Outline

• Equilibration and Thermalization (Overview)
• Classical Thermalization Review
• Issues with Quantum Thermalization
• A Random Matrix Theory Approach
• Eigenstate Thermalization Hypothesis (ETH)
• ETH and Quantum Information
What is Thermalization?

- Equilibration
  - Approaching a state and remaining near that state for most time

- Thermalization
  - Equilibrium state only depends on certain macroscopic quantities, and given by relevant ensemble
Types of Thermalization

Open System

Canonical Ensemble

$$\text{Tr} \frac{e^{-\beta H} \hat{O}}{Z}$$

Closed System

Subsystem

$$S = A \otimes B$$

Microcanonical Ensemble

$$P(A) \propto \Omega(A)$$
Observable vs Subsystem Thermalization

- Generally considering sets of observables

\[ O(t) \left\langle \psi(t)|\hat{O}|\psi(t)\rangle \]

\[ \hat{\rho}_S(t) = \text{Tr}_{S^c} \left( e^{-iHt} |\psi\rangle\langle\psi| e^{iHt} \right) \]

- This can be viewed as a special case of the above
Equilibration on Average

\[ O(t) \rightarrow O_{eq} \]

\[ \lim_{T \rightarrow \infty} \frac{1}{T} \int_{0}^{T} (O(t) - O_{eq})^2 dt \approx 0 \]

\[ O_{eq} \approx \lim_{T \rightarrow \infty} \frac{1}{T} \int_{0}^{T} O(t) dt \]

- Observable approaches an equilibrium value
- Observable stays close to equilibrium value
- Equilibrium value well-approximated by long-time average

Analogous expressions hold for subsystem equilibration
Thermalization and Information

• Thermalization necessarily implies a loss of information
• Where did it go?

Lost to Environment

Inaccessible to local observables

Inaccessible to macroscopic observables
Classical Thermalization

\[ H(p, q) \quad z = (q, p) \]

\[ H(p(t), q(t)) = H(p(0), q(0)) = E \]

Equations of Motion

\[ \dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \]
Classical Thermalization

Chaos → Mixing → Ergodicity

**Ergodicity**
"Time average = Phase space average"
Orbits fill the entire energy shell uniformly

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T O(t) dt = \frac{\int_{S_E} O(z) dz}{\int_{S_E} dz}$$

$$\frac{\int_{S_E} O(z) dz}{\int_{S_E} dz} \equiv \langle O \rangle_{\text{micro}}$$
Classical Thermalization (Closed System)

Consider a macroscopic observable $O$

$$\Omega(A) : \text{Number or measure of microstates corresponding to observable value } A$$

$$O_{eq} = \lim_{T \to \infty} \frac{1}{T} \int_0^T O(t) dt = \frac{\int_{S_E} O(z) dz}{\int_{S_E} dz} \approx \arg\max_A \Omega(A)$$

Information about initial conditions cannot be resolved by macroscopic observable
Assume: S is in microcanonical ensemble at energy E and A,B can exchange energy

\[ \Omega_{AB}(E) = \Omega_B(E - E_A)\Omega_A(E_A) \]

\[ \frac{\text{Pr}(E_A)}{\text{Pr}(E'_A)} = \frac{\Omega_A(E_A)}{\Omega_A(E'_A)} = \frac{\Omega_B(E - E'_A)}{\Omega_B(E - E_A)} \]

\[ = e^{\frac{1}{k}(S(E-E_A)-S(E-E'_A))} \approx e^{-\frac{1}{kT}(E_A-E'_A)} \]

\[ \implies \text{Pr}(E_A) = \frac{e^{-\frac{E_A}{kT}}}{Z} \quad \text{(Canonical Ensemble)} \]

The initial information contained in A is spread throughout the whole system and becomes inaccessible to local observables.
Summary

Chaos → Ergodicity → Microcanonical Ensemble

Closed System Thermalization w.r.t Macroscopic Observable

Subsystem Thermalization to Canonical Ensemble
Transition to Quantum Thermalization

Immediate Issues

• No phase space (position and momentum don't commute)
• No well-defined trajectories, initial wave packets will spread
• No clear integrable vs nonintegrable definition
• Difficult to define chaos \( \langle \psi(t) | \phi(t) \rangle = \langle \psi(0) | \phi(0) \rangle \)
Transition to Quantum Thermalization

\[ |\psi(t)\rangle = \sum_i c_i e^{-iE_i t} |E_i\rangle \]

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \langle \psi(t) | \hat{O} | \psi(t) \rangle = \sum_i |c_i|^2 \langle E_i | \hat{O} | E_i \rangle 
\]

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T dt |\psi(t)\rangle \langle \psi(t)| = \sum_i |c_i|^2 |E_i\rangle \langle E_i| \neq \text{Microcanonical ensemble}
\]

In general, long-time averages are sensitive to the initial conditions \( c_i = \langle E_i | \psi(0) \rangle \)
Equilibration Turns Out to be Quite General

$N(\epsilon)$: The maximal number of approximately degenerate energy gaps in an energy interval of width $\epsilon$

$$\langle \hat{O}(t) \rangle = \langle \psi(t) | \hat{O} | \psi(t) \rangle$$

$$O_{eq} = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \langle \hat{O}(t) \rangle$$

$$\frac{1}{T} \int_0^T dt \left( \langle \hat{O}(t) \rangle - O_{eq} \right)^2 \leq C(|\psi(0)\rangle) \| \hat{O} \|^2 N(\epsilon) \left( 1 + \frac{8 \log(\dim(H))}{\epsilon T} \right)$$

$$C(|\psi(0)\rangle) \sim \frac{1}{\dim(H)}$$

For states that populate a significant number of energy eigenstates

$$\langle \psi(t) | \hat{O} | \psi(t) \rangle = \sum_i |c_i|^2 \langle E_i | \hat{O} | E_i \rangle + \sum_{i \neq j} c_i c_j^* e^{-it(E_i - E_j)} \langle E_j | \hat{O} | E_i \rangle$$

While this addresses how quantum systems equilibrate, we still must show that they thermalize

[Gogolin and Eisert, 2016]
Random Matrix Theory

- Recall, The BGS conjecture states that quantum systems with chaotic classical counterparts have spectra with the same statistics as random matrices.

- Since classical chaotic systems thermalize, consider an RMT approach to quantum thermalization.

- The eigenvectors of a random matrix are essentially random unit vectors which are mutually orthogonal.
Random Matrix Theory

\[ \hat{O} = \sum_i O_i |O_i\rangle\langle O_i| \]

We can calculate the average matrix elements of observables in the random energy eigenbasis

\[ O_{mn} = \langle E_m | \hat{O} | E_n \rangle = \sum_i O_i \langle E_m | O_i \rangle \langle O_i | E_n \rangle \]

\[ \langle E_m | O_i \rangle \langle O_j | E_n \rangle = \frac{1}{D} \delta_{mn} \delta_{ij} \]  
\text{(Averaging $|E_m\rangle$ and $|E_n\rangle$ over the Haar measure)}

\[ \overline{O_{mm}} = \frac{1}{D} \sum_i O_i \]

\[ \overline{O_{mn}} = 0 \quad m \neq n \]
Random Matrix Theory

\[ O_{mn} = \langle E_m | \hat{O} | E_n \rangle = \sum_i O_i \langle E_m | O_i \rangle \langle O_i | E_n \rangle \]

The fluctuations about the average of observables in the random energy eigenbasis can also be calculated.

\[
\overline{O_{mm}^2} - \overline{O_{mm}}^2 = \overline{|O_{mn}|^2} - \overline{|O_{mn}|}^2 = \frac{1}{D^2} \sum_i O_i^2
\]

(For GUE)

For the detailed derivation, see [D'Alessio et al., 2016]
RMT Observable Ansatz

Define: \[ \overline{O} \equiv \frac{1}{D} \sum_{i} O_i \quad \overline{O}^2 \equiv \frac{1}{D} \sum_{i} O_i^2 \]

Ansatz: \[ O_{mn} \approx \overline{O} \delta_{mn} + \sqrt{\frac{\overline{O}^2}{D}} R_{mn} \]

\( R_{mn} \) is a zero mean, unit variance random variable

Thermalizes*: \[ O_{eq} = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} dt \langle O(t) \rangle = \sum_{m} |c_m|^2 O_{mm} \approx \overline{O} \sum_{m} |c_m|^2 = \overline{O} \]

No dependence on initial conditions!
RMT is Insufficient

\[ O_{eq} = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} dt \langle O(t) \rangle = \sum_{m} |c_m|^2 O_{mm} \approx \bar{O} \sum_{m} |c_m|^2 = \bar{O} \]

1. The equilibrium values in the RMT ansatz are independent of the system energy density

2. Relaxation times are observable dependent, and this information should be contained in off-diagonal matrix elements
Eigenstate Thermalization Hypothesis

Define: \[ \bar{E} \equiv \frac{1}{2}(E_m + E_n) \quad \omega \equiv E_n - E_m \]

Ansatz

\[ O_{mn} = \mathcal{O}(\bar{E})\delta_{mn} + e^{-S(\bar{E})/2} f(\bar{E}, \omega) R_{mn} \]

\( \mathcal{O}(\bar{E}), f(\bar{E}, \omega) \) are smooth functions of their arguments

\( R_{mn} \) is a zero mean, unit variance random variable

\[ e^{S(E)} = E \sum_{\alpha} \delta_\epsilon(E - E_\alpha) \] is the thermodynamic entropy
Comparing ETH to RMT

<table>
<thead>
<tr>
<th>RMT</th>
<th>ETH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_{mn} \approx \overline{O} \delta_{mn} + \sqrt{\frac{O^2}{D}} R_{mn}$</td>
<td>$O_{mn} = \mathcal{O}(\overline{E}) \delta_{mn} + e^{-S(\overline{E})/2} f(\overline{E}, \omega) R_{mn}$</td>
</tr>
</tbody>
</table>

1. The diagonal elements in ETH are not the same for all eigenstates, and more importantly, energy dependent.

2. The off-diagonal elements in ETH depend on the envelope function $f(\overline{E}, \omega)$ characterizing the relaxation time.

The results of the ETH ansatz agree with the semi-classical predictions of the BGS conjecture [D'Alessio et al., 2016]
ETH Thermalizes

\[ O_{mn} = \mathcal{O}(\bar{E}) \delta_{mn} + e^{-S(\bar{E})/2} f(\bar{E}, \omega) R_{mn} \]

\[ e^{S(E)} = E \sum_{\alpha} \delta_{\epsilon}(E - E_{\alpha}) \]

\[ \langle O \rangle_\beta \equiv \frac{\text{Tr} e^{-\beta H} O}{\text{Tr} e^{-\beta H}} = \frac{\int_0^\infty \frac{dE}{E} e^{S(E)-\beta E} \mathcal{O}(E)}{\int_0^\infty \frac{dE}{E} e^{S(E)-\beta E}} + O(e^{-S/2}) \]

Solving this for \( \mathcal{O}(E) \) (see [Srednicki, 1998] for proof)

\[ \mathcal{O}(E) = \langle O \rangle_\beta + O(N^{-1}) + O(e^{-S/2}) \]
The Envelope Function \( f(\overline{E}, \omega) \)

**Ansatz:**  
\[
O_{mn} = \mathcal{O}(\overline{E}) \delta_{mn} + e^{-S(\overline{E})/2} f(\overline{E}, \omega) R_{mn}
\]

\( f(\omega) \) is approximately constant

\( f(\omega) \propto \frac{\Gamma}{\omega^2 + \Gamma^2} \)

Markovian decay back to equilibrium from small fluctuations [Srednicki, 1998]

\( f(\omega) \) decays exponentially
When Will Systems Thermalize?

• In order for a system to thermalize in ETH, the energy variance of the initial state should be small.

• The energy eigenstates with appreciable populations should be contained in a region where $\mathcal{O}(E)$ does not vary significantly.

\[
|\psi(0)\rangle = \sum_{m: E_m \in \Delta E^1} c_m |E_m\rangle + \sum_{n: E_n \in \Delta E^2} c_n |E_n\rangle
\]

\[
O_{eq}^{(\psi_0)} \approx O(E^1) \sum_m |c_m|^2 + O(E^2) \sum_n |c_n|^2
\]

Depends on initial conditions, does not thermalize!
Validity of ETH

• For which observables?
  • ETH is expected to hold for all few-body observables
  • In [Garrison and Grover, 2015], it is conjectured ETH holds for observables with support on up to half of the system size

• For which parts of the spectrum?
  • ETH is expected to be valid for the bulk of the spectrum, not near the edges
  • Strong ETH: Holds everywhere in the bulk
  • Weak ETH: Holds for most eigenstates in the bulk
ETH Subsystem Formulation

- Consider the set of all local observable on a subsystem $A$ and assume ETH holds for each of them

$$\langle E_i | O | E_i \rangle = \langle E_j | O | E_j \rangle + O(e^{-S/2}) \quad \text{for } E_i, E_j \in \Delta E$$

$$\implies \text{Tr}_B(|E_i\rangle\langle E_i|) = \text{Tr}_B(|E_j\rangle\langle E_j|) + O(e^{-S/2})$$

$$\langle E_i | O | E_i \rangle = \langle O \rangle_\beta + O(N^{-1}) + O(e^{-S/2})$$

$$\implies \text{Tr}_B(|E_i\rangle\langle E_i|) \approx \text{Tr}_B \left( \frac{e^{-\beta H}}{Z} \right) + O(N^{-1}) + O(e^{-S/2})$$
ETH Subsystem Formulation

• "Excited eigenstates are thermal"

\[
\text{Tr}_B(|E_i\rangle\langle E_i|) \approx \text{Tr}_B \left( \frac{e^{-\beta H}}{Z} \right)
\]

\[
\bar{\rho}_T \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T |\psi(t)\rangle\langle \psi(t)| dt = \sum_i |c_i|^2 |E_i\rangle\langle E_i|
\]

\[
\text{Tr}_B(\bar{\rho}_T) = \sum_i |c_i|^2 \text{Tr}_B(|E_i\rangle\langle E_i|) \approx \text{Tr}_B \left( \frac{e^{-\beta H}}{Z} \right)
\]
Classical vs Quantum Ergodicity

Classical ergodicity is a direct result of the dynamics of the system.

Quantum ergodicity is not caused by the dynamics of the system, it is already present in the initial state.

\[ \langle \psi(t) | \hat{O} | \psi(t) \rangle = \sum_i |c_i|^2 \langle E_i | \hat{O} | E_i \rangle + \sum_{i \neq j} c_i c_j^* e^{-it(E_i - E_j)} \langle E_j | \hat{O} | E_i \rangle \]

This will start to dephase and destructively interfere.
ETH and Quantum Error Correction

• Recently, a connection between ETH, chaotic Hamiltonians, and quantum error correction was demonstrated in [Brandao, Crosson et al., 2018]

• If local errors are of the form of the ETH matrix ansatz, they satisfy the approximate Knill-Laflamme conditions where the codespace is the eigenstates contained in a small energy window

\[ O_{mn} = \mathcal{O}(E)\delta_{mn} + e^{-S(E)/2} f(E, \omega) R_{mn} \]

\[ \langle \psi_i | E | \psi_j \rangle = C_E \delta_{ij} + \epsilon_{ij} \]
ETH and Quantum Error Correction

• A more physical formulation of this idea is introduced in [Bao and Cheng, 2019] which extends to a more general definition of chaotic Hamiltonians

• Recall that information about initial conditions in the subsystem picture must become distributed throughout the whole system and becomes inaccessible locally

• The method by which this is achieved via ETH and RMT is that nearby energy eigenstates are already locally indistinguishable

\[ \text{Tr}_B(|E_i\rangle\langle E_i|) \approx \text{Tr}_B(|E_j\rangle\langle E_j|) \iff \langle E_i|O|E_i\rangle \approx \langle E_j|O|E_j\rangle \]
ETH and Quantum Error Correction

- If an adversarial environment cannot learn anything about the encoded information by local measurements, this implies an approximate quantum error correcting code [Beny and Oreshkov, 2010]

- Sets of nearby energy eigenstates in ETH form approximate quantum error correcting codes!
Further Topics

• Thermalization in integrable systems and the generalized Gibbs ensemble [D'Alessio et al., 2016, section 8] [Gogolin and Eisert, 2016, section 5.2]

• Other mechanisms of thermalization
  • Typicality [Gogolin and Eisert, 2016, section 6] [Deutsch, 2018]
  • Open System [Deutsch, 2018]
  • Maximum Entropy Principles [Gogolin and Eisert, 2016, section 5.1]
  • Quantum Ergodic Theorem [D'Alessio et al., 2016, section 4.1]
Summary

• In classical systems, there is a well-understood route to understanding thermalization through chaotic dynamics

• Notions of quantum chaos are quite different from classical chaos, requiring a different means of analyzing quantum thermalization

• Random matrix theory almost solves the thermalization problem, but does not contain any energy dependence or relaxation time information (non-physical)

• Modifying RMT yields the Eigenstate Thermalization Hypothesis, which characterizes thermalization for local observables
References