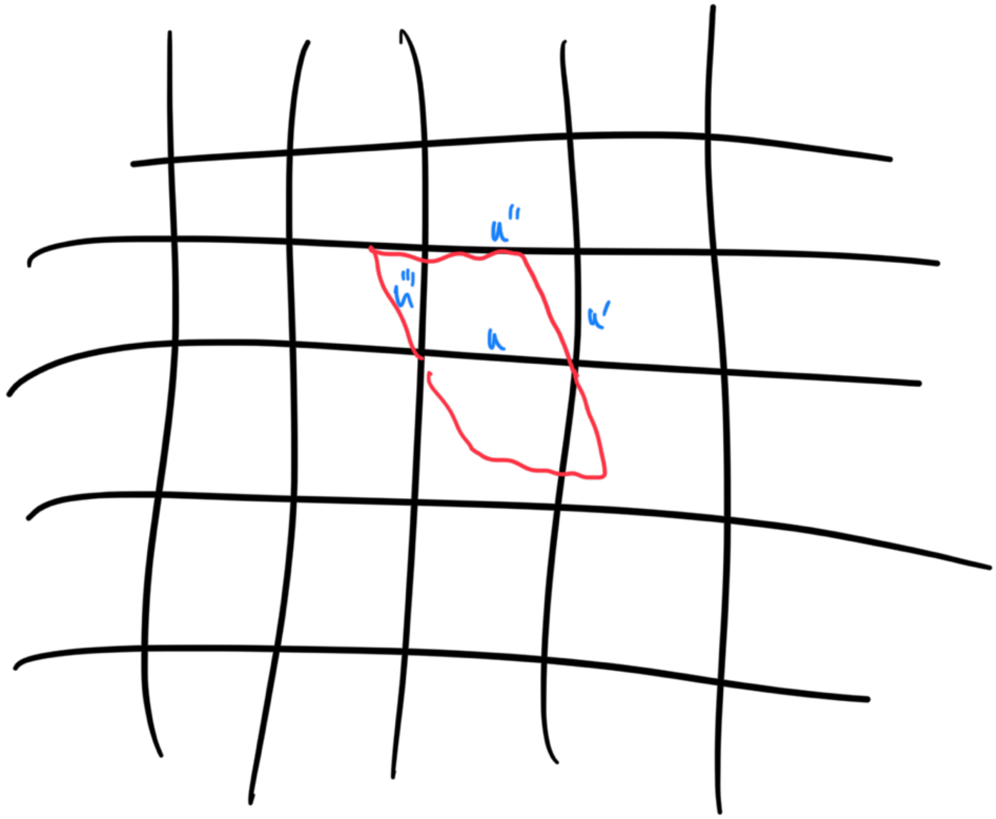


$d=4 \quad 10^4$ lattice points

$100^4 = 10^8$ points

$$\langle A \rangle = \frac{\int e^{-S_E(\phi)} A(\phi)\, D\phi}{\int e^{-S_E(\phi)}\, D\phi}$$

$$ds^2 = g_{ik}\, dx^i dx^k$$

$$g_{ik} \approx \eta_{ik} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$A^i{}_l B^l{}_j \equiv \sum_{l=0}^{3} A^i{}_l B^l{}_j$$

$$L \eta L = \eta$$

$$\eta^2 = 1$$

$$dx'^k = L^k{}_l\, dx^l$$

$$A'^k = L^k{}_l A^l$$

contra

$$\frac{\partial}{\partial x'^j} = \frac{\partial x^l}{\partial x'^j}\frac{\partial}{\partial x^l}$$

$$\int \nabla \times E \cdot da$$

$$\int_{\partial R} E \cdot dl = \int_R \nabla \times E \cdot dS$$

boundary of

$$\left(r^2 R_{k\ell}'(r)\right)' + \left[k^2 r^2 - \ell(\ell+1)\right] R_{k\ell}(r) = 0$$

$$R_{k\ell}(r) = j_\ell(kr)$$

$$\frac{d}{dr}\left(r^2 \frac{d}{dr} j_\ell(kr)\right) + \left[k^2 r^2 - \ell(\ell+1)\right] j_\ell(kr) = 0$$

$$k\frac{d}{dkr}\left(r^2 k \frac{d}{dkr} j_\ell(kr)\right) + \left[k^2 r^2 - \ell(\ell+1)\right] j_\ell(kr) = 0$$

$x = kr$

$$\frac{d}{dx}\left(x^2 \frac{d}{dx} j_\ell(k)\right) + \left[x^2 - \ell(\ell+1)\right] j_\ell(x) = 0$$

$-\Delta V = k^2 V$

$$\begin{cases} 0 & r < a \end{cases}$$

$$V(r) = \begin{cases} \infty & r > a \\ \end{cases}$$

So $\quad j_\ell(ka) = 0 = R_{k\ell}(a)$

so $\quad ka = z_{\ell,n} \qquad j_\ell(z_{\ell,n}) = 0$

$$-\frac{\hbar^2}{2m}\nabla^2 \psi_{\ell n} = E\,\psi_{\ell n}$$

$$+\nabla^2 \psi_{\ell n} = -\frac{2mE}{\hbar^2}\psi_{\ell n} = -k^2 \psi_{\ell n}$$

$$k^2 = \frac{2mE}{\hbar^2} = \left(\frac{z_{\ell,n}}{a}\right)^2$$

$$E_{\ell n} = \frac{\hbar^2}{2m}\left(\frac{z_{\ell n}}{a}\right)^2.$$

So the smaller the size of the volume in which the particle is free to move, the higher its energy.

$$E = h\nu$$

$x = (x^0, x^1, x^2, x^3)$ e.g. $x = (ct, \vec{r})$

dx^k $\qquad dx^1 = x^1 + \epsilon - x^1 = \epsilon$

$$dx'^i = \frac{\partial x'^i}{\partial x^k} dx^k$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \frac{\partial x'^0}{\partial x^0} & \cdots & \frac{\partial x'^0}{\partial x^3} \\ \vdots & \ddots & \vdots \\ \frac{\partial x'^3}{\partial x^0} & \cdots & \frac{\partial x'^3}{\partial x^3} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

contravariant vectors

$$A'^k(x') = \sum_{i=0}^{3} \frac{\partial x'^k}{\partial x^i} A^i(x)$$

field = something that's a function of $x = (t, \vec{x})$

$$\frac{\partial}{\partial x'^i} = \sum_{k=0}^{3} \frac{\partial x^k}{\partial x'^i} \frac{\partial}{\partial x^k}$$

chain rule

covariant vectors

$$\rightarrow C'_i(x') = \sum_{k=0}^{3} \frac{\partial x^k}{\partial x'^i} C_k(x)$$

covariant vector field.

$$F'_{jl}(x') = \sum_{i,k=0}^{3} \frac{\partial x^i}{\partial x'^j} \frac{\partial x^k}{\partial x'^l} F_{ik}(x)$$

covariant tensor
(of rank 2)

$$A_i B^i \equiv \sum_{i=0}^{3} A_i B^i$$

$\langle \psi | S | \psi \rangle$ = average value of S in state ψ.

$$e_i(x) \cdot e_k(x) = g_{ik}(x)$$

$$= \sum_{\alpha=1}^{\cancel{p}} \sigma_\alpha e_i^\alpha(x) e_k^\alpha(x)$$

empty space

$$g_{ik}(x) = \eta_{ik}(x_0)$$

$$g_{ik} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

$$\frac{\partial V}{\partial x^l} = \frac{\partial (V^i e_i)}{\partial x^l} = \underline{\frac{\partial V^i}{\partial x^l} e_i} + V^i \frac{\partial e_i}{\partial x^l}$$

covariant derivative

$$D_l V^k = e^k \cdot \frac{\partial V}{\partial x^l}$$

$$e_i \cdot e_k = g_{ik} \qquad e^i \cdot e_k = \delta^i_k$$

$$D_l V^k = e^k \cdot \left(\frac{\partial V^i}{\partial x^l} e_i + V^i \frac{\partial e_i}{\partial x^l} \right)$$

$$= \delta^k_i \frac{\partial V^i}{\partial x^l} + e^k \cdot \frac{\partial e_i}{\partial x^l} V^i$$

$$\frac{\partial V^k}{} \quad e^k \frac{\partial e_i}{} V^i$$

$$= \frac{\partial V}{\partial x^\ell} + \underbrace{e \cdot \frac{\partial}{\partial x^\ell}} V$$

$$D_\ell V^k = \frac{\partial V^k}{\partial x^\ell} + \underline{\Gamma^k_{i\ell} V^i}$$

$$V = V_k e^k$$

$$\frac{\partial V}{\partial x^\ell} = \frac{\partial V_k}{\partial x^\ell} e^k + V_k \frac{\partial e^k}{\partial x^\ell}$$

$$e_i \cdot \frac{\partial V}{\partial x^\ell} = \frac{\partial V_k}{\partial x^\ell} e_i \cdot e^k + V_k \, e_i \cdot \frac{\partial e^k}{\partial x^\ell}$$

$$= \frac{\partial V_i}{\partial x^\ell} + V_k \underbrace{e_i \cdot \frac{\partial e^k}{\partial x^\ell}}_{\Gamma}$$

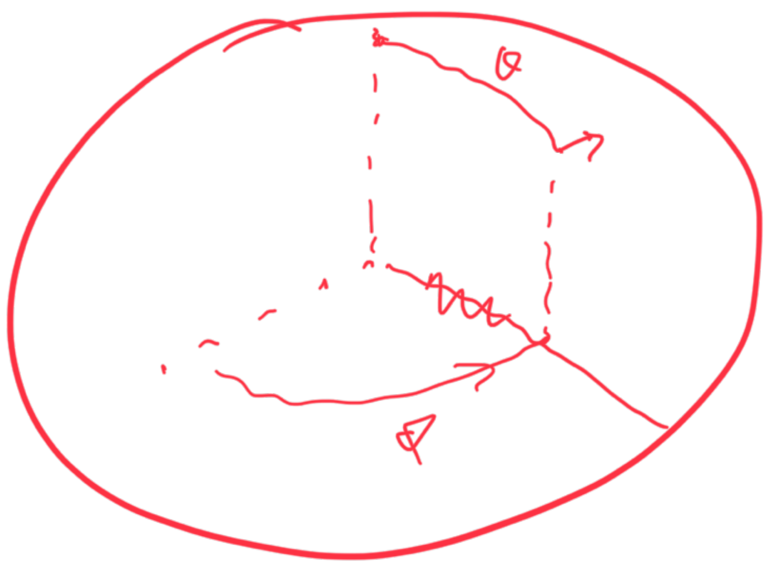
$$D_\ell V_i = \frac{\partial V_i}{\partial x^\ell} + V_k \, e_i \cdot \frac{\partial e^k}{\partial x^\ell}$$

$$\Gamma^k \quad V$$

$$= \frac{\partial V_i}{\partial x^e} - \Gamma^{\,k}_{ie} V_k$$

covariant derivative of a covariant vector.

$$D_e\, g_{ik} = 0$$



$$d\phi = \hat{\phi}\, r \sin\theta\, d\phi$$

in 3D

$$e_i(x) = \partial_i\, \phi(x)$$

$e_\theta(x) = \partial_\theta p(x)$

$e_\phi(x) = \partial_\phi p(x)$

complete on surface

but they are vectors in 3D

e_i basis vectors.

$$g_{ik}(x) = e_i(x) \cdot e_k(x)$$

$$g^{li} g_{ik} = \delta^l_k$$

$$A^i(x) = g^{ik} A_k(x)$$

$$A_k(x) = g_{kl} A^l(x)$$

$$e^i = g^{ik} e_k$$

$$V(x) = V^i(x)\, e_i(x)$$

$$V'(x') = V'^i(x')\, e'_i(x')$$

$$= \frac{\partial x'^i}{\partial x^j} V^j(x) \frac{\partial x^k}{\partial x'^i} e_k(x)$$

$$= \delta^k_j\, V^j(x)\, e_k(x)$$

$$= V^j(x)\, e_j(x) = V(x)$$

So $V(x)$ is a scalar.

$$\frac{\partial V}{\partial x^\ell} = \frac{\partial V^i}{\partial x^\ell} e_i(x) + V^i(x) \frac{\partial e_i(x)}{\partial x^\ell}$$

covariant vector

$$D_\ell V^i = \frac{\partial V^i}{\partial x^\ell} e_i(x) + \frac{\partial e_i(x)}{\partial x^\ell} V^i(x)$$

$$e^k \cdot D_\ell V^i = \frac{\partial V^k}{\partial x^\ell} + e^k \cdot \frac{\partial e_i(x)}{\partial x^\ell} V^i(x)$$

$$e^k \cdot e_i = \delta^k_i$$

$$\Gamma^k{}_{il} = e^k \cdot \frac{\partial e_i}{\partial x^l}$$

$$D_l V^k = \partial_l V^k + \Gamma^k{}_{il} V^i$$

$$D_l \psi = \partial_l \psi + A_l \psi$$

photon

$$D_l V_i = \partial_l V_i - \Gamma^k{}_{il} V_k$$

$$e_i \cdot e_k = e_i{}^\alpha \eta_{\alpha\beta} e^\beta{}_k$$

$$\eta = \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & +1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

$$g_{ik} = e_i \cdot e_k = e^\alpha{}_i \eta_{\alpha\beta} e^\beta{}_k$$

$$A_{il} = -A_{li}$$

$\partial_k A_{il} + \partial_l A_{ki} + \partial_i A_{lk}$

$= \partial_k A_{il} + \partial_l A_{ki} + \partial_i A_{lk}$

$F_{ik} = -F_{ki}$

$$\Gamma^k_{il} = \frac{1}{2} g^{kj} (\partial_l g_{ni} + \partial_i g_{nl} - \partial_n g_{il})$$

$$ds^2 = g_{ik} dx^i dx^k$$

4D geometry

$a(t)$: how space expands

space is maximally symmetric

$F_{ik} = [\partial_i + A_i, \partial_k + A_k]$

$$F_{ik} = [D_i, D_k]$$

$$= \partial_i A_k - \partial_k A_i + A_i A_k - A_k A_i$$

$$y' + xy = c\, e^{-x^2/2}$$

$$r(x) = x \qquad s(x) = c\, e^{-x^2/2}$$

$$\alpha(x) = e^{\int r(x)\,dx} = e^{x^2/2}$$

$$y(x) = y(0)\, e^{-x^2/2} + e^{-x^2/2} \int c\, dx$$

$$= \left(y(0) + cx\right) e^{-x^2/2}$$

$$y(0) = (y(0) + 0)\, e^{-0^2/2} = y(0).$$

$$V(r, \theta) = \sum \frac{b_l}{r^{l+1}} P_l(\cos\theta)$$

$$V(R,\theta) = \sum \frac{b_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$$= V_0 \cos^2\theta$$

$$x^2 = \frac{2P_2(x) + P_0(x)}{3}$$

$$\sum \frac{b_\ell}{R^{\ell+1}} P_\ell(x) = \frac{V_0}{3}\left(2P_2(x) + P_0(x)\right)$$

$$\int \sum \frac{b_\ell}{R^{\ell+1}} P_\ell(x) P_n(x)\,dx$$

$$= \sum_\ell \frac{b_\ell}{R^{\ell+1}} \frac{2}{2n+1} \delta_{n\ell}$$

$$= \frac{2}{2n+1} \cdot \frac{b_n}{R^{n+1}}$$

$$= \frac{V_0}{3}\left(2 \frac{2}{5}\delta_{n2} + 2\delta_{n0}\right)$$

$$b_n = \frac{(2n+1) V_0 R^{n+1}}{3}\left(\frac{2}{5}\delta_{n2} + \delta_{n0}\right)$$

$$b_0 = \frac{V_0 R}{3} \qquad b_2 = \frac{2 V_0 R^3}{3}$$

$$V(r,\theta) = \frac{V_0}{3}\left[\frac{R}{r} + \frac{R^3}{r^3}(3\cos^2\theta - 1)\right].$$

$$k \cdot r = kr\cos\theta$$

$$k = |\vec{k}|$$

$$r = |\vec{r}|$$

$$\langle \vec{r} | \vec{k} \rangle = \frac{e^{ik\cdot r}}{(2\pi)^{3/2}}$$

$$= \frac{e^{ikr\cos\theta}}{(2\pi)^{3/2}} = \frac{1}{(2\pi)^{3/2}} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta)\, i^l j_l(kr$$

$$= \frac{1}{(2\pi)^{3/2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} 4\pi i^l j_l(kr) Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta',\phi')$$

$$\theta\phi \leftrightarrow \vec{r} \qquad \theta'\phi' \leftrightarrow \vec{k}$$