Summer study course on many-body quantum chaos, Session 3 : Random Matrix Theory

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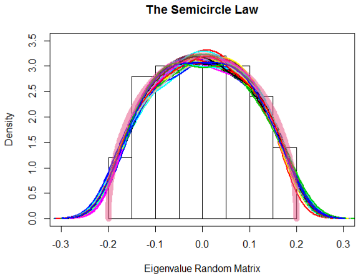
Outline

- Time reversal symmetry
- Gaussian matrix ensemble
- Level statistics
- Quantum kicked top
- Circular matrix ensemble
- Spectral form factor



Signatures of quantum chaos

- Level repulsion

$$P(s) \propto s e^{-\pi s^2/4}$$

- Eigenstate delocalization

$$IPR = \sum_k |\langle \psi_k | \phi \rangle|^4 \sim 1/N$$

- Loschmidt echo

$$F(t) = |\langle \psi_{rev}(t) | \psi \rangle|^2$$
$$|\psi_{rev}(t)\rangle = e^{i(H+\epsilon V)t} e^{-iHt} |\psi\rangle$$

- OTOC

$$F(t) = \langle V^\dagger(t) W^\dagger V(t) W \rangle$$

Symmetry of Hamiltonian systems

- In the study of quantum chaos, when a system is too complicated, we can no longer know the detail of the Hamiltonian. Instead, we represent it with a matrix ensemble $\{H, P(H)\}$.
- We want the matrix ensemble to be as random as possible, while still satisfies some kind of symmetry.
- Time translation symmetry:

$$\frac{dH}{dt} = 0$$

- Space translation symmetry:

$$[H, P] = 0$$

- Rotational symmetry:

$$[H, J] = 0$$

Symmetry of Hamiltonian systems

- Time-reversal symmetry : Hamiltonian $H(x, p, t)$ invariant under $t \to -t, p \to -p$.
- A general time reversal operator can be written as:

$$T = UK, \quad K : \sum_v \psi_v |v\rangle \to \sum_v \psi_v^* |v\rangle$$

  It further satisfies that $TJT^{-1} = -J$.
- For integer spin system:

$$[H, T] = 0, \quad T^2 = 1$$

  Then we show that:

$$H_{jk} = (THT)_{jk} = H_{jk}^*$$

  which means the matrix is real symmetric.

Symmetry of Hamiltonian systems

- Based on the condition of symmetry, we can classify the matrix ensembles into three classes: Gaussian orthogonal ensemble (GOE), Gaussian unitary ensemble (GUE) and Gaussian symplectic ensemble (GSE).

Ensemble	Time-reversal symmetry	Rotational symmetry	Invariant under
GOE	Yes	Yes	orthogonal transformation
GUE	No	Yes	unitary transformation
GSE	Yes	No	symplectic transformation

- We also want the entries of the matrix to be independent.

GOE

- GOE describes a probability distribution of real symmetric matrix $P(H)$, let's start with 2 dimensional matrix as an example:

$$H = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$$

- The entries of H are independent:

$$P(H)dH = p_1(x)p_2(y)p_3(z)dxdydz$$

- And the distribution is invariant under orthogonal transformation.

$$P(H)dH = P(H')dH', \quad H' = OHO^{-1}$$

GOE

- We first assume that the transformation is infinitesimal:

$$O = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}, \quad |\theta| \ll 1$$

- Then in the leading term of θ, the change of variables becomes:

$$H' = OHO^{-1} = \begin{pmatrix} x' & y' \\ y' & z' \end{pmatrix}$$

$$x' = x - 2\theta y, \quad y' = y + 2\theta x, \quad z' = z + \theta(x - y)$$

GOE

- The requirement that $P(H) = P(H')$ gives us a relation:

$$p_1(x')p_2(y')p_3(z') = p_1(x)p_2(y)p_3(z)$$

$$\frac{1}{y}\frac{d\ln p_2}{dy} - \frac{2}{x-z}\left(\frac{d\ln p_1}{dx} - \frac{d\ln p_3}{dz}\right) = 0$$

- Solve this relation, we obtain an expression for $P(H)$:

$$P(H) \propto \exp\left[-A(x^2 + z^2 + 2y^2) - B(x+z)\right]$$

- We can always shift the energy levels to make sure $\mathrm{Tr}(H) = 0$, thus:

$$P(H) \propto \exp[-A\mathrm{Tr}(H^2)]$$

GUE

- Similarly, we want $P(H)dH$ to be invariant under infinitesimal unitary transformation:

$$U = I + i\epsilon \cdot \sigma$$

- Analogously, to make sure $P(H) = P(H')$:

$$(\epsilon_x + i\epsilon_y)\left[y\left(-\frac{d \ln p_1}{dx} + \frac{d \ln p_3}{dz}\right) + (x - z)\frac{d \ln p_2}{dy}\right] + c.c = 0$$

- The result is exactly the same:

$$P(H) \propto \exp[-A\mathrm{Tr}(H^2)]$$

Eigenvalue distribution

- For GOE, change variables from (x, y, z) to $(\lambda_1, \lambda_2, \theta)$:

$$H = \begin{pmatrix} x & y \\ y & z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- Calculate the Jacobian:

$$\frac{D(x, y, z)}{D(\lambda_1, \lambda_2, \theta)} = \lambda_1 - \lambda_2$$

- Then we have:

$$dH \propto |\lambda_1 - \lambda_2| d\lambda_1 d\lambda_2 d\theta$$

$$\rho(\lambda_1, \lambda_2) = \int P(H) d\theta \propto |\lambda_1 - \lambda_2| e^{-A(\lambda_1^2 + \lambda_2^2)}$$

Eigenvalue distribution

- For GUE, change variables from (x, y, y^*, z) to $(\lambda_1, \lambda_2, \theta, \phi)$:

$$H = \begin{pmatrix} x & y \\ y^* & z \end{pmatrix} = \begin{pmatrix} \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta & (\lambda_1 - \lambda_2)e^{-i\phi}\cos\theta\sin\theta \\ (\lambda_1 - \lambda_2)e^{i\phi}\cos\theta\sin\theta & \lambda_1 \sin^2\theta + \lambda_2 \cos^2\theta \end{pmatrix}$$

- Calculate the Jacobian:

$$\frac{D(x, z, \mathrm{Re}(y), \mathrm{Im}(y))}{D(\lambda_1, \lambda_2, \theta, \phi)} = |\lambda_1 - \lambda_2|^2 \cos\theta\sin\theta$$

- Then we have:

$$dH \propto |\lambda_1 - \lambda_2|^2 \cos\theta\sin\theta d\lambda_1 d\lambda_2 d\theta d\phi$$

$$\rho(\lambda_1, \lambda_2) \propto |\lambda_1 - \lambda_2|^2 e^{-A(\lambda_1^2 + \lambda_2^2)}$$

Eigenvalue distribution

- For general N level system, it can be proved that (See [Anderson et.al, 10]):

$$\mathrm{GOE} \;:\quad dH \propto \prod_{i<j} |\lambda_i - \lambda_j| \prod_i d\lambda_i dO$$

$$\rho(\lambda_1, \lambda_2, \cdots, \lambda_n) \propto \prod_{i<j} |\lambda_i - \lambda_j| e^{-A\sum_i \lambda_i^2}$$

$$\mathrm{GUE} \;: dH \propto \prod_{i<j} |\lambda_i - \lambda_j|^2 \prod_i d\lambda_i dU$$

$$\rho(\lambda_1, \lambda_2, \cdots, \lambda_n) \propto \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-A\sum_i \lambda_i^2}$$

- Repulsion : $\prod_{i<j} |\lambda_i - \lambda_j|$. (Vandermonde determinant)
- Confinement : $e^{-A\sum_i \lambda_i^2}$.

Eigenvalue distribution

- GOE and GUE can both be included in a more general matrix ensemble, named Gaussian β ensemble:

$$\rho(\lambda_1, \lambda_2, \cdots, \lambda_n) = C_{n\beta} \prod_{i<j}^{n} |\lambda_i - \lambda_j|^{\beta} \exp[-\frac{\beta}{2} \sum_{i=1}^{n} \lambda_i^2]$$

- The parameter β quantifies the "strength" of repulsion.
- $\beta = 1, 2$ correspond to GOE and GUE separately.
- The limit $\beta \to 0$ represents Poisson statistics.

Level spacing statistics

- Given the joint distribution of energy levels, we can analyze level spacing distribution:

$$P(s) = \text{const} \int d\lambda_1 \int d\lambda_2 \delta(s - |\lambda_1 - \lambda_2|) |\lambda_1 - \lambda_2|^{\beta} e^{-\beta(\lambda_1^2 + \lambda_2^2)/2}$$

- The results are :

$$P(s) = \begin{cases} \frac{s\pi}{2} e^{-\pi s^2/4}, & GOE \\ \frac{s^2 32}{\pi^2} e^{-4s^2/\pi}, & GUE \end{cases}$$

- This level spacing matches with the prediction of BGS conjecture.

Level spacing statistics

- BGS conjecture : the eigenvalues of quantum system whose classical analogue is fully chaotic, obey the statistics of level spacing predicted by RMT.

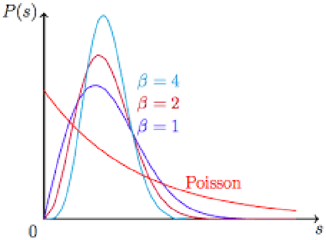


Figure: Wigner-Dyson statistics

Level spacing statistics

- $P(s)ds$: the probabilty that the spacing s between any neighboring energy levels lies in the interval $s \to s + ds$
- Another convenient formula:

$$P(s) = \frac{d^2E}{ds^2}, \quad E(s) = \int_{\Omega(s)^c} \rho(\lambda_1, \cdots, \lambda_n) d\lambda_1 d\lambda_2 \cdots \lambda_n$$

- $E(s)$: the probability that any interval of length s is empty of levels.
- $\Omega(s)$: the set of events that at least one eigenvalue stays in a certain region of length s.

Level spacing statistics

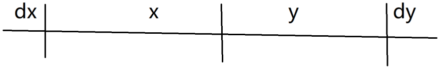


Figure: We want the probability that dx,dy has energy level while x+y is empty.

Level spacing statistics

- $E(x+y) - E(x+y+\delta x)$: the probability that $x+y$ is empty and δx is not.

$$E(x+y) - E(x+y+\delta x) \approx -\frac{\partial E(x+y)}{\partial x}\delta x$$

- The probability that $x+y$ is empty and $\delta x, \delta y$ are not empty:

$$\frac{\partial^2 E(x+y)}{\partial x \partial y}\delta x \delta y$$

- Given a level in δx, the probability of finding a level in interval $x+y \to x+y+\delta y$:

$$\frac{\partial^2 E(x+y)}{\partial x \partial y}\delta y = \frac{d^2 E(s)}{ds^2} ds = P(s) ds$$

Averaged level density

- The probability distribution of a single eigenvalue:

$$\rho(E) = \int dE_2 dE_3 \cdots dE_n \rho(E, E_2, \cdots, E_n)$$

- Semicircle law, for large systems $N \to \infty$:

$$\rho(E) = \begin{cases} \frac{2}{\pi}\sqrt{1 - E^2}, & |E| \leq 1 \\ 0, & |E| > 1 \end{cases}$$

- For rigorous proof, see [Anderson et.al, 10]

Distribution of eigenvectors

- In $P(H)dH \propto e^{-A\text{Tr}(H^2)}dH$, $H = O\Lambda O^{-1}$, the distribution of O is Haar random. Given eigenvector $v = (a_1, a_2, \cdots a_n)$, we have:

$$P(v) \propto \delta\left(\sum_{i=1}^{n} a_i^2 - 1\right)$$

- The distribution of the i th component of v is given by:

$$\begin{aligned} P(y) &= \int da_1 da_2 \cdots da_n \delta(y - a_i^2) P(v) \\ &\propto (1-y)^{(n-3)/2}/\sqrt{y} \end{aligned}$$

Distribution of eigenvectors

- In the limit $n \to \infty$, this becomes:

$$\lim_{n\to\infty} P(y) \approx \left(\frac{n}{2}\pi\right)^{1/2} \frac{1}{\sqrt{y}} e^{-ny/2}$$

- Let $\eta = ny$, it becomes:

$$P(\eta) = \left(\frac{1}{2\pi}\right)^{1/2} \frac{1}{\sqrt{\eta}} e^{-\eta/2}, \quad \text{(Porter-Thomas distribution)}$$

- By comparing to Porter-Thomas distribution, we can quantify how uniformly random a probability distribution of unit vectors is.

Periodic driven system

- Besides level statistics, RMT can also describe spectral fluctuations in some many body systems. A common model that exhibits spectral fluctuation is quantum kicked top.
- The dynamics of quantum kicked top is described by the time dependent Hamiltonian:

$$H(t) = H_X + TH_Z \sum_{n \in Z} \delta(t - nT)$$

- The evolution operator for any period $[nT, (n+1)T]$ is called a Floquet operator:

$$U_T = \exp_{\mathcal{T}}\left[-i \int_{nT}^{(n+1)T} H(t)\right] = \exp\left[-iH_X T\right] \exp[-iH_Z T]$$

- For this specific model, it's a lot easier to directly analyze the unitary operator U_T.

Circular matrix ensemble

- For large T, the spectral statistics of quantum kicked top matches with that of circular matrix ensembles.
- Circular orthogonal ensemble (COE) has time-reversal symmetry. It has eigenvalue distribution:
$$P(\phi_1, \phi_2, \cdots, \phi_n) = \prod_{j<k} |e^{-i\phi_j} - e^{-i\phi_k}|$$
- Circular unitary ensemble (CUE), on the other hand, is similar to GUE:
$$P(\phi_1, \phi_2, \cdots, \phi_n) = \prod_{j<k} |e^{-i\phi_j} - e^{-i\phi_k}|^2$$

Spectral form factor

- Given a Floquet operator U, its spectral density is defined as:

$$\rho(\psi) = \frac{2\pi}{N} \sum_n \delta(\psi - \psi_n)$$

- Here we use $\langle \cdot \rangle_\psi$ to denote mean level density.

$$\langle f(\psi) \rangle_\psi = \frac{1}{2\pi} \int_0^{2\pi} f(\psi) d\psi$$

- Spectral fluctuation is reflected by:

$$R(\theta) = \langle \rho(\psi + \theta/2)\rho(\psi - \theta/2) \rangle_\psi - \langle \rho \rangle_\psi^2$$

Spectral form factor

- The spectral form factor $K(t), t \in Z$, is the Fourier transformation of it.

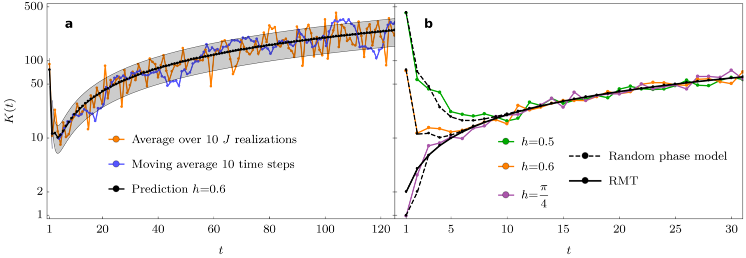
$$
\begin{aligned}
K(t) &= \frac{N^2}{2\pi}\int_0^{2\pi} d\theta R(\theta) e^{-i\theta t} \\
&= \sum_n e^{-it\phi_n} \cdot \sum_m e^{it\phi_m} - N^2 \delta_{t,0} \\
&\to \langle \mathrm{Tr}(U^t)\mathrm{Tr}(U^{-t})\rangle - N^2\delta_{t,0}
\end{aligned}
$$

- For COE and CUE, we have:

$$
K(t) = \begin{cases} 2t - t\ln(1+2t/N), & COE \\ t, & CUE \end{cases}
$$

Numerical results [Kos et.al, 10]

- Comparison between numerical results and RMT prediction.



- Kicked Ising model:

$$H_X = \sum_k J_k^{(1)} X_k + \sum_{k<l} J_{k,l}^{(2)} X_k X_l + \cdots$$

$$H_Z = h \sum_k Z_k$$

Relation to quantum simulation [Heyl et.al, 19]

- Trotter sequences as Floquet systems

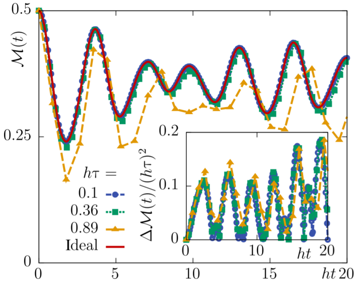
$$U^{(n)}(t) = [U_1(t/n)U_2(t/n)\cdots U_M(t/n)]^n$$

- Robustness of local observables

$$H = H_Z + H_X, \quad H_X = g\sum_l X_l$$

$$H_Z = J\sum_l Z_l Z_{l+1} + h\sum_l Z_l$$

$$M(t) = \langle \frac{1}{N}\sum_l Z_l \rangle$$



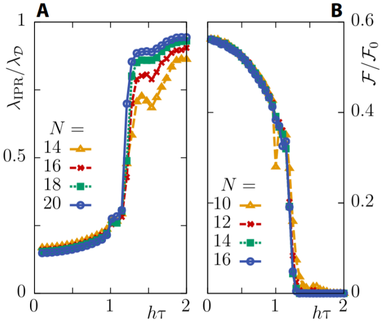
Relation to quantum simulation [Heyl et.al, 19]

- Inverse participation ratio:

$$IPR = \sum_{v} |\langle \psi_v | \phi_0 \rangle|^4$$

- OTOC

$$F(t) = \langle V^\dagger(t) W^\dagger V(t) W \rangle$$



Summary

- Use the condition of symmetry + independence of entries, the Gaussian matrix ensembles are derived.

$$P(H)dH \propto \exp[-A\mathrm{Tr}(H^2)]dH$$

- After change of variables, we derive the joint distribution of eigenvalues:

$$\rho(\lambda_1, \lambda_2, \cdots, \lambda_n) \propto \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} e^{-\beta \sum_i \lambda_i^2/2}$$

- Use this distribution, we can calculate many useful things, like level spacing distribution, averaged level density and correlation functions. The level spacing distribution derived matches with that of quantum chaotic systems.

Summary

- Periodic driven system is a useful and important model for quantum chaos.

$$U_T = \exp[-iH_X T] \exp[-iH_Z T]$$

  Its spectral statistics matches with that of circular ensembles.

- Spectral form factor directly reflects spectrum fluctuation.

$$K(t) = \langle \sum_{n,m} e^{it(\phi_n - \phi_m)} \rangle - N^2 \delta_{t,0}$$

- COE/CUE can explain the behavior of $K(t)$ as well.

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