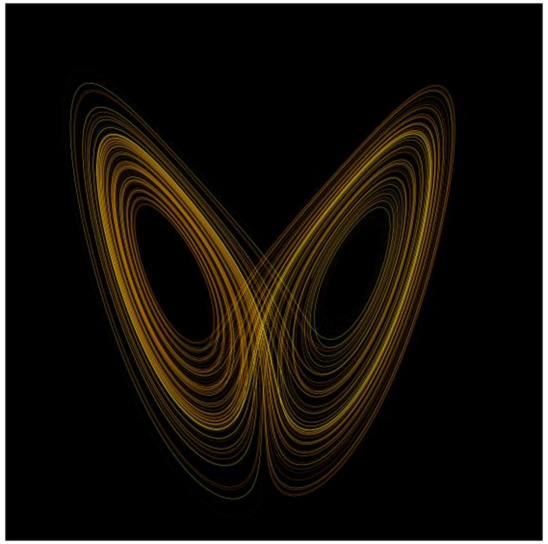
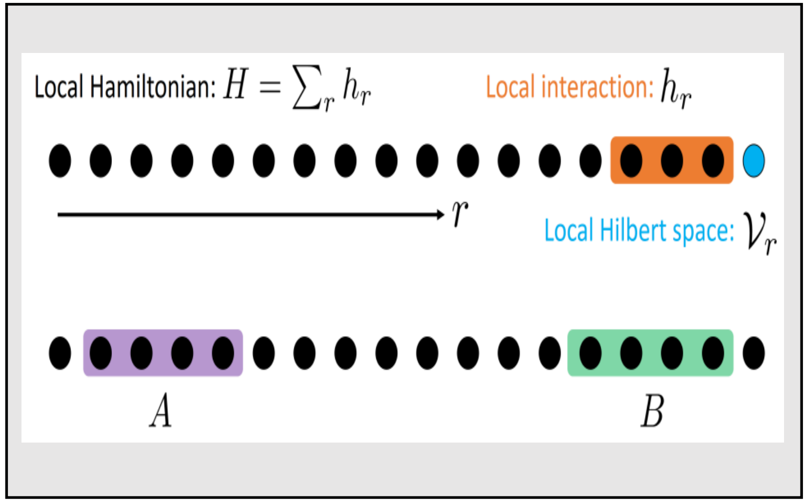


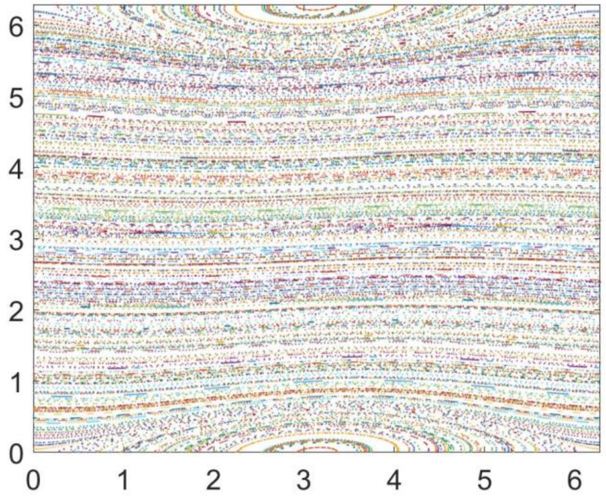
# Scrambling and out-of-time-order correlators I

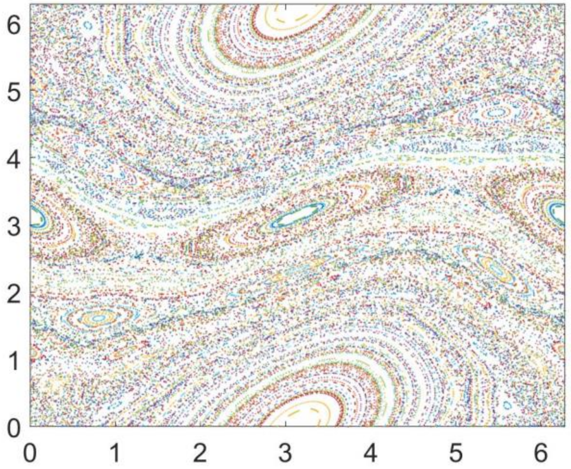


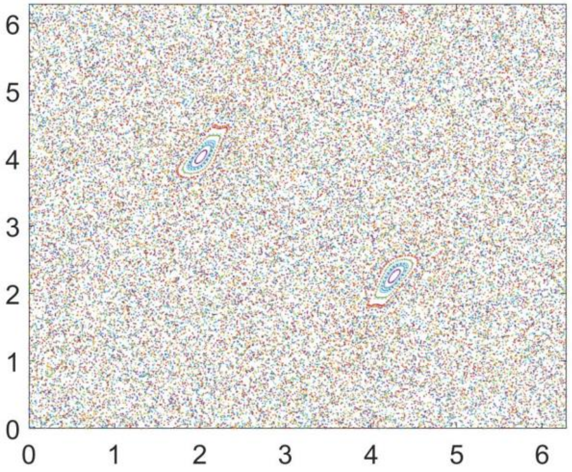

Sivaprasad Omanakuttan (CQuIC)

07/07/21

# Classical chaos



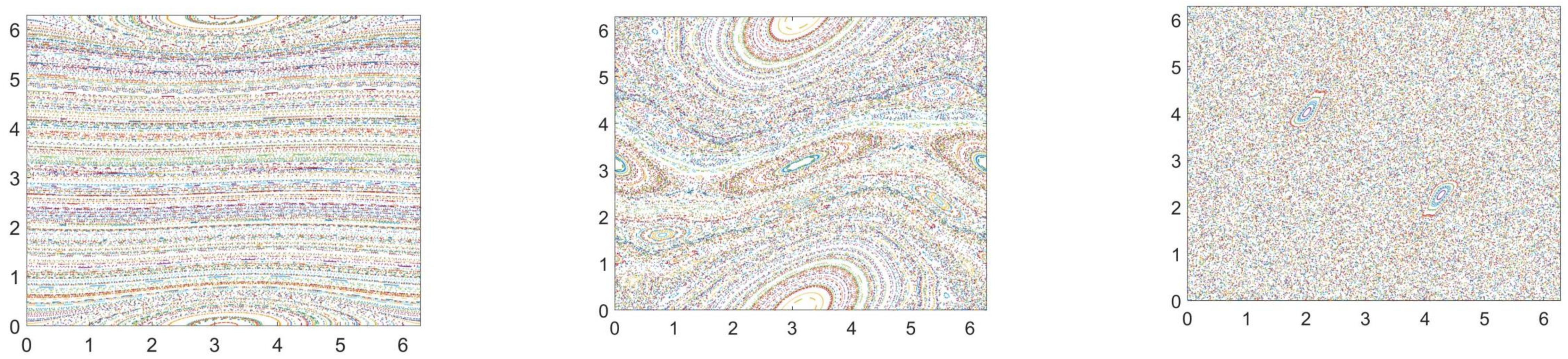




- Almost all systems with more than one degree of freedom

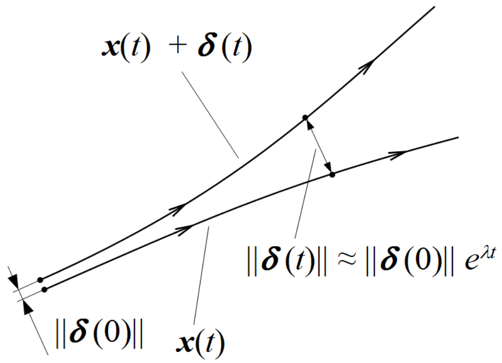
- Ergodic: Phase space average =time average

- Mixing

# Classical chaos



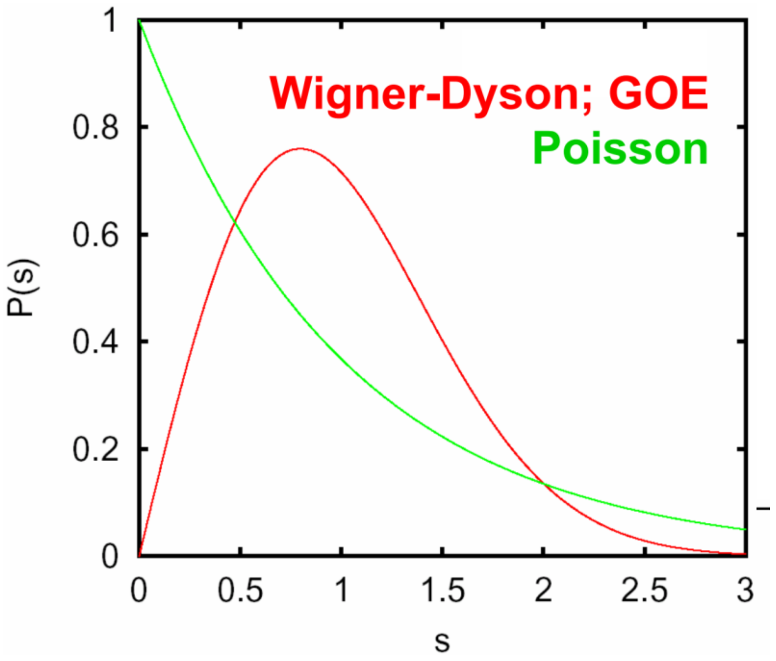

- Almost all systems with more than one degree of freedom
- Ergodic: Phase space average =time average
- Mixing
- Growth of Poisson brackets:




$$\{x(t), p(0)\}^2 = \left(\frac{\partial x(t)}{\partial x(0)}\right) \sim e^{2\lambda t}$$

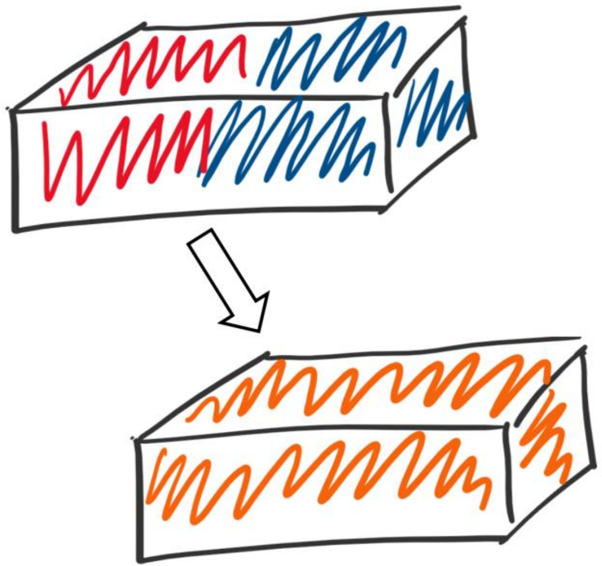
# Quantum Chaos

- We don't have Phase space in Quantum mechanics (x and p don't commute)
- We can study the spectral properties of the system (Changhao's lecture)



# Quantum Chaos

- We don't have Phase space in Quantum mechanics (x and p don't commute)
- We can study the spectral properties of the system (Changhao)
- Even understanding whether a system is integrable (given there is no classical limit) is a highly non-trivial (Manuel and Austin).
- There is a notion of thermalization of local observables in quantum chaotic systems (Sam and Mason)



# Quantum Scrambling

- Localized information spreads within typical many-body systems fast (exponentially) in the scrambling time scale defining a form of quantum chaos.
- Quantum scrambling consists of two different mechanisms: spreading of information and entanglement transport.
- Spreading of information refers to transfer of information localized in some part of the system to the entirety in some way (related to Lieb-Robinson bound).
- Entanglement transport: Entanglement which was localized in some part of the system could also move to different parts of the system.
- Now we are formally going to understand about these two concepts.

[1] X. Mi and et.al, Information scrambling in computationally complex quantum circuits, arXiv preprint arXiv:2101.08870 (2021)
[2] ] J. Karthik, A. Sharma, and A. Lakshminarayan, Phys. Rev. A 75, 022304 2007

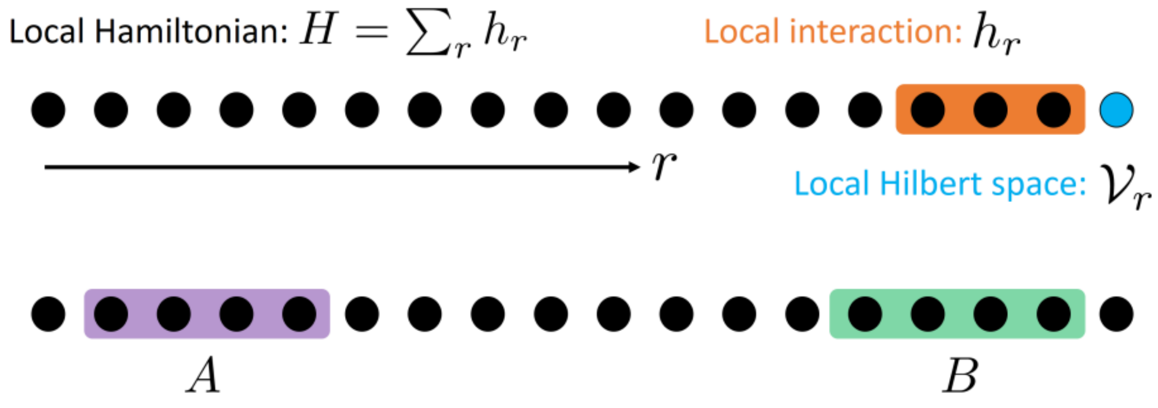
# Scrambling and Propagation of Quantum Information

- One interesting question in many body physics is understand the question of how information which was localized spreads.

- Consider a simple scenario, a discrete one-dimensional system with some local degrees of freedom. We have a Hilbert space $\mathcal{V}$ which can be written as,

$$\mathcal{V} = \otimes_r \mathcal{V}_r$$

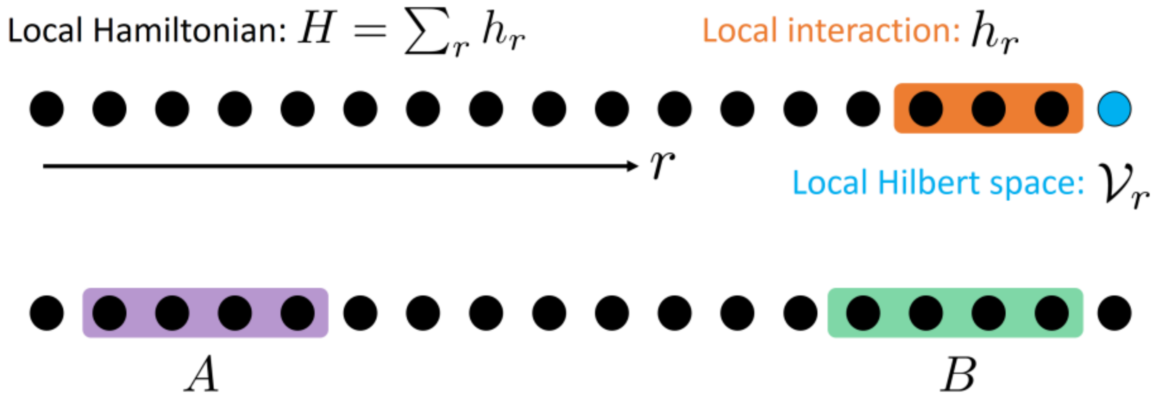
- Say we have Alice(A) and Bob(B), who are trying to communicate. Alice wants to send a bit $a \in \{0, 1\}$ and the way they have both access to a shared physical state $|\psi\rangle$




[3] B. Swingle. Lecture notes on Quantum information scrambling

# Scrambling and Propagation of Quantum Information

- Say we have Alice(A) and Bob(B), who are trying to communicate. Alice wants to send a bit $a \in \{0,1\}$ and the way they have both access to a shared physical state $|\psi\rangle$




- Depending on *a*, Alice applies a unitary $U_a$ alternatively, she can do nothing, The probability that Bob obtains output *b* = 0, 1 given that Alice sent *a* is,

$$\Pr(b|a) = \langle\psi| U_a^\dagger e^{iHt} M_b e^{-iHt} U_a |\psi\rangle$$

And using the fact that $\langle\psi| e^{-iHt} M_b e^{iHt} |\psi\rangle = 0$ (If Alice does nothing Bob does not measure anything, Bob makes a measurement $M_b$ of the system to try to learn Alice's bit)


[3] B. Swingle. Lecture notes on Quantum information scrambling

# Scrambling and Propagation of Quantum Information

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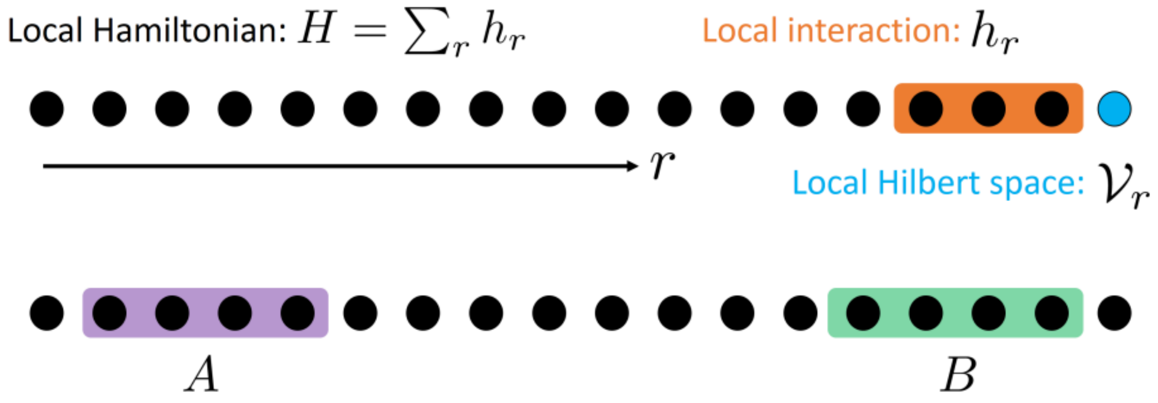


- Depending on $a$, Alice applies a unitary $U_a$, The probability that Bob obtains output $b$ = 0, 1 given that Alice sent $a$ is,

$$\mathrm{Pr}(b|a) = \langle\psi| U_a^\dagger [M_b(t), U_a] |\psi\rangle$$

- Where, $M_b$ is the measurement done by Bob to know about Alice's bit and

$$M_b(t) = e^{iHt} M_b e^{-iHt}$$

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# Scrambling and Propagation of Quantum Information

- Depending on *a*, Alice applies a unitary $U_a$, The probability that Bob obtains output *b* = 0, 1 given that Alice sent *a* is,

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- We Could think of it as having another measurement setting, Ø, corresponding to no signal from Alice. The measurement operators then obey MØ + M0 + M1 = I.

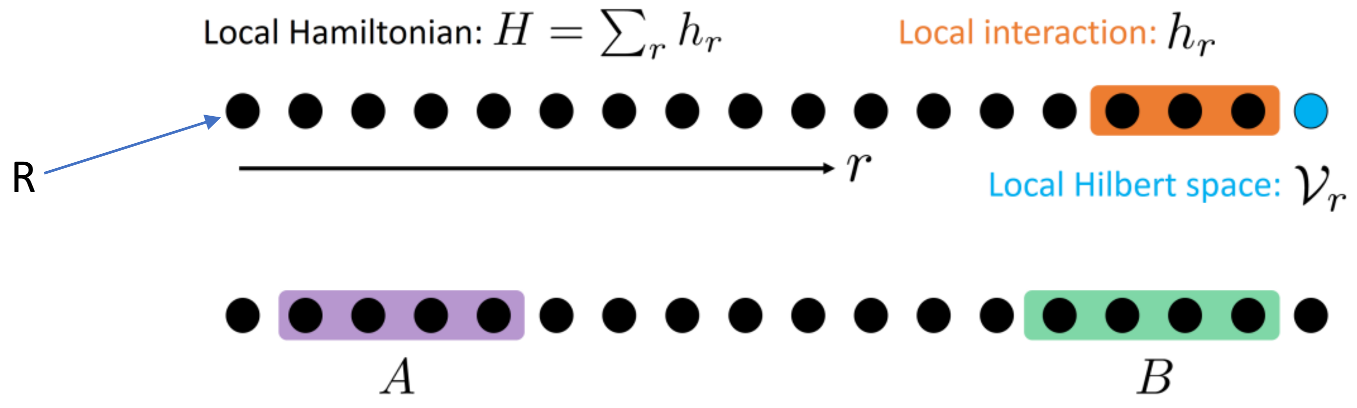
[3] B. Swingle. Lecture notes on Quantum information scrambling

# Entanglement spread

- So far, we thought of data being classical now we could assume a situation where there also entanglement in the picture.

- Consider two orthogonal initial states, $|\psi_1\rangle$ and $|\psi_2\rangle$ which differ by the application of some local unitary operator W at site $r_0$ . And we have a Reference system (auxiliary system) $R$ and we have the initial state

$$\frac{|\psi_1\rangle\,|0\rangle_R + |\psi_2\rangle\,|1\rangle_R}{\sqrt{2}}$$

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$$\frac{|\psi_1\rangle\,|0\rangle_R + |\psi_2\rangle\,|1\rangle_R}{\sqrt{2}}$$

- Note that if A and B are initially entangled, then even acting just on A conditioned on the state of R can lead to correlation between R and B.

- In other words, while at time zero the reference is entangled with a single spin in the chain, as time progresses, the reference will instead become entangled a complex collection of many spins.

[3] B. Swingle. Lecture notes on Quantum information scrambling

# Out of time ordered Correlators (OTOC)

- Localized information spreads within many-body systems exponentially fast in the scrambling time scale defining a form of quantum chaos.

- OTOC is a measure of scrambling in the system and thus becoming a measure of quantum chaos.

- OTOC is defined as, $F(t) = \langle W_t^\dagger V^\dagger W_t V \rangle; \;\; W_t = U(-t) W U(t)$

- Measures the overlap between two states,

$$|\psi_A\rangle = U^\dagger W U V |\psi\rangle ; |\psi_B\rangle = V U^\dagger W U |\psi\rangle$$

$$F(t) = \langle \psi | W_t^\dagger V^\dagger W_t V |\psi\rangle = \langle \psi_B | \psi_A \rangle .$$

"An out-of-time-order correlator (OTOC) is a four-point correlation function that probes the way in which (local) perturbation inhibit the cancellation between forward and backward evolution" by Swingle.

[4] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, Phys. Rev. A 94, 040302 (2016).

# Out of time ordered Correlators (OTOC)

- $F(t) = \langle W_t^\dagger V^\dagger W_t V \rangle$, diagnoses the spread of quantum information by measuring how quickly two commuting operators fail to commute.

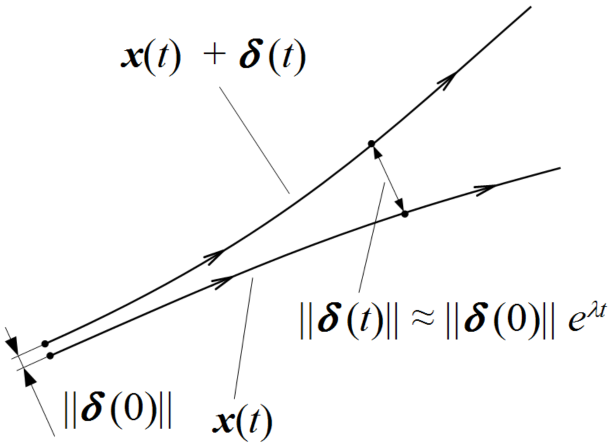
- Consider,

$$\begin{aligned} C(t) =& \langle |[W_t, V]|^2 \rangle \\ =& \langle (W_t V - V W_t)(V^\dagger W_t^\dagger - W_t^\dagger V^\dagger) \rangle \\ =& \langle W_t V V^\dagger W_t^\dagger + V W_t W_t^\dagger V^\dagger - W_t V W_t^\dagger V^\dagger - V W_t V^\dagger W_t^\dagger \rangle. \end{aligned}$$

- We have three cases,

$$\begin{aligned} C(t) =& 2(1 - \mathrm{Re}\,[F(t)]), \ W \text{and } V \text{are unitary} \\ C(t) =& \langle W_t V^2 W_t + V W_t^2 V \rangle - 2F(t), \ W \text{and } V \text{are Hermitian} \\ C(t) =& 2(1 - F(t)), \ W \text{and } V \text{are unitary and Hermitian} \end{aligned}$$

# OTOCs in the semiclassical limit.



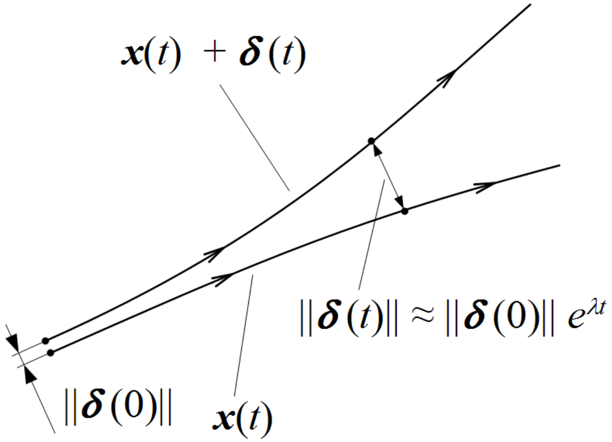

- For a classical chaotic system $\dfrac{\partial q_t}{\partial q_0} \sim e^{\lambda t}$ and we know $\dfrac{\partial q_t}{\partial q_0} = \{q_t, p_0\}_{q_0, p_0}$
- Now in the semiclassical limit,

$$[q_t, p_0] \sim i\hbar \{q_t, p_0\}$$

[4] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, Phys. Rev. A 94, 040302 (2016).

# OTOCs in the semiclassical limit.

$$\frac{\partial q_t}{\partial q_0} \sim e^{\lambda t}$$




- Now in the semiclassical limit, $[q_t, p_0] \sim i\hbar\{q_t, p_0\}$
- Consider the case $F(t) = \langle W_t^\dagger V^\dagger W_t V\rangle$, with $W = e^{iaq}$ and $V = e^{ibp}$
- Now in the limit of *a,b* are small and in a small-time limit using BCH formula we get,

$$W_t^\dagger V^\dagger W_t V \approx e^{i^2 ab[q_t,p]} \approx e^{-iab\hbar e^{\lambda t}}$$

[4] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, Phys. Rev. A 94, 040302 (2016).

# OTOCs in the semiclassical limit.

$$W_t^\dagger V^\dagger W_t V \approx e^{i^2 ab[q_t,p]} \approx e^{-iab\hbar e^{\lambda t}}$$

- Hence, we get

$$F(t) = \langle W_t^\dagger V^\dagger W_t V \rangle \sim e^{-iabe^{\lambda t}}$$

- Hence the phase of this correlation function initially diverges rapidly with t and, as higher order terms become important, the magnitude will also begin to decay.

[4] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, Phys. Rev. A 94, 040302 (2016).

# Scrambling, OTOC, and Loschmidt Echo

- For the case of two unitary operators, we have

$$C(t) = \langle |[W(t), V]|^2 \rangle = 2(1 - \mathrm{Re}[F(t)])$$

Where, $F(t) = \langle W_t^\dagger V^\dagger W_t V \rangle$

**Loschmidt echo** $F(t) = |\langle \psi_{\mathrm{rev}}(t) | \psi \rangle|^2$

$$|\psi_{\mathrm{rev}}(t)\rangle = e^{i(H+\epsilon V)t} e^{-iHt} |\psi\rangle$$

backwards forward


- Also, Loschmidt Echo (LE) can be written as a time ordered correlator,

$$G(t) = \langle V_t^\dagger V \rangle$$

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Where, $F(t) = \langle W_t^\dagger V^\dagger W_t V \rangle$

- Also, we have Loschmidt Echo (LE) which is a time ordered correlator,

$$G(t) = \langle V_t^\dagger V \rangle$$

- Which can be rewritten as,

$$G(t) = \langle \psi_B | \psi_A \rangle \,; \quad |\psi_A\rangle = V\,|\psi\rangle \,, |\psi_B\rangle = U^\dagger V U\,|\psi\rangle$$

- LE measures decay to mean Lyapunov exponent of a corresponding chaotic system.

[4] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, Phys. Rev. A 94, 040302 (2016).

# Scrambling, OTOC, and Loschmidt Echo

- LE measures decay to mean Lyapunov exponent of a corresponding chaotic system.
- Decay rate of OTOC depends not only on Lyapunov exponent but also on the number of degrees of freedom: higher the entropy slower the decay.
- Consider the case of n qubits (spin) under unitary evolution
- Fastest way to delocalize the information localized in any of the qubit is to apply a random two body unitaries between n/2 random spins for a time a length t.
- Time ordered auto correlation function decay requires a single step (relaxation of information)

[4] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, Phys. Rev. A 94, 040302 (2016).

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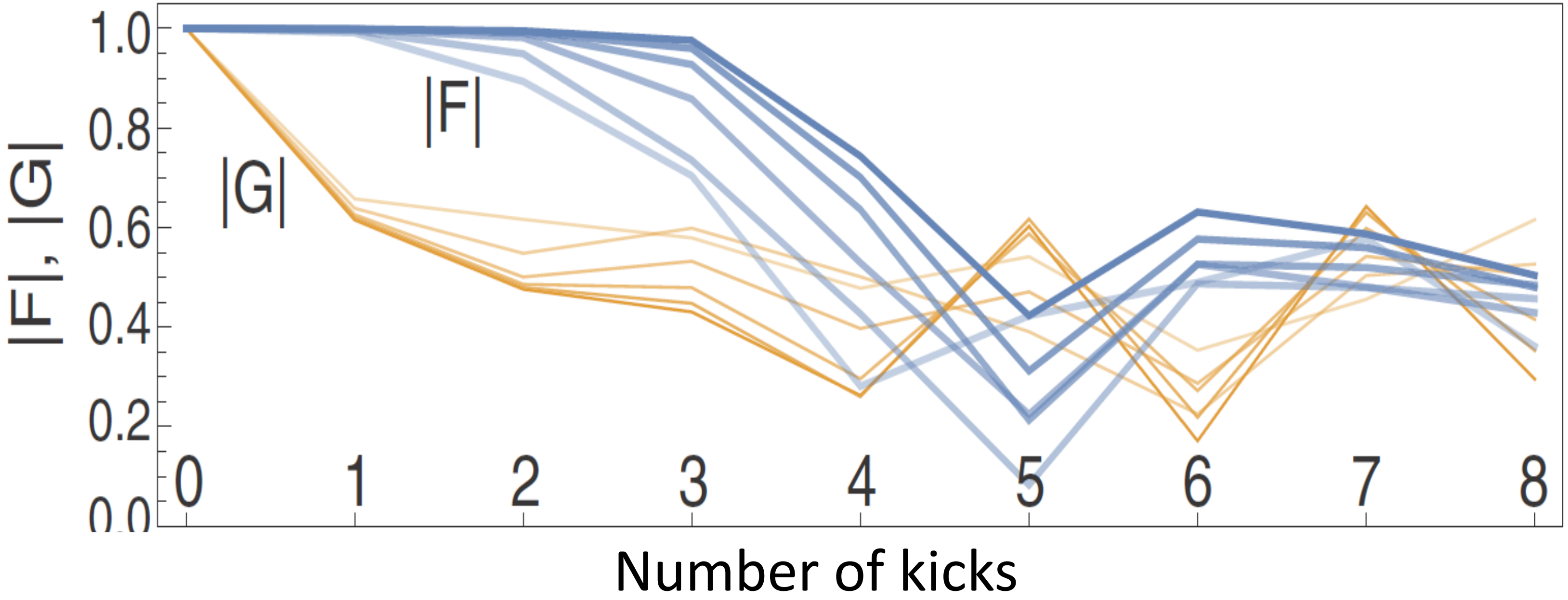
- Time ordered auto correlation function decay requires a single step (relaxation of information)

- Scrambling however requires information in one spin to spread exponentially fast to all spin chains and hence we require a total time

$$t^* = t \log_2 n$$

[4] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, Phys. Rev. A 94, 040302 (2016).

# Scrambling, OTOC, and Loschmidt Echo




$$F(t) = \langle W_t^\dagger V^\dagger W_t V \rangle$$

$$G(t) = \langle V_t^\dagger V \rangle$$

- Loschmidt Echo decays faster compared to OTOC, this is an indicator that OTOC also depends on the total degrees of freedom we need to access.
- LE goes like $\lambda^{-1}$ but $\lambda^{-1}\log(1/\epsilon)$ where $\epsilon \propto \hbar$

[4] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, Phys. Rev. A 94, 040302 (2016).

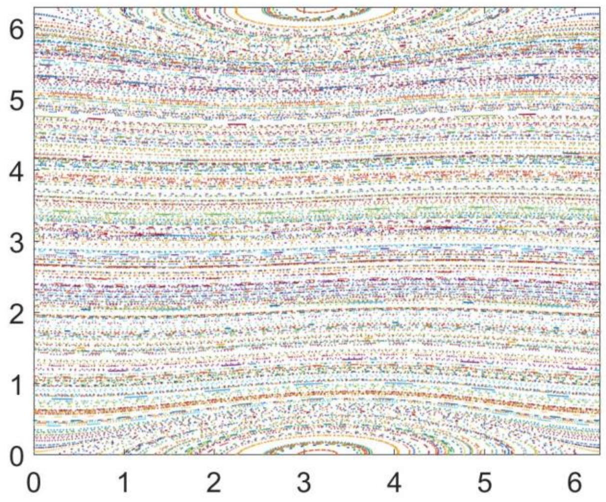
# Kicked Rotor: Classical dynamics

- The Hamiltonian of the Kicked Rotor is (*K* is the kicking strength)

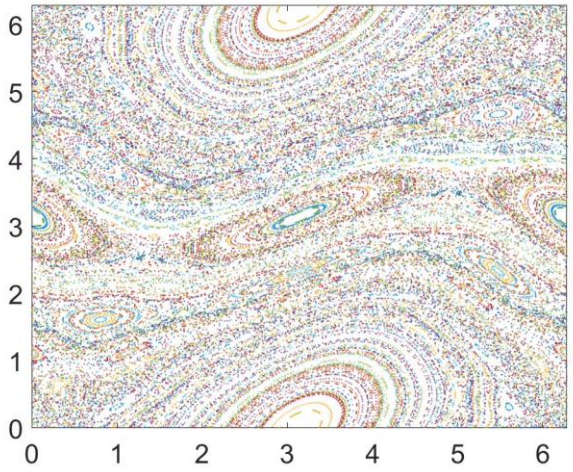
$$\mathcal{H}(p,x,t) = \frac{1}{2}p^2 + K\cos(x)\sum_{n=-\infty}^{\infty}\delta(t-n)$$

- It describes a particle that is constrained to move on a ring (equivalently: a rotating stick). The particle is kicked periodically by a homogeneous field (equivalently: the gravitation is switched on periodically in short pulses

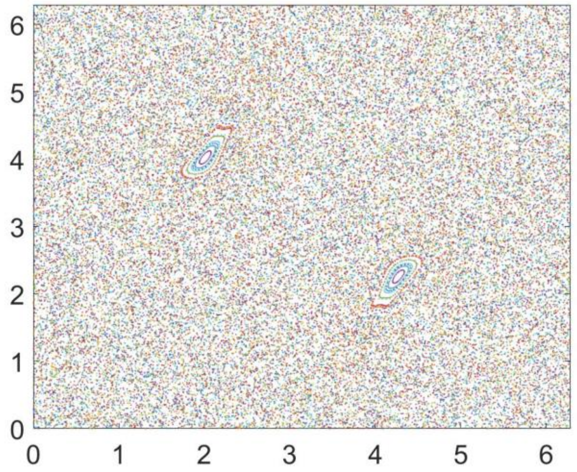
K=0.1




K=1




K=5




with $K_C = 0.971635$

# Kicked Rotor: Quantum case

- The Hamiltonian of the Kicked Rotor is (*K* is the kicking strength)

$$\mathcal{H}(\hat{p}, \hat{x}, t) = \frac{1}{2}\hat{p}^2 + K\cos(\hat{x}) \sum_{n=-\infty}^{\infty} \delta(t-n)$$

- The time evolution operator under this Hamiltonian is the Floquet map is,

$$U = \exp\left(-i\frac{1}{2}\hat{p}^2\right)\exp(-iK\cos(\hat{x}))$$

- We have these two quantities of interest,

$$C(t) = \langle [p(t), p(0)]^2 \rangle, \quad B(t) = \mathrm{Re}\left(\langle p(t)p(0) \rangle\right)$$

[5] E. B. Rozenbaum, S. Ganeshan, and V. Galitski, Phys. Rev. Lett. 118, 086801 (2017).

# OTOC in Kicked Rotor

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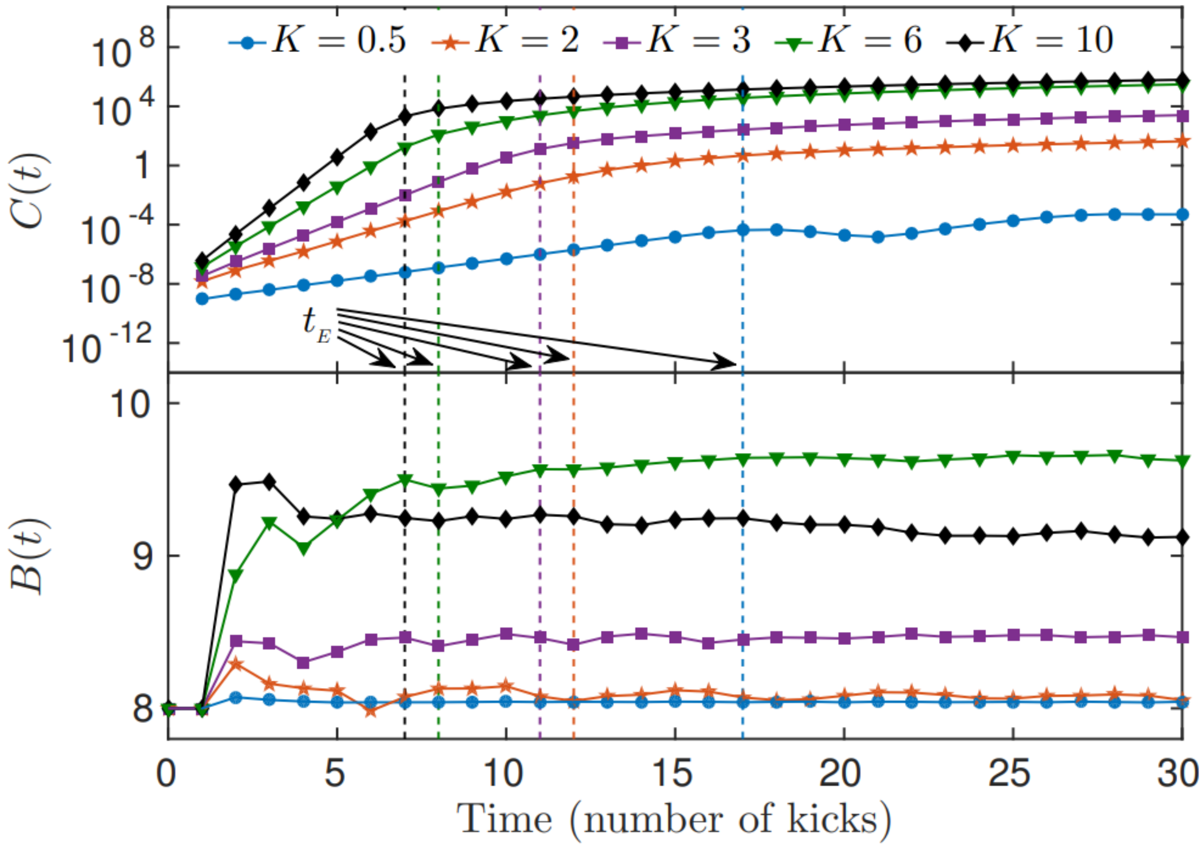
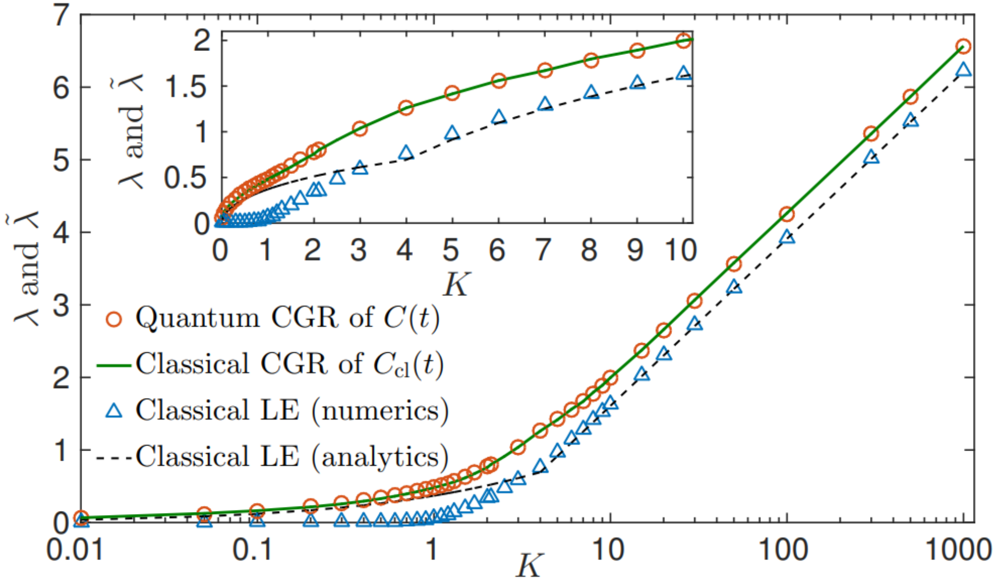
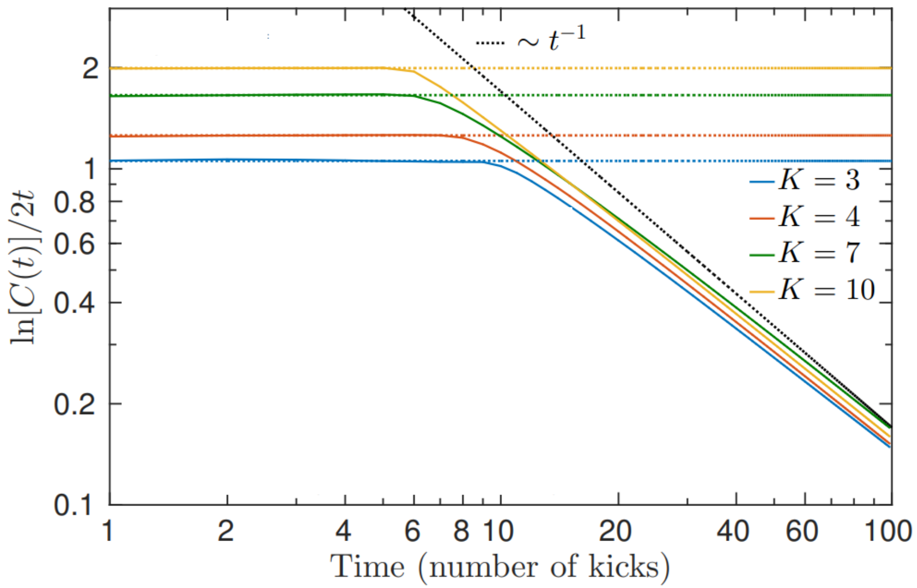



- We can see that there is an exponentially growth till the Ehrenfest time for C(t), and saturates and universal behavior we will see in lot of systems
- Where as the behavior of B(t) does not give such insights.


[5] E. B. Rozenbaum, S. Ganeshan, and V. Galitski, Phys. Rev. Lett. 118, 086801 (2017).

# OTOC in Kicked Rotor




$$C(t) = \langle [p(t), p(0)]^2 \rangle, \quad B(t) = \mathrm{Re}\left(\langle p(t) p(0) \rangle\right)$$

The correspondence with classical behavior is clearly illustrated in this figures as well the notion of Ehrenfest time (which would be useful if there is no classical analogue)


[5] E. B. Rozenbaum, S. Ganeshan, and V. Galitski, Phys. Rev. Lett. 118, 086801 (2017).

# OTOCs in a quantum Ising chain

- Consider the quantum Ising chain,

$$H = -\frac{J}{2}\left(\sum_{j=0}^{L-1} \sigma_j^z \sigma_{j+1}^z + g \sum_{j=0}^{L-1} \sigma_j^x\right)$$

- Interestingly or not, this is an integrable model and we know the solutions of if using free fermions.

- One of the reasons being that it is one of the most studied many body system,

$$C_{xx}(l,t) = \frac{1}{2}\langle |[\sigma_l^x(t), \sigma_0^x]|^2\rangle = 1 - \mathrm{Re}\left(\langle \sigma_l^x(t)\sigma_0\sigma_l^x(t)\sigma_0\rangle\right) = 1 - \mathrm{Re}\left(F(t)\right)$$

[6] C.-J. Lin and O. I. Motrunich, Phys. Rev. B 97, 144304 (2018)

# OTOCs in a quantum Ising chain

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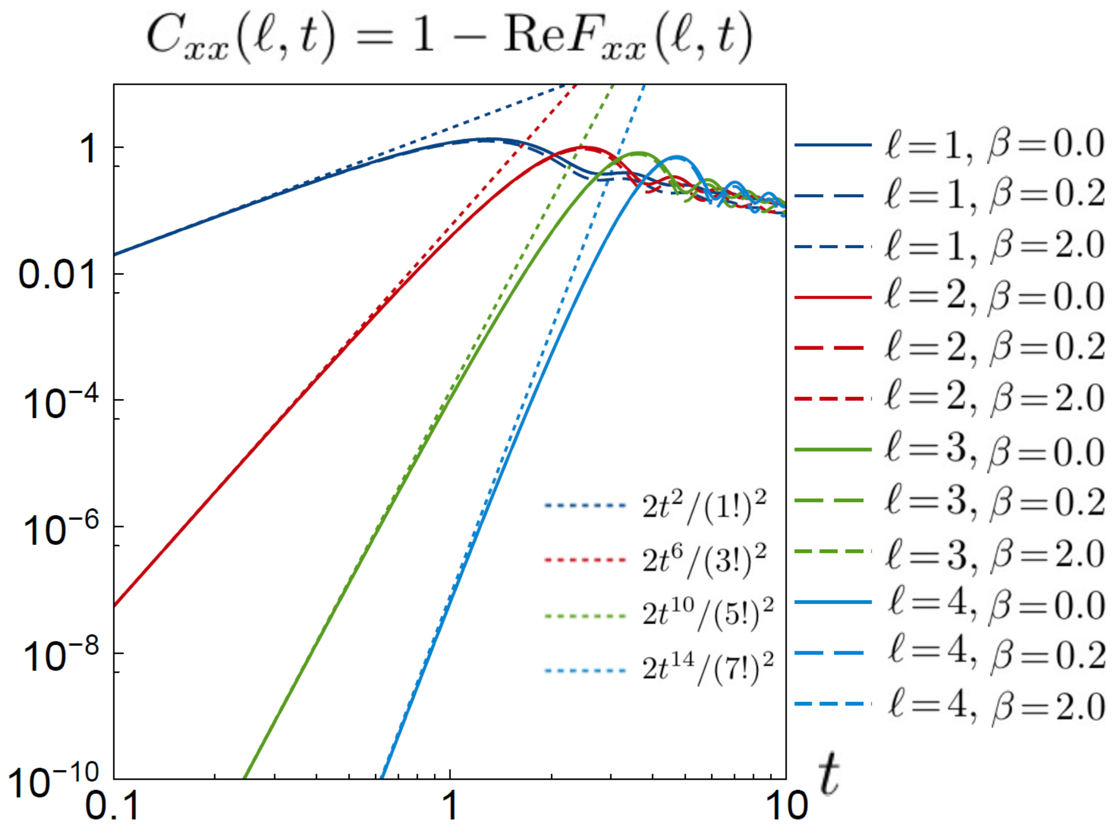



[6] C.-J. Lin and O. I. Motrunich, Phys. Rev. B 97, 144304 (2018)

Rather than the exponentially growth we have seen for the case of chaotic case we see an algebraic growth which also indicates there is no chaos

# OTOCs in a Tilted field Ising chain

- Consider the Hamiltonian,

$$H = -1/E_0 \left( J \sum_{r=1}^{L} \sigma_r \sigma_{r+1} + h_x \sum_{r=1}^{L} \sigma_x + h_z \sum_{r=1}^{L} \sigma_z \right)$$

with $E_0 = \sqrt{4J^2 + 2h_x^2 + 2h_z^2}$

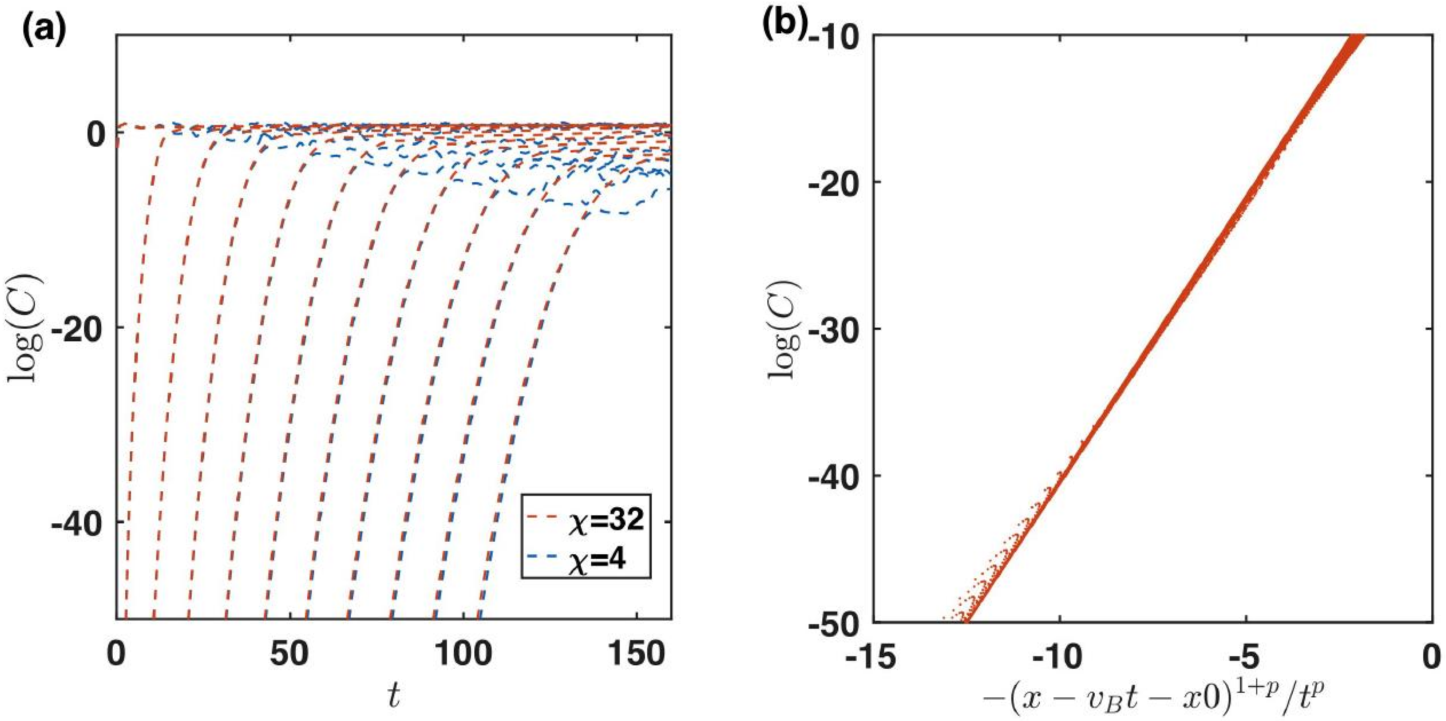
- Similar to the transverse Ising model we are studying,

$$C_{xx}(l,t) = \frac{1}{2} \langle |[\sigma_l^x(t), \sigma_0^x]|^2 \rangle = 1 - \mathrm{Re}\left(\langle \sigma_l^x(t) \sigma_0 \sigma_l^x(t) \sigma_0 \rangle\right) = 1 - \mathrm{Re}\left(F(t)\right)$$

[7] S. Xu and B. Swingle, "Accessing scrambling using matrix product operators", 1802.00801 (2018).

# OTOCs in a Tilted field Ising chain

$$H = -1/E_0 \left( J \sum_{r=1}^{L} \sigma_r \sigma_{r+1} + h_x \sum_{r=1}^{L} \sigma_x + h_z \sum_{r=1}^{L} \sigma_z \right) \qquad C_{xx}(l,t) = \frac{1}{2} \langle | [\sigma_l^x(t), \sigma_0^x] |^2 \rangle$$

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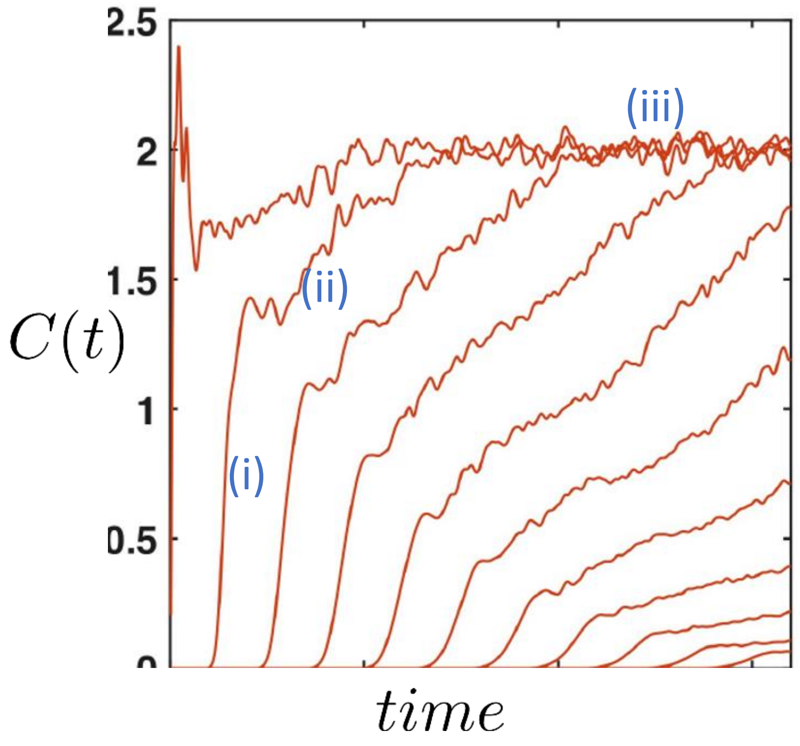



- There are three regions of C(t)

(i) The initial exponential growth related to the Lyapunov exponent

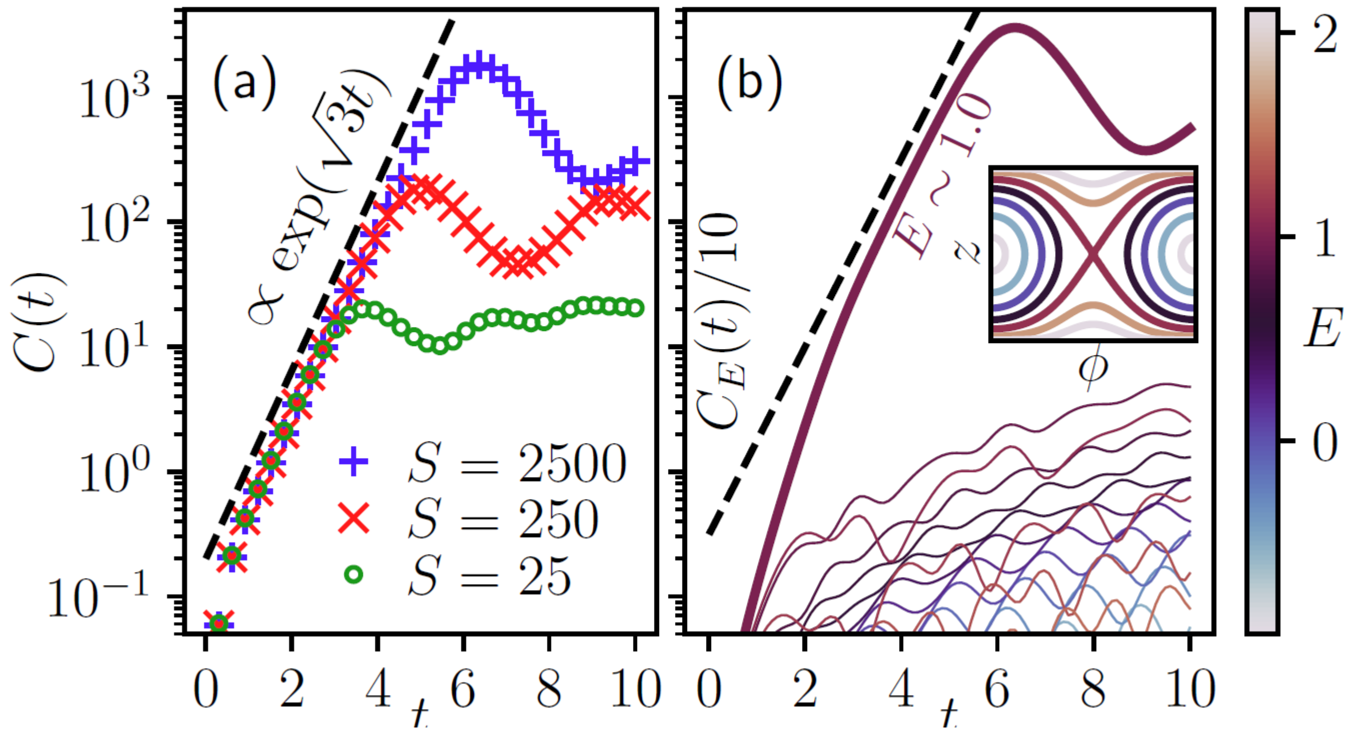
(ii) The intermediate exponential growth but with a different exponent due to conservation laws

(iii) The saturation regime


[7] S. Xu and B. Swingle, "Accessing scrambling using matrix product operators", 1802.00801 (2018).

# Does Scrambling equal Chaos?

- The unstable trajectories in a small neighborhood of a saddle can be enough for the OTOC to grow exponentially.




The parametrically long exponential growth of out-of-time order correlators (OTOCs), also known as scrambling, does not necessitate chaos. Indeed, scrambling can simply result from the presence of unstable fixed points in phase space, even in a classically integrable model.


[8] T. Xu, T. Scaffidi, and X. Cao, Does Scrambling Equal Chaos? Phys. Rev. Lett. 124, 140602 (2020)

# Conclusion

- Scrambling is the propagation of localized information in quantum systems and OTOC is very good indicator for the scrambling in the system.
- There is a well-defined classical limit for OTOC and OTOC has more to offer than Loschmidt Echo.
- Study of OTOC in one dimensional non-autonomous systems helps us in understanding the relation of OTOC to Lyapunov exponent
- The Many body systems : For integrable case we have algebraic growth and for non integrable there is well defined behavior of OTOC
- More to follow…. (wait for the next week! )

Thanks for Listening

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Acknowledgement:

(i)Pablo Poggi (CQuIC) (ii) Arul Lakshminarayan (IIT Madras)