equations, we see that the real part of ψ_i satisfies them, and by subtracting them, we see that the imaginary part of ψ_i also satisfies them. So it *might* seem that $\psi_i = u_i + iv_i$ is made of two real eigenfunctions with the same eigenvalue.

But each eigenfunction u_i in the domain D satisfies two homogeneous boundary conditions as well as its second-order differential equation

$$-\,(p\,u_i')' + q\,u_i = \lambda_i\,\rho\,u_i \tag{6.325}$$

and so u_i is the unique solution in D to this equation. There can be no other eigenfunction in D with the same eigenvalue. In a regular Sturm-Liouville system, **there is no degeneracy**. All the eigenfunctions u_i are orthogonal and can be normalized on the interval $[a,b]$ with weight function $\rho(x)$

$$\int_a^b u_j^*\,\rho\,u_i\,dx = \delta_{ij}. \tag{6.326}$$

They may be taken to be **real**.

It is true that the eigenfunctions of a second-order differential equation come in pairs because one can use Wronski's formula (6.268)

$$y_2(x) = y_1(x)\int^x \frac{dx'}{p(x')\,y_1^2(x')} \tag{6.327}$$

to find a linearly independent second solution with the same eigenvalue. But the second solutions don't obey the boundary conditions of the domain. Bessel functions of the second kind, for example, are infinite at the origin.

A set of eigenfunctions u_i is **complete in the mean** in a space S of functions if every function $f \in S$ can be represented as a series

$$f(x) = \sum_{i=1}^{\infty} a_i u_i(x) \tag{6.328}$$

(called a Fourier series) that converges **in the mean**, that is

$$\lim_{N\to\infty} \int_a^b \left| f(x) - \sum_{i=1}^{N} a_i u_i(x) \right|^2 \rho(x)\,dx = 0. \tag{6.329}$$

The natural space S is the space $\mathcal{L}_2(a,b)$ of all functions f that are square-integrable on the interval $[a,b]$

$$\int_a^b |f(x)|^2\,\rho(x)\,dx < \infty. \tag{6.330}$$

The orthonormal eigenfunctions of every regular Sturm-Liouville system on an interval $[a,b]$ are complete in the mean in $\mathcal{L}_2(a,b)$. The completeness of