Session 2: Quantum chaos in systems with few degrees of freedom
Wednesday June 9th 2021
1. Signatures of chaos in the energy spectrum
   1. Level repulsion
   2. Level spacing statistics
2. Signatures of chaos in the energy eigenstates
3. Signatures of chaos in quantum dynamics
   1. Ehrenfest time
   2. Loschmidt echo
Chaos and integrability in the energy spectrum

Integrable

- $n$ constants of motion
- Solvable H-J equation
- Motion on a $n$-dim. torus

Chaotic

- Motion in a $2n$-dim. phase space
- Instabilities and sensitivity to initial conditions
- Mixing and ergodicity

Classical

- Constraints
  - Less generic

- No constraints
  - More generic

Quantum

- Energy vs. Parameter $\lambda$
- Exact level crossing
- Level clustering
- Quantum ‘regular’

- Energy vs. Parameter $\lambda$
- Avoided level crossing
- Level repulsion
- Quantum ‘chaotic’
Model for level dynamics

Energy

Parameter \( \lambda \)

In some basis \( \{|1\rangle, |2\rangle\} \)

\[
H = \begin{pmatrix}
H_{11} & H_{12} \\
H_{12}^* & H_{22}
\end{pmatrix}
\]

Eigenvalues:

\[
E_{\pm} = \frac{1}{2} (H_{11} + H_{22}) \pm \frac{1}{2} \sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2}
\]

Difference:

\[
\Delta E = E_+ - E_- = \sqrt{(H_{11} - H_{22})^2 + 4|H_{12}|^2}
\]

When does \( \Delta E = 0 \)?

- If \( H_{12} = 0 \), \( H \) is a function of two real parameters, \( \Delta E = |H_{11} - H_{22}|^2 \)

  \( \Delta E = 0 \) by tuning \( k = 1 \) parameter

- If \( H_{12} \in \mathbb{R} \), \( H \) is a function of three real parameters, \( \Delta E = \sqrt{(H_{11} - H_{22})^2 + 4H_{12}^2} \)

  \( \Delta E = 0 \) by tuning \( k = 2 \) parameters

- If \( H_{12} \in \mathbb{C} \), \( H \) is a function of four real parameters, \( \Delta E = \sqrt{(H_{11} - H_{22})^2 + 4 \text{Re}(H_{12})^2 + 4 \text{Im}(H_{12})^2} \)

  \( \Delta E = 0 \) by tuning \( k = 3 \) parameters

Levelling crossing ‘codimension’ \( k \)

Less constraints

Exact crossing requires tuning more parameters

Increasing ‘level repulsion’
Model for level dynamics

**Level repulsion**

If $H$ is chosen at random, how likely is it that adjacent levels cross (are degenerate)?

**Level spacing distribution $P(s)$**

(for small $s$)

$s_i = \Delta E_i = E_{i+1} - E_i$

For our 2x2 model:

$$P(s) = \langle \delta(s - \Delta E) \rangle \quad \text{with} \quad \Delta E = \sqrt{(H_{11} - H_{22})^2 + 4 \operatorname{Re}(H_{12})^2 + 4 \operatorname{Im}(H_{12})^2}$$

$$\Rightarrow P(s) = \int dx \, dy \, dz \, P(x, y, z) \, \delta(s - r)$$

Approximating $P(r^2) \approx P_0$ constant near $s = 0$

- If $H_{12} = 0 \ (k = 1) \Rightarrow P(s) \sim \int dx \ \delta(s - x) \sim \text{const} \ (\text{independent of } s)$
- If $H_{12} \in \mathbb{R} \ (k = 2) \Rightarrow P(s) \sim \int dx \int dy \ \delta(s - r) \sim 2\pi \int dr \ r \ \delta(s - r) \sim s$
- If $H_{12} \in \mathbb{C}, \ (k = 3) \Rightarrow P(s) \sim \int dx \int dy \int dz \ \delta(s - r) \sim 4\pi \int dr \ r^2 \ \delta(s - r) \sim s^2$

Level repulsion implies **correlation** of the energy levels. In absence of level repulsion, the **levels are uncorrelated** with each other.
Level clustering in integrable systems

For integrable systems, semiclassical methods can be used to compute the energy levels.

**Einstein-Keller-Brillouin (EKB) quantization:**

\[ I_i = \frac{1}{2\pi} \int p \, dq = \hbar \left( m_i + \frac{\mu_i}{4} \right) \]

\[ \sim \text{Bohr-Sommerfeld} \]

Integrable systems with arbitrary dimension

Valid for large action \( I \gg \hbar \)

Energy levels are determined by a set of quantum numbers \( \vec{m} \). Levels with completely different \( m \)’s can have the same energy (typically, if # d.o.f. > 1) - there is no correlation.

**Example:** rectangular Billiard

\[ H = \frac{1}{2m} \left( p_x^2 + p_y^2 \right) = \frac{\pi^2}{2m} \left( \frac{l_x^2}{a_x^2} + \frac{l_y^2}{a_y^2} \right) \]

Actions:

\[ I_k = \frac{1}{2\pi} \int p_k \, dq_k = \frac{a_k p_k}{\pi} \]

**Quantization:**

\[ I_k = \hbar (m_k + 1) \equiv \hbar n_k \]

\[ E_{n_x,n_y} = \frac{\hbar^2 \pi^2}{2m} \left( \frac{n_x^2}{a_x^2} + \frac{n_y^2}{a_y^2} \right) \]

(particle in a 2D box)
Berry-Tabor conjecture

Rectangular Billiard: $E_{n_x,n_y} = \frac{\hbar^2 \pi^2}{2m} \left( \frac{n_x^2}{a_x^2} + \frac{n_y^2}{a_y^2} \right)$

Level spacing distribution $P(s)$
(for incommensurate $a_x^2$ and $a_y^2$)

**Berry – Tabor (B-T) conjecture:** In the limit of large energies (semiclassical limit), the level spacing statistics of the quantum spectra of classically integrable systems correspond to the prediction for *randomly* distributed energy levels, and follow the exponential distribution $P(s) = e^{-s}$


**Exceptions**
- Systems with one degree of freedom (all of them are integrable anyway)
- Linear systems (quadratic Hamiltonians)
- Systems with closed orbits (commensurate frequencies)
Level repulsion for nointegrable systems

- For nonintegrable systems, the semiclassical methods cannot be used to compute the energies anymore.
- Lifting constraints → **Level repulsion**

**Bohigas-Giannoni-Schmidt (BGS) conjecture**: the eigenvalues of a quantum system whose classical analogue is *fully* chaotic, obey the statistics of level spacing predicted by Random Matrix Theory, and in particular those from the Gaussian random ensembles.

\[
P(s) = \begin{cases} 
\frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right) & \text{GOE: (real, symmetric) random matrices, elements } \sim \text{ Gaussian} \\
32 \pi s^2 \exp\left(-\frac{4}{\pi} s^2\right) & \text{GUE: (complex, hermitian) random matrices, elements } \sim \text{ Gaussian}
\end{cases}
\]

**Wigner-Dyson distributions**

Next week (June 16\textsuperscript{th}): Random Matrix Theory!

Extensions to general systems

Level spacing statistics is often taken as the definition of quantum chaos

Integrable (Bethe ansatz)

\[
H_0 = \frac{j}{2} \sum_{i=1}^{L-1} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \alpha_+ \sigma_i^z \sigma_{i+1}^z \quad \text{(Heisenberg XXZ)}
\]

\[
H_1 = \frac{j}{2} \sum_{i=1}^{L-2} \sigma_i^x \sigma_{i+2}^x + \sigma_i^y \sigma_{i+2}^y + \alpha_- \sigma_i^z \sigma_{i+2}^z \quad \text{and} \quad H_{01} = H_0 + \Gamma H_1
\]

Nonintegrable


\[
H = H_z + H_{NN},
\]

where

\[
H_z = \sum_{i=1}^{L} \omega_i S_i^z = \left( \sum_{i=1}^{L} \omega_i S_i^z \right) + \epsilon_d S_d^z,
\]

\[
H_{NN} = \sum_{i=1}^{L-1} \left[ J_{xy} \left( S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z S_i^z S_{i+1}^z \right].
\]

Eigenstates of integrable systems

- In integrable systems, semiclassical methods can also be used to approximate eigenstates (WKB theory)

- Eigenstates tend to localize around the regular structures
  \[ W_{\vec{n}}(q, p) \approx \delta \left( I(q, p) - I_{\vec{n}} \right) + \mathcal{O}(\hbar) \]

  - This is explained in Wimberger’s book 4.2.2, 4.2.4 and 4.4

- In chaotic systems, there is no tori, and eigenstates tend to be irregular, and smeared out over chaotic regions

Inverse participation ratio (IPR): \( \eta_{IPR} = \sum_k |\langle \phi_k | \psi \rangle|^4 \)

- Measures concentration of \( |\psi\rangle \) on a basis \( \{|\phi_k\rangle\} \)
- In phase space, these could be coherent states
- In general settings, other choices are possible, for instance site basis or mean field basis

\[ \frac{\delta}{d^{-1}} \rightarrow \eta_{IPR} \rightarrow \frac{\delta}{d} \]

- Averaged ‘participation ratio’ (\( \eta_{IPR}/d \))
- Here, \( |\psi\rangle \) is an eigenstate of \( H \), and \( |\phi_k\rangle \) eigenstate of \( J_y \).
- Average is over all \( |\psi\rangle \)

\[ H = \frac{\alpha}{\tau} J_y + \frac{k}{p J_y - 1} f(t) J_z^p \]

\[ f(t) = \sum_{m=-\infty}^{m=+\infty} \delta(t - m\tau) \]

Ehrenfest time

- Classical chaos → exponential separation of trajectories in phase space
- Quantum dynamics → linear evolution of vectors in Hilbert space

**What about expectation values?**

**Ehrenfest theorem**

\[
\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})
\]

\[
\begin{aligned}
\frac{d \langle q(t) \rangle}{dt} &= \frac{\langle p(t) \rangle}{m} \\
\frac{d \langle p(t) \rangle}{dt} &= \langle F(q(t)) \rangle
\end{aligned}
\]

where \( \langle A(t) \rangle = \langle \psi(t) | \hat{A} | \psi(t) \rangle \)

In general \( \langle F(q(t)) \rangle \neq F(\langle q(t) \rangle) \)

One can expand \( F(q) \) to obtain

\[
\langle F(q) \rangle = F(\langle q \rangle) + \frac{1}{2} (\Delta q)^2 \frac{d^2 F}{dq^2} \bigg|_{q=\langle q \rangle}
\]

\[
(\Delta q)^2 = \langle (q - \langle q \rangle)^2 \rangle
\]

In time, an initially localized wave packet will diffuse → at some point, the correspondence breaks down

**Ehrenfest time** \( t_E \)

i.e. a minimum uncertainty Gaussian wavepacket, free evolution

\[
\begin{aligned}
\sigma_x^2(T) &= \sigma_x^2(0) \left( 1 + \left( \frac{T}{\tau} \right)^2 \right) \\
T &\sim \frac{\sigma_x(T)}{\sigma_x(0)} \frac{L}{\sigma_p(0)} \sim \frac{S_0}{\hbar}
\end{aligned}
\]

\( t_E \) for regular systems scales as \( \left( \frac{S_0}{\hbar} \right)^\alpha \) → for ‘macroscopic’ action, these times are very large
Ehrenfest time for chaotic systems

Chaotic systems have exponential instabilities

\[ \sigma_u(t) \approx \sigma_u(0)e^{\lambda t} \Rightarrow t = \lambda^{-1}\log(\sigma_u(t)/\sigma_u(0)) \]

\[ \Rightarrow t_E \approx \lambda^{-1} \log\left(\frac{S_0}{\hbar}\right) \]

Lyapunov exponent

In chaotic systems, the Ehrenfest correspondence breaks down in a shorter timescale than for regular systems.

Hyperion \(\rightarrow\) ‘potato-shaped’ moon of Saturn with chaotic motion (tumbling), \(\lambda = 100 \text{ days}^{-1}\)

\[ \Rightarrow \text{Berry estimates } \frac{S_0}{\hbar} \sim 10^{58} \rightarrow t_E \sim 100 \text{ days} \times \log 10^{58} \sim 37 \text{ years} \]

J. Wisdom, S. Peale and F. Mignard, Icarus 58, 137-152 (May 1984)
M. V. Berry, ‘Chaos and the semiclassical limit of quantum mechanics’ (2001)

1984 + 37 = 2021!

Liouville correspondence

- Comparing evolution of distributions in phase space
- Classical: \(\frac{\partial \rho(z,t)}{\partial t} = \{\rho(z,t), H(z)\}_{PB}\)
- Quantum: \(\frac{\partial W(z,t)}{\partial t} = \{W(z,t), H(z)\}_{MB} \approx \{W(z,t), H(z)\}_{PB} + O(\hbar)\)

Classical and quantum disagree when Wigner function becomes negative (typically, interference)

Correspondence is typically more accurate, but break timescale in the same way with \(S_0/\hbar\)

Chaotic dynamics makes classical states turn quantum very quickly!

Sensitivity to perturbations

- Evolution in Hilbert space is linear → trajectories can’t separate ‘exponentially’
- Unitarity implies $d_{12} = |\langle \psi_1(t) | \psi_2(t) \rangle| = |\langle \psi_1(0) | \psi_2(0) \rangle|$


$F(t) = |\langle \psi_\text{rev}(t) | \psi_\text{ev} \rangle|^2$

Loschmidt echo $F(t) = |\langle \psi_\text{rev}(t) | \psi_\text{ev} \rangle|^2$

$|\psi_\text{rev}(t)\rangle = e^{i(H+\epsilon V)t} e^{-iHt} |\psi\rangle$

backwards   forward
Simple model for fidelity decay


Assume $V$ is a Gaussian random matrix

$$e^{-iVt}V = \exp(-\frac{\epsilon^2 t^2}{2})$$

Inverse participation ratio (IPR)

$$\eta_{IPR} = \sum_k |\langle \phi_k | \psi \rangle|^4$$: measures how localized a state is in a given basis

Notice: here $|\psi\rangle$ is the initial state, and $\{|\phi_k\rangle\}$ are the eigenstates of $H$

**Eigenstuff of $H$**

$$|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle = \sum_k \langle \phi_k | \psi_0 \rangle e^{-iE_k t} |\phi_k\rangle$$

$$|\psi_\epsilon(t)\rangle = e^{-i(H+\epsilon V)t}|\psi_0\rangle = \sum_k \langle \tilde{\phi}_k | \psi_0 \rangle e^{-i\tilde{E}_k t} |\tilde{\phi}_k\rangle$$

**Fidelity**

$$F(t) = |\langle \psi(t) | \psi_\epsilon(t) \rangle|^2 = \left| \sum_k |\langle \phi_k | \psi_0 \rangle|^2 e^{-i\epsilon V_{kk}t} \right|^2 + O(\epsilon)$$

$$= \sum_k |\langle \phi_k | \psi_0 \rangle|^4 + \sum_{k \neq l} |\langle \phi_k | \psi_0 \rangle|^2 |\langle \phi_l | \psi_0 \rangle|^2 e^{-2i\epsilon (V_{kk} - V_{ll})t}$$

So, $F(t) \simeq \eta_{IPR} + e^{-\epsilon^2 t^2} (1 - \eta_{IPR})$

High IPR (localized) - **Stable**

Low IPR (delocalized) - **Unstable**
Fidelity decay and Loschmidt echo

Recall the kicked top: \( H = \frac{\alpha}{\tau} J_z + \frac{k}{2J} f(t) J_z^2 \)  
\[ f(t) = \sum_{m=-\infty}^{m=+\infty} \delta(t - m\tau) \]

Intermediate values of \( k \) giving a ‘mixed’ phase space (with both regular and chaotic structures)

\[ \text{Experimental results using cold atoms – Poul Jessen’s lab U. Arizona} \]

- Regular I.C.s more stable
- Chaotic I.C.s less stable

Initial condition in regular part – high IPR (localized in energy)

Initial condition in chaotic part – small IPR (delocalized in energy)

Decay of the Loschmidt echo

- Gaussian decay is typical of the perturbative regime, breaks down in the semiclassical limit of chaotic systems
- There, the decay is typically exponential. The rate is perturbation-dependent first (intermediate \( \epsilon \)), and then perturbation-independent, and given by the largest Lyapunov exponent.

A. Goussev et al, Scholarpedia 7, 11687 (2012)

Relation to OTOCs

- OTOCs - Newly rediscovered metrics for quantum chaos are also known to show intrinsic Lyapunov decay independently of the presence of a perturbation


Summary

• The behavior of the level spacing statistics is usually considered as the defining feature of quantum chaos. In the semiclassical regime, the BT and BGS conjectures provide formal links between quantum and classical integrability and chaos.

• Properties of eigenstates are also an important tool to diagnose quantum chaos. When expanded on a ‘physical’ basis, chaos can be interpreted as the average delocalization of energy eigenstates.

• Signatures of chaos in the dynamics of quantum systems can be seen through i) the fast breakdown of the quantum-to-classical correspondence (Ehrenfest time) and ii) the sensitivity of the evolution of a quantum state to small deviations in the Hamiltonian (fidelity decay)

References

• S. Wimberger, Nonlinear dynamics and quantum chaos: an Introduction (Chap. 4)
• F. Haake, Quantum signatures of chaos (Chap. 2 and 3)
• J. Emerson, PhD Thesis: Quantum chaos and quantum-classical correspondence

Further reading

• D. Poulin, A rough guide to quantum chaos - https://epiq.physique.usherbrooke.ca/pdf/Pou02a.pdf
• M. V. Berry, https://michaelberryphysics.files.wordpress.com/2013/07/berry337.pdf

Next week (June 16th): Random Matrix Theory (Changhao Yi) – Main reference: Haake’s book, chapter 4
Extra stuff
WKB approximation

Highest weight near turning points

\[ |\psi(x)|^2 \Delta x \propto \text{time spent in a given interval } [x, x + \Delta x] \]

Harmonic oscillator
(taken from wiki

Chaotic trajectories have fixed energy, but are extremely delocalized in phase space!

Ergodicity – they spend roughly the same amount of time everywhere
\[ H_B = \sum_{i=1}^{L} \left[ -t (b_i^\dagger b_{i+1} + \text{H.c.}) + V \left( n_i^b - \frac{1}{2} \right) \left( n_{i+1}^b - \frac{1}{2} \right) \right] \\
- t' (b_i^\dagger b_{i+2} + \text{H.c.}) + V' \left( n_i^b - \frac{1}{2} \right) \left( n_{i+2}^b - \frac{1}{2} \right), \]

Hilbert space for the classical Liouville equation.— Consider the time-independent classical Hamiltonian $H(z)$, where $z = (x_1, \ldots, x_N, p_1, \ldots, p_N)$ specifies a point in $N$-dimensional phase space and $H(z)$ belongs to an integrable or nonintegrable system. The evolution of the phase space distribution $\rho(z,t)$ obeys the classical Liouville equation:

$$i \frac{\partial \rho(z,t)}{\partial t} = \hat{L} \rho(z,t) \equiv i\{H(z), \rho(z,t)\}, \quad (5)$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket, $\hat{L}$ is called the Liouvillian, and $\rho(z,t)$ is normalized as $\int d\mathbf{z} \rho(\mathbf{z}, t) = 1$. The Liouvillian is a Hermitian operator with respect to the given inner product $\langle \rho_1 | \rho_2 \rangle = \int d\mathbf{z} \rho_1^*(\mathbf{z}) \rho_2(\mathbf{z})$. Then, using the eigenstate $|n\rangle$ of $\hat{L}$, we can expand $\rho(z,t)$ as

$$|\rho(t)\rangle = \sum_n c_n e^{-i\lambda_n t} |n\rangle, \quad (6)$$

where $c_n$ is a time-independent constant and $\langle \rho(t)|\rho(t)\rangle \neq 1$. We note that $\langle \rho|\hat{L}|\rho\rangle = 0$ and, if $\lambda_n$ is an eigenvalue of $\hat{L}$, then $-\lambda_n$ is also an eigenvalue of $\hat{L}$ (see Supplemental Material [34] for the proof).

In the following, we obtain the CSL for the classical Liouville equation by using the Hilbert space for the classical Liouville equation.