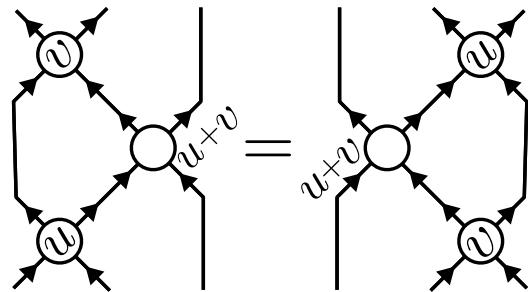
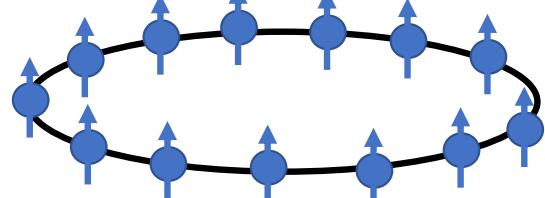




# Quantum Integrability

Part 2: Algebraic Bethe Ansatz

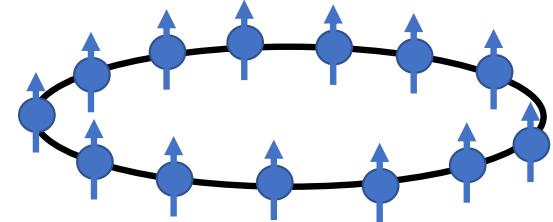
Following [VieiraCostelloCourse] (Lectures 1-3) and [arXiv:1804.07350]



# Spin-1/2 models of magnetism

$$H_{XYZ} = - \left( J_X \sum_{j=1}^L X_j X_{j+1} + J_Y \sum_{j=1}^L Y_j Y_{j+1} + J_Z \sum_{j=1}^L Z_j Z_{j+1} \right)$$

$J_X = J_Y \Rightarrow H_{XXZ}$       Integrable family of Hamiltonians!



$L$  spins on a ring

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

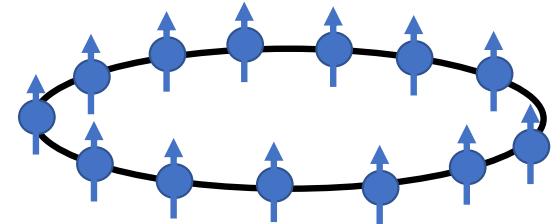
$$Z|\uparrow\rangle = (+1)|\uparrow\rangle$$

$$X_j = I \otimes I \otimes \cdots \otimes X \otimes \underset{2}{\cdots} \otimes I$$

# Spin-1/2 models of magnetism

$$H_{XYZ} = - \left( J_X \sum_{j=1}^L X_j X_{j+1} + J_Y \sum_{j=1}^L Y_j Y_{j+1} + J_Z \sum_{j=1}^L Z_j Z_{j+1} \right)$$

$J_X = J_Y \Rightarrow H_{XXZ}$       Integrable family of Hamiltonians!



L-spins on a ring

Special case XXX-model (this lecture)

$$J_X = J_Y = J_Z \Rightarrow H_{XXX}$$

$$\begin{aligned}\boldsymbol{\sigma}_j &= (X_j, Y_j, Z_j) \\ \boldsymbol{\sigma}_1 &= \boldsymbol{\sigma}_{L+1} \quad \text{PBC}\end{aligned}$$

$$H_{XXX} = -J \sum_{j=1}^L \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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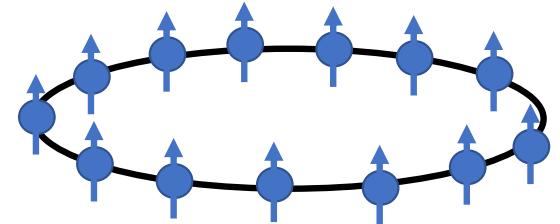
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L-spins on a ring

## Special case XXX-model (this lecture)

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$$H_{XXX} = -J \sum_{j=1}^L \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1}$$

Recall:  
 $\text{SWAP}_{1,2} = \frac{1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{2}$

Rescale ground state energy to zero

$$H_{XXX} = -\frac{J}{2} \sum_{j=1}^L (\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1} - 1) = -J \sum_{j=1}^L (\text{SWAP}_{j,j+1} - 1)$$

$$H_{XXX} |\uparrow\rangle^{\otimes L} = 0$$

Ground state space is really symmetric subspace

# A clever way to write the XXX-Hamiltonian

$$H_{XXX} = -iJ\partial_u \log T(u) \Big|_{u=0} + JL$$

$$T(u) = \text{Tr}_0 \left( \prod_{j=1}^L u I_{0,j} + i \text{SWAP}_{0,j} \right)$$

Recall:

$$\partial_u \log T(u) \Big|_{u=0} = T(0)^{-1}(\partial_u T)(0)$$

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Transfer matrix      R-matrix       $R_{0,j}(u)$

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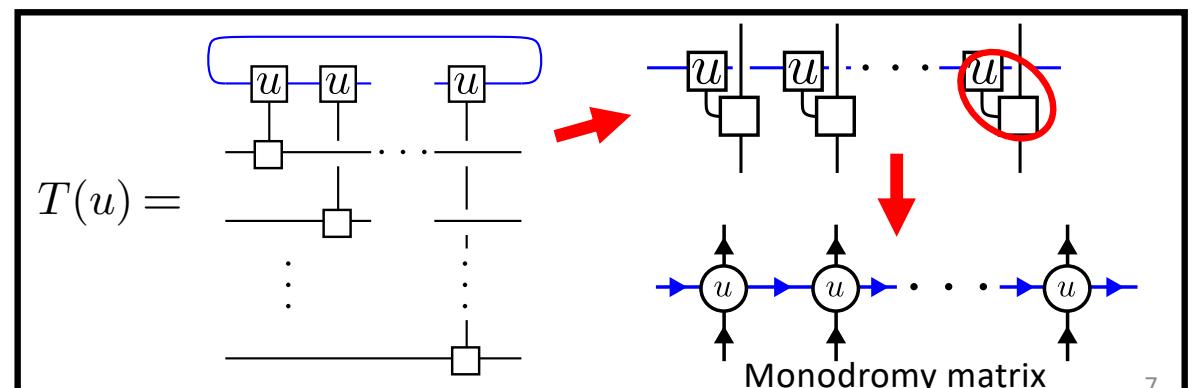
$$R_{0,j}(u) = \begin{array}{c} \square \\ \downarrow \\ \square \end{array} = u \begin{array}{c} \rule[1.5ex]{0.8em}{0.1ex} \\ \rule[1.5ex]{0.8em}{0.1ex} \end{array} + i \begin{array}{c} \times \\ \diagup \quad \diagdown \end{array}$$

$$R_{0,j}(u) = \begin{array}{c} \uparrow \\ \square \\ \rightarrow \end{array} = u \begin{array}{c} | \\ | \end{array} + i \begin{array}{c} \diagup \\ \diagdown \end{array}$$

Quick note on diagramatics:

$$BA|\psi\rangle = |\psi\rangle \begin{array}{c} \square \\ \square \end{array} \begin{array}{c} \square \\ \square \end{array} \dots$$

$$\text{Tr}(A) = \boxed{\begin{array}{c} \square \\ A \end{array}}$$



Proof

$$T(u) = \begin{array}{c} \text{---} \\ | \\ \textcircled{u} \\ | \\ \text{---} \end{array} \rightarrow \begin{array}{c} \text{---} \\ | \\ \textcircled{u} \\ | \\ \text{---} \end{array} \rightarrow \dots \rightarrow \begin{array}{c} \text{---} \\ | \\ \textcircled{u} \\ | \\ \text{---} \end{array}$$

$$H_{XXX} = -iJ\partial_u \log T(u) \Big|_{u=0} + JL$$

$$T(u) = \text{Tr}_0 \left( \prod_{j=1}^L u I_{0,j} + i \text{SWAP}_{0,j} \right)$$

$$\begin{array}{c} \text{---} \\ | \\ \textcircled{u} \\ | \\ \text{---} \end{array} = u \begin{array}{c} | \\ | \end{array} + i \begin{array}{c} | \\ \curvearrowleft \end{array}$$

Proof

$$T(u) = \begin{array}{c} \text{Diagram of } T(u) \\ \text{A sequence of } L \text{ circles labeled } u, \text{ each with an upward arrow.} \\ \text{Horizontal arrows connect } u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_L. \\ \text{Vertical arrows point from } u_1 \text{ up to } u_2, u_2 \text{ up to } u_3, \dots, u_{L-1} \text{ up to } u_L. \\ \text{A curved arrow connects } u_L \text{ back to } u_1. \end{array}$$
$$T(0) = i^L \quad \begin{array}{c} \text{Diagram of } T(0) \\ \text{A sequence of } L \text{ circles labeled } u, \text{ each with an upward arrow.} \\ \text{Horizontal arrows connect } u_1 \text{ to } u_2, u_2 \text{ to } u_3, \dots, u_{L-1} \text{ to } u_L. \\ \text{Vertical arrows point from } u_1 \text{ up to } u_2, u_2 \text{ up to } u_3, \dots, u_{L-1} \text{ up to } u_L. \end{array}$$

$$H_{XXX} = -iJ\partial_u \log T(u) \Big|_{u=0} + JL$$

$$T(u) = \text{Tr}_0 \left( \prod_{j=1}^L u I_{0,j} + i \text{SWAP}_{0,j} \right)$$

$$\begin{array}{c} \text{Diagram of } T(u) \\ \text{A circle labeled } u \text{ with an upward arrow.} \\ \text{Horizontal arrows point right from the top and left to the bottom.} \\ \text{Vertical arrows point up from the bottom and down to the top.} \end{array} = u \quad \begin{array}{c} | \\ | \end{array} + i \quad \begin{array}{c} \diagup \\ \diagdown \end{array}$$

Proof

$$T(u) = \begin{array}{c} \text{Diagram of } T(u) \\ \text{A sequence of } L \text{ nodes, each labeled } u, \text{ connected by horizontal arrows pointing right.} \\ \text{Each node has two vertical arrows pointing up from below it.} \\ \text{A curved arrow at the top connects the first node to the last node.} \end{array}$$

$$T(0) = i^L \quad \begin{array}{c} \text{Diagram of } T(0) \\ \text{A sequence of } L \text{ nodes, each labeled } u, \text{ connected by horizontal arrows pointing right.} \\ \text{Each node has two vertical arrows pointing up from below it.} \\ \text{A curved arrow at the top connects the first node to the last node.} \end{array}$$

$$T(0)^{-1} = i^{-L} \quad \begin{array}{c} \text{Diagram of } T(0)^{-1} \\ \text{A sequence of } L \text{ nodes, each labeled } u, \text{ connected by horizontal arrows pointing right.} \\ \text{Each node has two vertical arrows pointing up from below it.} \\ \text{A curved arrow at the top connects the first node to the last node.} \end{array}$$

$$H_{XXX} = -iJ\partial_u \log T(u) \Big|_{u=0} + JL$$

$$T(u) = \text{Tr}_0 \left( \prod_{j=1}^L u I_{0,j} + i \text{SWAP}_{0,j} \right)$$

$$\begin{array}{c} \text{Diagram of } T(u) \\ \text{A single node labeled } u, \text{ with four arrows pointing outwards: up, down, left, and right.} \\ \text{A vertical line segment is positioned to its right.} \\ \text{A curved arrow at the top connects the node to the vertical line.} \end{array} = u \quad \begin{array}{c} | \\ | \end{array} + i \quad \begin{array}{c} \text{Diagram of } i \text{ SWAP} \\ \text{A single node labeled } u, \text{ with four arrows pointing outwards: up, down, left, and right.} \\ \text{A vertical line segment is positioned to its right.} \\ \text{A curved arrow at the top connects the node to the vertical line.} \end{array}$$

# Proof

$$T(u) = \begin{array}{c} \text{---} \\ | \\ u \\ | \\ \text{---} \end{array} \rightarrow \begin{array}{c} \text{---} \\ | \\ u \\ | \\ \text{---} \end{array} \rightarrow \cdots \rightarrow \begin{array}{c} \text{---} \\ | \\ u \\ | \\ \text{---} \end{array}$$

$$T(0) = i^L \quad | \quad \begin{array}{c} \diagup \\ \diagdown \end{array} \quad | \quad \cdots$$

$$T(0)^{-1} = i^{-L} \left| \begin{array}{c} \nearrow \\ \nearrow \\ \nearrow \\ \dots \\ \nearrow \\ \nearrow \\ \nearrow \end{array} \right|$$

$$\partial_u T(u) \Big|_{u=0} = i^{L-1} \sum_{k=1}^L \begin{array}{ccccccccc} & \swarrow & \searrow & & \cdots & \swarrow & | & \searrow & \cdots & \swarrow \\ & 1 & 2 & & k-1 & k & k+1 & & L & \end{array}$$

$$H_{XXX} = -iJ\partial_u \log T(u) \Big|_{u=0} + JL$$

$$T(u) = \text{Tr}_0 \left( \prod_{j=1}^L u I_{0,j} + i \text{SWAP}_{0,j} \right)$$

$$= u \text{ } \begin{array}{|c|}\hline\end{array} + i \text{ } \begin{array}{|c|}\hline\end{array}$$

Proof

$$T(u) = \begin{array}{c} \text{Diagram of } T(u) \text{ showing } L \text{ nodes } u \text{ connected by horizontal arrows, with vertical arrows pointing up from each node.} \end{array}$$

$$T(0) = i^L \quad \begin{array}{c} \text{Diagram of } T(0) \text{ showing } L \text{ nodes connected by curved arrows forming a loop.} \end{array}$$

$$T(0)^{-1} = i^{-L} \quad \begin{array}{c} \text{Diagram of } T(0)^{-1} \text{ showing } L \text{ nodes connected by curved arrows forming a loop, with a different orientation than } T(0). \end{array}$$

$$\partial_u T(u) \Big|_{u=0} = i^{L-1} \sum_{k=1}^L \begin{array}{c} \text{Diagram of } \partial_u T(u) \Big|_{u=0} \text{ showing } L \text{ nodes labeled } 1, 2, \dots, k-1, k, k+1, \dots, L. \text{ Node } k \text{ has a vertical arrow pointing up and a horizontal arrow pointing right. Other nodes have horizontal arrows pointing right. Curved arrows connect nodes } 1, 2, \dots, k-1, k+1, \dots, L. \end{array}$$

$$-iJ \left( T(0)^{-1} \partial_u T(u) \Big|_{u=0} \right) = -J \sum_{k=1}^L \begin{array}{c} \text{Diagram of } -iJ \left( T(0)^{-1} \partial_u T(u) \Big|_{u=0} \right) \text{ showing } L \text{ nodes labeled } 1, 2, \dots, k-1, k, k+1, \dots, L. \text{ Nodes } 1, 2, \dots, k-1, k+1, \dots, L \text{ have horizontal arrows pointing right. Node } k \text{ has a vertical arrow pointing up and a horizontal arrow pointing right. Curved arrows connect nodes } 1, 2, \dots, k-1, k+1, \dots, L. \end{array}$$

$$H_{XXX} = -iJ \partial_u \log T(u) \Big|_{u=0} + JL$$

$$T(u) = \text{Tr}_0 \left( \prod_{j=1}^L u I_{0,j} + i \text{SWAP}_{0,j} \right)$$

$$\begin{array}{c} \text{Diagram of } T(u) \text{ showing } L \text{ nodes labeled } 1, 2, \dots, k-1, k, k+1, \dots, L. \text{ Node } k \text{ has a vertical arrow pointing up and a horizontal arrow pointing right. Other nodes have horizontal arrows pointing right. Curved arrows connect nodes } 1, 2, \dots, k-1, k+1, \dots, L. \end{array}$$

# Proof

**Proof**  $T(u) =$  

$$T(0)^{-1} = i^{-L} \left| \begin{array}{c} \nearrow \\ \nearrow \\ \nearrow \\ \dots \\ \nearrow \\ \nearrow \end{array} \right|$$

$$\partial_u T(u) \Big|_{u=0} = i^{L-1} \sum_{k=1}^L \begin{array}{c} \text{Diagram showing } L \text{ points labeled } 1, 2, \dots, k-1, k, k+1, \dots, L \\ \text{with } k \text{ points connected by } k-1 \text{ curved arcs.} \end{array}$$

$$-iJ \left( T(0)^{-1} \partial_u T(u) \Big|_{u=0} \right) = -J \sum_{k=1}^L \text{Diagram}_k$$

1    2    ...    k-1    k    k+1    ...    L

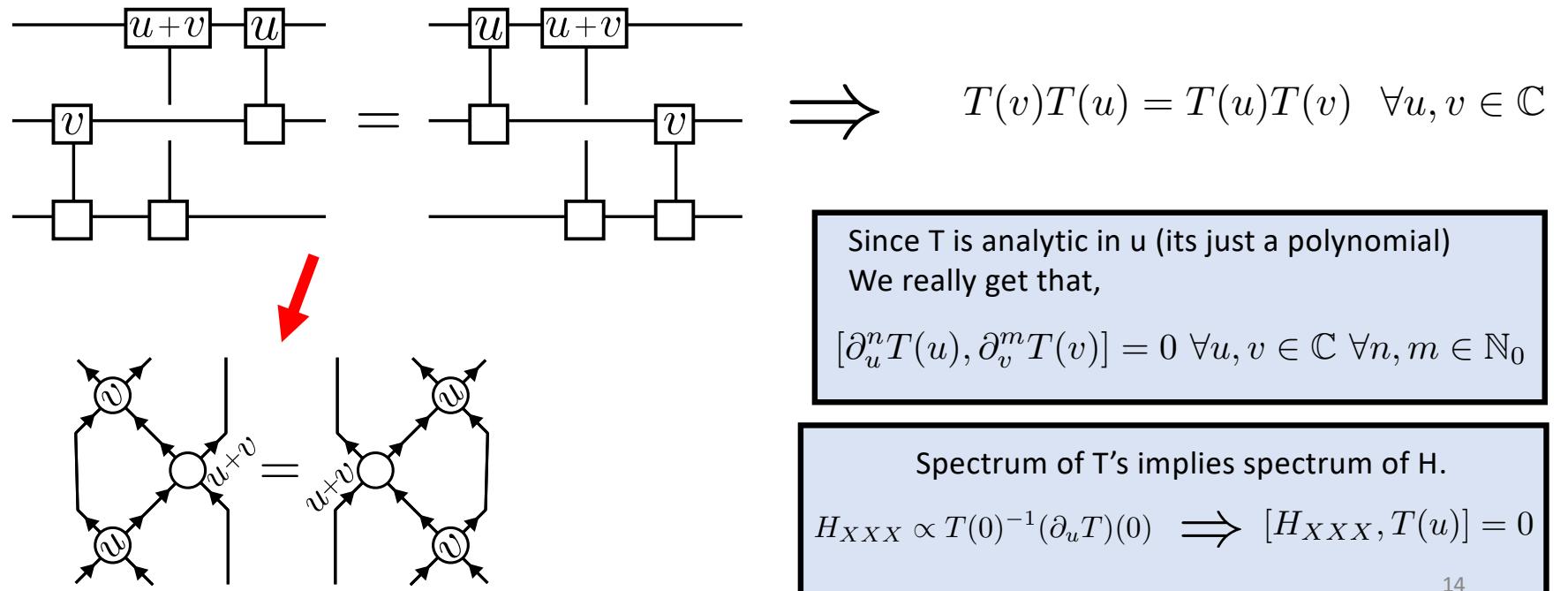
$$= -J \sum_{k=1}^L \text{SWAP}_{k-1,k} = H_{XXX} - JL \quad \square$$

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# Conserved quantities and Yang-Baxter

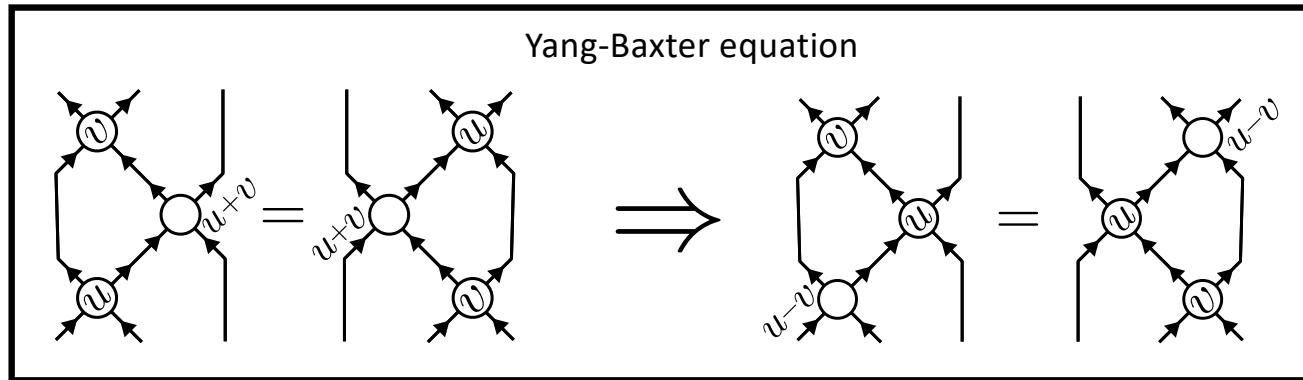
- R-matrix satisfies Yang-Baxter equation  $\Rightarrow$  Many conserved quantities

$$R_{1,2}(u)R_{1,3}(u+v)R_{2,3}(v) = R_{2,3}(v)R_{1,3}(u+v)R_{1,2}(u)$$



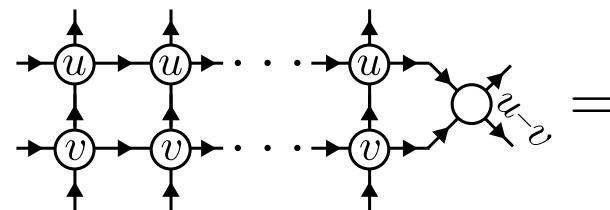
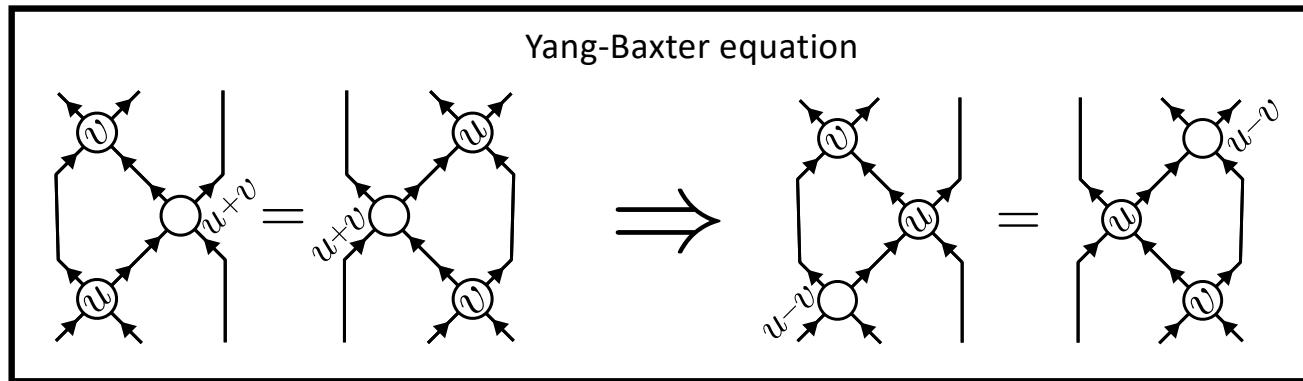
# Proof

Yang-Baxter  
⇒  
Conserved quantities



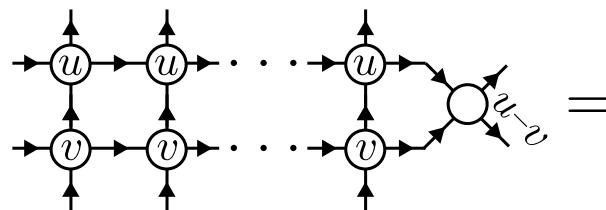
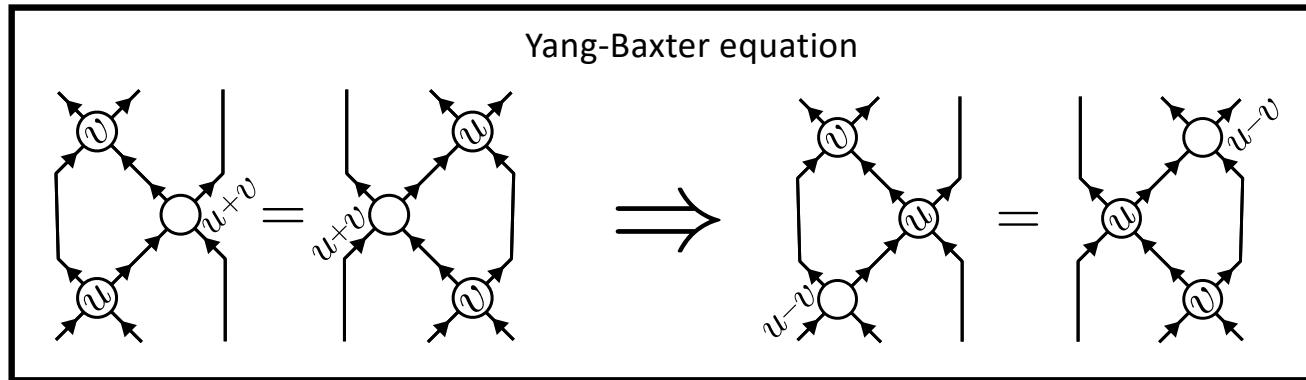
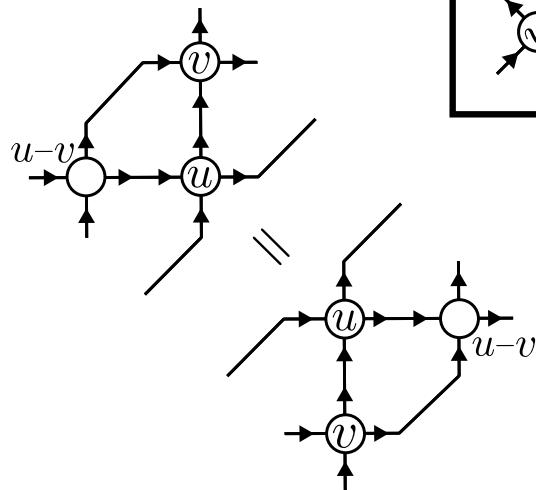
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Yang-Baxter  
⇒  
Conserved quantities



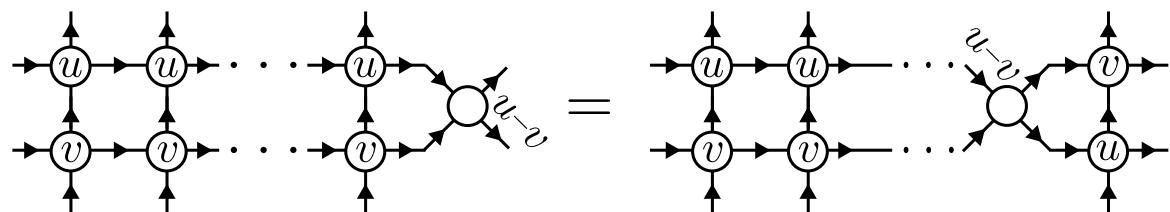
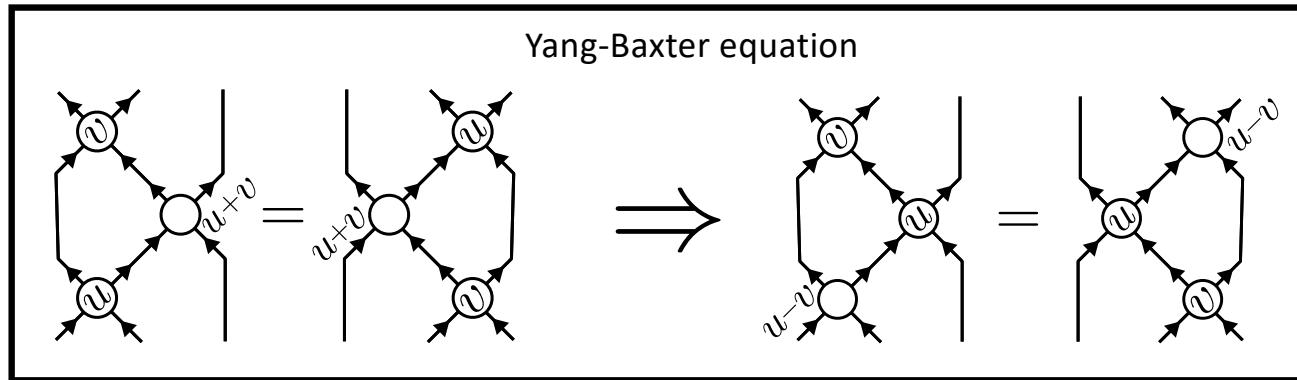
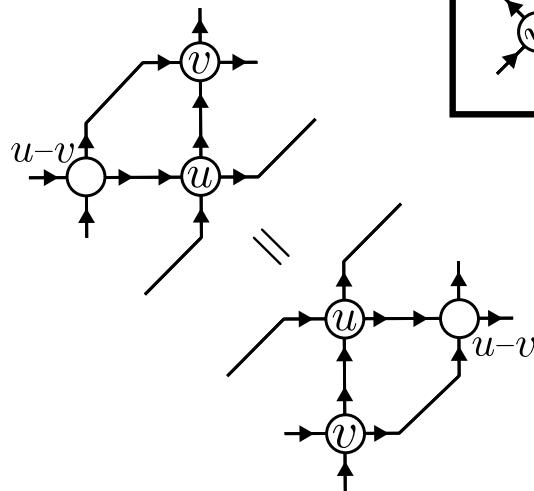
# Proof

Yang-Baxter  
⇒  
Conserved quantities



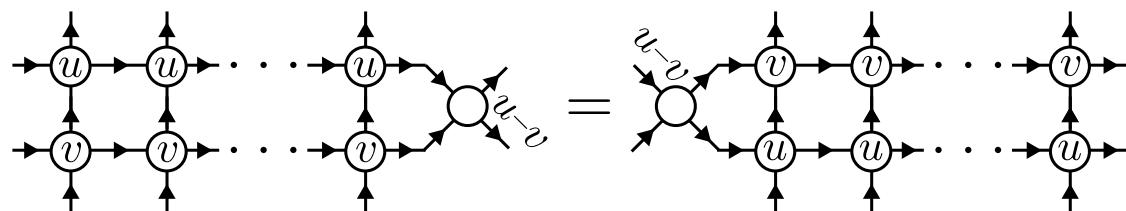
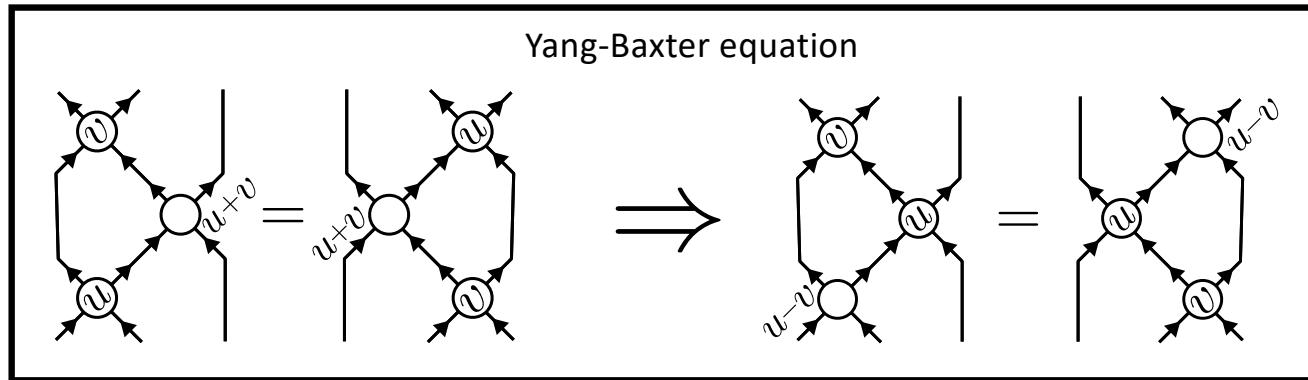
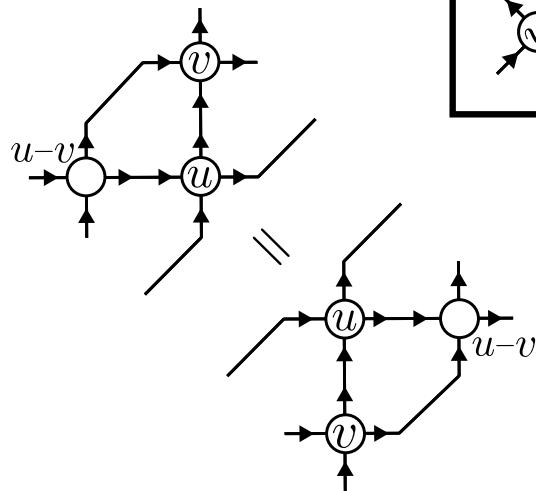
# Proof

Yang-Baxter  
⇒  
Conserved quantities



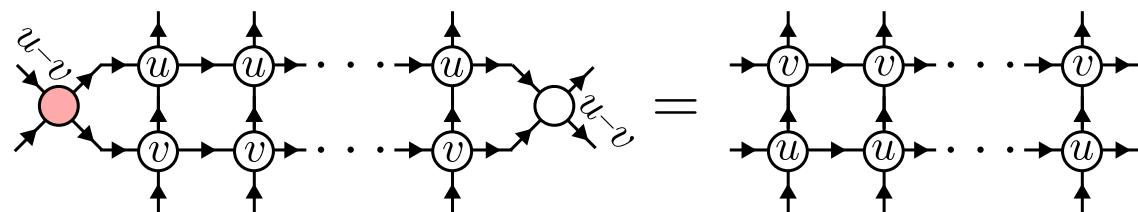
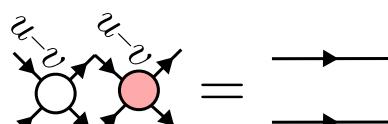
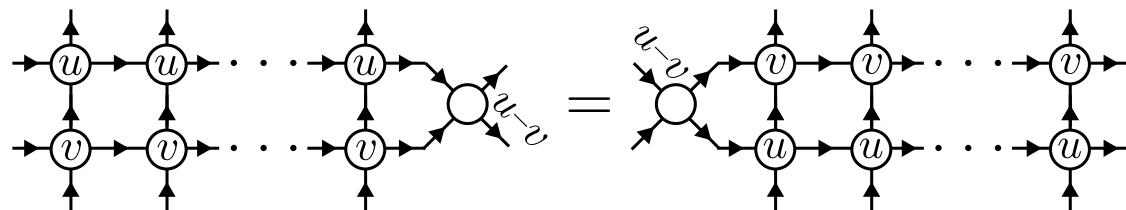
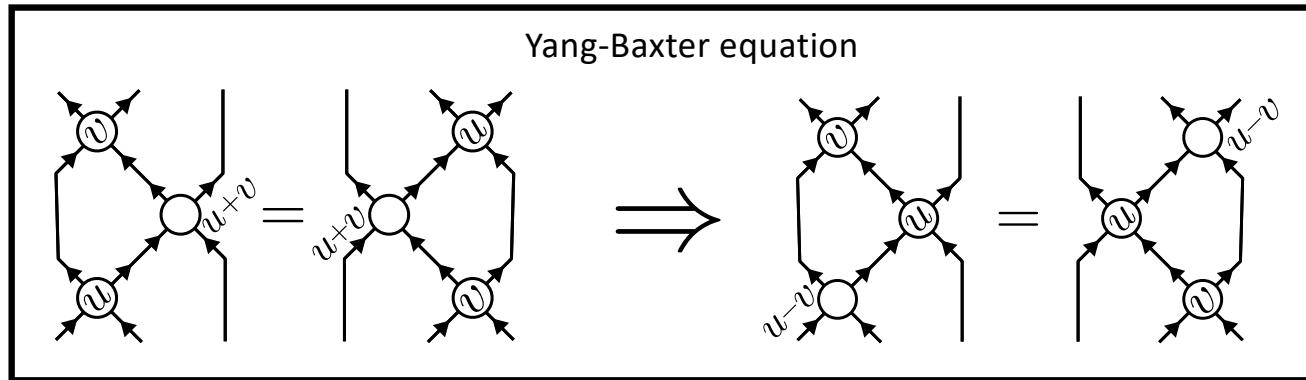
# Proof

Yang-Baxter  
⇒  
Conserved quantities



# Proof

Yang-Baxter  
 $\Rightarrow$   
 Conserved quantities

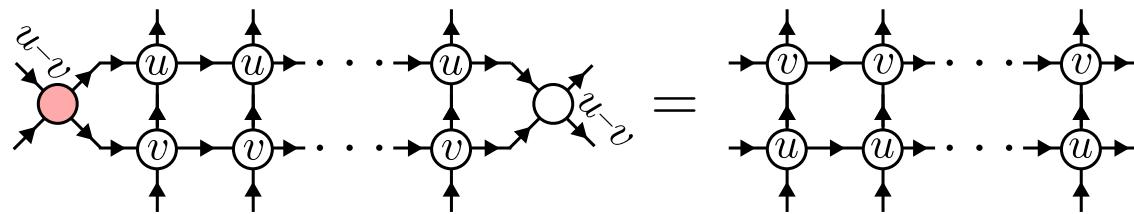


# Proof

Yang-Baxter

$\Rightarrow$

Conserved quantities

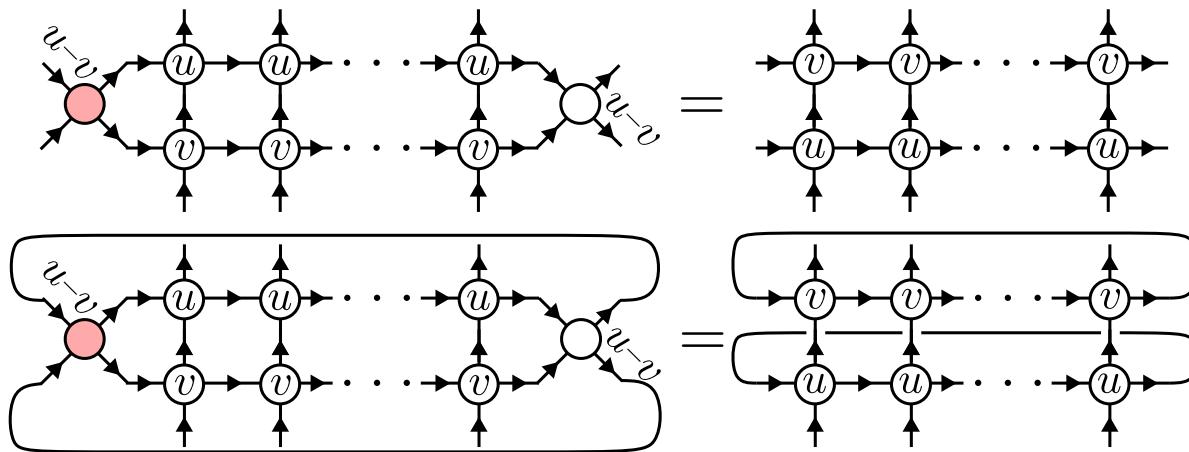


# Proof

Yang-Baxter



Conserved quantities

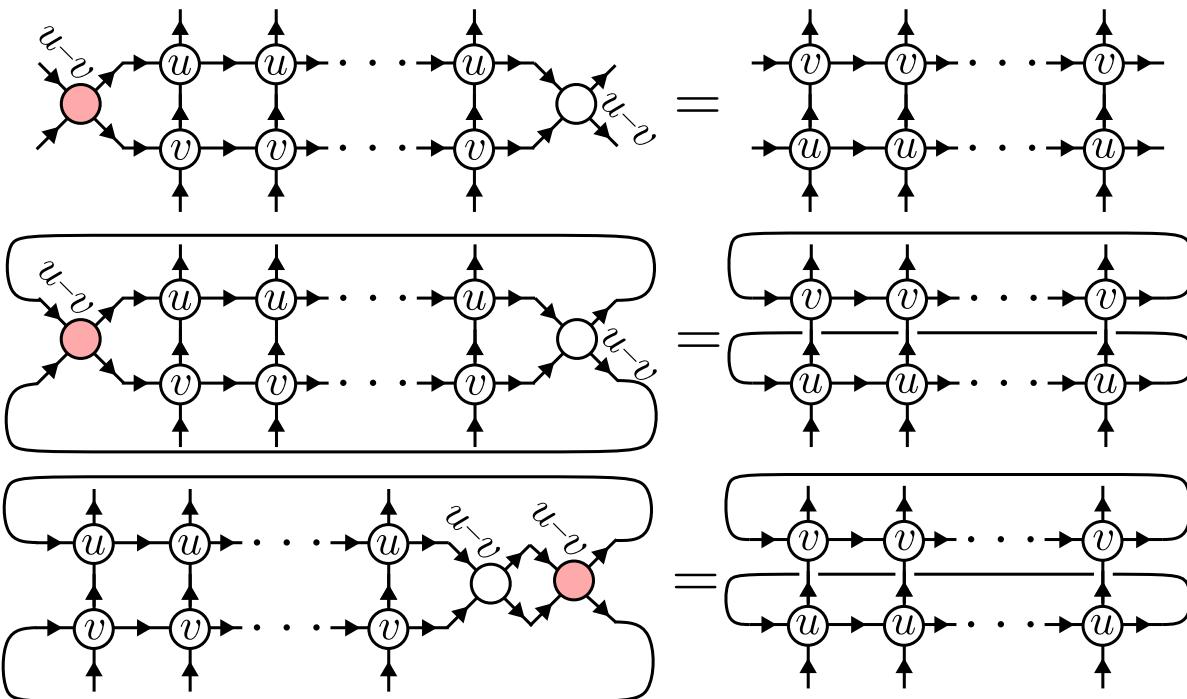
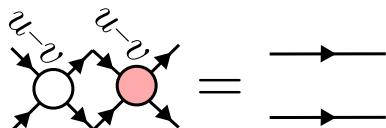


# Proof

Yang-Baxter

$\Rightarrow$

Conserved quantities

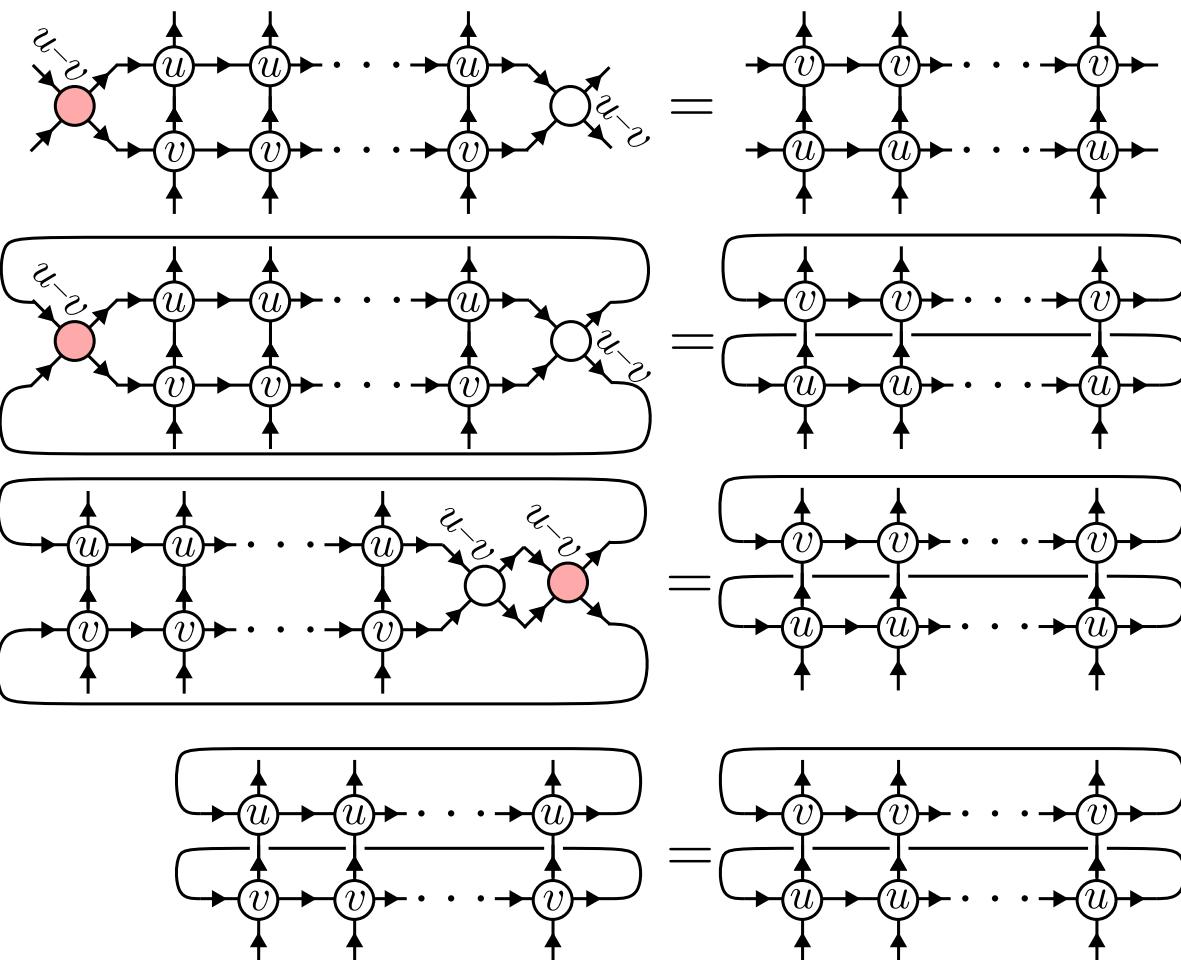
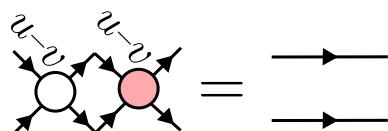


# Proof

Yang-Baxter

$\Rightarrow$

Conserved quantities

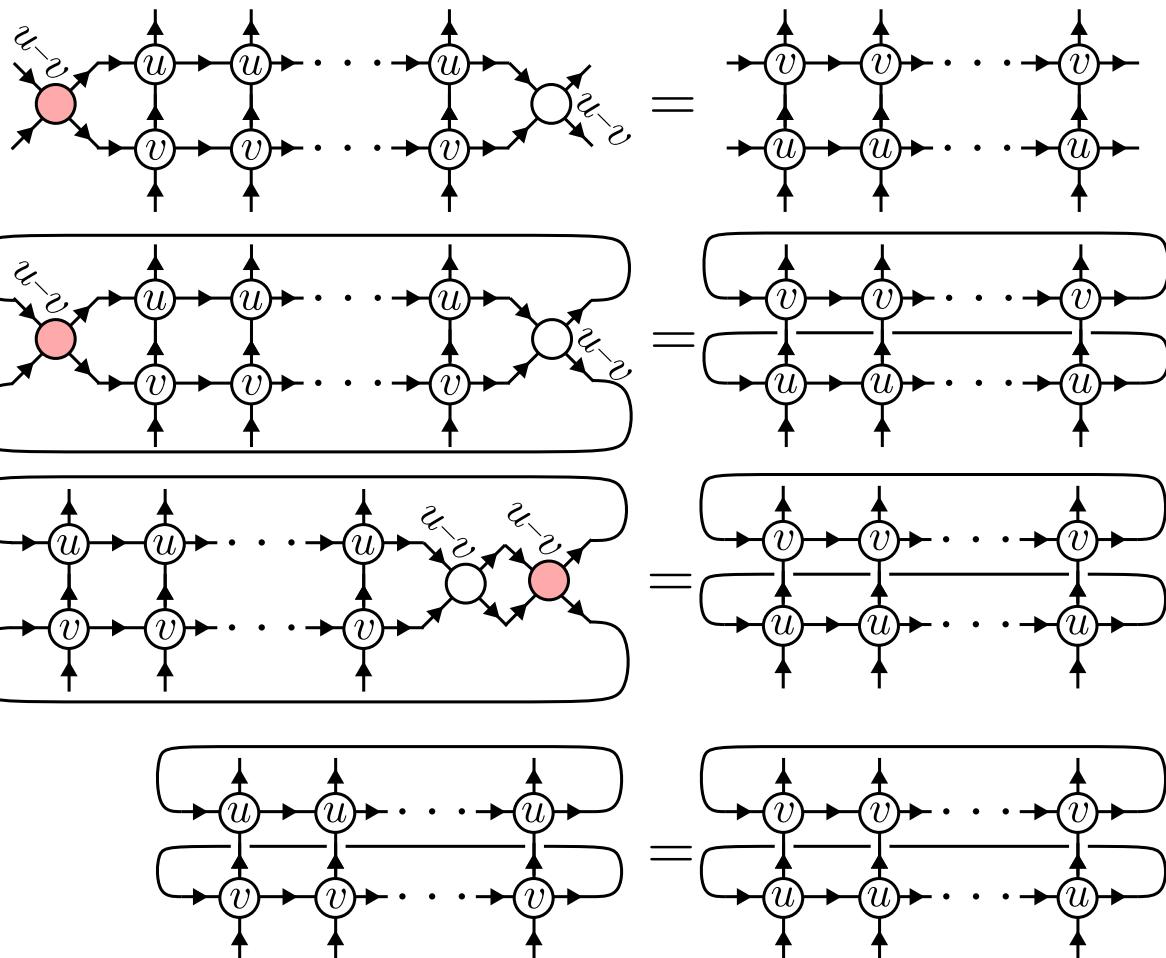
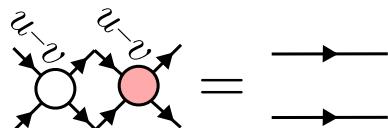


# Proof

Yang-Baxter

$\Rightarrow$

Conserved quantities

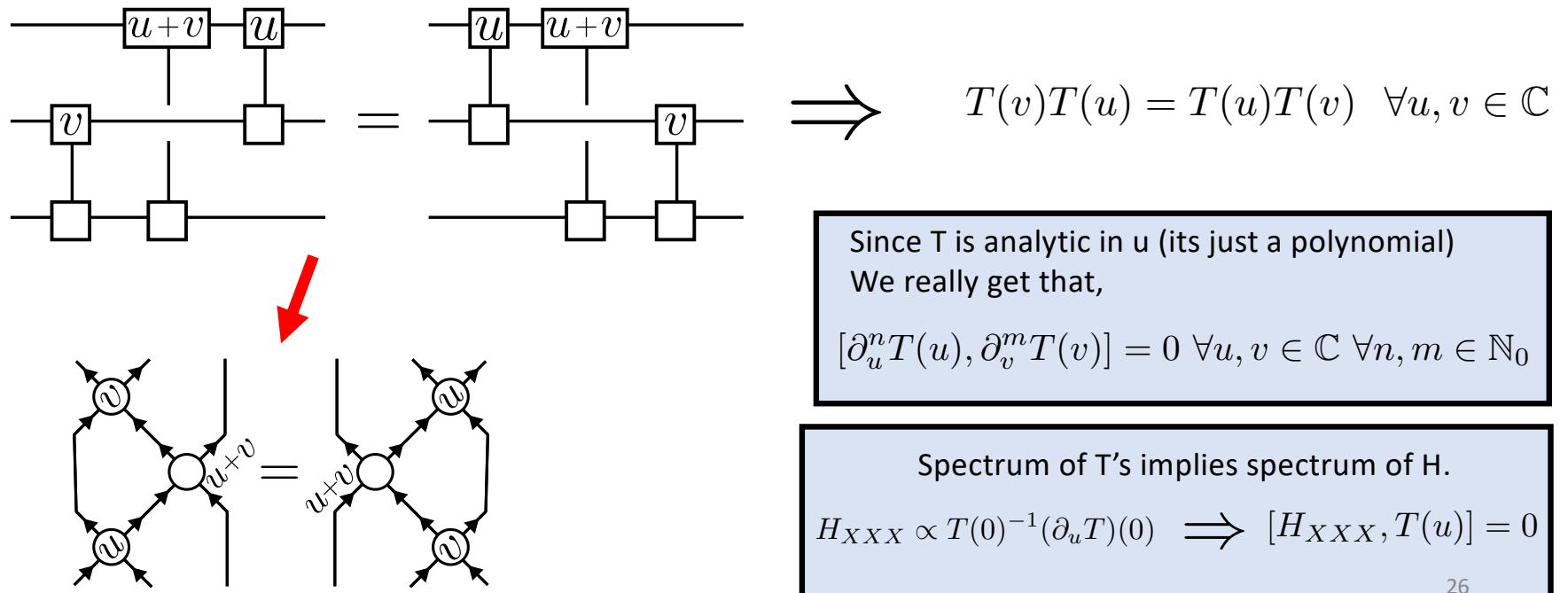


$$T(u)T(v) = T(v)T(u) \quad \forall u, v \in \mathbb{C} \quad \square$$

# Conserved quantities and Yang-Baxter

- R-matrix satisfies Yang-Baxter equation  $\Rightarrow$  Many conserved quantities

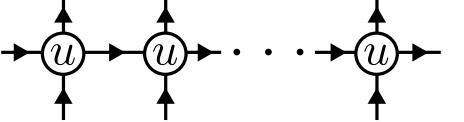
$$R_{1,2}(u)R_{1,3}(u+v)R_{2,3}(v) = R_{2,3}(v)R_{1,3}(u+v)R_{1,2}(u)$$



# The Bethe Ansatz

Basic idea: Use monodromy matrices to construct spectrum of the transfer operator.

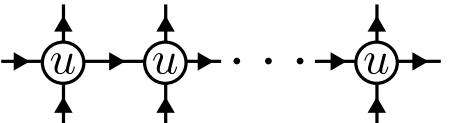
Monodromy matrix is a  $2 \times 2$  matrix of  $2^L \times 2^L$  matrices


$$\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \quad A(u), B(u), C(u), D(u) : \mathbb{C}^{2^L} \rightarrow \mathbb{C}^{2^L}$$

# The Bethe Ansatz

Basic idea: Use monodromy matrices to construct spectrum of the transfer operator.

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$$= \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \quad A(u), B(u), C(u), D(u) : \mathbb{C}^{2^L} \rightarrow \mathbb{C}^{2^L}$$

$$T(u) = \text{Tr} \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = A(u) + D(u)$$

Transfer matrix: Gives us the Hamiltonian via

$$H_{XXX} = -iJ\partial_u \log T(u) \Big|_{u=0} + JL$$

$$|p_1, \dots, p_N\rangle \propto \prod_{j=1}^N B(u_j) |\uparrow\rangle^{\otimes L}$$

The Bethe ansatz: Eigenstates of  $H_{XXX}$  understood as  $N$  excitations with momenta  $p_1, \dots, p_N$ .

Technique: Y-B eqn.  
implies an algebra  
(commutators!)

$$T(u)|u_1, \dots, u_N\rangle = (A(u) + D(u)) \prod_{j=1}^N B(u_j) |\uparrow\rangle^{\otimes L}$$

Fix the  $u$ 's such that this is zero

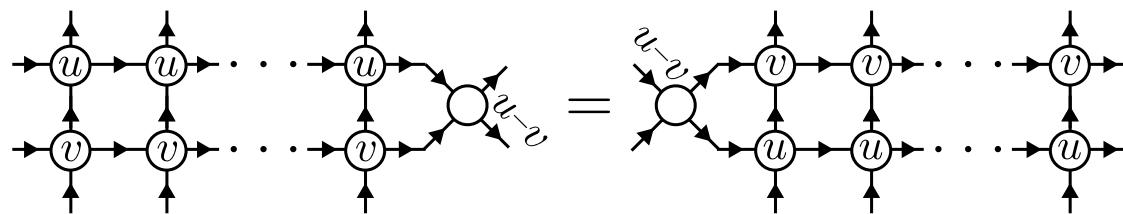
$$= \tau(u, u_1, \dots, u_n) \prod_{j=1}^N B(u_j) |\uparrow\rangle^{\otimes L} + |\text{stuff}\rangle$$

# Zamolodchikov-Faddeev Algebra

- Yang-Baxter equation implies algebra on monodromy matrices

Monodromy matrix is a  $2 \times 2$  matrix of  $2^L \times 2^L$  matrices

$$\begin{array}{c} \text{Diagram showing a chain of } u \text{-circles connected by horizontal arrows, with vertical arrows pointing up and down from each circle.} \\ = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \quad A(u), B(u), C(u), D(u) : \mathbb{C}^{2^L} \rightarrow \mathbb{C}^{2^L} \end{array}$$

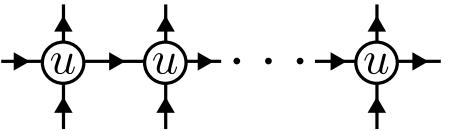


$$\begin{pmatrix} u - v + i & 0 & 0 & 0 \\ 0 & i & u - v & 0 \\ 0 & u - v & i & 0 \\ 0 & 0 & 0 & u - v + i \end{pmatrix} \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \otimes \begin{pmatrix} A(v) & B(v) \\ C(v) & D(v) \end{pmatrix} = \begin{pmatrix} A(v) & B(v) \\ C(v) & D(v) \end{pmatrix} \otimes \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \begin{pmatrix} u - v + i & 0 & 0 & 0 \\ 0 & i & u - v & 0 \\ 0 & u - v & i & 0 \\ 0 & 0 & 0 & u - v + i \end{pmatrix}$$

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- Yang-Baxter equation implies algebra on monodromy matrices

Monodromy matrix is a  $2 \times 2$  matrix of  $2^L \times 2^L$  matrices


$$\begin{array}{c} \xrightarrow{\hspace{1cm}} \\ \circlearrowleft u \circlearrowright \\ \xrightarrow{\hspace{1cm}} \end{array} \quad \dots \quad \begin{array}{c} \xrightarrow{\hspace{1cm}} \\ \circlearrowleft u \circlearrowright \\ \xrightarrow{\hspace{1cm}} \end{array} = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \quad A(u), B(u), C(u), D(u) : \mathbb{C}^{2^L} \rightarrow \mathbb{C}^{2^L}$$

$$A(v)B(u) = \left(1 + \frac{i}{u-v}\right) B(u)A(v) - \frac{i}{u-v} B(v)A(u)$$
$$D(v)B(u) = \left(1 - \frac{i}{u-v}\right) B(u)D(v) + \frac{i}{u-v} B(v)D(u)$$
$$[B(u), B(v)] = 0$$

# Exact eigenstates and energies

## Z-F algebra

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## Approach to the problem

$$\begin{aligned} T(u)|u_1, \dots, u_N\rangle &= (A(u) + D(u)) \prod_{j=1}^N B(u_j) |\uparrow\rangle^{\otimes L} \\ &= \tau(u, u_1, \dots, u_n) \prod_{j=1}^N B(u_j) |\uparrow\rangle^{\otimes L} + |\text{stuff}\rangle \end{aligned}$$

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## Result of taking N commutators

$$A(u) \prod_{j=1}^N B(u_j) |\uparrow\rangle^{\otimes L} = \left( \prod_{j=1}^N 1 + \frac{i}{u_j - u} \right) \left( \prod_{j=1}^N B(u_j) \right) A(u) |\uparrow\rangle^{\otimes L} + \sum_{j=1}^N \frac{-i}{u_j - u} \left( \prod_{k \neq j} 1 + \frac{i}{u_k - u_j} \right) B(u) \left( \prod_{k \neq j} B(u_k) \right) A(u_j) |\uparrow\rangle^{\otimes L}$$

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## Fact

$$A(u) |\uparrow\rangle^{\otimes L} = (u + i)^L |\uparrow\rangle^{\otimes L}$$

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$$\begin{aligned} &+ \sum_{j=1}^N \frac{-i}{u_j - u} \left( \prod_{k \neq j} 1 + \frac{i}{u_k - u_j} \right) B(u) \left( \prod_{k \neq j} B(u_k) \right) A(u_j) |\uparrow\rangle^{\otimes L} \\ &+ \sum_{j=1}^N \frac{i}{u_j - u} \left( \prod_{k \neq j} 1 + \frac{-i}{u_k - u_j} \right) B(u) \left( \prod_{k \neq j} B(u_k) \right) D(u_j) |\uparrow\rangle^{\otimes L} \end{aligned}$$

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Unwanted terms cancel if...

$$\left( \frac{u_j + i}{u_j} \right)^L \prod_{k \neq j} \left( \frac{u_k - u_j + i}{u_k - u_j - i} \right) = 1$$

# Exact eigenstates and energies

Eigenvalues!

$$T(u)|u_1, \dots, u_N\rangle = \left[ (u+i)^L \prod_{j=1}^N \left( \frac{u_j - u + i}{u_j - u} \right) + u^L \prod_{j=1}^N \left( \frac{u_j - u - i}{u_j - u} \right) \right] |u_1, \dots, u_N\rangle$$

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It is convenient to redefine

$$u'_j = u_j + i/2$$

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Energies!

$$E_{XXX}(u'_1, \dots, u'_N) = -iJ(\partial_u \log \tau)(0) + JL = \sum_{j=1}^N \frac{J}{u'^2_j + 1/4}$$

# Exact eigenstates and energies

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Yet another redefinition

$$\frac{u'_j + i/2}{u'_j - i/2} = e^{ip_j} \Rightarrow u'_j = \frac{1}{2} \cot(p_j/2)$$

Energies!

$$E_{XXX}(u'_1, \dots, u'_N) = -iJ(\partial_u \log \tau)(0) + JL = \sum_{j=1}^N \frac{J}{u'^2_j + 1/4}$$

# Exact eigenstates and energies

Energies:

$$E_{XXX}(p_1, \dots, p_N) = 2J \sum_{j=1}^N (1 - \cos(p_j))$$

Allowed momenta:

$$e^{ip_j L} \prod_{k \neq j} \frac{\cot(p_j/2) - \cot(p_k/2) + 2i}{\cot(p_j/2) - \cot(p_k/2) - 2i} = 1$$

$S(p_j, p_k)$  S-matrix

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$S(p_j, p_k)$  S-matrix

What are these particles?

N=1:  $B(p_1)|\uparrow\rangle^{\otimes L} \propto \frac{1}{\sqrt{L}} \sum_{j=1}^L e^{-ip_1 j} \sigma_j^- |\uparrow\rangle^{\otimes L}$

$$e^{ip_1 L} = 1 \Rightarrow p_1 = \frac{2\pi n}{L}$$

One spin flip propagating with momentum  $p_1$ .  
Called a “magnon”.

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N=2:

$$B(p_1)B(p_2)|\uparrow\rangle^{\otimes L} \propto \sum_{j < k} \left( e^{-i(p_1 j + p_2 k)} + S(p_1, p_2) e^{-i(p_2 j + p_1 k)} \right) \sigma_j^- \sigma_k^- |\uparrow\rangle^{\otimes L}$$

$$e^{ip_1 L} S(p_1, p_2) = e^{ip_2 L} S(p_2, p_1) = 1$$

Two spin flips propagating with momentum  $p_1$  and  $p_2$ .  
Relative phase is indicative of a scattering event.

# Other models

- XXZ model

$$R(u-v) = \frac{1}{\sinh(u-v)} \begin{pmatrix} \sinh(u-v+\eta) & 0 & 0 & 0 \\ 0 & \sinh(u-v) & \sinh(\eta) & 0 \\ 0 & \sinh(\eta) & \sinh(u-v) & 0 \\ 0 & 0 & 0 & \sinh(u-v+\eta) \end{pmatrix}$$

$$H_{XXZ} = 2 \sinh(\eta) \partial_u \log T(u) \Big|_{u=\eta/2} - N\Delta = \sum_{j=1}^L X_j X_{j+1} + Y_j Y_{j+1} + \Delta Z_j Z_{j+1}$$

$$\Delta = \cosh(\eta) \quad \text{Only works for } \Delta \geq 1$$

- Lieb-Linger model (nonlinear Schrodinger eqn.)[arXiv:1804.07350]
- Inhomogenous XXX model [arXiv:1804.07350]

Thanks for listening!

