Scrambling in Black Holes

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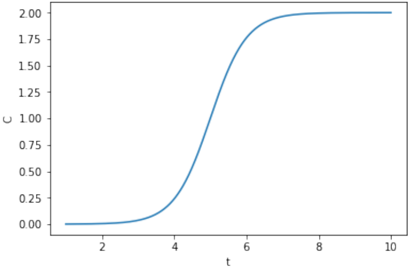
Squared Commutators and Chaos

- The squared commutator can function as diagnostic of chaos

$$\begin{aligned} C(x,y,t) &= -\langle [W(y), V(x,t)]^2 \rangle_\beta \\ &= 2(1 - \mathrm{Re}\langle W(y)V(x,t)W(y)V(x,t)\rangle_\beta) \end{aligned}$$

  where both operators are unitary and Hermitian
- The second term is the out of time order correlator (OTOC)
- In chaotic systems this quantity is expected to start small and then becomes ≈ 1 at the scrambling time t_*



Early Time Behavior of OTOCs

- Lieb-Robinson bounds force commutators of distantly separated operators to be small

$$\|[W(y), V(x,t)]\| \leq K_0 \|W\| \|V\| e^{\zeta_0 (t - |x-y|/v_{LR})} \tag{1}$$

- In many chaotic systems the early time behavior of $C(x,t)$ is exponential

$$C(x,y,t) \sim e^{\lambda_L (t - |x-y|/v_B)} \tag{2}$$

- λ_L is the correlator growth rate ('quantum Lyapunov exponent')
- The butterfly velocity v_B is in some ways like a state dependent Lieb-Robinson velocity
- size(V) is the volume of the region such that for y in that region C is at least 1
- How large may λ_L be?

Fast Scramblers

- Consider the thermally regulated OTOC

$$F(t) = \text{Tr}\left[\rho_\beta^{1/4} W \rho_\beta^{1/4} V(t) \rho_\beta^{1/4} W \rho_\beta^{1/4} V(t)\right] \tag{3}$$

- Before the scrambling time $F(t)$ should factorize

$$F(t) \approx F_d \equiv \text{Tr}[\rho_\beta^{1/2} W \rho_\beta^{1/2} W]\text{Tr}[\rho_\beta^{1/2} V(t) \rho_\beta^{1/2} V(t)] \tag{4}$$

- In chaotic systems OTOCs are expected to decay expoonentially

$$F_d - F(t) \sim e^{\lambda_L t} \tag{5}$$

- It was conjectured in 2015 by Maldancena, Shenker, and Stanford that for all systems

$$\frac{d}{dt}(F_d - F(t)) \leq \frac{2\pi}{\beta}(F_d - F(t)) \implies \lambda_L \leq \frac{2\pi}{\beta} \tag{6}$$

Maldacena, Shenker, Stanford, A bound on chaos, arxiv 2015

Proof Sketch

- Suppose $f(t + i\tau)$
  - is analytic for $t > 0$ and $-\frac{\beta}{4} \leq \tau \leq \frac{\beta}{4}$ and
  - that $|f(t + i\tau)| \leq 1$ in this region, then

$$\frac{1}{1-f}\left|\frac{df}{dt}\right| \leq \frac{2\pi}{\beta} \tag{7}$$

- The conjecture would be implied if $F(t)/F_d$

$$\frac{F_d}{F_d - F(t)}\frac{d}{dt}\left|\frac{F(t)}{F_d}\right| = \frac{1}{F_d - F(t)}\frac{d}{dt}(F_d - F(t)) \leq \frac{2\pi}{\beta} \tag{8}$$

- The ratio $F(t)/F_d$ is analytic in this strip
- The second condition causes issues. For example, how to bound the OTOC on the $\tau = |\frac{\beta}{4}|$ edge

$$F(t - i\beta/4) = \mathrm{Tr}[\rho_\beta^{1/2} WV(t) \rho_\beta^{1/2} WV(t)] \tag{9}$$

Maldacena, Shenker, Stanford, A bound on chaos, arxiv 2015

Proof Sketch

- The Cauchy-Schwarz inequality bounds $|F(t - i\beta/4)|$

$$|\mathrm{Tr}[\rho_\beta^{1/2} W V(t) \rho_\beta^{1/2} W V(t)]| \leq \mathrm{Tr}[\rho_\beta^{1/2} V(t) W \rho_\beta^{1/2} W V(t)]$$

- Define t_0 and ϵ such that for all times $t \geq t_0$

$$\mathrm{Tr}[\rho_\beta^{1/2} V(t) W \rho_\beta^{1/2} W V(t)] \leq F_d + \epsilon \tag{10}$$

- The inequality obtained for $t \geq t_0$ is

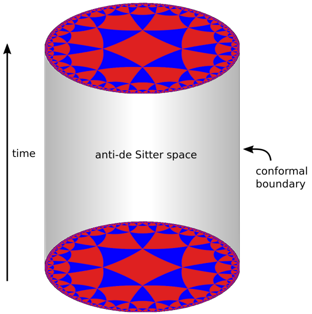
$$\frac{d}{dt}(F_d - F(t)) \leq \frac{2\pi}{\beta}(F_d - F(t) + \epsilon) \tag{11}$$

- Can we find systems in nature which saturate this bound? Black holes!

Maldacena, Shenker, Stanford, A bound on chaos, arxiv 2015

The Anti de-Sitter Space/ Conformal Field Theory Correspondence

- These ideas provide useful tools for studying scrambling in black holes
- A theory without gravity on the boundary is equivalent to a theory with gravity in the bulk
- Properties of the bulk and boundary are related via a 'dictionary'
- States on the boundary correspond to geometry in the bulk



Boulder Lectures, Brian Swingle; Image credit Wikipedia, Alex Dunkel

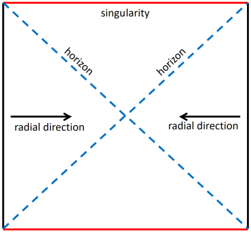
Black Holes AdS

- Schwarzschild black holes in asymptotically AdS space have metric

$$ds^2 = \frac{\ell_{\text{AdS}}^2}{r^2}\left[-f(r)dt^2 + f^{-1}(r)dr^2 + (dx^j)^2\right] \tag{12}$$

$$f(r) = 1 - (r/r_H)^{d+1} \tag{13}$$

- Holographically, this is dual to a thermal state on the boundary ρ_β
- As in asymptotically Minkowski spacetime, this can be extended to a two-side black hole



- The two-side black hole is dual the thermofield double state

Quantum Information Scrambling: Boulder Lectures, Brian Swingle

The Thermofield Double State

- The thermofield double state is

$$|\text{TFD}\rangle = \frac{1}{Z^{1/2}} \sum_n e^{-\beta E_n/2} |\tilde{n}\rangle_L |n\rangle_R \tag{14}$$

- It is an entangled state of two systems R and L such that the reduced density operators are thermal

$$\rho_{TFD} = |\text{TFD}\rangle\langle\text{TFD}| \tag{15}$$

$$\rho_R = Tr_L[\rho_{TFD}] = \frac{1}{Z} \sum_n e^{-\beta E_n} |n\rangle \tag{16}$$

- Holographically, it is dual to two entangled black holes

$$ds^2 = \frac{\ell_{\text{AdS}}^2}{r^2} \left[-f(r)dt^2 + f^{-1}(r)dr^2 + (dx^j)^2 \right] \tag{17}$$

$$f(r) = 1 - (r/r_H)^{d+1} \tag{18}$$

TFD, OTOCs and the Geometric Dual

- Operator pushing property:

$$\mathcal{O}_R|\mathrm{TFD}\rangle = \left(e^{\beta H/2}\mathcal{O}^T e^{-\beta H/2}\right)_L |\mathrm{TFD}\rangle \tag{19}$$

- Perturb one side (L) of the TFD:

$$|W\rangle \equiv W(-t)_L|\mathrm{TFD}\rangle \tag{20}$$

- Perturb both sides with a different operator

$$\langle W|V_L \otimes V_R^T|W\rangle_{T=\mathrm{inf}} = \langle TFD|W^\dagger(-t_W)V_L W_L(-t_W) \otimes V_R^T|TFD\rangle \tag{21}$$

$$= \langle TFD|W^\dagger(-t_W)V_L W_L(-t_W)V_L|TFD\rangle \tag{22}$$

OTOCs and Shockwaves

- An infalling particle at time -t_w accelerates to the speed of light as it approaches the horizon.
- For a Black Hole it turns out that the momentum of this particle is equivalent to the growth of operator $W(-t_W)$

$$p \approx e^{\tau} \tag{23}$$

$$size(W) = |W| \approx e^{\frac{2\pi}{\beta}t} \tag{24}$$

where

$$\lambda_L = \frac{2\pi}{\beta} \tag{25}$$

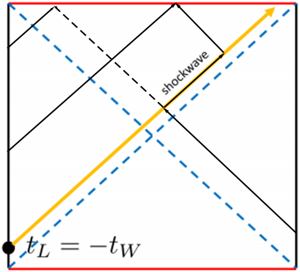
- τ is the Rindler time given by the metric of a non-extremal black hole which is related to the time on the boundary by

$$t = \frac{\beta}{2\pi}\tau \tag{26}$$

Why do Things Fall?, Susskind, arxiv:1802.01198

OTOCs and Shockwaves

- This warps the geometry of the black hole causing a "shockwave".
- We can then "patch together" the two sides of the wormhole, but the perturbed side's shockwave elongates the throat of the wormhole, decreasing the correlation between the two sides of the TFD



Black Holes and the Butterfly Effect, Shenker and Stanford, arxiv:1306.0622

The Black Hole Information Problem

- In 1971, Bekenstein proposed that black holes have entropy

$$S = \frac{k_B A}{4\ell_p^2} \tag{27}$$

- In 1974, Hawking showed that black holes emit thermal radiation with temperature

$$T = \frac{\hbar\kappa}{2\pi k_B} \tag{28}$$

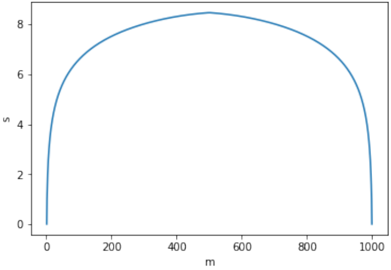
- Black holes evaporate! Do they destroy information?!
- Hawking-Thorne-Preskill bet
- Evidence from string theory that black hole evaporation is unitary
  - In 1996, Strominger and Vafa provided a microscopic origin for the Bekenstein-Hawking entropy
  - In 1997, Maldacena established the AdS/CFT correspondence

The Page Curve

- In 1993, Page showed that if one subsystem is m dimensional and the other is n dimensional them the average entropy is

$$S \approx \ln m - \frac{m}{2n} \quad \text{for} \quad m << n \tag{29}$$

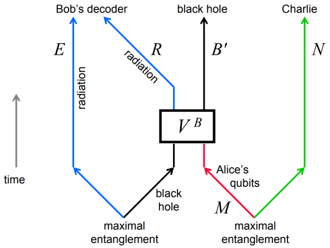
- Before the evaporation process is halfway over no information has escaped



D. N Page, "Average entropy of a subsystem,"

The Hayden-Preskill Thought Experiment

- Suppose Alice would like to hide some information from Bob by tossing it into a black hole BUT Bob has been collecting all of the emitted radiation from the black hole



- How quickly can bob recover the information Alice has attempted to hide?

Hayden and Preskill, Black Holes as Mirrors, JHEP 2007

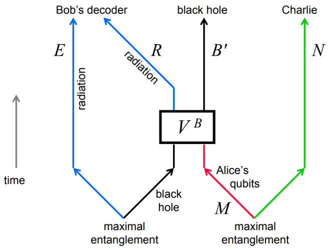
The Decoupling Principle

- It is possible for Bob to recover a state maximally entangled with Charlie's state once the black hole is decoupled from the radiation

$$\rho_{NB'} \approx \rho_N \otimes \rho_{B'} \tag{30}$$

- This is equivalent to the mutual information between Charlie and the Hawking radiation reaching its maximal value

$$I^{(1)}(N : ER) \equiv S_N + S_{ER} - S_{NER} \approx 2 \log \dim N \tag{31}$$



Entropies and Mutual Informations

- The nth Rényi entropy is defined by

$$S^{(n)}(\rho) = \frac{1}{1-n} \log \mathrm{Tr}[\rho^n] \tag{32}$$

- The von Neumann entropy can be recovered as a special case

$$S(\rho) \equiv S^{(1)}(\rho) \equiv \lim_{n\to 1} \frac{1}{1-n} \log \mathrm{Tr}[\rho^n] \tag{33}$$

- An important inequality satisfied by these entropies is

$$S^{(n)}(\rho) \geq S^{(n+1)}(\rho) \tag{34}$$

- Rényi mutual informations can also be defined

$$I^{(n)}(A:B) = S_A^{(n)} + S_B^{(n)} - S_{AB}^{(n)} \tag{35}$$

Scrambling and Decoupling

- It is convenient to average the OTOC over an operator basis for Charlie's (N_j) system and for the new radiation (R_j)

$$\langle \mathcal{O}_N \mathcal{O}_R(t) \mathcal{O}_N \mathcal{O}_R(t) \rangle_{\text{ave}} \equiv \frac{1}{2(\dim N + \dim R)} \sum_{j,k} \text{Tr}[N_j R_k(t) N_j R_k(t)]$$

- If the black hole is old (so it is maximally entangled with the radiation) it can be shown

$$\langle \mathcal{O}_N \mathcal{O}_R(t) \mathcal{O}_N \mathcal{O}_R(t) \rangle_{\text{ave}} = 2^{-I^{(2)}(N:ER)} \tag{36}$$

- Since black holes are the fastest possible scramblers $I^{(2)}(N : ER)$ must be large!
- But wait... decoding depended on the size of $I^{(1)}(N : ER)$...

Hosur, Qi, Roberts and Yoshida, Chaos in quantum channels, JHEP 2016

Second Rényi Mutual Information Implies Decoupling

- If ρ is maximally mixed the first and second Rényi entropies are equal

$$S^{(1)}(\rho) = \log d = S^{(2)}(\rho) \tag{37}$$

- Charlie's system and the remaining black hole are in maximally mixed states so

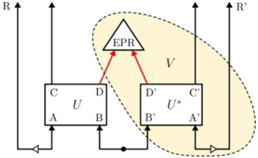
$$\begin{aligned} I^{(2)}(N:ER) &= S^{(2)}(\rho_N) + S^{(2)}(\rho_{ER}) - S^{(2)}(\rho_{NER}) && (38) \\ &= S^{(2)}(\rho_N) + S^{(2)}(\rho_{ER}) - S^{(2)}(\rho_{B'}) && (39) \\ &= S^{(1)}(\rho_N) + S^{(2)}(\rho_{ER}) - S^{(1)}(\rho_{B'}) && (40) \\ &\leq I^{(1)}(N:ER) && (41) \end{aligned}$$

- Black holes can be quantum information mirrors!

Hosur, Qi, Roberts and Yoshida, Chaos in quantum channels, JHEP 2016

Probabilistic Decoding

- Step 1: Bob prepares a maximally entangled state like the one Alice and Charlie shared and 'simulates' the black hole
- Step 2: Bob preforms a 'Bell' measurement on the physical and 'simulated' new radiation



- Bob postselects on the outcome

$$|\text{EPR}\rangle_{DD'} = \frac{1}{\sqrt{\dim D}} \sum_m |m\rangle_D |m\rangle_{D'} \tag{42}$$

which occurs with probability at least $(\dim A)^{-2}$

- The state of R and R' are now have fidelity ≈ 1 with $|\text{EPR}\rangle_{RR'}$ and Bob has successfully decoded

Beni Yoshida and Alexei Kitaev, Efficient decoding for the Hayden-Preskill protocol, arxiv: 1710.03363

Measuring OTOCs in the Lab

- Expanding the OTOC

$$C = -\langle [W(t), V(0)]^2 \rangle$$

$$= \langle W(t)VVW(t)\rangle + \langle VW(t)W(t)V\rangle$$
$$\text{-}\langle VW(t)VW(t)\rangle - \langle W(t)VW(t)V\rangle (43)$$

- The second line requires backwards time evolution in order to perform a measurement which is only possible in certain systems

$$|\psi\rangle_1 = VW(t)|\psi\rangle \tag{44}$$

$$|\psi\rangle_2 = W(t)V|\psi\rangle \tag{45}$$

- Instead, we will see how to use NTOCs on the TFD to measure OTOCs on a single side of the TFD

TFDs and OTOCs

- To prepare the TFD we prepare two identical Hamiltonians and couple them with a small interaction term:

$$H = H_L + H_R + H_{int} \tag{46}$$

- The ground state of this Hamiltonian is approximately the TFD
- NTOC on the double system:

$$F(t, t') = \langle T[V_L(t) W_R(t) V_R(t') W_L(t')] \rangle_{TFD} \tag{47}$$

$$F(t, t') = \frac{1}{Z_\beta} \sum_{n,m} e^{-\beta(E_n+E_m)/2} \langle \tilde{n}| V_L(t) W_L(t') |\tilde{m}\rangle_L \\ x \langle n| W_R(t) V_R(t') |m\rangle_R \tag{48}$$

How to Build the Thermofield Double, Cottrell et al.,arxiv:1811.11528

TFD and OTOCs Cont.

- We note that the because the subsystems L and R are identical, we can drop the subscripts:

$$F(t,t') = \frac{1}{Z_\beta}\sum_{n,m} e^{-\beta(E_n+E_m)/2}\langle m|W(-t')V(-t)|n\rangle \\ x\langle n|W(t)V(t')|m\rangle \tag{49}$$

- We have used the fact that:

$$\langle \tilde{n}|O(t)|\tilde{m}\rangle = \langle m|O(-t)|n\rangle \tag{50}$$

Diagnosing QC in Many-Body Systems Using Entanglement, Franz et al., arxiv:1907.01628

TFDs and OTOCs Cont.

- Now, we want to use the completeness relation so we note that:

$$e^{-\beta(E_n+E_m)/2}\langle m|W(t)V(t')|n\rangle = Z_\beta\langle n|\rho_\beta^{\frac{1}{2}}W(t)V(t')\rho_\beta^{\frac{1}{2}}|m\rangle \quad (51)$$

where,

$$\rho_\beta = e^{-\beta H}/Z_\beta \quad (52)$$

- Then, we have:

$$F(t,t') = Tr[W(-t')V(-t)\rho_\beta^{\frac{1}{2}}W(t)V(t')\rho_\beta^{\frac{1}{2}}] \quad (53)$$

- Setting t' = -t, we recover a "regularized" OTOC on one side of the TFD. Note that this trace is only over a single side:

$$F(t,t') = Tr[W(2t)V(0)\rho_\beta^{\frac{1}{2}}W(2t)V(0)\rho_\beta^{\frac{1}{2}}] \quad (54)$$

Why does this work?

- The TFD has the property:

$$(H_L - H_R)|TFD\rangle = 0 \tag{55}$$

meaning

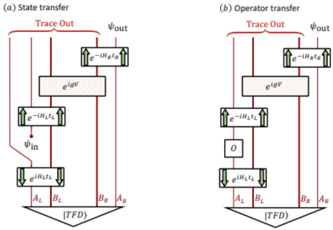
$$e^{-it(H_L - H_R)}|TFD\rangle = |TFD\rangle \tag{56}$$

and for the expectation value of an operator on both sides:

$$\langle O_L(t_1)O_R(t_2)\rangle_{TFD} = \langle O_L(t_1 + t)O_R(t_2 - t)\rangle_{TFD} \tag{57}$$

Quantum Gravity in the Lab

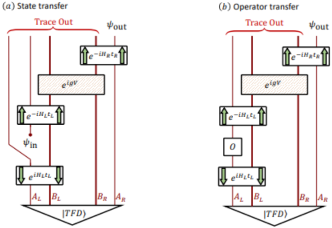
- First we roll back time on subsystem L, input our message and roll time forward. This scrambles the message amongst subsystem L
- Then we couple the two sides with a small coupling term, $V = \frac{1}{(n-m)} \sum_{i \in carrier} (\sigma_z)_i^L (\sigma_z)_i^R$
- Finally, we roll subsystem R forward and the message reappears on ψ_{out}



Quantum Gravity in the Lab, Brown et al., https://arxiv.org/pdf/1911.06314.pdf

Quantum Gravity in the Lab

- The surprise here is not that we can teleport the information, but that the experiment scrambles and unscrambles the message. Given the right parameters, all we have to do is apply a Y gate to recover the message
- The simplest way to explain this is to imagine the message passed through a wormhole made traversable by the coupling term e^{igV}



Quantum Gravity in the Lab

- First we note that the TFD is slightly different in this case:

$$|\text{TFD}\rangle = \frac{1}{Z^{1/2}} \sum_n e^{-\beta E_n/2} |n\rangle_L |\bar{n}\rangle_R \tag{58}$$

- Operator Pushing Property:

$$\frac{1}{2^{\frac{N}{2}}} O_L^T(-t)|TFD\rangle = \rho_{\beta}^{\frac{1}{2}}{}_R O_R(t)|\phi^+\rangle \tag{59}$$

where,

$$\rho_\beta = e^{-\beta H} \tag{60}$$

and $|\phi^+\rangle$ is the maximally entangled state.

Quantum Gravity in the Lab

- Expand $\rho_{\beta}^{\frac{1}{2}}{}_R O_R(t)$ in the Pauli basis:

$$\rho_{\beta}^{\frac{1}{2}}{}_R O_R(t) = \frac{1}{2^{\frac{N}{2}}} \sum_P c_P P = 2^{-\frac{N}{2}} \sum_P e^{i\alpha|P|/n} r_P P \tag{61}$$

if we let,

$$|P\rangle_{LR} = P_R|\phi^+\rangle \tag{62}$$

then,

$$O_L^T(-t)|TFD\rangle \approx \sum_P e^{i\alpha|P|} r_P |P\rangle \tag{63}$$

$$O_R(t)|TFD\rangle \approx \sum_P e^{-i\alpha|P|} r_P |P\rangle \tag{64}$$

Quantum Gravity in the Lab

- We can make a similar approximation for e^{igV}

$$e^{igV} \approx e^{-i\frac{4}{3}g|P|/n}|P\rangle \tag{65}$$

- Then we can expand the full action of the coupling and operator O_L^T:

$$e^{igV}O_L^T \approx \sum_P e^{i(\alpha-\frac{4}{3}g)|P|/n} r_P|P\rangle \tag{66}$$

- And now, by choosing the right value of g, we can teleport O_L^T to the other side in the form of O_R.

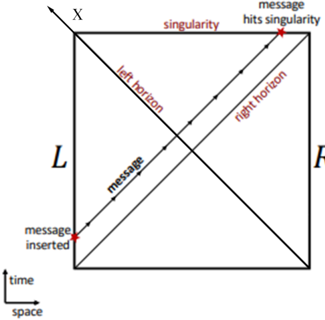
Quantum Gravity in the Lab

- Draw an x axis and look at the particle in momentum space:

$$\psi(P) \approx e^{iP|X_0|} \tag{67}$$

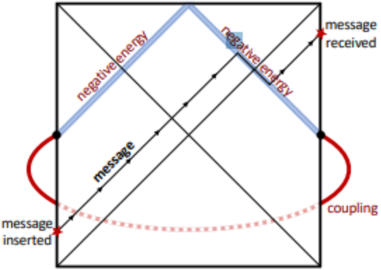
- The question then becomes, how do we move the particle into the -X quadrant?

$$e^{iP|X_0|} \rightarrow e^{-iP|X_0|} \tag{68}$$



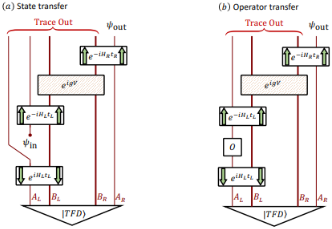
Quantum Gravity in the Lab

- In the QGitL experiment, this coupling is given by the term e^{igV} and we can think of it as a special case of quantum teleportation
- In the geometric picture, this coupling creates a negative energy shockwave that imparts a time advance to whatever it encounters, taking the particle from momentum $e^{iP|X_0|}$ to $e^{-iP|X_0|}$ and effectively crossing the wormhole to the other side of the TFD



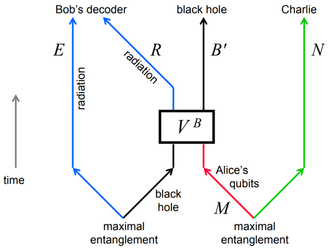
What's the point?

- For small systems there is probably no emergent geometry
- We can also imagine it would be possible to design a similar experiment in which the geometry of the "wormhole" would warp the message and if the message output is warped in this way, it would imply that there is an emergent geometry and a wormhole through which the message traveled.
- Whether or not there is any emergent geometry in quantum systems, the ADS/CFT correspondence seems to be a fruitful perspective in designing experiments and understanding quantum systems



Overview and Questions

- There is bound on how quickly squared commutators can grow $\lambda_L \leq \frac{2\pi}{\beta}$
- Black holes saturate this bound (they are fast scramblers!)
- As a result black holes can act as quantum information mirrors
- We've seen two experiments we can run in the near future with the Thermofield Double
- Questions?



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