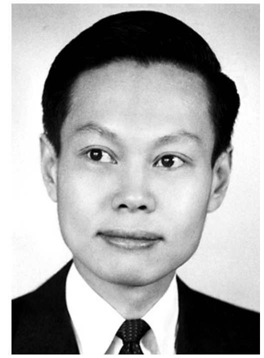
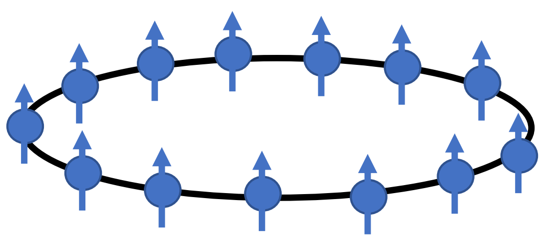
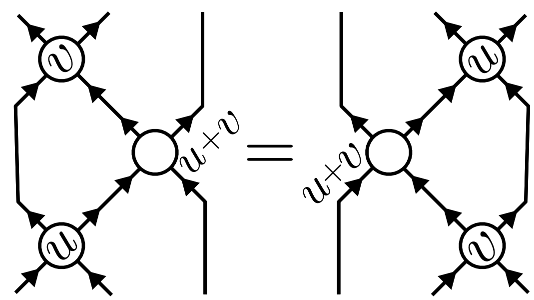
   

Quantum Integrability

Part 2: Algebraic Bethe Ansatz

Following [VieiraCostelloCourse] (Lectures 1-3) and [arXiv:1804.07350]

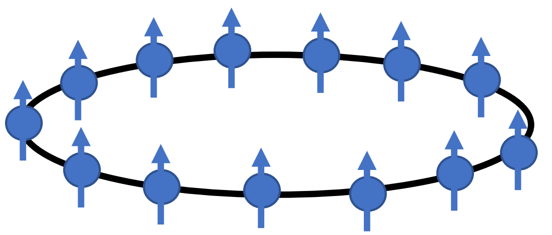




Spin-1/2 models of magnetism

$$H_{XYZ} = -\left(J_X \sum_{j=1}^{L} X_j X_{j+1} + J_Y \sum_{j=1}^{L} Y_j Y_{j+1} + J_Z \sum_{j=1}^{L} Z_j Z_{j+1} \right)$$

$$J_X = J_Y \Rightarrow H_{XXZ}$$ Integrable family of Hamiltonians!



L spins on a ring

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z|\uparrow\rangle = (+1)|\uparrow\rangle$$

$$X_j = I \otimes I \otimes \cdots \otimes X \otimes \cdots \otimes I$$

Spin-1/2 models of magnetism

$$H_{XYZ} = -\left(J_X \sum_{j=1}^{L} X_j X_{j+1} + J_Y \sum_{j=1}^{L} Y_j Y_{j+1} + J_Z \sum_{j=1}^{L} Z_j Z_{j+1} \right)$$

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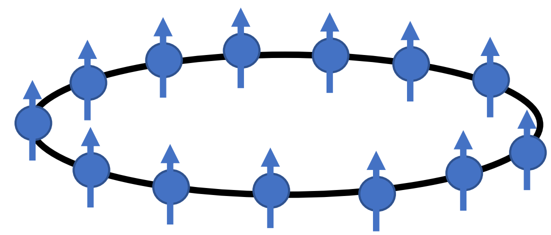

L-spins on a ring

Special case XXX-model (this lecture)

$$J_X = J_Y = J_Z \Rightarrow H_{XXX}$$

$$\boldsymbol{\sigma}_j = (X_j, Y_j, Z_j)$$

$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_{L+1}$ PBC

$$H_{XXX} = -J \sum_{j=1}^{L} \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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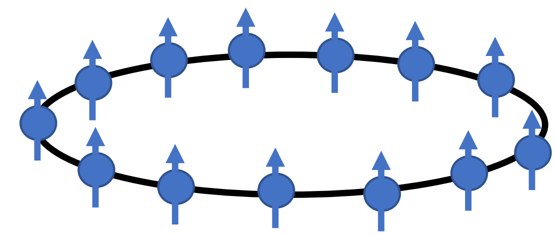

L-spins on a ring

<u>Special case XXX-model (this lecture)</u>

$J_X = J_Y = J_Z \Rightarrow H_{XXX}$

$\boldsymbol{\sigma}_j = (X_j, Y_j, Z_j)$

$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_{L+1}$ PBC

$$H_{XXX} = -J \sum_{j=1}^{L} \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1}$$

Recall:

$$\mathrm{SWAP}_{1,2} = \frac{1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{2}$$

<u>Rescale ground state energy to zero</u>

$$H_{XXX} = -\frac{J}{2} \sum_{j=1}^{L} (\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1} - 1) = -J \sum_{j=1}^{L} (\mathrm{SWAP}_{j,j+1} - 1)$$

$$H_{XXX} |\uparrow\rangle^{\otimes L} = 0$$

Ground state space is really symmetric subspace

A clever way to write the XXX-Hamiltonian

$$H_{XXX} = -iJ\partial_u \log T(u)\Big|_{u=0} + JL$$

$$T(u) = \mathrm{Tr}_0 \left(\prod_{j=1}^{L} uI_{0,j} + i\mathrm{SWAP}_{0,j} \right)$$

Recall:

$$\partial_u \log T(u)\Big|_{u=0} = T(0)^{-1}(\partial_u T)(0)$$

A clever way to write the XXX-Hamiltonian

$$H_{XXX} = -iJ\partial_u \log T(u)\Big|_{u=0} + JL$$

$$\underbrace{T(u)}_{\text{Transfer matrix}} = \mathrm{Tr}_0 \left(\prod_{j=1}^{L} \underbrace{uI_{0,j} + i\mathrm{SWAP}_{0,j}}_{\text{R-matrix } R_{0,j}(u)} \right)$$

Recall:

$$\partial_u \log T(u)\Big|_{u=0} = T(0)^{-1}(\partial_u T)(0)$$

A clever way to write the XXX-Hamiltonian

$$H_{XXX} = -iJ\partial_u \log T(u)\Big|_{u=0} + JL$$

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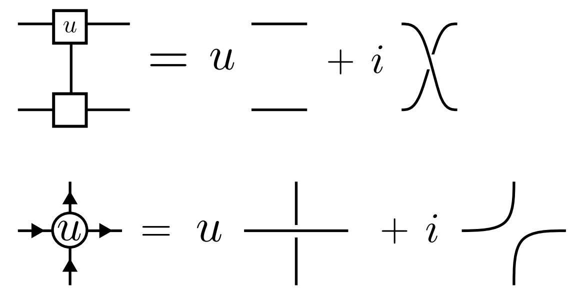
Transfer matrix

R-matrix $R_{0,j}(u)$

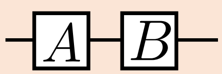
Recall:

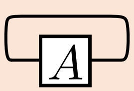
$$\partial_u \log T(u)\Big|_{u=0} = T(0)^{-1}(\partial_u T)(0)$$

$R_{0,j}(u) =$

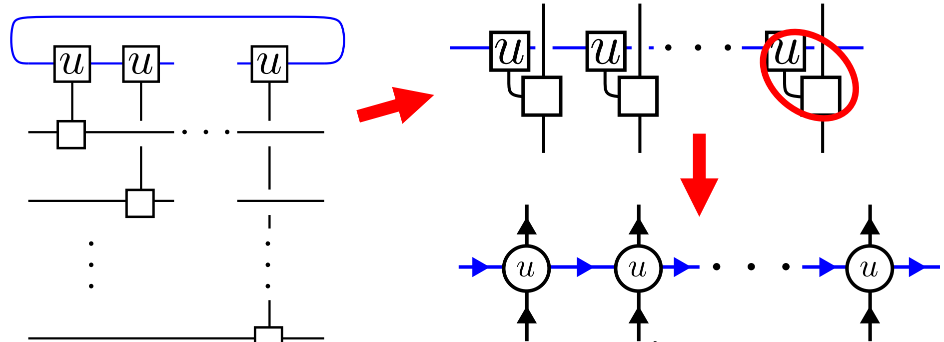


Quick note on diagramatics:

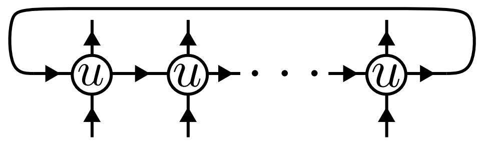
$BA|\psi\rangle = |\psi\rangle$ 

$\mathrm{Tr}(A) =$ 

$T(u) =$

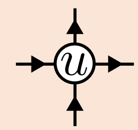


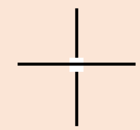
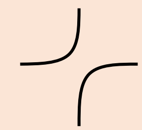
Proof

$T(u) =$ 

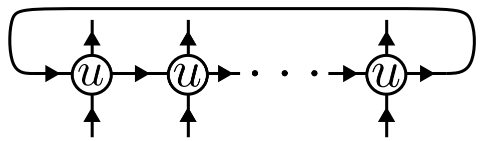
$$H_{XXX} = -iJ\partial_u \log T(u)\Big|_{u=0} + JL$$

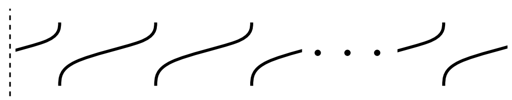
$$T(u) = \mathrm{Tr}_0\left(\prod_{j=1}^{L} uI_{0,j} + i\mathrm{SWAP}_{0,j}\right)$$



$= u$  $+\ i$ 

Proof

$T(u) =$ 

$T(0) \; = \; i^L$ 

$$H_{XXX} = -iJ\partial_u \log T(u)\Big|_{u=0} + JL$$

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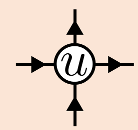
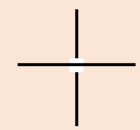
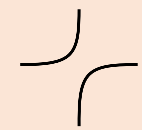
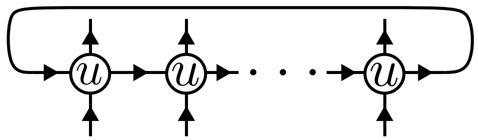
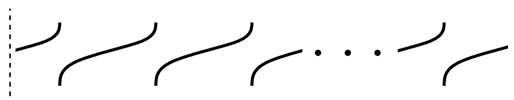
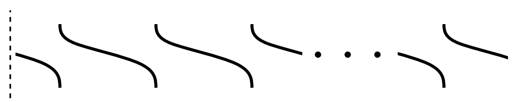
 $= \; u$  $+ \; i$ 

Proof

$T(u) =$ 

$T(0) \;=\; i^L$ 

$T(0)^{-1} \;=\; i^{-L}$ 

$$H_{XXX} = -iJ\partial_u \log T(u)\Big|_{u=0} + JL$$

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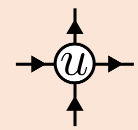
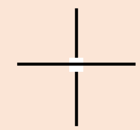
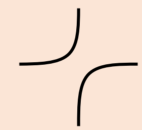
 $= \; u$  $+ \; i$ 

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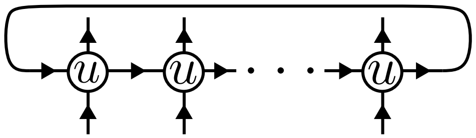
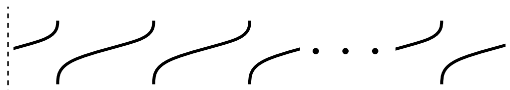
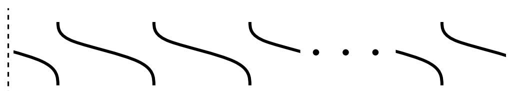
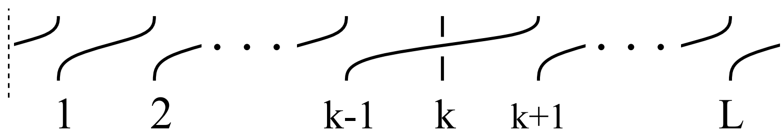
$T(u) =$ 

$T(0) \;=\; i^{L}$ 

$T(0)^{-1} \;=\; i^{-L}$ 

$\partial_u T(u)\Big|_{u=0} = i^{L-1} \sum_{k=1}^{L}$ 

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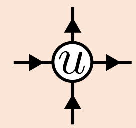
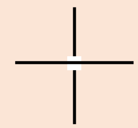
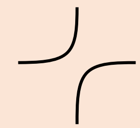
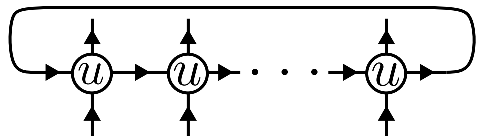


$= u$  $+\ i$ 

Proof

$T(u) =$ 

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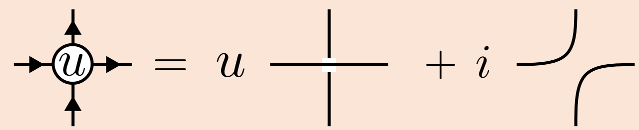
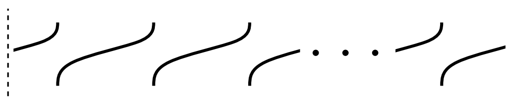
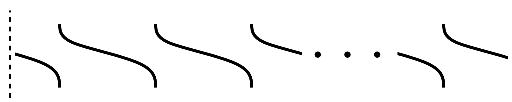
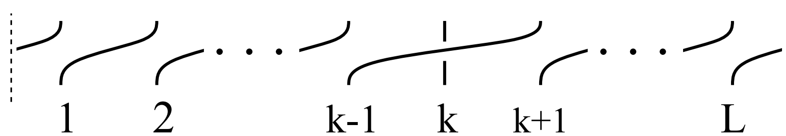
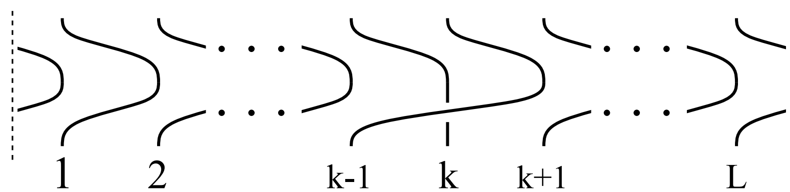


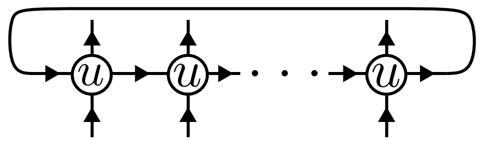
$T(0) \;=\; i^L$ 

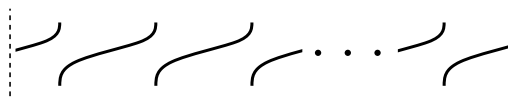
$T(0)^{-1} \;=\; i^{-L}$ 

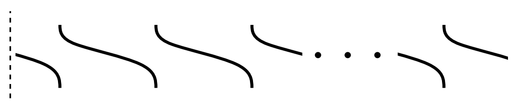
$\partial_u T(u)\Big|_{u=0} = i^{L-1} \sum_{k=1}^{L}$ 

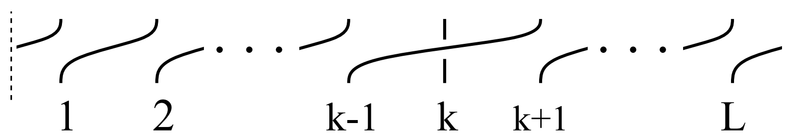
$-iJ\left(T(0)^{-1}\partial_u T(u)\Big|_{u=0}\right) \;=\; -J\sum_{k=1}^{L}$ 

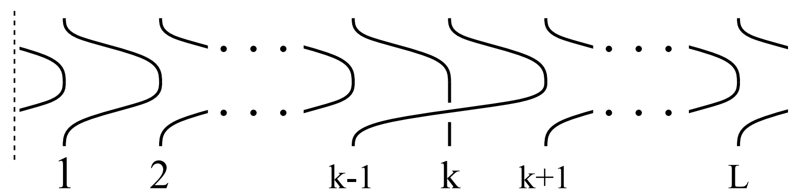
Proof

$T(u) =$ 

$T(0) \; = \; i^L$ 

$T(0)^{-1} \; = i^{-L}$ 

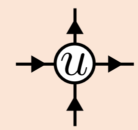
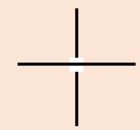
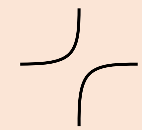
$\partial_u T(u)\Big|_{u=0} = i^{L-1} \sum_{k=1}^{L}$ 

$-iJ\left(T(0)^{-1}\partial_u T(u)\Big|_{u=0}\right) \; = \; -J \sum_{k=1}^{L}$ 

$$= -J \sum_{k=1}^{L} \mathrm{SWAP}_{k-1,k} = H_{XXX} - JL \quad \square$$

$$H_{XXX} = -iJ\partial_u \log T(u)\Big|_{u=0} + JL$$

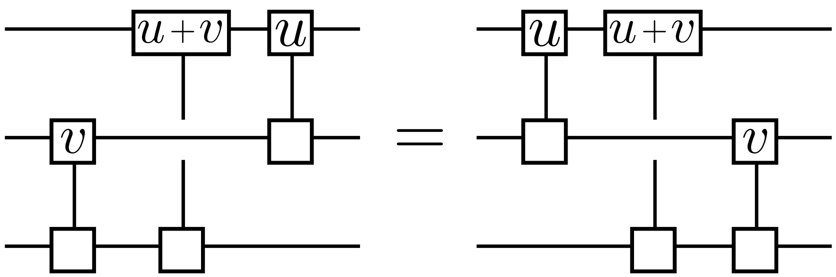
$$T(u) = \mathrm{Tr}_0 \left(\prod_{j=1}^{L} uI_{0,j} + i\mathrm{SWAP}_{0,j} \right)$$

 $= \; u$  $+ \; i$ 

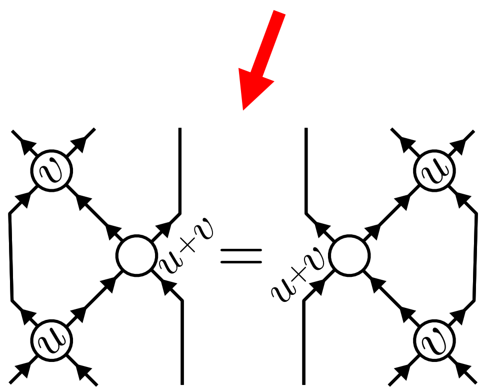
Conserved quantities and Yang-Baxter

- R-matrix satisfies Yang-Baxter equation ⇒ Many conserved quantities

$$R_{1,2}(u)R_{1,3}(u+v)R_{2,3}(v) = R_{2,3}(v)R_{1,3}(u+v)R_{1,2}(u)$$



$$\Longrightarrow \quad T(v)T(u) = T(u)T(v) \quad \forall u, v \in \mathbb{C}$$



Since T is analytic in u (its just a polynomial)
We really get that,

$$[\partial_u^n T(u), \partial_v^m T(v)] = 0 \ \forall u, v \in \mathbb{C} \ \forall n, m \in \mathbb{N}_0$$

Spectrum of T's implies spectrum of H.

$$H_{XXX} \propto T(0)^{-1}(\partial_u T)(0) \quad \Longrightarrow \quad [H_{XXX}, T(u)] = 0$$

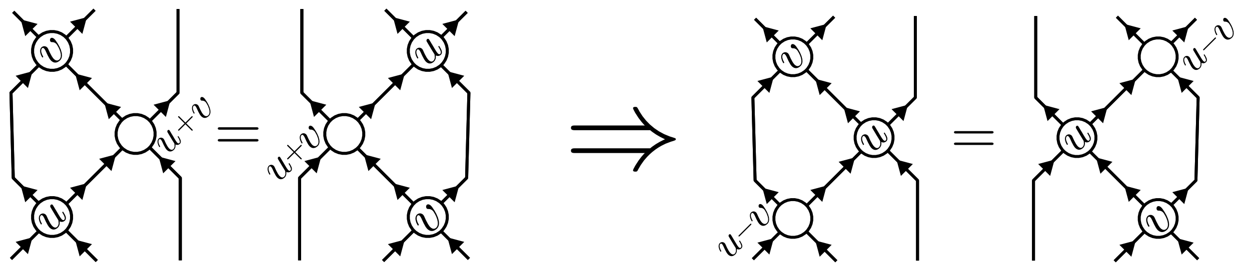
Proof

Yang-Baxter

⇒

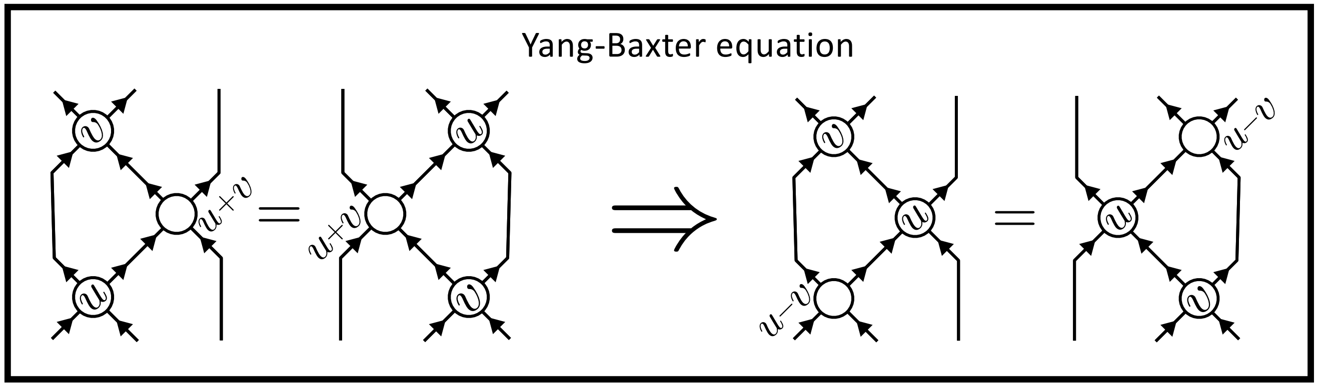
Conserved quantities

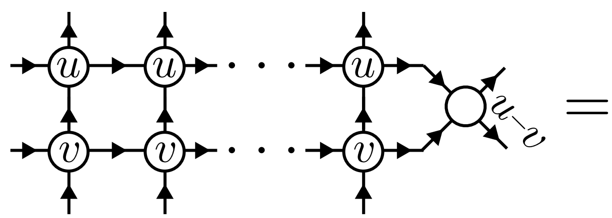
Yang-Baxter equation



Proof

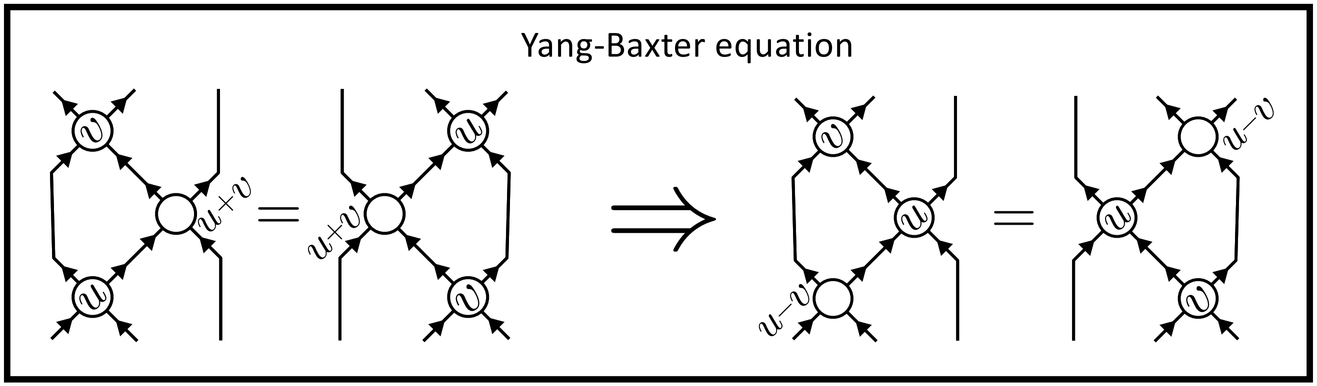
Yang-Baxter
⇒
Conserved quantities

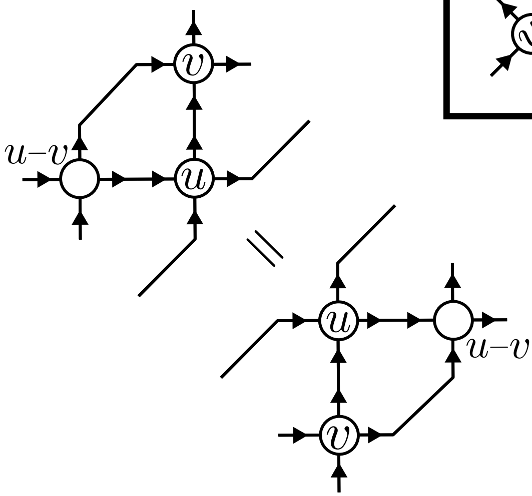


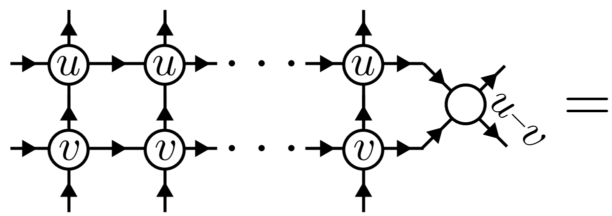


Proof

Yang-Baxter
⇒
Conserved quantities



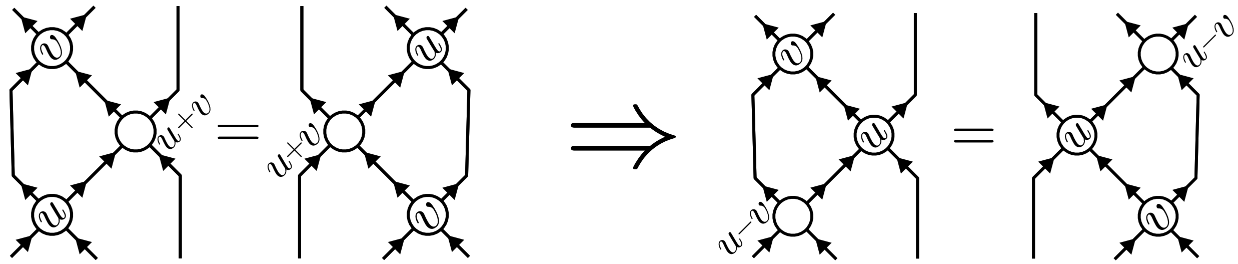


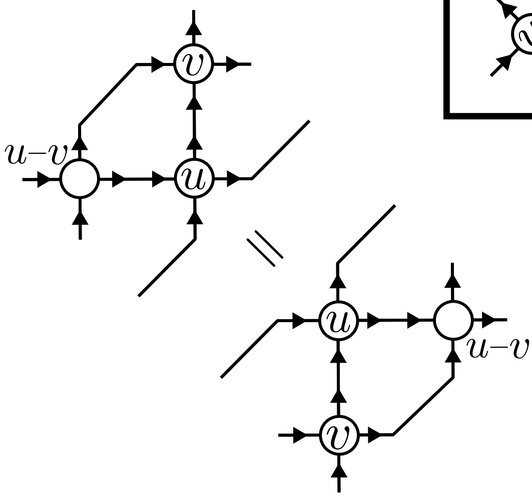


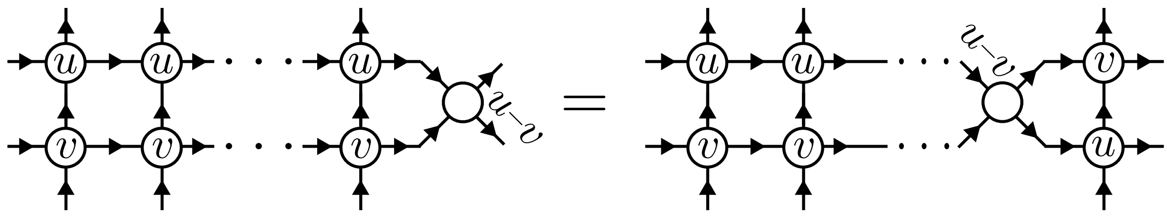
Proof

Yang-Baxter
⇒
Conserved quantities

Yang-Baxter equation







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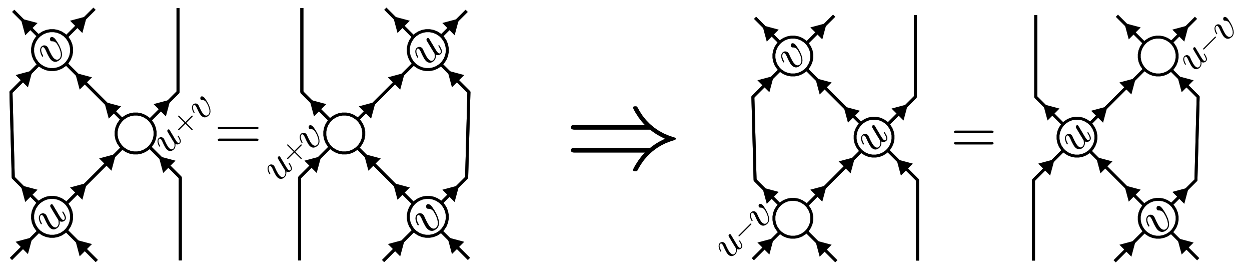

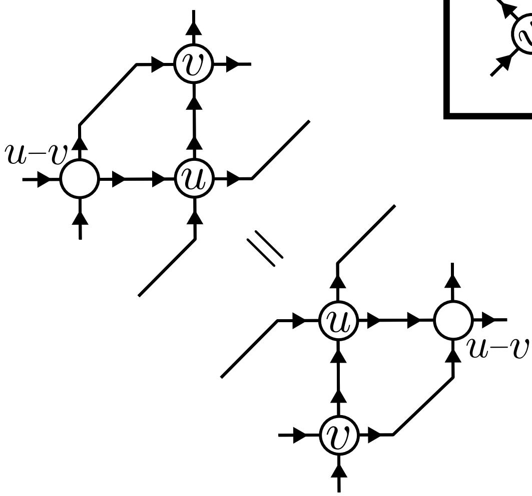

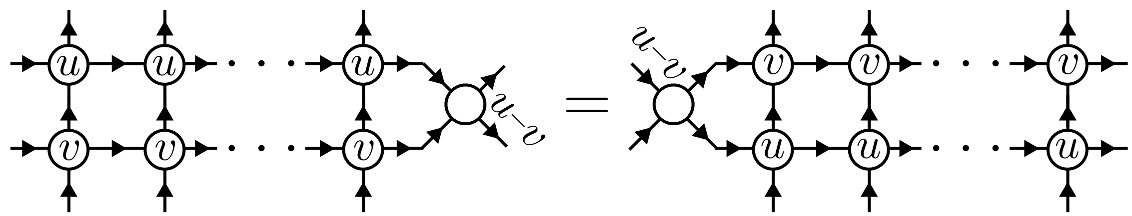

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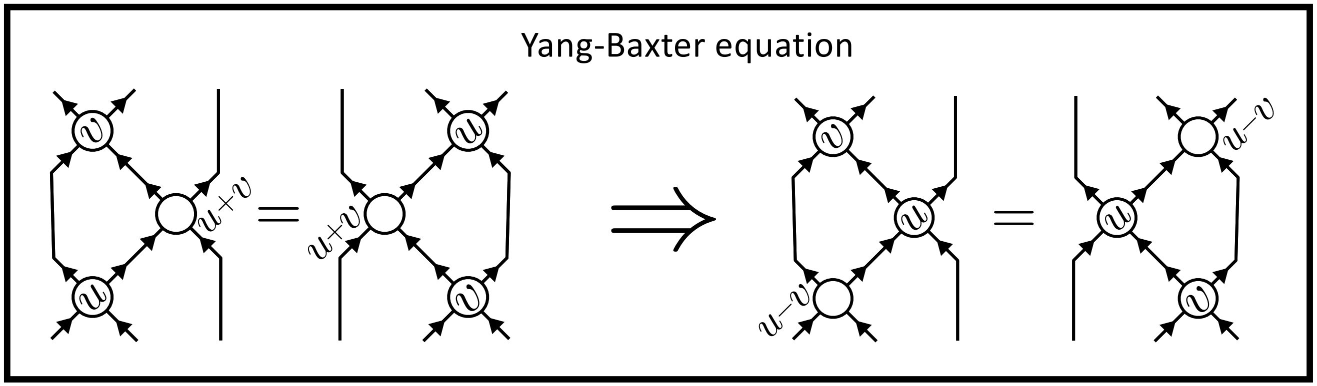

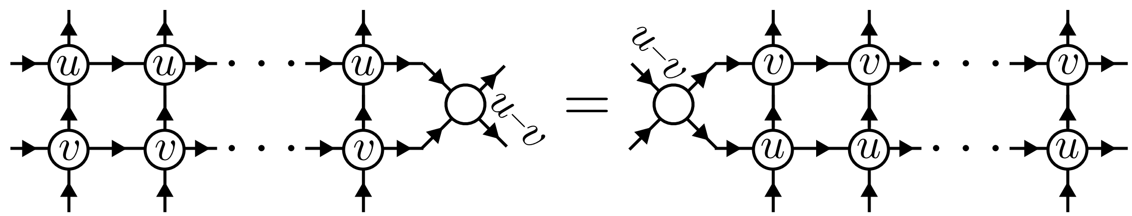

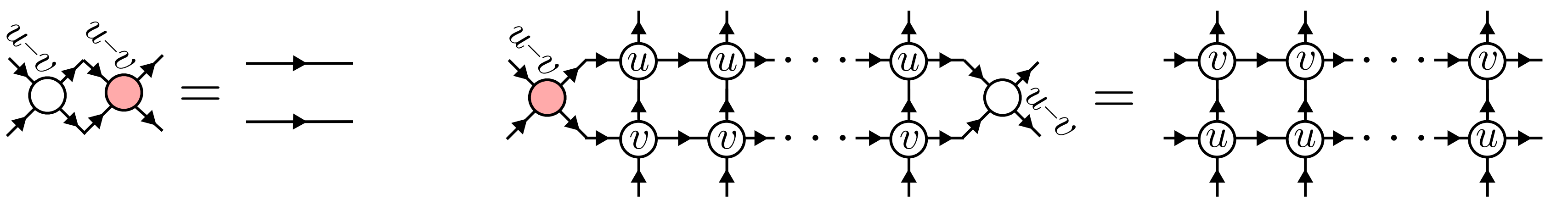

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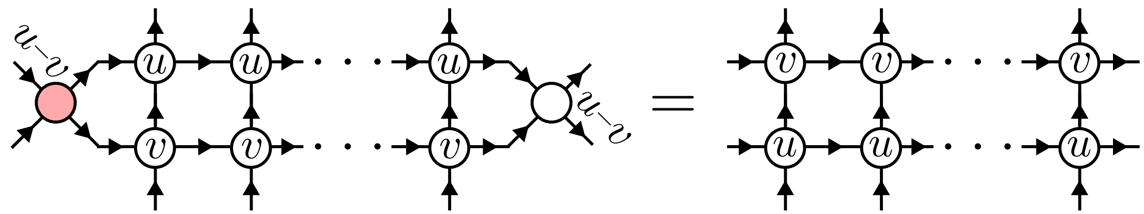

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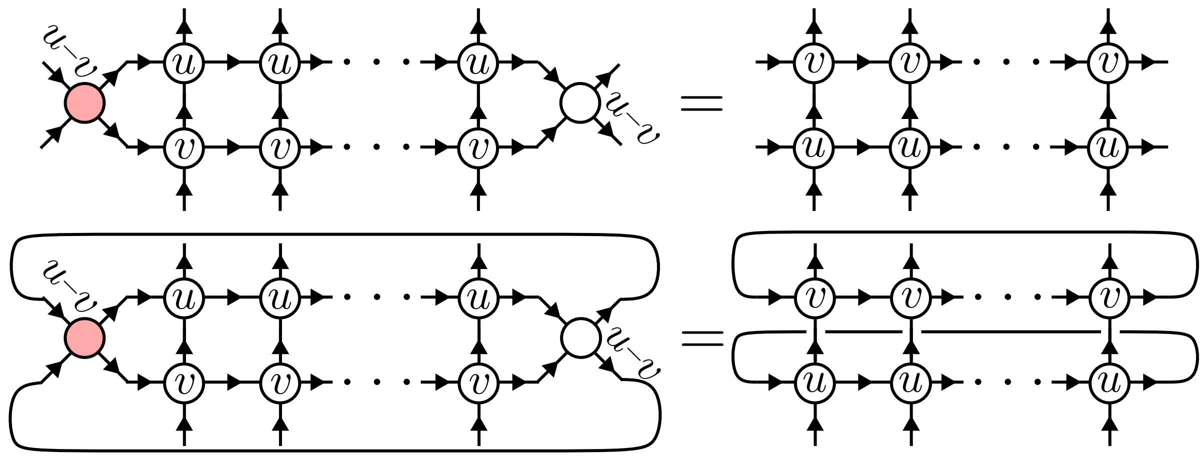

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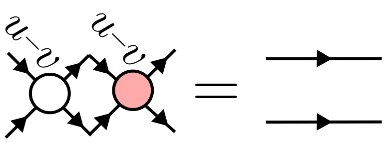

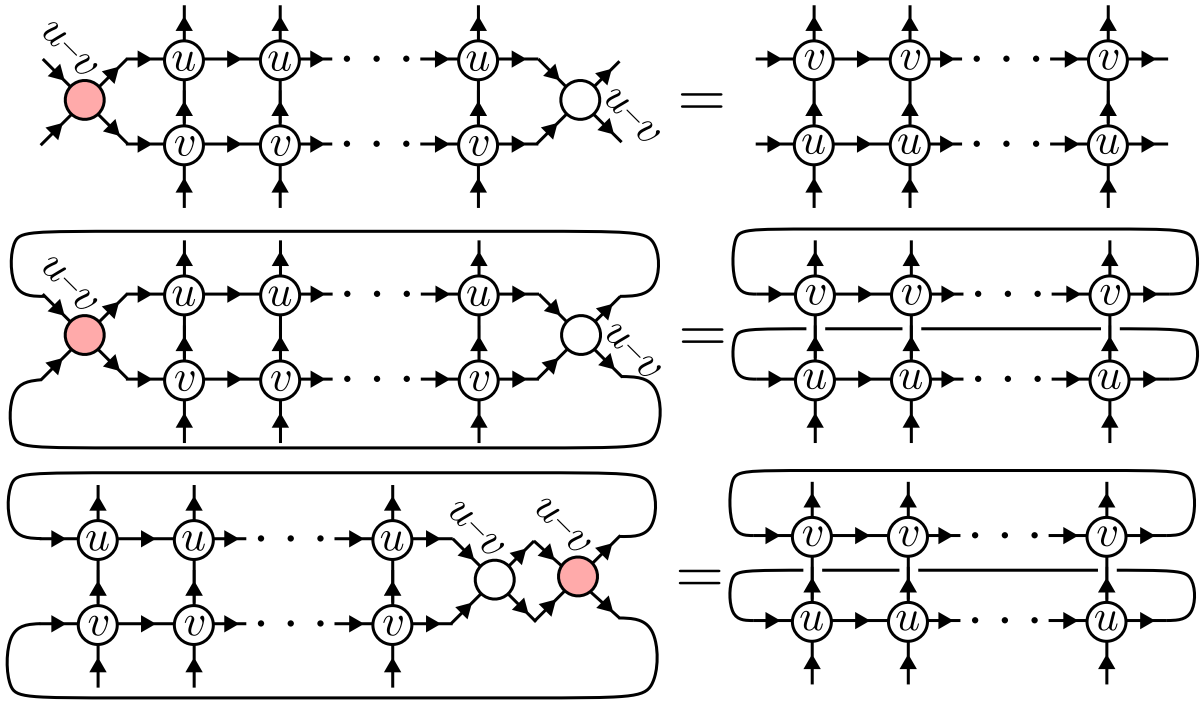

Proof

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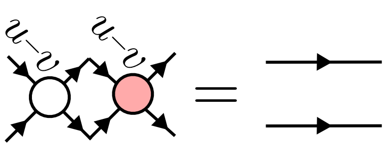

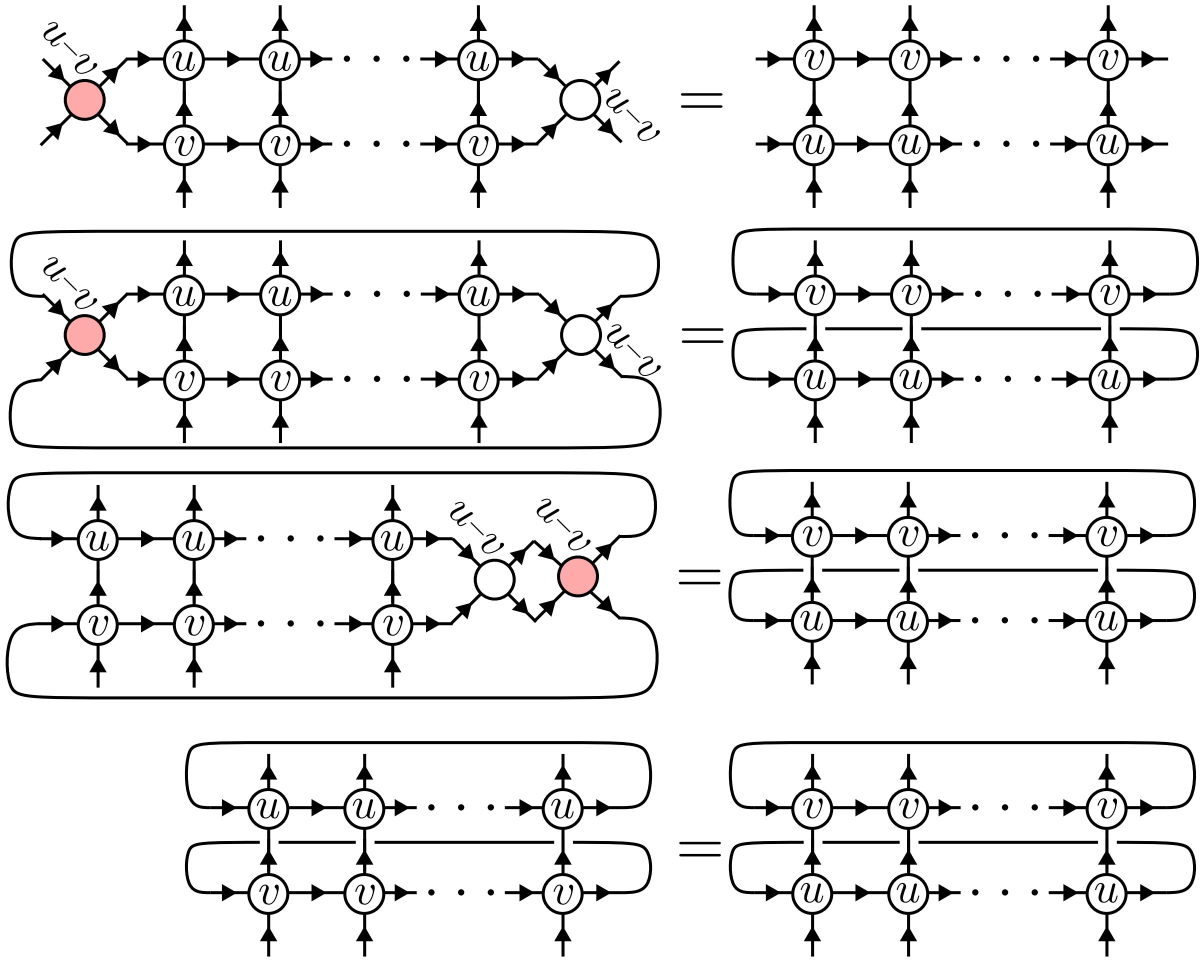

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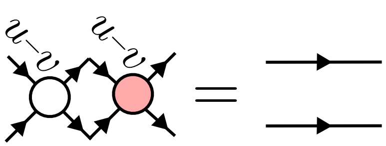
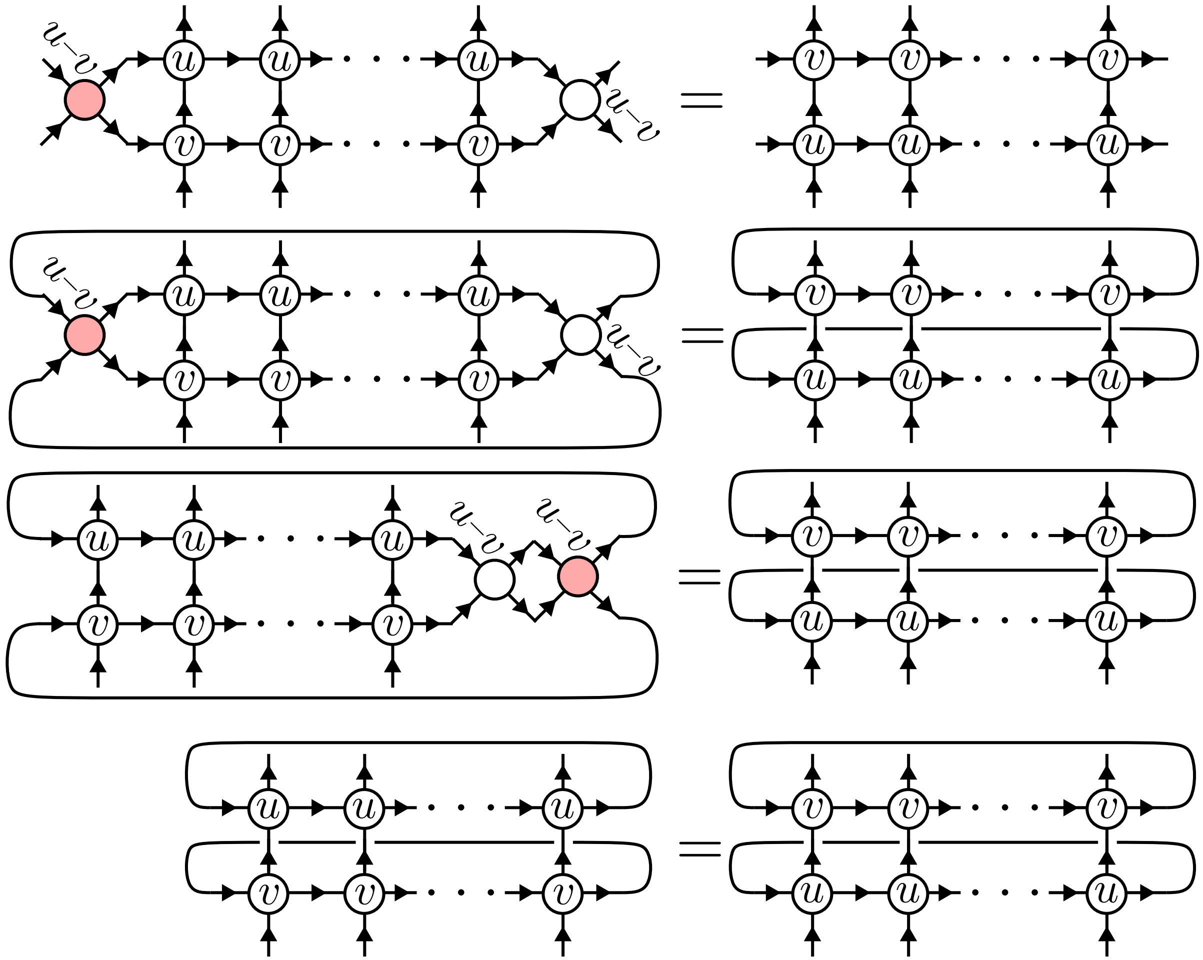


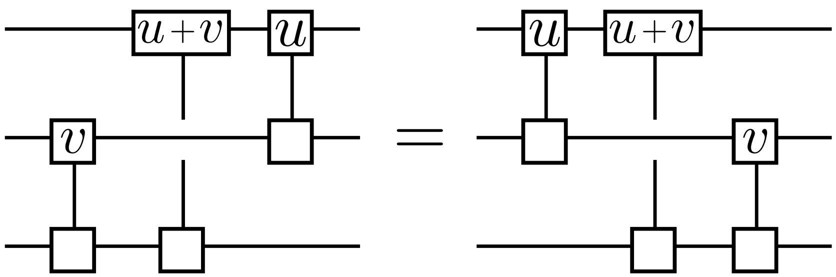


$$T(u)T(v) = T(v)T(u) \quad \forall u, v \in \mathbb{C} \; \square$$

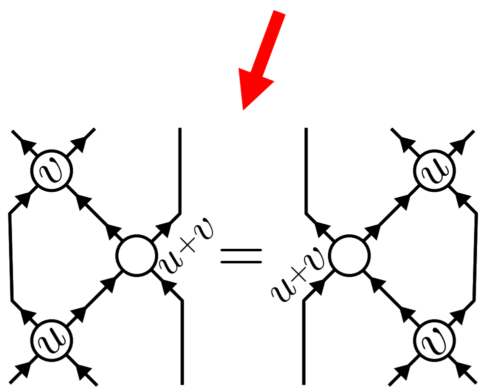
Conserved quantities and Yang-Baxter

- R-matrix satisfies Yang-Baxter equation ⇒ Many conserved quantities

$$R_{1,2}(u)R_{1,3}(u+v)R_{2,3}(v) = R_{2,3}(v)R_{1,3}(u+v)R_{1,2}(u)$$



$$\Longrightarrow \quad T(v)T(u) = T(u)T(v) \quad \forall u, v \in \mathbb{C}$$



Since T is analytic in u (its just a polynomial)
We really get that,

$$[\partial_u^n T(u), \partial_v^m T(v)] = 0 \ \forall u, v \in \mathbb{C} \ \forall n, m \in \mathbb{N}_0$$

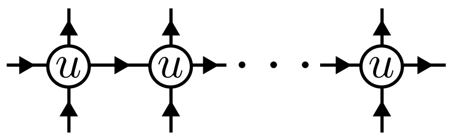
Spectrum of T's implies spectrum of H.

$$H_{XXX} \propto T(0)^{-1}(\partial_u T)(0) \quad \Longrightarrow \quad [H_{XXX}, T(u)] = 0$$

The Bethe Ansatz

Basic idea: Use monodromy matrices to construct spectrum of the transfer operator.

Monodromy matrix is a 2x2 matrix of 2^L x 2^L matrices



$$= \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \qquad A(u), B(u), C(u), D(u) : \mathbb{C}^{2^L} \to \mathbb{C}^{2^L}$$

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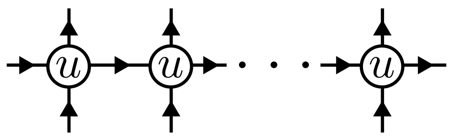

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$$T(u) = \mathrm{Tr} \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = A(u) + D(u)$$

Transfer matrix: Gives us the Hamiltonian via

$$H_{XXX} = -iJ\partial_u \log T(u)\Big|_{u=0} + JL$$

$$|p_1, \ldots, p_N\rangle \propto \prod_{j=1}^{N} B(u_j)|\uparrow\rangle^{\otimes L}$$

The Bethe ansatz: Eigenstates of H_XXX understood as N excitations with momenta p_1,…,p_N.

Technique: Y-B eqn. implies an algebra (commutators!)

$$T(u)|u_1, \ldots, u_N\rangle = (A(u) + D(u)) \prod_{j=1}^{N} B(u_j)|\uparrow\rangle^{\otimes L}$$

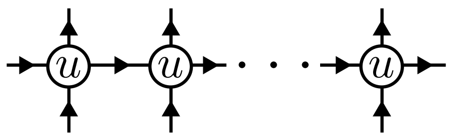
$$= \tau(u, u_1, \ldots, u_n) \prod_{j=1}^{N} B(u_j)|\uparrow\rangle^{\otimes L} + |\mathrm{stuff}\rangle$$

Fix the u's such that this is zero

Zamolodchikov-Faddeev Algebra

- Yang-Baxter equation implies algebra on monodromy matrices

<u>Monodromy matrix</u> is a 2x2 matrix of 2^L x 2^L matrices



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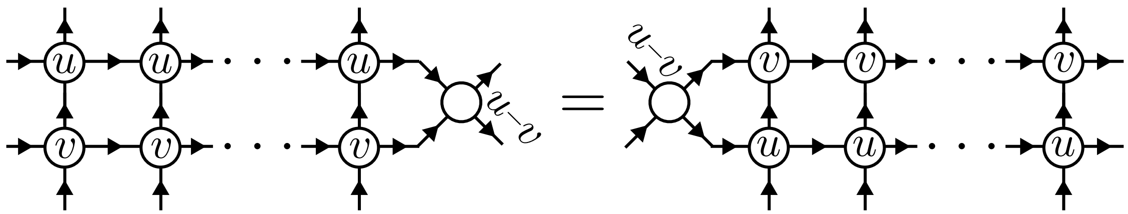

$$\begin{pmatrix} u-v+i & 0 & 0 & 0 \\ 0 & i & u-v & 0 \\ 0 & u-v & i & 0 \\ 0 & 0 & 0 & u-v+i \end{pmatrix} \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \otimes \begin{pmatrix} A(v) & B(v) \\ C(v) & D(v) \end{pmatrix} = \begin{pmatrix} A(v) & B(v) \\ C(v) & D(v) \end{pmatrix} \otimes \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \begin{pmatrix} u-v+i & 0 & 0 & 0 \\ 0 & i & u-v & 0 \\ 0 & u-v & i & 0 \\ 0 & 0 & 0 & u-v+i \end{pmatrix}$$

Zamolodchikov-Faddeev Algebra

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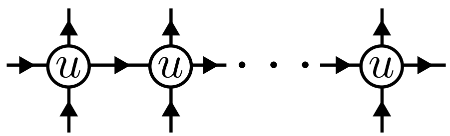

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$$D(v)B(u) = \left(1 - \frac{i}{u-v}\right) B(u)D(v) + \frac{i}{u-v} B(v)D(u)$$

$$[B(u), B(v)] = 0$$

Exact eigenstates and energies

<u>Z-F algebra</u>

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<u>Approach to the problem</u>

$$\begin{aligned} T(u)|u_1, ..., u_N\rangle &= (A(u) + D(u)) \prod_{j=1}^{N} B(u_j)|\uparrow\rangle^{\otimes L} \\ &= \tau(u, u_1, ..., u_n) \prod_{j=1}^{N} B(u_j)|\uparrow\rangle^{\otimes L} + |\text{stuff}\rangle \end{aligned}$$

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Result of taking N commutators

$$A(u)\prod_{j=1}^{N} B(u_j)|\uparrow\rangle^{\otimes L} = \left(\prod_{j=1}^{N} 1 + \frac{i}{u_j - u}\right)\left(\prod_{j=1}^{N} B(u_j)\right) A(u)|\uparrow\rangle^{\otimes L} + \sum_{j=1}^{N} \frac{-i}{u_j - u}\left(\prod_{k\neq j} 1 + \frac{i}{u_k - u_j}\right) B(u)\left(\prod_{k\neq j} B(u_k)\right) A(u_j)|\uparrow\rangle^{\otimes L}$$

$$D(u)\prod_{j=1}^{N} B(u_j)|\uparrow\rangle^{\otimes L} = \left(\prod_{j=1}^{N} 1 + \frac{-i}{u_j - u}\right)\left(\prod_{j=1}^{N} B(u_j)\right) D(u)|\uparrow\rangle^{\otimes L} + \sum_{j=1}^{N} \frac{i}{u_j - u}\left(\prod_{k\neq j} 1 + \frac{-i}{u_k - u_j}\right) B(u)\left(\prod_{k\neq j} B(u_k)\right) D(u_j)|\uparrow\rangle^{\otimes L}$$

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Fact

$$A(u)|\uparrow\rangle^{\otimes L} = (u+i)^L|\uparrow\rangle^{\otimes L}$$

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Unwanted terms cancel if…

$$\left(\frac{u_j + i}{u_j}\right)^L \prod_{k \neq j} \left(\frac{u_k - u_j + i}{u_k - u_j - i}\right) = 1$$

Exact eigenstates and energies

Eigenvalues!

$$T(u)|u_1, ..., u_N\rangle = \left[(u+i)^L \prod_{j=1}^{N} \left(\frac{u_j - u + i}{u_j - u} \right) + u^L \prod_{j=1}^{N} \left(\frac{u_j - u - i}{u_j - u} \right) \right] |u_1, ..., u_N\rangle$$

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It is convenient to redefine

$$u_j' = u_j + i/2$$

Exact eigenstates and energies

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$$T(u)|u'_1,...,u'_N\rangle = \left[(u+i)^L\prod_{j=1}^{N}\left(\frac{u'_j-u+i/2}{u'_j-u-i/2}\right)+u^L\prod_{j=1}^{N}\left(\frac{u'_j-u-3i/2}{u'_j-u-i/2}\right)\right]|u'_1,...,u'_N\rangle$$

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Eigenvalues!

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Energies!

$$E_{XXX}(u'_1,\ldots,u'_N) = -iJ(\partial_u \log\tau)(0) + JL = \sum_{j=1}^{N}\frac{J}{u'^2_j + 1/4}$$

Exact eigenstates and energies

Eigenvalues!

$$T(u)|u'_1,\ldots,u'_N\rangle = \underbrace{\left[(u+i)^L \prod_{j=1}^{N}\left(\frac{u'_j-u+i/2}{u'_j-u-i/2}\right)+u^L\prod_{j=1}^{N}\left(\frac{u'_j-u-3i/2}{u'_j-u-i/2}\right)\right]}_{\tau(u)}|u'_1,\ldots,u'_N\rangle$$

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Yet another redefinition

$$\frac{u'_j+i/2}{u'_j-i/2}=e^{ip_j} \Rightarrow\rightarrow u'_j=\frac{1}{2}\cot(p_j/2)$$

Energies!

$$E_{XXX}(u'_1,\ldots,u'_N)=-iJ(\partial_u\log\tau)(0)+JL=\sum_{j=1}^{N}\frac{J}{u'^2_j+1/4}$$

Exact eigenstates and energies

Energies:

$$E_{XXX}(p_1, ..., p_N) = 2J \sum_{j=1}^{N} (1 - \cos(p_j))$$

Allowed momenta:

$$e^{ip_j L} \prod_{k \neq j} \underbrace{\frac{\cot(p_j/2) - \cot(p_k/2) + 2i}{\cot(p_j/2) - \cot(p_k/2) - 2i}}_{S(p_j, p_k)} = 1$$

$S(p_j, p_k)$ S-matrix

Exact eigenstates and energies

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$S(p_j, p_k)$ S-matrix

What are these particles?

<u>N=1:</u> $B(p_1)|\uparrow\rangle^{\otimes L} \propto \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{-ip_1 j} \sigma_j^- |\uparrow\rangle^{\otimes L}$

$$e^{ip_1 L} = 1 \Rightarrow p_1 = \frac{2\pi n}{L}$$

One spin flip propagating with momentum p_1.
Called a “magnon”.

Exact eigenstates and energies

Energies:

$$E_{XXX}(p_1, ..., p_N) = 2J \sum_{j=1}^{N} (1 - \cos(p_j))$$

Allowed momenta:

$$e^{ip_j L} \prod_{k \neq j} \underbrace{\frac{\cot(p_j/2) - \cot(p_k/2) + 2i}{\cot(p_j/2) - \cot(p_k/2) - 2i}}_{S(p_j, p_k)} = 1$$

S-matrix

What are these particles?

<u>N=1:</u> $$B(p_1)|\uparrow\rangle^{\otimes L} \propto \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{-ip_1 j} \sigma_j^- |\uparrow\rangle^{\otimes L}$$

$$e^{ip_1 L} = 1 \Rightarrow p_1 = \frac{2\pi n}{L}$$

One spin flip propagating with momentum p_1.
Called a “magnon”.

<u>N=2:</u>

$$B(p_1)B(p_2)|\uparrow\rangle^{\otimes L} \propto \sum_{j<k} \left(e^{-i(p_1 j + p_2 k)} + S(p_1, p_2) e^{-i(p_2 j + p_1 k)} \right) \sigma_j^- \sigma_k^- |\uparrow\rangle^{\otimes L}$$

$$e^{ip_1 L} S(p_1, p_2) = e^{ip_2 L} S(p_2, p_1) = 1$$

Two spin flips propagating with momentum p_1 and p_2.
Relative phase is indicative of a scattering event.

Other models

- XXZ model

$$R(u-v) = \frac{1}{\sinh(u-v)} \begin{pmatrix} \sinh(u-v+\eta) & 0 & 0 & 0 \\ 0 & \sinh(u-v) & \sinh(\eta) & 0 \\ 0 & \sinh(\eta) & \sinh(u-v) & 0 \\ 0 & 0 & 0 & \sinh(u-v+\eta) \end{pmatrix}$$

$$H_{XXZ} = 2\sinh(\eta)\partial_u \log T(u)\Big|_{u=\eta/2} - N\Delta = \sum_{j=1}^{L} X_j X_{j+1} + Y_j Y_{j+1} + \Delta Z_j Z_{j+1}$$

$\Delta = \cosh(\eta)$ Only works for $\Delta \geq 1$

- Lieb-Linger model (nonlinear Schrodinger eqn.)[arXiv:1804.07350]
- Inhomogenous XXX model [arXiv:1804.07350]

Thanks for listening!