Example 11.14 (Exterior Derivatives Anticommute with Differentials) The exterior derivative acting on two one-forms $A = A_i dx^i$ and $B = B_j dx^j$ is

$$
\begin{aligned}
d(A \wedge B) &= d(A_i dx^i \wedge B_j dx^j) = \partial_k (A_i B_j)\, dx^k \wedge dx^i \wedge dx^j \qquad (11.121)\\
&= (\partial_k A_i) B_j \, dx^k \wedge dx^i \wedge dx^j + A_i (\partial_k B_j)\, dx^k \wedge dx^i \wedge dx^j \\
&= (\partial_k A_i) B_j \, dx^k \wedge dx^i \wedge dx^j - A_i (\partial_k B_j)\, dx^i \wedge dx^k \wedge dx^j \\
&= (\partial_k A_i) dx^k \wedge dx^i \wedge B_j dx^j - A_i dx^i \wedge (\partial_k B_j)\, dx^k \wedge dx^j \\
&= dA \wedge B - A \wedge dB.
\end{aligned}
$$

If A is a p-form, then $d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB$ (exercise 11.10). □

11.14 Tensor Equations

Maxwell's homogeneous equations (11.93) relate the derivatives of the field-strength tensor to each other as

$$0 = \partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki}. \qquad (11.122)$$

They are generally covariant **tensor equations** (sections 11.31 & 11.32). In terms of invariant forms, they are the Bianchi identity (11.114)

$$dF = ddA = 0. \qquad (11.123)$$

Maxwell's inhomegneous equations (11.94) relate the derivatives of the field-strength tensor to the current density j^i and to the square-root of the modulus g of the determinant of the metric tensor g_{ij} (section 11.16)

$$\frac{\partial(\sqrt{g}\, F^{ik})}{\partial x^k} = \mu_0 \sqrt{g}\, j^i. \qquad (11.124)$$

We'll write them as invariant forms in section 11.26 and derive them from an action principle in section 11.38.

If we can write a physical law in one coordinate system as a tensor equation

$$K^{kl} = 0 \qquad (11.125)$$

then in any other coordinate system, the corresponding tensor equation

$$K'^{ij} = 0 \qquad (11.126)$$

also is valid since

$$K'^{ij} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} K^{kl} = 0. \qquad (11.127)$$