r \right) + \boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \int \frac{\boldsymbol{V}(\boldsymbol{r})}{|\boldsymbol{r} - \boldsymbol{x}|} \, d^3 r \right). \tag{2.4}$$

Thus any suitably smooth 3-dimensional vector field $\boldsymbol{V}(\boldsymbol{x})$ can be written as the sum

$$\boldsymbol{V}(\boldsymbol{x}) = \boldsymbol{\nabla} \phi(\boldsymbol{x}) + \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{x}) \tag{2.5}$$

of the gradient of a scalar field

$$\phi(\boldsymbol{x}) = -\boldsymbol{\nabla} \cdot \int \frac{\boldsymbol{V}(\boldsymbol{r})}{|\boldsymbol{r} - \boldsymbol{x}|} \, d^3 r \tag{2.6}$$

and the curl of a vector field

$$\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{\nabla} \times \int \frac{\boldsymbol{V}(\boldsymbol{r})}{|\boldsymbol{r} - \boldsymbol{x}|} \, d^3 r \tag{2.7}$$

(Hermann von Helmholtz, 1821–1894).

3

Examples for Chapter 3, Fourier Series

3.1 Fourier and Dirac

If we combine the Fourier series (3.2)

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \, \frac{e^{inx}}{\sqrt{2\pi}} \tag{3.1}$$

with the formula (3.3) for the Fourier coefficients

$$f_n = \int_0^{2\pi} \frac{e^{-inx}}{\sqrt{2\pi}} \, f(x) \, dx, \tag{3.2}$$

then we get equation (3.120):

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \, \frac{e^{inx}}{\sqrt{2\pi}} = \sum_{n=-\infty}^{\infty} \int_0^{2\pi} \frac{e^{-iny}}{\sqrt{2\pi}} \, f(y) \frac{e^{inx}}{\sqrt{2\pi}} \, dy. \tag{3.3}$$

If the function $f(x)$ is suitably smooth, then we may change the order of summation and the integration and get for $0 \le x \le 2\pi$

$$f(x) = \int_0^{2\pi} \left(\sum_{n=-\infty}^{\infty} \frac{e^{in(x-y)}}{2\pi} \right) f(y) \, dy \tag{3.4}$$

which is (3.121). This equation says that for $0 \le x, y \le 2\pi$

$$\sum_{n=-\infty}^{\infty} \frac{e^{in(x-y)}}{2\pi} = \delta(x-y). \tag{3.5}$$

But the right-hand side of (2.4) is periodic in x with period 2π, and it defines the periodic extension $f_p(x)$ of the function $f(x)$ from the interval $[0, 2\pi]$ to the whole real line

$$f_p(x) = f_p(x + 2\pi m) = \int_0^{2\pi} \left(\sum_{n=-\infty}^{\infty} \frac{e^{in(x-y)}}{2\pi} \right) f(y)\, dy \tag{3.6}$$

in which m is an integer. So the sum of phases is a sum of delta functions

$$\sum_{n=-\infty}^{\infty} \frac{e^{in(x-y)}}{2\pi} = \sum_{m=-\infty}^{\infty} \delta(x - y - 2\pi m) \tag{3.7}$$

which is (3.123). This is the Dirac comb. It is illustrated in Fig. 3.11.

3.2 Hilbert and Dirac

The Fourier series is an example of a much more general class of series. Suppose $H_n(x), n = 0, 1, 2, \ldots, \infty$ is a set of orthonormal functions

$$\int_a^b H_n^*(x)\, H_m(x)\, dx = \delta_{nm}. \tag{3.8}$$

These functions span a vector space S of functions

$$f(x) = \sum_{n=0}^{\infty} f_n\, H_n(x). \tag{3.9}$$

The orthonormality (3.8) of these functions implies that the coefficients f_n of the expansion (3.9) are

$$\int_a^b f(x)\, H_n^*(x)\, dx = \int_a^b \left(\sum_{m=0}^{\infty} f_m\, H_m(x) \right) H_n^*(x)\, dx = \sum_{m=0}^{\infty} f_m\, \delta_{nm} = f_n. \tag{3.10}$$

These three equations (3.8–3.10) are analogous to the three basic equations (3.1–3.3) of Fourier series.

By combining the expansion (3.9) of $f(x)$ with the formula (3.10) for its coefficients f_n, we have

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} f_n\, H_n(x) = \sum_{n=0}^{\infty} \int_a^b f(y)\, H_n^*(y)\, dy\, H_n(x) \\ &= \int_a^b \left(\sum_{n=0}^{\infty} H_n^*(y)\, H_n(x) \right) f(y)\, dy \end{aligned} \tag{3.11}$$

for all points $a \le x \le b$. Thus the orthonormal functions $H_n(x)$ provide a representation for the delta function suitable for functions in the space S for $a \le x, y \le b$

$$\delta(x-y) = \sum_{n=0}^{\infty} H_n^*(y)\, H_n(x) \tag{3.12}$$

which is analogous to the expansion (3.5) of the delta function. It differs from the Dirac comb (3.7) because the orthonormal functions $H_n(x)$ may not be periodic.

3.3 Example 3.8 again

Example 3.8 derived the Fourier series for the function that is $1 + \cos 2x$ for $|x| \le \pi/2$ and zero otherwise. The series for the function that is $1 + \cos 2x$ for all x is much simpler:

$$1 + \cos 2x = 1 + \frac{1}{2}\left(e^{2ix} + e^{-2ix}\right). \tag{3.13}$$

Its coefficients f_n are nonzero only for $n = -2, 0, 2$.

3.4 Fourier series for $(x^2 - \pi^2)^2$ on $(-\pi, \pi]$

To find the Fourier series for the function $(x^2 - \pi^2)^2$ on the interval $(-\pi, \pi]$, we use the fact that the function $(x^2 - \pi^2)^2$ is even on that interval. Since the function is even, only its a_n Fourier coefficients are nonzero. To do the integrals

$$\begin{aligned} a_n &= \int_{-\pi}^{\pi} \cos nx\, (x^2 - \pi^2)^2 \frac{dx}{\pi} \\ a_0 &= \int_{-\pi}^{\pi} (x^2 - \pi^2)^2 \frac{dx}{\pi}, \end{aligned} \tag{3.14}$$

we can use Helen Yang's Matlab script:

```
format rat % Keep all the results in fraction
syms x n p % variables in the function
% (p for pi if you don't want pi to be calculated)

an = cos(n*x)*(x^2-p^2)^2
an_integral = int(an,x,-p,p) %int is the integration function

a0 = (x^2-p^2)^2
a0_integral = int(a0,x,-p,p) .
```

Matlab then gives us

```
an_integral =
-(16*(n^2*p^2*sin(n*p) - 3*sin(n*p) + 3*n*p*cos(n*p)))/n^5
a0_integral = (16*p^5)/15
```

in which p stands for π. In latex, this is

$$a_n = \int_{-\pi}^{\pi} \cos nx\,(x^2-\pi^2)^2 \frac{dx}{\pi} = -\frac{16}{n^5}\left[3n\cos(n\pi) + (n^2\pi - 3/\pi)\sin(n\pi)\right]$$
$$= \frac{48}{n^4}(-1)^{n+1}$$

and

$$a_0 = \int_{-\pi}^{\pi} (x^2-\pi^2)^2 \frac{dx}{\pi} = \frac{16\pi^4}{15}.$$

So the Fourier series is

$$(x^2-\pi^2)^2 = \frac{8\pi^4}{15} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{48}{n^4}\,\cos(nx).$$

4
Examples for Chapter 4, Fourier Transforms

4.1 Fourier derivation of Helmholtz decomposition

The Levi-Civita identity (2.56)

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{4.1}$$

implies that

$$\begin{aligned} \boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{V}) &= \sum_{j,k,\ell=1}^{3} \epsilon_{ijk}\epsilon_{k\ell m}\, k_j\, k_\ell\, V_m \\ &= \sum_{j,\ell=1}^{3} (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\, k_j\, k_\ell\, V_m \\ &= (\boldsymbol{k}\cdot\boldsymbol{V})\, k_i - (\boldsymbol{k}\cdot\boldsymbol{k})\, V_i. \end{aligned} \tag{4.2}$$

Thus we can use any nonzero 3-vector $\boldsymbol{k}$ to write every 3-vector $\boldsymbol{V}$ as

$$\boldsymbol{V} = \frac{1}{\boldsymbol{k}\cdot\boldsymbol{k}}\Big((\boldsymbol{k}\cdot\boldsymbol{V})\,\boldsymbol{k} - \boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{V})\Big). \tag{4.3}$$

This expansion lets us write the Fourier transform $\boldsymbol{V}(\boldsymbol{k})$ of any square-integrable 3-vector field $\boldsymbol{V}(\boldsymbol{x})$ as

$$\begin{aligned} \boldsymbol{V}(\boldsymbol{x}) &= \int \boldsymbol{V}(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d^3x = \int \Big(\frac{\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{V}(\boldsymbol{k}))}{\boldsymbol{k}^2} - \frac{\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{V}(\boldsymbol{k}))}{\boldsymbol{k}^2}\Big)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d^3x \\ &= \int \frac{\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{V}(\boldsymbol{k}))}{\boldsymbol{k}^2}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} - \frac{\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{V}(\boldsymbol{k}))}{\boldsymbol{k}^2}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d^3x \\ &= \nabla \int \frac{-i\,\boldsymbol{k}\cdot\boldsymbol{V}(\boldsymbol{k})}{\boldsymbol{k}^2}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d^3x + \nabla\times\int \frac{i\,\boldsymbol{k}\times\boldsymbol{V}(\boldsymbol{k})}{\boldsymbol{k}^2}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d^3x \end{aligned} \tag{4.4}$$

which is the sum of the gradient of a scalar field plus the curl of a vector field.

4.2 3D Delta Function

The 3-dimensional delta function is

$$\delta(\boldsymbol{x}-\boldsymbol{y}) = \int e^{\pm i(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}}\, \frac{d^3k}{(2\pi)^3} \tag{4.5}$$

in which you can use either + or −. The n-dimensional delta function is

$$\delta(\boldsymbol{x}-\boldsymbol{y}) = \int e^{\pm i(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}}\, \frac{d^nk}{(2\pi)^n}. \tag{4.6}$$

The Laplace transform of t^{s-1} is related to the gamma function (5.58)

$$s^{-z}\,\Gamma(s) = \int_0^\infty dt\, e^{-st}\, t^{s-1}. \tag{4.7}$$

4.3 Composite Objects

The sum identity

$$\sum_{i=1}^{n} p_i\, x_i = P\,\bar{x} + \sum_{i=1}^{n} (p_i - \bar{p})(x_i - \bar{x}) \tag{4.8}$$

expresses the dot product $p_i\, x_i$ in terms of the total momentum P, the average position $\bar{x}$, and the average momentum $\bar{p}$

$$P = \sum_{i=1}^{n} p_i, \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad \bar{p} = \frac{P}{n} = \frac{1}{n}\sum_{i=1}^{n} p_i. \tag{4.9}$$

We may verify the sum identity (4.8) by using repeated indexes to mean summation:

$$p_i x_i = P\bar{x} + p_i x_i - n\bar{p}\bar{x} - n\bar{p}\bar{x} + n\bar{p}\bar{x} = p_i x_i. \tag{4.10}$$

A special case of the sum identity (4.8) lets us write the kinetic energy of n particles as

$$T = \sum_{i=1}^{n} \frac{m_i}{2}\dot{q}_i^2 = \frac{1}{2}\left[P\bar{\dot{q}} + \sum_{i=1}^{n}(p_i - \bar{p})(\dot{q}_i - \bar{\dot{q}})\right] \tag{4.11}$$

in which

$$P = \sum_{i=1}^{n} m_i \dot{q}_i, \quad \bar{p} = \frac{P}{n}, \quad \text{and} \quad \bar{\dot{q}} = \frac{1}{n}\sum_{i=1}^{n} \dot{q}_i. \tag{4.12}$$

Thus the kinetic energy of n particles that are bound together as one object

$$\dot{q}_i - \bar{\dot{q}} = \epsilon\, \sin(\omega_i t) \tag{4.13}$$

in which $|\epsilon| \ll 1$ consists mainly of the term $P\bar{q}/2$ plus terms of order ϵ^2 .

We also may use the sum identity (4.8) to write the phase of the wave function of n particles as

$$e^{i\sum_i p_i\, x_i} = e^{iP\,\bar{x}}\, e^{i\sum_i (p_i - \bar{p})(x_i - \bar{x})}. \tag{4.14}$$

So if the n particles are bound together as one object

$$x_i = \bar{x} + \epsilon \sin(\omega_i t) \quad \text{and} \quad p_i = \bar{p} + \delta \cos(\omega_i t), \tag{4.15}$$

then the exponential form (4.14) of the sum identity (4.8) shows that the departure of the phase from $\exp(iP\bar{x})$ is of second order in small quantities ϵ and δ

$$e^{i\sum_i p_i\, x_i} = e^{iP\,\bar{x}}\, e^{i\sum_i \epsilon\, \delta \sin(2\omega_i t)/2} \tag{4.16}$$

as illustrated in Fig. 4.1

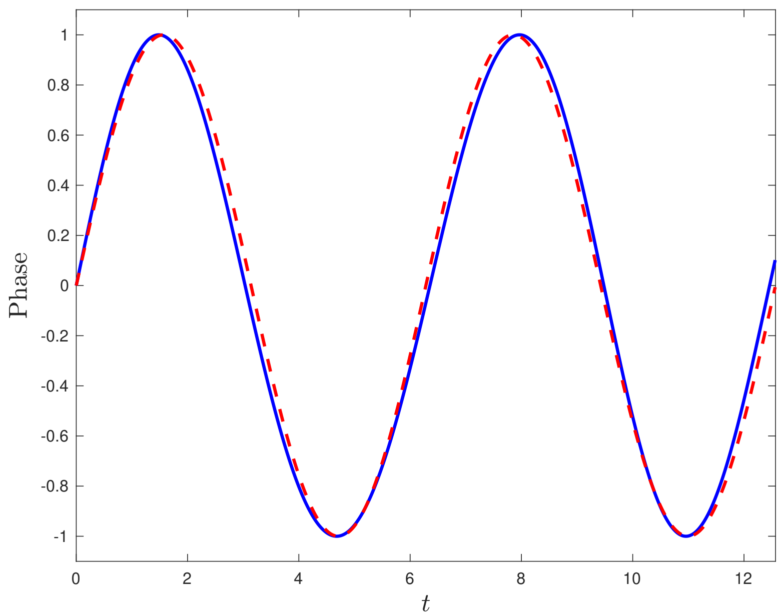


Figure 4.1 The sin of the sum of 100 phases $p_i x_i$ is plotted (solid, blue) for $\epsilon = \delta = 1/3$ and 100 random frequencies $0 < \omega_i < 1$. It closely follows the phase of the whole object, $\sin(P\bar{x})$, (dashes, red).

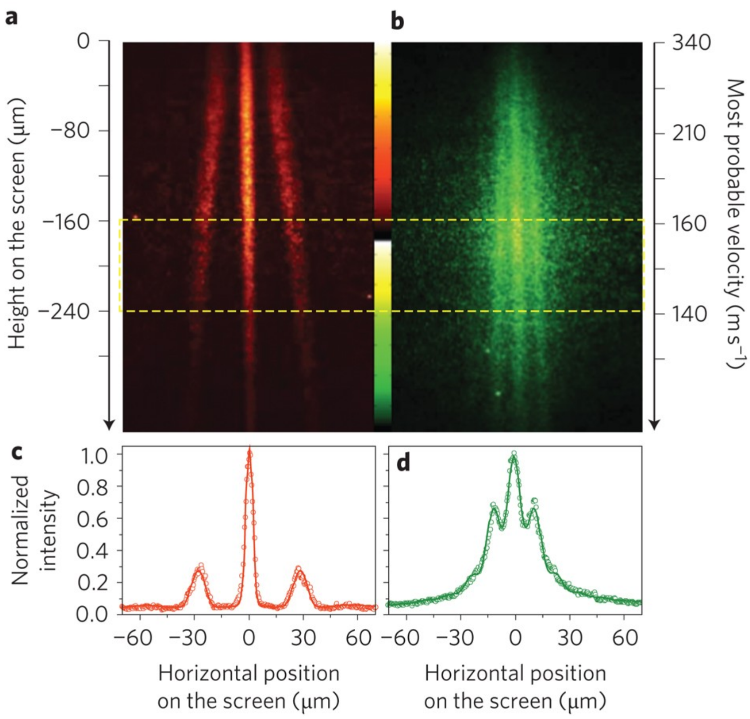
This is why an object made of many particles bound together but vibrating in their own ways appears quantum mechanically and approximately as a single object. The effective wavelength is very small, however,

$$\lambda \simeq \frac{\hbar}{P} \tag{4.17}$$

so experiments that show their interference are difficult. Arndt has exhibited the interference of complex macromolecules such as ones derived from phthalocyanine as illustrated in Fig. 4.2 (`https://www.nature.com/articles/nnano.2012.34`).

Figure 4: Comparison of interference patterns for PcH_2 and $F_{24}PcH_2$.

From: Real-time single-molecule imaging of quantum interference



a,**b**, False-colour fluorescence images of the quantum interference patterns of PcH_2 (**a**) and $F_{24}PcH_2$ (**b**). We can deduce both the mass and the velocity of the molecules from these images, because diffraction spreads out the molecular beam in the horizontal direction, and the effects of gravity mean that the height h on the screen (left axes) depends on the velocity v of the molecule (right axes). The colour bar ranges from −20 to 400 photons in **a**, and from −20 to 600 photons in **b**. The true fluorescence of both molecules starts at wavelengths above 700 nm. **c**,**d**, One-dimensional diffraction curves obtained by integrating the fluorescence images in **a** and **b** between $h = -160\ \mu$m and $h = -240\ \mu$m (dashed yellow lines in **a** and **b**), which corresponds to velocity spread $\Delta v/v = 0.27$. All imaging settings are specified in Supplementary Table S1. Collimation slit S (Fig. 1) was 3 μm wide (defined by a pair of steel razor blades with 300 nm edge width). Diffraction grating G was cut into a 10-nm-thick SiN membrane and had dimensions of 3 μm (width) and 100 μm (height), with period $d = 100$ nm (width of individual slits $s = 75$ nm). $L_1 = 566$ mm, $L_2 = 564$ mm.

Figure 4.2

The Matlab script for Fig. 4.1 is:

```
epsilon = 1/3; delta = 1/3;
P = 1; v = 1;
phase = 0;
```

```
for i = 1:100
    om(i) = rand ;
end
t = 0:.01:4*pi;
    for k = 1: 400*pi
        phase = P*v*t + epsilon*delta*sin(om(i).*t);
    end
    plot(t,sin(phase),'-b','LineWidth',2)
hold on
plot(t,sin(P*v*t),'--r','LineWidth',2)
axis([0 4*pi -1.1 1.1])
xlabel('$t$','Interpreter','latex','fontsize',16)
ylabel('Phase','Interpreter','latex','fontsize',16)
print -depsc /Users/kevin/papers/math/phaseSum
```

4.4 General solution of the diffusion equation

Since the density $\rho(\boldsymbol{x},t)$ due to $\rho(\boldsymbol{x},0) = \delta(\boldsymbol{x}-\boldsymbol{y})$ is

$$\rho(\boldsymbol{x},t) = \frac{1}{(4\pi D t)^{3/2}}\, e^{-(\boldsymbol{x}-\boldsymbol{y})^2/(4Dt)},$$

it follows from the linearity of the diffusion equation that the density $\rho(\boldsymbol{x},t)$ due to a linear combination

$$\rho(\boldsymbol{x},0) = \int \delta(\boldsymbol{x}-\boldsymbol{y})\rho(\boldsymbol{y},0)\, d^3y$$

is

$$\rho(\boldsymbol{x},t) = \frac{1}{(4\pi D t)^{3/2}} \int e^{-(\boldsymbol{x}-\boldsymbol{y})^2/(4Dt)}\rho(\boldsymbol{y},0)\, d^3y.$$

5
Examples for Chapter 5, Series

5.1 Convergence

The trace is cyclic so we expect that

$$\mathrm{Tr}(q\,p) = \mathrm{Tr}(p\,q). \tag{5.1}$$

But then we'd have

$$\mathrm{Tr}\big([q,p]\big) = 0. \tag{5.2}$$

But we know that

$$[q,p] = i\hbar, \tag{5.3}$$

so now we have

$$0 = \mathrm{Tr}\big([q,p]\big) = \mathrm{Tr}(i\hbar) = i\hbar\mathrm{Tr}(I) = i\hbar\,\infty. \tag{5.4}$$

The raising and lowering operators explained in Section 3.12 offer an equivalent paradox

$$0 = \mathrm{Tr}\big([a,a^\dagger]\big) = \mathrm{Tr}(1) = \mathrm{Tr}(I) = \infty. \tag{5.5}$$

These equations don't represent a breakdown of quantum mechanics. They present us with divergent series

$$\begin{aligned}
\mathrm{Tr}\big([a,a^\dagger]\big) &= \mathrm{Tr}\big(aa^\dagger\big) - \mathrm{Tr}\big(a^\dagger a\big) \\
&= \sum_{n=0}^{\infty} \langle n|aa^\dagger|n\rangle - \sum_{n=0}^{\infty} \langle n|a^\dagger a|n\rangle \\
&= \sum_{n=0}^{\infty} n+1 - \sum_{n=0}^{\infty} n = \sum_{n=0}^{\infty} 1 = \infty.
\end{aligned} \tag{5.6}$$

They tell us to keep in mind that not all series converge appropriately in every equation.

5.2 Geometric Series

Example 5.1 (Geometric series) The sum of the first n terms of a geometric series (5.10) is

$$S_n(z) = \sum_{k=0}^{n} z^k = \frac{1 - z^{k+1}}{1 - z}. \tag{5.7}$$

This Matlab script shows how this series goes for $z = 1/2$ in blue and $z = 2$ in red:

```
n=0:1:200;
z = 2;
s = (1-z.^(n+1))./(1-z);
plot(n,s,'-r','LineWidth',2)
hold on
z = 1.1;
s = (1-z.^(n+1))./(1-z);
plot(n,s,'-m','LineWidth',2)
z = 1.01;
s = (1-z.^(n+1))./(1-z);
plot(n,s,'-','LineWidth',2,'Color',[.8 0 .5])
z = 1.001;
s = (1-z.^(n+1))./(1-z);
plot(n,s,'-','LineWidth',2,'Color',[.5 0 .5])
z = 0.5;
s = (1-z.^(n+1))./(1-z);
plot(n,s,'-b','LineWidth',2)
z = 0.9;
s = (1-z.^(n+1))./(1-z);
plot(n,s,'-g','LineWidth',2)
z = 0.99;
s = (1-z.^(n+1))./(1-z);
plot(n,s,'-','LineWidth',2,'Color',[0 .5 .5])
z = 0.999;
s = (1-z.^(n+1))./(1-z);
plot(n,s,'-','LineWidth',2,'Color',[.05,0,.55])
axis([0 100 0 100])
textx='$n$';
xlabel(textx,'Interpreter','latex')
texty='Sum';
ylabel(texty)
```

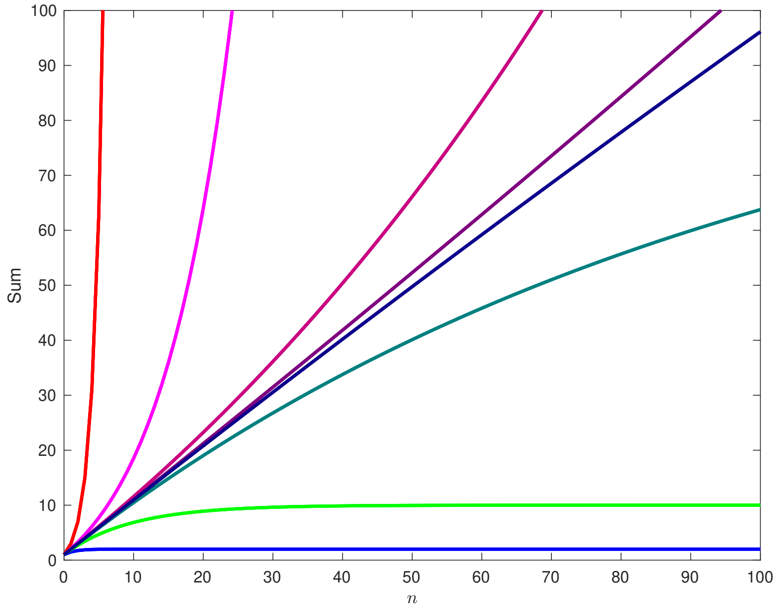


Figure 5.1 The geometric series (5.7) for $z = 2$ red, 1.1 magenta, 1.01 dark red, 1.001 0.9 purple, 0.5 blue, 0.9 green, 0.99 blue green, and 0.999 navy blue.

```
print -dpdf ~/papers/math/PowerSum
print -depsc ~/papers/math/PowerSum
```

The resulting plot is Fig. 5.1.

5.3 Leibniz's Rule

In the notation

$$f^{(n)}(x) \equiv \frac{d^n}{dx^n} f(x) \tag{5.8}$$

for derivatives Leibniz's rule for differentiating the product of two functions is

$$\frac{d^n}{dx^n} [f(x)\, g(x)] = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)\, g^{(n-k)}(x) \tag{5.9}$$

(Gottfried Leibniz, 1646–1716).

The rule is obviously true for $n = 0$ and $n = 1$, and one may use mathematical induction to prove it. Keeping in mind that $(-1)! = \infty$, we find for

$0 \le k \le n$

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left[\frac{1}{k} + \frac{1}{n-k}\right] \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left[\frac{n}{k(n-k)}\right] \\
&= \frac{n!}{k!(n-k)!} = \binom{n}{k}.
\end{aligned} \tag{5.10}$$

Let's assume that for $\ell = 0 \ldots n-1$ that

$$\frac{d^\ell}{dx^\ell}\left[f(x)\, g(x)\right] = \sum_{k=0}^{\ell} \binom{\ell}{k} f^{(k)}(x)\, g^{(\ell-k)}(x).$$

So for $\ell = n-1$, we have

$$\frac{d^{n-1}}{dx^{n-1}}\left[f(x)\, g(x)\right] = \sum_{k=0}^{n-1} \binom{n-1}{k} f^{(k)}(x)\, g^{(n-1-k)}(x).$$

So using the result (5.10) of the first part of this exercise, we get

$$\begin{aligned}
\frac{d^n}{dx^n}(f\, g) &= \frac{d}{dx}\sum_{k=0}^{n-1} \binom{n-1}{k} f^{(k)}\, g^{(n-1-k)} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \left[f^{(k+1)}\, g^{(n-1-k)} + f^{(k)}\, g^{(n-k)}\right] \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} f^{(k+1)}\, g^{(n-1-k)} + \sum_{k=0}^{n-1} \binom{n-1}{k} f^{(k)}\, g^{(n-k)}.
\end{aligned}$$

Since

$$\binom{n-1}{n} = 0, \tag{5.11}$$

we can replace $n-1$ with n in the second sum

$$\frac{d^n}{dx^n}(f\, g) = \sum_{k=0}^{n-1} \binom{n-1}{k} f^{(k+1)}\, g^{(n-1-k)} + \sum_{k=0}^{n} \binom{n-1}{k} f^{(k)}\, g^{(n-k)}.$$

In the first sum, we set $j = k+1$ and so $k = j-1$, and in the second sum, we set $j = k$

$$\frac{d^n}{dx^n}(f\,g) = \sum_{j=1}^{n} \binom{n-1}{j-1} f^{(j)}\, g^{(n-j)} + \sum_{j=0}^{n} \binom{n-1}{j} f^{(j)}\, g^{(n-j)}.$$

Since $(-1)! = \infty$, the binomial coefficient

$$\binom{n-1}{-1} = 0,$$

and so we can start the first sum at $j = 0$ and use the identity (5.10) to write

$$\begin{aligned}\frac{d^n}{dx^n}(f\,g) &= \sum_{j=0}^{n} \binom{n-1}{j-1} f^{(j)}\, g^{(n-j)} + \sum_{j=0}^{n} \binom{n-1}{j} f^{(j)}\, g^{(n-j)} \\ &= \sum_{j=0}^{n} \left[\binom{n-1}{j-1} + \binom{n-1}{j}\right] f^{(j)}\, g^{(n-j)} = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}\, g^{(n-j)}\end{aligned}$$

which is Leibniz's rule. □

5.4 Radii of convergence

Find the radii of convergence of the series (5.97) and (5.98).

The radius of convergence of the series (5.97) for $\sqrt{1+x}$ is

$$\begin{aligned}R &= \left(\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}\right)^{-1} \\ &= \left(\lim_{n\to\infty} \frac{(\frac{1}{2} - n)}{(n+1)(\frac{1}{2} - n + 1)}\right)^{-1} \\ &= \left(\lim_{n\to\infty} \frac{1}{n}\right)^{-1} = \infty.\end{aligned}$$

Similarly, the radius of convergence of the series (5.98) for $1/\sqrt{1+x}$ is

$$\begin{aligned}R &= \left(\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}\right)^{-1} \\ &= \left(\lim_{n\to\infty} \frac{(-\frac{1}{2} - n)}{(n+1)(-\frac{1}{2} - n + 1)}\right)^{-1} = \infty.\end{aligned}$$

6

Examples for Chapter 6, Complex-Variable Theory

6.1 Analyticity

Example 6.1 $(z = x + iy)$ Is the function $f(x, y) = x + iy = z$ analytic? If we compute its derivative at $(x, y) = (0, 0)$ by setting $x = \epsilon$ and $y = 0$, then the limit is

$$\lim_{\epsilon \to 0} \frac{f(\epsilon, 0) - f(0, 0)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\epsilon}{\epsilon} = 1, \tag{6.1}$$

while if we instead set $x = 0$ and $y = \epsilon$, then the limit is

$$\lim_{\epsilon \to 0} \frac{f(0, \epsilon) - f(0, 0)}{i\epsilon} = \lim_{\epsilon \to 0} \frac{i\epsilon}{i\epsilon} = 1. \tag{6.2}$$

So $f(x, y) = x - iy$ may be differentiable at $z = 0$. □

Example 6.2 (z^*) Is the function $f(x, y) = x - iy = z^*$ analytic? If we compute its derivative at $(x, y) = (0, 0)$ by setting $x = \epsilon$ and $y = 0$, then the limit is

$$\lim_{\epsilon \to 0} \frac{f(\epsilon, 0) - f(0, 0)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\epsilon}{\epsilon} = 1, \tag{6.3}$$

while if we instead set $x = 0$ and $y = \epsilon$, then the limit is

$$\lim_{\epsilon \to 0} \frac{f(0, \epsilon) - f(0, 0)}{i\epsilon} = \lim_{\epsilon \to 0} \frac{-i\epsilon}{i\epsilon} = -1. \tag{6.4}$$

So $f(x, y) = x - iy = z^*$ is not an analytic function of z. □

We can't apply the definition (6.1) of differentiability to every point z for every function $f(z)$. We need a better test of analyticity.

6.2 Cauchy-Riemann Conditions

If $f(x,y) = u(x,y) + iv(x,y)$ with u and v real, is analytic then $df = f'(z)dz$ with $dz = dx + idy$ and

$$df = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) dx = f'(z)\, dx \;\text{ and }\; df = \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) dy = f'(z)\, idy, \tag{6.5}$$

or

$$f'(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right). \tag{6.6}$$

These complex equations imply the two real equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{6.7}$$

or more succinctly

$$u_x = v_y \quad \text{and} \quad v_x = -u_y \tag{6.8}$$

which are the Cauchy-Riemann conditions.

Example 6.3 ($f(x,y) = x^2 - y^2$) For the function $f(x,y) = u(x,y) + iv(x,y)$, the real and imaginary parts are $u = x^2 - y^2$ and $v = 0$, and so the Cauchy-Riemann conditions require that

$$2x = 0 \quad \text{and} \quad 0 = 2y. \tag{6.9}$$

So $f(x,y) = x^2 - y^2$ is not analytic. □

6.3 Calculus of residues

Example 6.4 (Isolated pole) Let's consider the integral

$$I = \oint_{\mathcal{C}} a_n(w)(z-w)^n\, dz \tag{6.10}$$

along a closed, counterclockwise contour $\mathcal{C}$ around the point w. Setting $z = w + \epsilon e^{i\theta}$, we find

$$\begin{aligned} I &= a_n(w)\int_0^{2\pi} \left(\epsilon e^{i\theta}\right)^n i\epsilon\, e^{i\theta}\, d\theta = i a_n(w)\epsilon^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\, d\theta \\ &= 2\pi i\, a_{-1}(w)\, \delta_{n,-1}. \end{aligned} \tag{6.11}$$

This is why if $f(z)$ has a Laurent series (6.89-6.90), only the $n = -1$ term $a_{-1}(z_0)$ contributes

$$\oint_{\mathcal{C}} f(z)\, dz = \oint_{\mathcal{C}} \sum_{n=-\infty}^{\infty} a_n(z_0)\,(z - z_0)^n\, dz = 2\pi i\, a_{-1}(z_0). \tag{6.12}$$

□

6.4 Ghost Contours

Example 6.5 (Yukawa potential) An intermediate formula in the derivation of the Yukawa potential (6.154) was

$$\begin{aligned} G_Y(\boldsymbol{x}) &= \int \frac{d^3k}{(2\pi)^3}\, \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{\boldsymbol{k}^2 + m^2} \\ &= \frac{1}{ir} \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^2}\, \frac{k}{(k - im)(k + im)}\, e^{ikr} \end{aligned} \tag{6.13}$$

in which $k = |\boldsymbol{k}|$ and $r = |\boldsymbol{x}|$. Since $r > 0$, we add a ghost contour that goes over the north pole of the upper half plane wherein $ikr = i(k_r + ik_i)r$ has a negative real part

$$G_Y(\boldsymbol{x}) = \frac{1}{ir} \oint_{\mathcal{C}} \frac{dk}{(2\pi)^2}\, \frac{k}{(k - im)(k + im)}\, e^{ikr}. \tag{6.14}$$

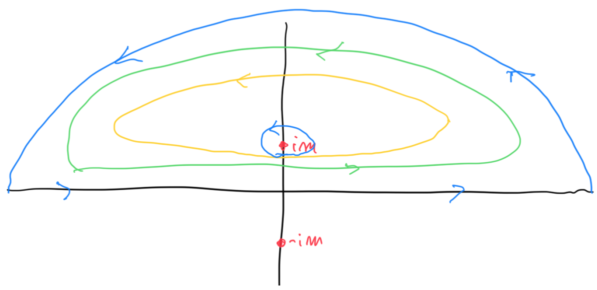


Figure 6.1 We shrink the ghost contour to a tiny loop about $k = im$.

The contour $\mathcal{C}$ encircles the point $k = im$ in a counter-clockwise sense, and so Cauchy's integral formula (6.40) gives

$$G_Y(\boldsymbol{x}) = \frac{1}{ir}\, 2\pi i \left[\frac{1}{(2\pi)^2}\, \frac{k}{k + im}\, e^{ikr} \right]\Bigg|_{k=im} = \frac{1}{4\pi r}\, e^{-mr} \tag{6.15}$$

in units with $\hbar = c = 1$. Since $\hbar = c = 1$ in this formula (6.16), we can season it with factors of $\hbar$ and c until the argument of the exponential becomes dimensionless standard units. We then get

$$G_Y(\boldsymbol{x}) = \frac{1}{4\pi r}\, e^{-cmr/\hbar}. \tag{6.16}$$

This potential is the Green's function for the differential operator $-\triangle+m^2$ because

$$\begin{aligned}(-\triangle + m^2)\, G_Y(\boldsymbol{x}) &= (-\triangle + m^2) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{\boldsymbol{k}^2 + m^2} \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{(\boldsymbol{k}^2 + m^2)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{\boldsymbol{k}^2 + m^2} = \int \frac{d^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \delta(\boldsymbol{x}).\end{aligned} \tag{6.17}$$

If we turn this last equation into a convolution

$$(-\triangle + m^2) \int d^3y\, G_Y(\boldsymbol{x} - \boldsymbol{y})\, j(\boldsymbol{y}) = \int d^3y\, \delta(\boldsymbol{x} - \boldsymbol{y}) j(\boldsymbol{y}) = j(\boldsymbol{x}), \tag{6.18}$$

we find that the solution to the equation

$$(-\triangle + m^2)\, f(\boldsymbol{x}) = j(\boldsymbol{x}) \tag{6.19}$$

is

$$f(\boldsymbol{x}) = \int d^3y\, G_Y(\boldsymbol{x} - \boldsymbol{y})\, j(\boldsymbol{y}). \tag{6.20}$$

□

Example 6.6 (Ghost countour)

$$I = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{-\infty}^{\infty} \frac{dx}{(x + i)(x - i)} \tag{6.21}$$

Add a ghost contour in UHP

$$I = \oint \frac{dx}{(x + i)(x - i)} = \frac{1}{2\pi i}\frac{1}{2i} = -\frac{1}{4\pi}. \tag{6.22}$$

□

6.5 Contour integral

We can compute the value of the nth Legendre polynomial $P_n(x)$ at $x = -1$ by using Cauchy's integral formula (6.44) and Schlaefli's formula (6.45):

$$\begin{aligned}
P_n(-1) &= \frac{1}{2^n\,2\pi i} \oint \frac{(z'^2-1)^n}{(z'+1)^{n+1}}\,dz' \\
&= \frac{1}{2^n\,2\pi i} \oint \frac{(z'-1)^n(z'+1)^n}{(z'+1)^{n+1}}\,dz' \\
&= \frac{1}{2^n\,2\pi i} \oint \frac{(z'-1)^n}{(z'+1)}\,dz' \\
&= \frac{1}{2^n}(-2)^n = (-1)^n.
\end{aligned}$$

7

Examples for Chapter 7, Differential Equations

7.1 Exact differentials and the Cauchy-Riemann equations

A differential is exact if it is the change dG in some function

$$dG = G_x dx + G_y dy. \tag{7.1}$$

If $F(x, y) = U(x, y) + iV(x, y)$, then

$$\begin{aligned} dF &= dU + idV \\ &= U_x dx + U_y dy + i(V_x dx + V_y dy). \end{aligned} \tag{7.2}$$

If, further, dF is proportional to $dz = dx + idy$, then

$$\begin{aligned} dF &= (u + iv)(dx + idy) \\ &= (udx - vdy) + i(vdx + udy). \end{aligned} \tag{7.3}$$

So now we have

$$\begin{aligned} u = U_x \quad &\text{and} \quad v = -U_y \\ v = V_x \quad &\text{and} \quad u = V_y \end{aligned} \tag{7.4}$$

which imply that

$$\begin{gathered} U_x = V_y \quad \text{and} \quad V_x = -U_y \\ \text{as well as} \\ u_y = -v_x \quad \text{and} \quad v_y = u_x \end{gathered} \tag{7.5}$$

which are the Cauchy-Riemann conditions for both F and its derivative

$$F = U + iV \quad \text{and} \quad F' = \frac{dF}{dz} = u + iv. \tag{7.6}$$

All integrals such as

$$'F = \int_{z_0}^{z} dz' F(z') \tag{7.7}$$

and derivatives of a function that is analytic in a simply connected region are also analytic there.

7.2 Examples of Frobenius's method

Example 7.1 (Legendre's equation) If one rewrites Legendre's equation $(1-x^2)y'' - 2xy' + \lambda y = 0$ as $x^2 y'' + xpy' + qy = 0$, then one finds $p(x) = -2x^2/(1-x^2)$ and $q(x) = x^2\lambda/(1-x^2)$, which are analytic but not polynomials. In this case, it is simpler to substitute the expansions (7.222–7.224)

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n\, x^{r+n} \\ y'(x) &= \sum_{n=0}^{\infty} (r+n)\, a_n\, x^{r+n-1} \\ y''(x) &= \sum_{n=0}^{\infty} (r+n)(r+n-1)\, a_n\, x^{r+n-2} \end{aligned} \tag{7.8}$$

directly into Legendre's equation $(1-x^2)y'' - 2xy' + \lambda y = 0$. We then find

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1)(1-x^2)x^{n+r-2} - 2(n+r)x^{n+r} + \lambda x^{n+r}\right] a_n = 0. \tag{7.9}$$

The coefficient of the lowest power of x is $r(r-1)a_0$, and so the indicial equation is $r(r-1) = 0$.

For $r = 0$, the condition (7.9) is

$$\sum_{n=0}^{\infty} \left[n(n-1)(1-x^2)x^{n-2} - 2nx^n + \lambda x^n\right] a_n = 0. \tag{7.10}$$

We shift the index n on the term $n(n-1)x^{n-2}a_n$ to $n = j+2$ and replace n by j in the other terms:

$$\sum_{j=0}^{\infty} \left\{(j+2)(j+1)\, a_{j+2} - [j(j-1) + 2j - \lambda]\, a_j\right\} x^j = 0. \tag{7.11}$$

Since the coefficient of x^j must vanish, we get the recursion relation

$$a_{j+2} = \frac{j(j+1) - \lambda}{(j+2)(j+1)} a_j \tag{7.12}$$

which for big j says that $a_{j+2} \approx a_j$. Thus the series (7.8)

$$y(x) = \sum_{n=0}^{\infty} a_n \, x^n \tag{7.13}$$

does not converge for $|x| \ge 1$ unless $\lambda = j(j+1)$ for some integer j in which case this series (7.13) is a Legendre polynomial (chapter 9). □

Example 7.2 (Hermite polynomials) The interval for Hermite's equation $y'' + (\lambda - x^2)y = 0$ runs from $-\infty$ to $+\infty$, so we must first determine how y behaves as $|x| \to \infty$. The ratio $Q(1/z)/z^2 = (\lambda - z^{-2})/z^2$ diverges as $z \to 0$, so the equation is essentially singular at $|x| = \infty$. When x^2 is huge, Hermite's equation is $y'' \approx x^2 y$ which is approximately satisfied by $y = e^{-x^2/2}$. So we set $y(x) = e^{-x^2/2}h(x)$. The equation for h then is $x^2h'' - 2x^3h' + (\lambda - 1)x^2h = 0$ which is a regular at all x and has $p(x) = -2x^2$ and $q(x) = (\lambda - 1)x^2$. Since $p(0) = 0 = q(0)$, the indicial equation (7.226) is $r(r-1) = 0$ with roots $r = 0$ and $r = 1$. If $r = 0$, then $h(x)$ obeys

$$\sum_{n=0}^{\infty} [n(n-1)x^n - 2nx^{n+2} + (\lambda - 1)x^{n+2}]a_n = 0. \tag{7.14}$$

Shifting n on the first sum, we get

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - (2n+1-\lambda)a_n]x^{n+2} = 0. \tag{7.15}$$

The recursion relation is

$$a_{n+2} = \frac{2n+1-\lambda}{(n+2)(n+1)} a_n. \tag{7.16}$$

The series is a polynomial only if it terminates which it will if the eigenvalue $\lambda = 2n + 1 = 1, 3, 5, \ldots$ These are the even polynomials.

We get the odd polynomials by setting $r = 1$. We then find

$$\sum_{n=0}^{\infty} [(n+3)(n+2)a_{n+3} - (2n+3-\lambda)a_n]x^{n+3} = 0 \tag{7.17}$$

and

$$a_{n+2} = \frac{2n+3-\lambda}{(n+3)(n+2)} a_n \tag{7.18}$$

which gives $\lambda = 2n + 3 = 3, 5, \ldots$. □

Example 7.3 (Hydrogen atom) The nonrelativistic radial equation for the hydrogen atom is $(r^2R')' + (\alpha r^2 + \beta r + \gamma)R = 0$ which has a regular singular point at $r = 0$ and an essentially singular point at infinity. To compensate at $r = 0$, we set $R = r^\ell S$ and find that near $r = 0$ we need $\gamma = -\ell(\ell + 1)$. To fix the essential singularity at infinity, we set $S = e^{-\delta r} s$ and find that for huge r we need $\delta = \sqrt{\alpha}$. One then sets $R = r^\ell e^{-\sqrt{-\alpha} r} s$ and uses the method of Frobenius to find $s(r)$ (exercise 7.19). □

7.3 Why there's a leading minus sign in self-adjoint operators

You may wonder why a differential operator is said to be in self-adjoint form when it has a leading minus sign

$$L = -\frac{d}{dx}\left(p\frac{d}{dx}\right) + q. \tag{7.19}$$

The reason is that the differential operator

$$-\frac{d^2}{dx^2} \tag{7.20}$$

is intrinsically positive when acting on square-integrable functions.

The way I think of it is to apply $-\partial_x^2$ to a a function $f(x)$ expressed as its Fourier transform

$$-\frac{d^2}{dx^2} f(x) = -\frac{d^2}{dx^2} \int dk\, e^{ikx}\, \tilde{f}(k) = \int dk\, k^2\, e^{ikx}\, \tilde{f}(k). \tag{7.21}$$

We then see that

$$-\frac{d^2}{dx^2} \tag{7.22}$$

amounts to multiplication by k^2 which is positive for real k.

Another way is to consider examples. For instance,

$$-\frac{d^2}{dx^2}\, e^{-x^2} = \frac{d}{dx}\, 2x\, e^{-x^2} = -4x^2\, e^{-x^2} + 2\, e^{-x^2} = (2 - 4x^2)\, e^{-x^2}. \tag{7.23}$$

The integral of this function over the real line is

$$\int_{-\infty}^{\infty} dx\, (2 - 4x^2)\, e^{-x^2} = 0, \tag{7.24}$$

but the function is positive where it is biggest as seen in Fig. 7.1.

7.4 Completeness of Eigenfunctions

To get equation 7.395, we use the orthonormality of the basis functions

$$\int_a^b u_j^*\, \rho\, u_i\, dx = \delta_{ij}. \tag{7.25}$$

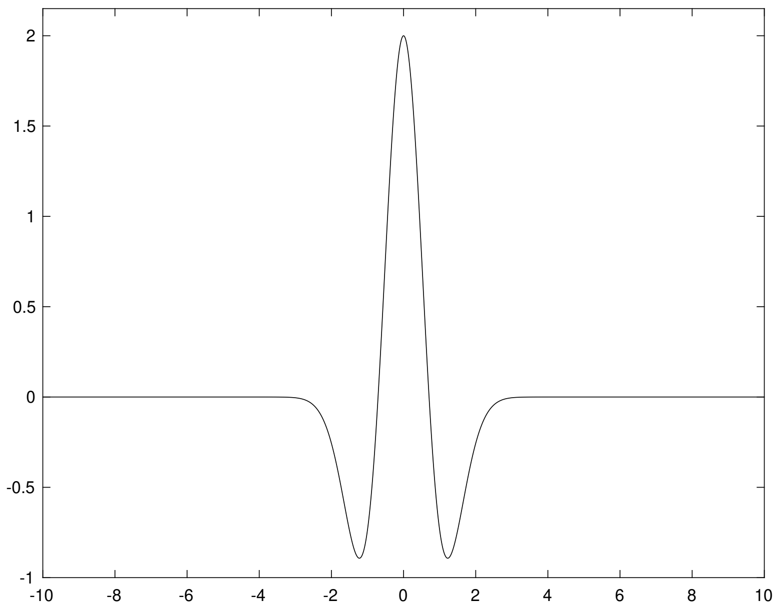


Figure 7.1 The function $\exp(-x^2)''$ is positive where it's biggest.

We then have

$$\begin{aligned} N[r_n, u_k] &= \int_a^b \rho(x)\, r_n(x)\, u_k(x)\, dx \\ &= \int_a^b \rho(x) \Big[f(x) - \sum_{j=1}^n c_j\, u_j(x) \Big] u_k(x)\, dx \\ &= c_k - \sum_{j=1}^n c_j\, \delta_{j,k} \\ &= 0 \quad \text{for} \quad k = 1, \ldots, n. \end{aligned} \tag{7.26}$$

7.5 Hydrogen atom

For a hydrogen atom with potential

$$V(r) = -e^2/4\pi\epsilon_0 r \equiv -q^2/r, \tag{7.27}$$

the radial equation is

$$(r^2 R'_{n,\ell})' + \left[(2m/\hbar^2)\left(E_{n,\ell} + Zq^2/r\right) r^2 - \ell(\ell+1)\right] R_{n,\ell} = 0. \tag{7.28}$$

So at big r, $R''_{n,\ell} \approx -2mE_{n,\ell}R_{n,\ell}/\hbar^2$ and $R_{n,\ell} \sim \exp(-\sqrt{-2mE_{n,\ell}}\,r/\hbar)$. At tiny r, $(r^2R'_{n,\ell})' \approx \ell(\ell+1)R_{n,\ell}$ and $R_{n,\ell}(r) \sim r^\ell$. So we set

$$R_{n,\ell}(r) = r^\ell \exp(-\sqrt{-2mE_{n,\ell}}\,r/\hbar)P_{n,\ell}(r) \tag{7.29}$$

and apply the method of Frobenius to find the values of $E_{n,\ell}$ for which $R_{n,\ell}$ is suitably normalizable.

We set $\beta = \sqrt{-2mE_{n,\ell}}/\hbar$ so that the radial wave function is

$$R_{n,\ell}(r) = r^\ell \,\exp(-\sqrt{-2mE_{n,\ell}}\,r/\hbar)\,P_{n,\ell}(r) = r^\ell\, e^{-\beta r}\,P_{n,\ell}(r).$$

Then we compute the first term in the radial equation

$$\begin{aligned}
(r^2R'_{n,\ell})' &= r^2R''_{n,\ell} + 2rR'_{n,\ell} \\
&= r^2\left[r^\ell e^{-\beta r}P_{n,\ell}\right]'' + 2r\left[r^\ell e^{-\beta r}P_{n,\ell}\right]' \\
&= r^2\left[\ell r^{\ell-1}e^{-\beta r}P_{n,\ell} - \beta r^\ell e^{-\beta r}P_{n,\ell} + r^\ell e^{-\beta r}P'_{n,\ell}\right]' \\
&\quad + 2r\left[\ell r^{\ell-1}e^{-\beta r}P_{n,\ell} - \beta r^\ell e^{-\beta r}P_{n,\ell} + r^\ell e^{-\beta r}P'_{n,\ell}\right] \\
&= r^2\Big[\ell(\ell-1)r^{\ell-2}e^{-\beta r}P_{n,\ell} + \beta^2 r^\ell e^{-\beta r}P_{n,\ell} + r^\ell e^{-\beta r}P''_{n,\ell} \\
&\quad +2\ell r^{\ell-1}e^{-\beta r}P'_{n,\ell} - 2\beta\ell r^{\ell-1}e^{-\beta r}P_{n,\ell} - 2\beta r^\ell e^{-\beta r}P'_{n,\ell}\Big] \\
&\quad + 2r\left[\ell r^{\ell-1}e^{-\beta r}P_{n,\ell} - \beta r^\ell e^{-\beta r}P_{n,\ell} + r^\ell e^{-\beta r}P'_{n,\ell}\right] \\
&= r^{\ell+2}e^{-\beta r}P''_{n,\ell} + 2(\ell - \beta r + 1)r^{\ell+1}e^{-\beta r}P'_{n,\ell} \\
&\quad + \left[\beta^2r^2 - 2(\ell+1)\beta r + \ell(\ell+1)\right]r^\ell e^{-\beta r}P_{n,\ell}.
\end{aligned}$$

Setting $b = 2mZq^2/\hbar^2$, we substitute $(r^2R'_{n,\ell})'$ into the radial equation

$$(r^2R'_{n,\ell})' + \left[-\beta^2r^2 + br - \ell(\ell+1)\right]R_{n,\ell} = 0$$

and get

$$\begin{aligned}
0 = r^{\ell+2}e^{-\beta r}P''_{n,\ell} &+ 2(\ell - \beta r + 1)r^{\ell+1}e^{-\beta r}P'_{n,\ell} \\
&+ \left[\beta^2r^2 - 2(\ell+1)\beta r + \ell(\ell+1)\right]r^\ell e^{-\beta r}P_{n,\ell} \\
&+ \left[-\beta^2r^2 + br - \ell(\ell+1)\right]r^\ell e^{-\beta r}P_{n,\ell}
\end{aligned}$$

or

$$0 = r\,P''_{n,\ell} + 2(\ell+1-\beta r)\,P'_{n,\ell} + (b - 2(\ell+1)\beta)P_{n,\ell}.$$

So we put

$$P_{n,\ell}(r) = r^k\sum_{n=0}^{\infty} a_n r^n$$

into the last equation and get

$$0 = \sum_{n=0}^{\infty} \Big[(n+k)(n+k-1)a_n r^{n+k-1} + 2(\ell+1)(n+k)a_n r^{n+k-1}$$
$$-2\beta(n+k)a_n r^{n+k} + (b - 2(\ell+1)\beta)a_n r^{n+k}\Big] .$$

We set the coefficient of the lowest power of x to zero

$$[k(k-1) + 2k(\ell+1)]a_0 = 0$$

and find $k = 0$ or $k = -(2\ell+1)$. Following Fuchs, we choose the larger root $k = 0$. We then have

$$0 = \sum_{n=0}^{\infty} n\,[n - 1 + 2(\ell+1)]\,a_n r^{n-1} + [b - 2(n+\ell+1)\beta]\,a_n r^n$$

or

$$0 = \sum_{n=0}^{\infty} \{(n+1)\,[n + 2(\ell+1)]\,a_{n+1} + [b - 2(n+\ell+1)\beta]\,a_n\}\,r^n$$

which gives us the recursion relation

$$a_{n+1} = -\frac{b - 2(n+\ell+1)\beta}{(n+1)\,[n+2(\ell+1)]}\,a_n.$$

As $n \to \infty$, this recursion relation tends to

$$a_{n+1} \sim \frac{(2\beta)^{n+1}}{(n+1)!}$$

so that asymptotically

$$P_{n,\ell}(r) \sim e^{2\beta r}.$$

The wave function then would be

$$R_{n,\ell}(r) \sim r^{\ell}\, e^{-\beta r}\, e^{2\beta r}$$

which is not normalizable. The recursion relation therefore must terminate. That is, for some n, we must have

$$b - 2(n+\ell+1)\beta = 0$$

or

$$\beta = \frac{\sqrt{-2mE_{n,\ell}}}{\hbar} = \frac{b}{2\,[n+\ell+1]} = \frac{2mZq^2}{2\hbar^2\,[n+\ell+1]}.$$

Thus, the energy levels of atomic hydrogen are

$$\begin{aligned} E_{n,\ell} &= -\frac{mZ^2q^4}{2\hbar^2[n+\ell+1]^2} \\ &= -\frac{1}{2}mc^2\frac{(Z\alpha)^2}{[n+\ell+1]^2} \end{aligned}$$

in which $\alpha \approx 1/137.036$ is the **fine-structure constant**, $\alpha = e^2/4\pi\epsilon_0\hbar c$ in which $-e$ is the charge of the electron.

8
Examples for Chapter 8, Integral Equations

8.1 Bessel Functions

The differential operator L for Bessel's equation (7.354)

$$z^2\,u'' + z\,u' + z^2\,u - \lambda^2\,u = 0 \tag{8.1}$$

is

$$L = z^2\,\frac{d^2}{dz^2} + z\,\frac{d}{dz} + z^2 \tag{8.2}$$

and the constant c is $-\lambda^2$. We choose the kernel $K(z,w) = e^{\pm iz\sin w}$which is entire in both variables and seek a differential operator M that satisfies $M\,K = L\,K$. We try $M = -\,d^2/dw^2$ and find (exercise 8.3) that

$$-K_{ww} = z^2\,K_{zz} + z\,K_z + z^2\,K \tag{8.3}$$

in which subscripts indicate differentiation as in (2.7).

In terms of M and K, our integral equation (8.35) for v is

$$\int_C \left[K_{ww}(z,w) + \lambda^2\,K(z,w)\right]\,v(w)\,dw = 0. \tag{8.4}$$

We integrate by parts once

$$\int_C \left[-\,K_w\,v' + \lambda^2\,K\,v + \frac{dK_w\,v}{dw}\right]\,dw \tag{8.5}$$

and then again

$$\int_C \left[K\,\left(v'' + \lambda^2\,v\right) + \frac{d(K_w\,v - Kv')}{dw}\right]\,dw. \tag{8.6}$$

So if we choose the contour so that $K_w\,v - Kv'$ vanishes at both ends, then the unknown function v need only satisfy the differential equation

$$v'' + \lambda^2\,v = 0 \tag{8.7}$$

which is much simpler than Bessel's equation (8.1). The solution $v(w) = \exp(i\lambda w)$ is an entire function of w for every complex λ.

The contour integral (8.32) now gives us Bessel's function as the integral transform

$$u(z) = \int_C K(z, w)\, v(w)\, dw = \int_C e^{\pm iz \sin w}\, e^{i\lambda w}\, dw. \tag{8.8}$$

For $\mathrm{Re}(z) > 0$ and any complex λ, the contour C_1 that runs from $-i\infty$ to the origin $w = 0$, then to $w = -\pi$, and finally up to $-\pi + i\infty$ has $K_w\, v - K v' = 0$ at its ends (exercise 8.4) provided we use the minus sign in the exponential. The function defined by this choice

$$H^{(1)}_\lambda(z) = -\frac{1}{\pi} \int_{C_1} e^{-iz \sin w + i\lambda w}\, dw \tag{8.9}$$

is the **first Hankel function** (Hermann Hankel, 1839–1873). The **second Hankel function** is defined for $\mathrm{Re}(z) > 0$ and any complex λ by a contour C_2 that runs from $\pi + i\infty$ to $w = \pi$, then to $w = 0$, and lastly to $-i\infty$

$$H^{(2)}_\lambda(z) = -\frac{1}{\pi} \int_{C_2} e^{-iz \sin w + i\lambda w}\, dw. \tag{8.10}$$

Because the integrand $\exp(-iz \sin w + i\lambda w)$ is an entire function of z and w, one may deform the contours C_1 and C_2 and analytically continue the Hankel functions beyond the right half-plane (Courant and Hilbert, 1955, chap. VII). One may verify (exercise 8.5) that the Hankel functions are related by complex conjugation

$$H^{(1)}_\lambda(z) = H^{(2)*}_\lambda(z) \tag{8.11}$$

when both $z > 0$ and λ are real.

9

Examples for Chapter 9, Legendre Polynomials

9.1 Electrostatic Potential inside a Hollow Sphere

Find the electrostatic potential $V(r,\theta)$ inside a hollow sphere of radius R if the potential on the sphere is $V(R,\theta) = V_0 \cos^2\theta$.

Inside a hollow sphere of radius R, the potential obeys Laplace's equation $\triangle V(r,\theta) = 0$. So we use the expansion (9.86)

$$f(r,\theta) = \sum_{\ell=0}^{\infty} \left[a_\ell \, r^\ell + b_\ell \, r^{-\ell-1} \right] \, P_\ell(\cos\theta) \tag{9.1}$$

in terms of Legendre's polynomials $P_\ell(\cos\theta)$ multiplied by $a_\ell r^\ell + b_\ell / r^{\ell+1}$. Since the potential is finite at the origin, all the b_ℓ's vanish. We then have

$$V(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell \, r^\ell \, P_\ell(\cos\theta).$$

To satisfy the boundary condition, we set

$$V(R,\theta) = \sum_{\ell=0}^{\infty} a_\ell \, R^\ell \, P_\ell(\cos\theta) = V(R,\theta) = V_0 \cos^2\theta.$$

Using the explicit formulas of example 9.4 with $x = \cos\theta$, we find $x^2 = (2P_2(x) + P_0(x))/3$, and so the boundary condition is

$$\sum_{\ell=0}^{\infty} a_\ell \, R^\ell \, P_\ell(x) = \frac{V_0}{3} \left(2P_2(x) + P_0(x) \right).$$

The orthogonality relation (9.29) now gives us

$$\sum_{\ell=0}^{\infty} a_\ell \, R^\ell \int_{-1}^{1} P_\ell(x) P_n(x) \, dx = \sum_{\ell=0}^{\infty} a_\ell \, R^\ell \, \frac{2}{2n+1} \, \delta_{n\ell} = \frac{2a_n \, R^n}{2n+1}$$
$$= \frac{V_0}{3} \left(2\frac{2}{5}\delta_{n2} + 2\delta_{n0} \right)$$

and lets us identify the coefficients a_n as

$$a_n = \frac{(2n+1)V_0}{3R^n} \left(\frac{2}{5}\delta_{n2} + \delta_{n0} \right).$$

That is, $a_0 = V_0/3$, and $a_2 = 2V_0/R^2$, and so the potential inside the sphere is

$$V(r,\theta) = \frac{V_0}{3} \left[1 + \frac{r^2}{R^2} \left(3\cos^2\theta - 1 \right) \right].$$

9.2 Electrostatic Potential outside a Hollow Sphere

Find the electrostatic potential $V(r,\theta)$ outside a hollow sphere of radius R if the potential on the sphere is $V(R,\theta) = V_0 \cos^2\theta$.

Outside the hollow sphere, the potential obeys Laplace's equation $\triangle V(r,\theta) = 0$. So we use the expansion (9.1) in terms of Legendre's polynomials $P_\ell(\cos\theta)$ multiplied by $a_\ell r^\ell + b_\ell / r^{\ell+1}$. Since the potential is zero as $r \to \infty$, all the a_ℓ's vanish. We then have

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{b_\ell}{r^{\ell+1}} \, P_\ell(\cos\theta).$$

To satisfy the boundary condition, we set

$$V(R,\theta) = \sum_{\ell=0}^{\infty} \frac{b_\ell}{R^{\ell+1}} \, P_\ell(\cos\theta) = V(R,\theta) = V_0 \, \cos^2\theta.$$

Using the explicit formulas of example 9.4 with $x = \cos\theta$, we find $x^2 = (2P_2(x) + P_0(x))/3$, and so the boundary condition is

$$\sum_{\ell=0}^{\infty} \frac{b_\ell}{R^{\ell+1}} \, P_\ell(x) = \frac{V_0}{3} \left(2P_2(x) + P_0(x) \right).$$

The orthogonality relation (9.29) now gives us

$$\sum_{\ell=0}^{\infty} \frac{b_\ell}{R^{\ell+1}} \int_{-1}^{1} P_\ell(x) P_n(x)\, dx = \sum_{\ell=0}^{\infty} \frac{b_\ell}{R^{\ell+1}} \frac{2}{2n+1}\, \delta_{n\ell} = \frac{2}{2n+1} \frac{b_n}{R^{n+1}}$$
$$= \frac{V_0}{3} \left(2 \frac{2}{5} \delta_{n2} + 2\delta_{n0} \right)$$

and lets us identify the coefficients b_n as

$$b_n = \frac{(2n+1)\, V_0\, R^{n+1}}{3} \left(\frac{2}{5} \delta_{n2} + \delta_{n0} \right).$$

That is, $b_0 = V_0\, R/3$, and $b_2 = 2V_0\, R^3/3$, and so the potential outside the sphere is

$$V(r, \theta) = \frac{V_0}{3} \left[\frac{R}{r} + \frac{R^3}{r^3} \left(3\cos^2\theta - 1 \right) \right].$$

9.3 Comoving and Physical Coordinates

In a flat Friedmann-Lemaître-Robinson-Walker cosmology, the invariant squared spacetime distance (aka, the line element) (7.489) with $k = 0$ is

$$ds^2 = -\, c^2 dt^2 + a^2(t) \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right) \tag{9.2}$$

in which r, θ, and ϕ are **comoving coordinates**, and the magnitude of the **scale factor** $a(t)$ describes the expansion of space. An element of **physical distance** is $cdt = a(t)dr$, the distance light goes in time dt as it traverses a comoving distance dr

$$c\, dt = a(t)\, dr. \tag{9.3}$$

The corresponding element of **comoving distance** is $dr = c\, dt/a(t)$.

9.4 The Distance from the Surface of Decoupling

For instance, the comoving distance from which the CMB photons come to us at the present time t_0 from the time of the decoupling t_d of photons and electrons is

$$r_d = \int_{t_d}^{t_0} \frac{c\, dt}{a(t)}. \tag{9.4}$$

If in accord with observations, we assume that space is flat ($k = 0$), then the first-order Friedmann equation (7.490) for the scale factor$a(t)$ is

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\,\rho(t) \tag{9.5}$$

in which G is Newton's constant

$$G = 6.6743 \times 10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}}, \tag{9.6}$$

and ρ is the mass density. The mass density of visible and invisible matter ρ_m is a fraction $\Omega_m = 0.315$ of the critical density

$$\rho_c = (1.87834 \times 0.674^2) \times 10^{-26} = 8.5328 \times 10^{-27}\,\mathrm{kg\ m^{-3}}. \tag{9.7}$$

So the product $G\,\rho_c$ is

$$G\,\rho_c = 6.6743 \times 10^{-11} \times 8.5328 \times 10^{-27} = 5.6950 \times 10^{-37}. \tag{9.8}$$

Numerically, the mass density is

$$\rho_m = \Omega_m \rho_c = 0.3143\,\rho_c = 2.68186 \times 10^{-27}\ \mathrm{kg\ m^{-3}}. \tag{9.9}$$

The mass density ρ_r of radiation, both photons and neutrinos, is a tiny fraction $\Omega_r = 9.153 \times 10^{-5}$ of the critical density

$$\rho_r = \Omega_r \rho_c = 9.153 \times 10^{-5}\,\rho_c. \tag{9.10}$$

Finally, the mass density of dark energy is a big fraction $\Omega_\Lambda = 0.685$ of the critical density

$$\rho_\Lambda = \Omega_\Lambda\,\rho_c = 0.685\,\rho_c. \tag{9.11}$$

As far as we know, the mass density of dark energy is independent of the scale factor. But the mass density of matter varies with the expansion of space as $\rho_m(t) = \rho_m/a^3(t)$ because it is approximately the number of particles times their average mass, and that of radiation as $\rho_r(t) = \rho_r/a^4(t)$ because wavelengths stretch with the scale factor. Setting the density equal to the sum

$$\rho(t) = \rho_\Lambda(t) + \rho_m(t) + \rho_r(t) = \rho_\Lambda + \frac{\rho_m}{a^3(t)} + \frac{\rho_r}{a^4(t)}, \tag{9.12}$$

we can write Friedmann's equation (9.5) as

$$\frac{dt}{a} = \frac{\sqrt{3}\,da}{\sqrt{8\pi G\rho_c\,(\Omega_\Lambda\,a^4 + \Omega_m\,a + \Omega_r)}}. \tag{9.13}$$

Thus in terms of the present value of the scale factor $a(t_0) = 1$ and its value at the time of decoupling $a(t_d) = 1/1091$, the distance r_d (9.4) is

$$\begin{aligned} r_d &= \int_{t_d}^{t_0} \frac{cdt}{a(t)} = c \int_{1/1091}^{1} \frac{\sqrt{3}\, da}{\sqrt{8\pi G \rho_c \left(\Omega_\Lambda\, a^4 + \Omega_m\, a + \Omega_r\right)}} \\ &= c \int_{1/1091}^{1} \frac{\sqrt{3}\, da}{\sqrt{8\pi\, 5.695 \times 10^{-37} \left(0.685\, a^4 + 0.3143\, a + 9.153 \times 10^{-5}\right)}} \\ &= 4.282 \times 10^{26}\ \mathrm{m}. \end{aligned} \tag{9.14}$$

But this number is not the same as the one I published in my EJP article.

The distance from the big bang (the comoving distance is the physical distance now when $a(t_0) = 1$) is the integral (9.14) with $t_d = 0$

$$\begin{aligned} r_{bb} &= \int_{0}^{t_0} \frac{cdt}{a(t)} = c \int_{0}^{1} \frac{\sqrt{3}\, da}{\sqrt{8\pi G \rho_c \left(\Omega_\Lambda\, a^2 + \Omega_m\, a + \Omega_r\right)}} \\ &= c \int_{0}^{1} \frac{\sqrt{3}\, da}{\sqrt{8\pi\, 5.695 \times 10^{-37} \left(0.685\, a^2 + 0.3143\, a + 9.153 \times 10^{-5}\right)}} \\ &= 4.369 \times 10^{26}\ \mathrm{m}. \end{aligned} \tag{9.15}$$

9.5 Scale Factor before Decoupling and the Sound Horizon

During the era before decoupling ($t_d \sim 380,000$ yr), the maximum physical (as opposed to the comoving) speed of a pressure wave, called the“speed of sound” v_s, is related to the scale factor a and to the speed of light c by the formula [1, chap. 2]

$$v_s = \frac{c}{\sqrt{3}} \frac{1}{\sqrt{1 + f\, a}} = \frac{c}{\sqrt{3}} \frac{1}{\sqrt{1 + 678.435\, a}}. \tag{9.16}$$

Before decoupling when $a < 1/1091$, the effect of dark energy is negligible and we may approximate Friedmann’s first-order equation (7.490) as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho_c}{3} \left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4}\right) \tag{9.17}$$

in which ρ_c is the critical density

$$\rho_c = (1.87834 \times 0.674^2) \times 10^{-26} = 8.5328 \times 10^{-27}\ \mathrm{kg\ m^{-3}} \tag{9.18}$$

and we set the curvature parameter $k = 0$ as is consistent with all experiments. After a few manipulations, we get

$$\frac{dt}{a} = \frac{\sqrt{3}\,da}{\sqrt{8\pi G\rho_c\,(\Omega_m\,a + \Omega_r)}}. \tag{9.19}$$

To find the maximum radius r_s of a region of higher pressure in the fluid of electrons, ions, and photons before decoupling, we integrate our formula (9.16) for the speed of sound

$$v_s = \frac{c}{\sqrt{3}}\frac{1}{\sqrt{1+f\,a}} = \frac{c}{\sqrt{3}}\frac{1}{\sqrt{1+678.435\,a}} \tag{9.20}$$

to get for the comoving distance r_s

$$\begin{aligned} r_s &= \int_0^{t_d} \frac{v_s}{a(t)} dt = c\int_0^{1/1091} \frac{da}{\sqrt{8\pi G\rho_c(1+fa)\,(\Omega_m\,a+\Omega_r)}} \\ &= 10^{18} c \int_0^{1/1091} \frac{da}{\sqrt{14.3132(1+678.435\,a)\,(\Omega_m\,a+\Omega_r)}}. \end{aligned} \tag{9.21}$$

Approximate values of the ratios of the total (visible and invisible) matter density and of the total (photons and neutrinos) radiation mass density to the critical density are

$$\Omega_m = 0.315 \quad \text{and} \quad \Omega_r = 9.153\times 10^{-5}. \tag{9.22}$$

So

$$\begin{aligned} r_s &= 10^{18} c \int_0^{1/1091} \frac{da}{\sqrt{14.3132(1+678.435\,a)\,(0.315\,a+9.153\times 10^{-5})}} \\ &= 0.01487\times 10^{18} c = 4.458\times 10^{24}\text{ m}. \end{aligned} \tag{9.23}$$

A different estimate of astronomical parameters gives $r_s = 4.4685\times 10^{24}$ m.

The ratio of this comoving radius r_s (9.23) to the comoving distance r_d (9.14) from the surface of last scattering at decoupling is

$$\theta = \frac{r_s}{r_d} = \frac{4.458\times 10^{24}}{4.29171\times 10^{26}} = 0.01039. \tag{9.24}$$

The Planck value is $\theta = 0.01041$. This angle is the principal maximum in the CMB plot of Fig. (9.4) of PM.

9.6 Density and Pressure

As explained in Section 13.44 of PM, if the density and pressure in a phase with one dominant component are related by

$$p_i = c^2 w_i \rho_i \tag{9.25}$$

in which w_i is a constant, then the 0-th component of the vanishing of the covariant divergence of the energy-momentum tensor is

$$0 = T^{0a}_{;a} = D_a T^{0a}. \tag{9.26}$$

This is not a conservation law because the divergence is covariant. In flat space, it becomes the conservation law $0 = \partial_a T^{0a}$.

The analog of T^{ab} for the gravitational field is not a tensor. Dirac calls it a pseudotensor t^{ab}. The reference is to pages 58–63 of his book, *General Theory of Relativity.* The divergence of $(T^{ab} + t^{ab})\sqrt{g}$ vanishes because the total action does not depend explicitly upon an external spacetime point x. But the resulting integrals for the energy and momentum of the gravitational field may lie at spatial infinity or diverge.

In any case, the vanishing (9.26) implies that in a phase with $w = p/c^2\rho$

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}. \tag{9.27}$$

Ijjas and Steinhardt use the notation

$$\epsilon_\pm = \frac{3}{2}\left(1 + \frac{p}{c^2\rho}\right) = \frac{3}{2}\,(1+w) \tag{9.28}$$

in which the subscript $\pm$ refers to expanding (+) and contracting (-) phases.

Ijjas and Steinhardt set $k = 0$, so we have

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\,\rho = \frac{8\pi G}{3}\,\rho_0\left(\frac{a_0}{a}\right)^{3(1+w)} \tag{9.29}$$

which means that

$$\frac{\dot a}{a} = \pm\,\alpha\, a^{3(1+w)/2} = \pm\alpha\, a^{\varepsilon} \tag{9.30}$$

with

$$\alpha = \sqrt{\frac{8\pi G}{3}\,\rho_0} \quad \text{and} \quad a_0 = 1. \tag{9.31}$$

Choosing the plus sign in (9.30), which is appropriate for $\dot a > 0$, and integrating, we find

$$\frac{da}{adt} = \alpha\, a^{\epsilon}, \quad \alpha\, dt = a^{-(1+\epsilon)} da, \quad \alpha\, t = a^{-\epsilon} \quad \text{and} \quad a = (\alpha\, t)^{1/\epsilon}. \tag{9.32}$$

To cover both cases, $\pm\dot a > 0$, I & S write this as $a \sim |t|^{1/\epsilon}$.

10
Examples for Chapter 10, Bessel Functions

10.1 Cylindrical Wave Guides

When reading the last paragraph of Example 10.3 of PM, one might wonder why the derivative of the Bessel function $J_1(\rho)$ has a lower zero than that of $J_0(\rho)$. That is, it may seem odd that the first zero of a first derivative of a Bessel function is $z'_{1,1} \approx 1.8412$. What about the first zero of J'_0? Well, a glance at Fig. PM10.1 shows that the first zero of J'_0 is nearly 4, while $z'_{0,1}$ is less than 2. Values accurate to 15 digits are in the tables of Fig. 10.1 which is taken from `http://wwwal.kuicr.kyoto-u.ac.jp/www/accelerator/a4/besselroot.htmlx`.

10.2 Generating function for cylindrical Bessel functions

By differentiating the generating function (10.5) with respect to u and identifying the coefficients of powers of u, one may derive the recursion relation

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z} J_n(z). \tag{10.1}$$

To do this, we differentiate the expansion

$$\exp\left[\frac{z}{2}(u - 1/u)\right] = \sum_{n=-\infty}^{\infty} u^n J_n(z)$$

and find

$$\begin{aligned}\frac{z}{2}\left(1+u^{-2}\right)\exp\left[\frac{z}{2}(u-1/u)\right] &= \frac{z}{2}\left(1+u^{-2}\right)\sum_{n=-\infty}^{\infty} u^n J_n(z)\\ &= \sum_{n=-\infty}^{\infty} n\, u^{n-1} J_n(z).\end{aligned}$$

Roots of Bessel functions (15 digits)

The n-th roots of Jm(x)=0.

m＼n	n=1	n=2	n=3	n=4	n=5
m=0	2.40482555769577	5.52007811028631	8.65372791291101	11.7915344390142	14.9309177084877
m=1	3.83170597020751	7.01558666981561	10.1734681350627	13.3236919363142	16.4706300508776
m=2	5.13562230184068	8.41724414039986	11.6198411721490	14.7959517823512	17.9598194949878
m=3	6.38016189592398	9.76102312998166	13.0152007216984	16.2234661603187	19.4094152264350
m=4	7.58834243450380	11.0647094885011	14.3725366716175	17.6159660498048	20.8269329569623
m=5	8.77148381595995	12.3386041974669	15.7001740797116	18.9801338751799	22.2177998965612
m=6	9.93610952421768	13.5892901705412	17.0038196678160	20.3207892135665	23.5860844355813
m=7	11.0863700192450	14.8212687270131	18.2875828324817	21.6415410198484	24.9349278876730
m=8	12.2250922640046	16.0377741908877	19.5545364309970	22.9451731318746	26.2668146411766
m=9	13.3543004774353	17.2412203824891	20.8070477892641	24.2338852577505	27.5837489635730
m=10	14.4755006865545	18.4334636669665	22.0469853646978	25.5094505541828	28.8873750635304

Roots of Derivatives of Bessel functions

The n-th roots of Jm'(x)=0.

m＼n	n=1	n=2	n=3	n=4	n=5
m=0	3.83170597020751	7.01558666981561	10.1734681350627	13.3236919363142	16.4706300508776
m=1	1.84118378134065	5.33144277352503	8.53631636634628	11.7060049025920	14.8635886339090
m=2	3.05423692822714	6.70613319415845	9.96946782308759	13.1703708560161	16.3475223183217
m=3	4.20118894121052	8.01523659837595	11.3459243107430	14.5858482861670	17.7887478660664
m=4	5.31755312608399	9.28239628524161	12.6819084426388	15.9641070377315	19.1960288000489
m=5	6.41561637570024	10.5198608737723	13.9871886301403	17.3128424878846	20.5755145213868
m=6	7.50126614468414	11.7349359530427	15.2681814610978	18.6374430096662	21.9317150178022
m=7	8.57783648971407	12.9323862370895	16.5293658843669	19.9418533665273	23.2680529264575
m=8	9.64742165199721	14.1155189078946	17.7740123669152	21.2290626228531	24.5871974863176
m=9	10.7114339706999	15.2867376673329	19.0045935379460	22.5013987267772	25.8912772768391
m=10	11.7708766749555	16.4478527484865	20.2230314126817	23.7607158603274	27.1820215271905

Last modified at Wed Apr 10 21:22:34 2013.

Figure 10.1 wwwal.kuicr.kyoto-u.ac.jp/www/accelerator/a4/besselroot.htmlx

So we have

$$\frac{z}{2}\sum_{n=-\infty}^{\infty}\left[u^n\,J_n(z)+u^{n-2}\,J_n(z)\right]=\sum_{n=-\infty}^{\infty}n\,u^{n-1}\,J_n(z).$$

Shifting the index n, we get

$$\frac{z}{2}\sum_{n=-\infty}^{\infty}\left[u^{n-1}\,J_{n-1}(z)+u^{n-1}\,J_{n+1}(z)\right]=\sum_{n=-\infty}^{\infty}n\,u^{n-1}\,J_n(z)$$

which implies that

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z}\, J_n(z).$$

10.3 Normalization of Bessel functions

With $y = J_n(ax)$ and with a for k, Bessel's equation (10.11) is

$$(xy')' + (xa^2 - n^2/x)y = 0. \tag{10.2}$$

We multiply this equation by xy', integrate from 0 to b, and so show that if $ab = z_{n,m}$ and $J_n(z_{n,m}) = 0$, then

$$2\int_0^b x\, J_n^2(ax)\, dx = b^2 J_n'^2(z_{n,m}) \tag{10.3}$$

which is the normalization condition (10.14).

We multiply this equation

$$-\frac{d}{dx}\left(x\,\frac{dJ_n(ax)}{dx}\right) + \frac{n^2}{x} J_n(ax) = a^2 x\, J_n(ax)$$

by $xJ_n'(ax)$

$$-xJ_n'(ax)\left[x\, J_n'(ax)\right]' + n^2 J_n'(ax)\, J_n(ax) = a^2x^2\, J_n'(ax)\, J_n(ax)$$

and integrate from $x = 0$ to $x = b = z_{n,m}/a$ where

$$nJ_n(0) = nJ_n(ab) = nJ_n(z_{n,m}) = 0.$$

We get

$$-\frac{1}{2}\left|\left[x\, J_n'(ax)\right]^2\right|_0^b + \frac{n^2}{2}\left[J_n^2(ax)\right]_0^b = a^2\int_0^b x^2\, J_n'(ax)\, J_n(ax)\, dx.$$

Since $nJ_n(0) = 0$, and $J_n(z_{n,m}) = 0$, we have, integrating by parts and dropping the vanishing surface terms,

$$\begin{aligned}
-\frac{1}{2}\left|\left[x\frac{dJ(ax)}{dx}\right]^2\right|_{x=z_{n,m}/a} &= a^2\int_0^b x^2\, J_n'(ax)\, J_n(ax)\, dx \\
-\frac{1}{2}\left|\left[ax\frac{dJ(z_{n,m})}{dz}\right]^2\right|_{x=z_{n,m}/a} &= a^2\int_0^b x^2\,\frac{1}{2}\left(J_n^2(ax)\right)'\, dx \\
-\frac{1}{2}\, z_{n,m}^2\, J_n'^2(z_{n,m}) &= -\, a^2\int_0^b x\, J_n^2(ax)\, dx.
\end{aligned}$$

That is,

$$\int_0^b x\, J_n^2(ax)\, dx = \frac{z_{n,m}^2}{2a^2}\, J_n'^2(z_{n,m}).$$

10.4 Traveling-wave-guide solutions

We want to show that

$$E_z \equiv E_z(\rho,\phi,z,t) = J_n(\sqrt{\omega^2/c^2 - k^2}\,\rho) e^{in\phi} e^{i(kz-\omega t)}$$

is a traveling-wave solution (10.56) of the wave equation (10.61)

$$-\triangle E_z = -\ddot{E}_z/c^2 = \omega^2 E_z/c^2.$$

Since

$$\frac{d^2}{dt^2} e^{i(kz-\omega t)} = -\omega^2\, e^{i(kz-\omega t)}$$

it follows that

$$-\frac{d^2}{c^2 dt^2} E_z = \omega^2 E_z/c^2.$$

So we need to show that

$$-\triangle E_z = \omega^2 E_z/c^2.$$

To show this, we use our formula (2.32) for the laplacian in cylindrical coordinates to write

$$\begin{aligned} -\triangle E_z &= -\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2} + \frac{\partial^2 E_z}{\partial z^2}\right] \\ &= -\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right) - \frac{n^2 E_z}{\rho^2} - k^2 E_z\right]. \end{aligned}$$

So we must show that

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right) + \frac{n^2 E_z}{\rho^2} = \left(\frac{\omega^2}{c^2} - k^2\right) E_z.$$

But this is Bessel's equation (10.11) for $J_n(a\rho)$ with $a = \sqrt{\omega^2/c^2 - k^2}$.

10.5 Spherical Bessel Functions

The spherical Bessel function $j_\ell(\rho)$ is

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}}\, J_{\ell+1/2}(x). \tag{10.4}$$

At small x, the series (10.10) for the cylindrical Bessel function says that

$$J_\nu(z) \approx \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu . \tag{10.5}$$

So

$$\begin{aligned} j_\ell(x) &= \sqrt{\frac{\pi}{2x}}\, J_{\ell+1/2}(x) \\ &\approx \sqrt{\frac{\pi}{2x}}\, \frac{1}{\Gamma(\ell+3/2)} \left(\frac{x}{2}\right)^{\ell+1/2} \\ &= \frac{\sqrt{\pi}}{2^{\ell+1}}\, \frac{x^\ell}{\Gamma(\ell+3/2)}. \end{aligned} \tag{10.6}$$

Use of the formulas (5.67 and 5.68)

$$\Gamma(n+\tfrac{1}{2}) = \frac{(2n)!}{n!\,2^{2n}}\,\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\,\sqrt{\pi} \tag{10.7}$$

for $n = \ell + 1$ now gives

$$\begin{aligned} j_\ell(x) &\approx \frac{\sqrt{\pi}}{2^{\ell+1}}\, \frac{(\ell+1)!\,2^{2\ell+2} x^\ell}{(2\ell+2)!\sqrt{\pi}} = \frac{(\ell+1)!\,2^{\ell+1}\,x^\ell}{(2\ell+2)!} \\ &= \frac{\ell!\,(2x)^\ell}{(2\ell+1)!} = \frac{x^\ell}{(2\ell+1)!!} \end{aligned} \tag{10.8}$$

which is the approximation (10.73).

Rayleigh's formula for the spherical Bessel function $j_\ell(\rho)$ is the ℓth derivative of $\sin\rho/\rho$

$$j_\ell(\rho) = (-1)^\ell\,\rho^\ell \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^\ell \left(\frac{\sin\rho}{\rho}\right). \tag{10.9}$$

At large $|\rho|$, the spherical Bessel function $j_\ell(\rho)$ is dominated by the terms in which all the ℓ derivatives act on the sine. We find

$$\begin{aligned} (-1)^\ell \frac{d^\ell \sin\rho}{d\rho^\ell} &= (-1)^\ell \frac{d^\ell}{d\rho^\ell} \frac{\left(e^{i\rho} - e^{-i\rho}\right)}{2i} = \frac{\left[(-i)^\ell e^{i\rho} - (i)^\ell e^{-i\rho}\right]}{2i} \\ &= \frac{\left[\left(e^{-i\pi/2}\right)^\ell e^{i\rho} - \left(e^{i\pi/2}\right)^\ell e^{-i\rho}\right]}{2i} \\ &= \frac{\left(e^{i(\rho-\ell\pi/2)} - e^{-i(\rho-\ell\pi/2)}\right)}{2i} = \sin(\rho - \ell\pi/2). \end{aligned} \tag{10.10}$$

So as $|\rho| \to \infty$,

$$
\begin{aligned}
j_\ell(\rho) &= (-1)^\ell \, \rho^\ell \left(\frac{1}{\rho} \frac{d}{d\rho} \right)^\ell \left(\frac{\sin \rho}{\rho} \right) \\
&\approx (-1)^\ell \, \frac{1}{\rho} \left(\frac{d}{d\rho} \right)^\ell \sin \rho \\
&= \frac{\sin(\rho - \ell\pi/2)}{\rho}
\end{aligned} \tag{10.11}
$$

which is the approximation (10.75).

11

Examples for Chapter 11, Group Theory

11.1 Little Group

A state of a particle of momentum p can be defined in terms of a standard Lorentz transformation of a state of a standard fiducial momentum k. For a particle of mass $m > 0$, the standard fiducial momentum is

$$k = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{11.1}$$

in units with $\hbar = c = 1$. For a massless particle, the standard fiducial momentum is

$$k = \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix} \tag{11.2}$$

in which k is an arbitrary momentum.

A state of a particle of fiducial momentum k may carry another label s related to the intrinsic spin of the particle

$$|k, s\rangle. \tag{11.3}$$

A state of momentum p is then defined in terms of a standard Lorentz transformation $L(p)$ that takes k to p

$$p = L(p)\, k \tag{11.4}$$

as

$$|p, s\rangle = n(p)\, U(L(p))\, |k, s\rangle \tag{11.5}$$

in which $n(p)$ is a factor of normalization, and $U(L(p))$ is a unitary operator that implements standard Lorentz transformation $L(p)$.

An arbitrary Lorentz transformation Λ takes p to $\Lambda\, p$ and $U(\Lambda)$ takes the state $|p, s\rangle$ to

$$U(\Lambda)\,|p, s\rangle = n(p)\,U(\Lambda)\,U(L(p))\,|k, s\rangle \tag{11.6}$$

The Wigner rotation $W(\lambda, p)$ is defined by the equation

$$\Lambda\, L(p) = L(\Lambda p) W(\lambda, p) \tag{11.7}$$

as

$$W(\Lambda, p) = \big(L(\Lambda p)\big)^{-1}\,\Lambda\, L(p). \tag{11.8}$$

We see that $W(\Lambda, p)$ takes a fiducial momentum k to

$$W(\Lambda, p)\, k = \big(L(\Lambda p)\big)^{-1}\,\Lambda\, L(p)\, k = k. \tag{11.9}$$

The **little group** of the fiducial momentum k is the group of Lorentz transformations that leave k invariant. For particles of mass $m > 0$, the fiducial momentum k (11.1 is unchanged by the group of rotations. So for massive particles, the little group is the group of rotations. For massless particles, the little group is the group of Lorentz transformations that leave the fiducial momentum k (11.2 unchanged. Rotations about the z axis leave the fiducial momentum k (11.2 unchanged. The 4×4 matrices J, X, Y of Section 1.4 leave k invariant.

12
Examples for Chapter 13, General Relativity

12.1 Cartan's tetrads

The Levi-Civita affine connection is

$$\Gamma^k{}_{i\ell} = \tfrac{1}{2}\, g^{kj}\,(\partial_\ell g_{ji} + \partial_i g_{j\ell} - \partial_j g_{i\ell}) = \Gamma^k{}_{\ell i}. \tag{12.1}$$

In this expression, the metric $g_{i\ell}$ is the Minkowski dot product of two tetrads

$$g_{i\ell} = c^a{}_i\, \eta_{ab}\, c^b{}_\ell = c_i \cdot c_\ell. \tag{12.2}$$

So the connection $\Gamma^k{}_{i\ell}$ is

$$\begin{aligned}\Gamma^k{}_{i\ell} &= \tfrac{1}{2} c^k \cdot c^j \left[\, \partial_\ell(c_j \cdot c_i) + \partial_i(c_j \cdot c_\ell) - \partial_j(c_i \cdot c_\ell)\,\right] \\ &= \tfrac{1}{2} c^k \cdot c^j \;(\partial_\ell c_j) \cdot c_i + \tfrac{1}{2} c^k \cdot c^j \;\; c_j \cdot \partial_\ell c_i \\ &\quad + \tfrac{1}{2} c^k \cdot c^j \;(\partial_i c_j) \cdot c_\ell + \tfrac{1}{2} c^k \cdot c^j \;\; c_j \cdot \partial_i c_\ell \\ &\quad - \tfrac{1}{2} c^k \cdot c^j \;(\partial_j c_i) \cdot c_\ell - \tfrac{1}{2} c^k \cdot c^j \;\; c_i \cdot \partial_j c_\ell.\end{aligned} \tag{12.3}$$

This is a mess.

The connection is much simpler when written in terms of the basis or tangent vectors $e_i(x)$ defined (13.31) as the derivatives of the point $p(x)$ labelled by the coordinates x

$$e_i(x) = \partial_k p(x) = \frac{\partial p(x)}{\partial x^i}. \tag{12.4}$$

The cotangent vectors $e^k(x)$ are the contravariant forms of the tangent vectors

$$e^k(x) = g^{ki}(x) e_i(x). \tag{12.5}$$

In terms of the tangent and cotangent vectors, the connection (12.3) is

$$\Gamma^k{}_{i\ell} = e^k \cdot \frac{\partial e_i}{\partial x^\ell} = e^k \cdot \partial_\ell e_i. \tag{12.6}$$

The cost of this simplicity is that a point $p(x)$ of our curved physical spacetime lies in a flat spacetime that usually has more than four dimensions some of which are dimensions of time. This higher-dimensional spacetime is flat in the sense of being semi-euclidian. That is, a point $p(x)$ can be written as a linear combination of fixed orthogonal vectors v^α

$$p(x) = v^\alpha \, p_\alpha(x). \tag{12.7}$$

So the tangent vectors are

$$e_i(x) = v^\alpha \, \partial_i p_\alpha(x), \tag{12.8}$$

and the metric is

$$g_{ik}(x) = e_i(x) \cdot e_k(x). \tag{12.9}$$

The vectors v^α are orthogonal and do not depend upon the spacetime coordinates x. But their inner product is indefinite; if the embedding spacetime has d dimensions, then

$$v^\alpha \cdot v^\beta = \zeta^{\alpha\beta} = \begin{cases} \sigma^\alpha = \pm 1 & \text{if}\, \alpha = \beta \\ 0 & \text{if}\, \alpha \neq \beta \end{cases} \tag{12.10}$$

for $\alpha, \beta = 1, \ldots, d$. So the inner product of two basis or tangent vectors is

$$\begin{aligned} g_{ik}(x) &= e_i(x) \cdot e_k(x) = v^\alpha \, \partial_i p_\alpha(x) \cdot v^\beta \, \partial_k p_\beta(x) \\ &= \zeta^{\alpha\beta} \, \partial_i p_\alpha(x) \, \partial_k p_\beta(x) = \sigma^\alpha \, \partial_i p_\alpha(x) \, \partial_k p_\alpha(x). \end{aligned} \tag{12.11}$$

In this formulation, the metric is symmetric

$$g_{ik} = g_{ki}$$

and depends on the several coordinates $p_\alpha(x)$. So instead of 16 functions $g_{ik}(x)$ or 10 when $g_{ik}(x) = g_{ki}(x)$, there are only as many functions $p_\alpha(x)$, $\alpha = 1, \ldots d$, as the number of dimensions d of the embedding spacetime.

In the open and closed FLRW cosmologies, the embedding space has $d = 5$ dimensions. The closed, $k = 1$, cosmology has

$$P(t, \chi, \theta, \phi) = \Big(A(t),\, B(t)\, p(\chi, \theta, \phi)\Big) \tag{12.12}$$

in which

$$p(\chi, \theta, \phi) = \big(\sin\chi \sin\theta \cos\phi,\, \sin\chi \sin\theta \sin\phi,\, \sin\chi \cos\theta,\, \cos\chi\big). \tag{12.13}$$

So

$$e_t = \Big(\dot{A},\ \dot{B}\,p\Big) \tag{12.14}$$

$$e_\chi = B\Big(0, \cos\chi\sin\theta\cos\phi, \cos\chi\sin\theta\sin\phi, \cos\chi\cos\theta, -\sin\chi\Big) \tag{12.15}$$

$$e_\theta = B\Big(0, \sin\chi\cos\theta\cos\phi, \sin\chi\cos\theta\sin\phi, -\sin\chi\sin\theta, 0\Big) \tag{12.16}$$

$$e_\phi = B\Big(0, -\sin\chi\sin\theta\sin\phi, \sin\chi\sin\theta\cos\phi, 0, 0\Big). \tag{12.17}$$

The metric is diagonal, $\zeta = \mathrm{diag}(-1,1,1,1,1)$. So we need

$$e_t \cdot e_t = -c^2t^2 = -\dot{A}^2 + \dot{B}^2 \tag{12.18}$$

$$e_\chi \cdot e_\chi = L^2\,a^2 = B^2 \tag{12.19}$$

$$e_\theta \cdot e_\theta = L^2\,a^2\,\sin^2\chi = B^2\,\sin^2\chi \tag{12.20}$$

$$e_\phi \cdot e_\phi = L^2\,a^2\,\sin^2\chi\,\sin^2\theta = B^2\,\sin^2\chi\,\sin^2\theta. \tag{12.21}$$

So $B = L\,a$, and

$$-c^2t^2 = -\dot{A}^2 + \dot{B}^2 = -\dot{A}^2 + L^2\dot{a}^2. \tag{12.22}$$

So

$$\dot{A}^2 = c^2t^2 + L^2\,\dot{a}^2, \tag{12.23}$$

and

$$A(t) = \int_0^t dt'\,\dot{A}(t') = \int_0^t dt'\,\sqrt{c^2t'^2 + L^2\,\dot{a}^2(t')}. \tag{12.24}$$

One then sets $r = L\sin\chi$ to get the usual $k=1$ metric.

The points of the open FLRW cosmology are

$$P(t,\chi,\theta,\phi) = \Big(A(t),\ B(t)\,p(\chi,\theta,\phi)\Big) \tag{12.25}$$

in which

$$p(\chi,\theta,\phi) = \big(\sinh\chi\sin\theta\cos\phi,\ \sinh\chi\sin\theta\sin\phi,\ \sinh\chi\cos\theta,\ \cosh\chi\big). \tag{12.26}$$

So

$$e_t = \Big(\dot{A},\ \dot{B}\,p\Big) \tag{12.27}$$

$$e_\chi = B\Big(0, \cosh\chi\sin\theta\cos\phi, \cosh\chi\sin\theta\sin\phi, \cosh\chi\cos\theta, \sinh\chi\Big) \tag{12.28}$$

$$e_\theta = B\Big(0, \sinh\chi\cos\theta\cos\phi, \sinh\chi\cos\theta\sin\phi, -\sinh\chi\sin\theta, 0\Big) \tag{12.29}$$

$$e_\phi = B\Big(0, -\sinh\chi\sin\theta\sin\phi, \sinh\chi\sin\theta\cos\phi, 0, 0\Big). \tag{12.30}$$

The metric is diagonal, $\zeta = \mathrm{diag}(-1,1,1,1,-1)$. So we need

$$e_t \cdot e_t = -\,c^2t^2 = -\dot{A}^2 + \dot{B}^2 \tag{12.31}$$

$$e_\chi \cdot e_\chi = L^2a^2 = B^2 \tag{12.32}$$

$$e_\theta \cdot e_\theta = L^2a^2 \sinh^2\chi = B^2 \sinh^2\chi \tag{12.33}$$

$$e_\phi \cdot e_\phi = L^2a^2 \sinh^2\chi \, \sin^2\theta = B^2 \sinh^2\chi \, \sin^2\theta. \tag{12.34}$$

So $B = L\,a$, and

$$-c^2t^2 = -\,\dot{A}^2 + \dot{B}^2 = -\dot{A}^2 + L^2\dot{a}^2. \tag{12.35}$$

So

$$\dot{A}^2 = c^2t^2 + L^2\,\dot{a}^2, \tag{12.36}$$

and

$$A(t) = \int_0^t dt'\,\dot{A}(t') = \int_0^t dt'\,\sqrt{c^2t'^2 + L^2\,\dot{a}^2(t')}. \tag{12.37}$$

One then sets $r = L\sinh\chi$ to get the usual $k = -\,1$ metric.

13
Probability and Statistics

13.1 Binomial Distribution

If the probability of success is p on each try, then we expect that in N tries the mean number of successes will be

$$\langle n \rangle = N\, p. \tag{13.1}$$

The probability of failure on each try is $q = 1 - p$. So the probability of a particular sequence of successes and failures, such as n successes followed by $N - n$ failures is $p^n\, q^{N-n}$. There are $N!/n!\,(N-n)!$ different sequences of n successes and $N - n$ failures, all with the same probability $p^n\, q^{N-n}$. So the probability of n successes (and $N - n$ failures) in N tries is

$$P_b(n, p, N) = \frac{N!}{n!\,(N-n)!}\, p^n\, q^{N-n} = \binom{N}{n}\, p^n\, (1-p)^{N-n}. \tag{13.2}$$

This **binomial distribution** also is called **Bernoulli's distribution** (Jacob Bernoulli, 1654–1705).

13.2 Poisson's Distribution

Poisson approximated the formula (13.2) for the binomial distribution $P_b(n, p, N)$ by taking the two limits $N \to \infty$ and $p = \langle n \rangle / N \to 0$ while keeping n and the product $pN = \langle n \rangle$ constant. Using Stirling's formula $n! \approx \sqrt{2\pi n}\,(n/e)^n$ (6.336) for the two huge factorials $N!$ and $(N-n)!$, we get as $n/N \to 0$ and

$\langle n \rangle / N \to 0$ with $\langle n \rangle = pN$ kept fixed

$$\begin{aligned} P_b(n,p,N) &= \binom{N}{n} p^n (1-p)^{N-n} = \frac{N!}{(N-n)!} \frac{p^n}{n!} (1-p)^{N-n} \\ &\approx \sqrt{\frac{N}{N-n}} \left(\frac{N}{e}\right)^N \left(\frac{e}{N-n}\right)^{N-n} \frac{(pN)^n}{n!} (1-p)^{N-n} \\ &\approx e^{-n} \left(1 - \frac{n}{N}\right)^{-N+n} \frac{\langle n \rangle^n}{n!} \left(1 - \frac{\langle n \rangle}{N}\right)^{N-n} . \end{aligned} \tag{13.3}$$

So using the definition $\exp(-x) = \lim_{N\to\infty} (1 - x/N)^N$ to take the limits

$$\left(1 - \frac{n}{N}\right)^{-N} \left(1 - \frac{n}{N}\right)^{n} \to e^n \quad \text{and} \quad \left(1 - \frac{\langle n \rangle}{N}\right)^{N} \left(1 - \frac{\langle n \rangle}{N}\right)^{-n} \to e^{\langle n \rangle}, \tag{13.4}$$

we get from the binomial distribution Poisson's estimate

$$P_P(n, \langle n \rangle) = \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle} \tag{13.5}$$

of the probability of n successes in a very large number N of tries, each with a tiny chance $p = \langle n \rangle / N$ of success. (Siméon-Denis Poisson, 1781–1840.

13.3 Gauss's distribution

Gauss considered the binomial distribution in the limits $n \to \infty$ and $N \to \infty$ with the probability p fixed. In this limit, all three factorials are huge, and we may apply Stirling's formula to each of them

$$\begin{aligned} P_b(n,p,N) &= \frac{N!}{n!\,(N-n)!} p^n q^{N-n} \\ &\approx \sqrt{\frac{N}{2\pi n(N-n)}} \left(\frac{N}{e}\right)^N \left(\frac{e}{n}\right)^n \left(\frac{e}{N-n}\right)^{N-n} p^n q^{N-n} \\ &= \sqrt{\frac{N}{2\pi n(N-n)}} \left(\frac{pN}{n}\right)^n \left(\frac{qN}{N-n}\right)^{N-n} . \end{aligned} \tag{13.6}$$

This probability $P_b(n,p,N)$ is tiny unless n is near pN which means that $n \approx pN$ and $N - n \approx (1-p)N = qN$ are comparable. So we set $y = n - pN$ and treat y/N as small. Since $n = pN + y$ and $N - n = (1-p)N + pN - n =$

$qN - y$, we can write the square root as

$$\sqrt{\frac{N}{2\pi\, n\, (N-n)}} = \frac{1}{\sqrt{2\pi\, N\, [(pN+y)/N]\, [(qN-y)/N]}}$$
$$= \frac{1}{\sqrt{2\pi\, pqN\, (1+y/pN)\, (1-y/qN)}}. \tag{13.7}$$

Because y remains finite as $N \to \infty$, the limit of the square root is

$$\lim_{N\to\infty} \sqrt{\frac{N}{2\pi\, n\, (N-n)}} = \frac{1}{\sqrt{2\pi\, pqN}}. \tag{13.8}$$

Substituting $pN + y$ for n and $qN - y$ for $N - n$ in (13.6), we find

$$P_b(n,p,N) \approx \frac{1}{\sqrt{2\pi\, pqN}} \left(\frac{pN}{pN+y}\right)^{pN+y} \left(\frac{qN}{qN-y}\right)^{qN-y}$$
$$= \frac{1}{\sqrt{2\pi\, pqN}} \left(1 + \frac{y}{pN}\right)^{-(pN+y)} \left(1 - \frac{y}{qN}\right)^{-(qN-y)} \tag{13.9}$$

which implies

$$\ln\left[P_b(n,p,N)\sqrt{2\pi\, pqN}\right] \approx -(pN+y)\ln\left[1 + \frac{y}{pN}\right] - (qN-y)\ln\left[1 - \frac{y}{qN}\right]. \tag{13.10}$$

The first two terms of the power series (5.101) for $\ln(1+\epsilon)$ are

$$\ln(1+\epsilon) \approx \epsilon - \frac{1}{2}\epsilon^2. \tag{13.11}$$

So applying this expansion to the two logarithms and using the relation $1/p + 1/q = (p+q)/pq = 1/pq$, we get

$$\ln\left(P_b(n,p,N)\sqrt{2\pi\, pqN}\right) \approx -(pN+y)\left[\frac{y}{pN} - \frac{1}{2}\left(\frac{y}{pN}\right)^2\right] \tag{13.12}$$
$$- (qN-y)\left[-\frac{y}{qN} - \frac{1}{2}\left(\frac{y}{qN}\right)^2\right] \approx -\frac{y^2}{2pqN}.$$

Remembering that $y = n - pN$, we get Gauss's approximation to the binomial probability distribution

$$P_{bG}(n,p,N) = \frac{1}{\sqrt{2\pi pqN}} \exp\left(-\frac{(n-pN)^2}{2pqN}\right). \tag{13.13}$$

This probability distribution is normalized

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi pqN}} \exp\left(-\frac{(n-pN)^2}{2pqN}\right) = 1 \tag{13.14}$$

almost exactly for $pN > 100$.

Extending the integer n to a continuous variable x, we have

$$P_G(x,p,N) = \frac{1}{\sqrt{2\pi pqN}} \exp\left(-\frac{(x-pN)^2}{2pqN}\right) \tag{13.15}$$

which on the real line $(-\infty, \infty)$ is a normalized probability distribution with mean $\langle x \rangle = \mu = pN$ and variance $\langle (x-\mu)^2 \rangle = \sigma^2 = pqN$. Replacing pN by μ and pqN by σ^2, we get the Standard form of **Gauss's distribution**

$$P_G(x,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \tag{13.16}$$

This distribution occurs so often in mathematics and in nature that it is often called **the normal distribution**. Its odd central moments all vanish $\nu_{2n+1} = 0$, and its even ones are $\nu_{2n} = (2n-1)!!\,\sigma^{2n}$.

Example 13.1 (Acccuracy of Gauss's distribution) If $p = 0.1$ and $N = 10^4$, then Gauss's approximation to the probability that $n = 10^3$ is $1/(30\sqrt{2\pi})$. The exact binomial probability is $P_b(10^3, 0.1, 10^4) = 0.013297$, and Gauss's estimate is $P_G(10^3, 0.1, 10^4) = 0.013298$. □

References

[1] Steven Weinberg. *Cosmology*. Oxford University Press, 2010.